Perturbations of Hydrodynamic Attractors

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NCBJ

Motivation

QGP evolution starts far from equilibrium

- Characteristics of heavy-ion collisions:

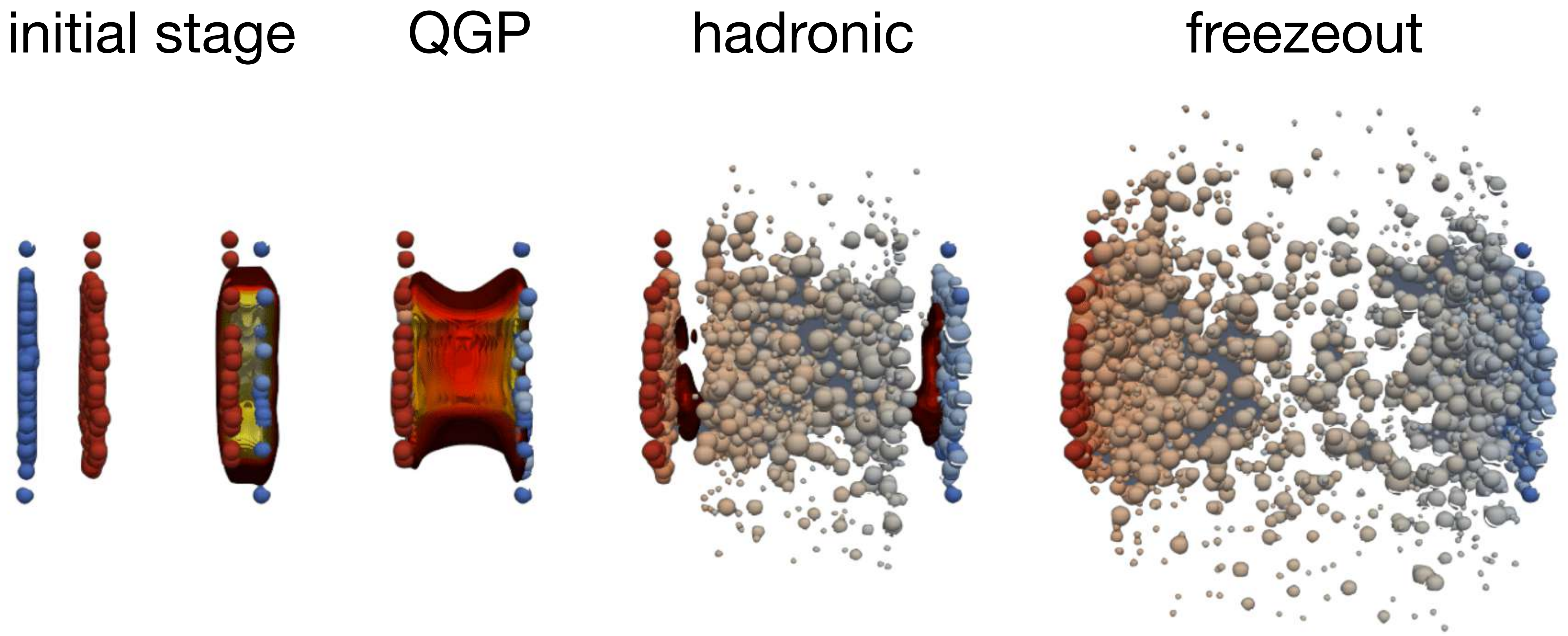
Speed ~ 1 fast

Energy $\sim 10 - 10^4$ GeV high

Collision time $\sim 0.01 - 1$ fm short

Size ~ 10 fm small

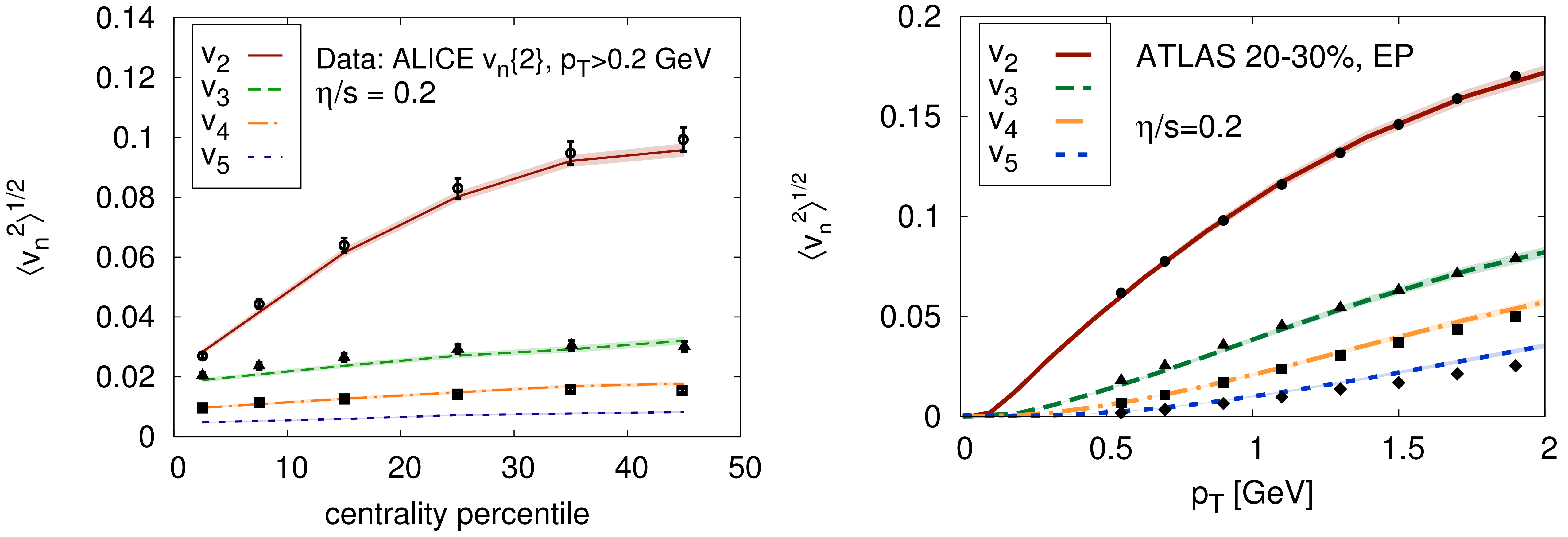
Particles $\sim 10^2 - 10^4$ few



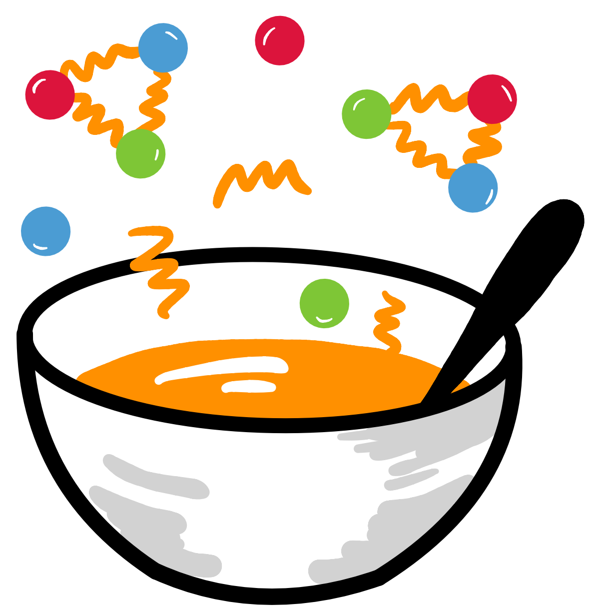
History of the little bang

QGP is well described by hydrodynamics

- Flow collectivity manifests QGP as a *nearly perfect fluid*.

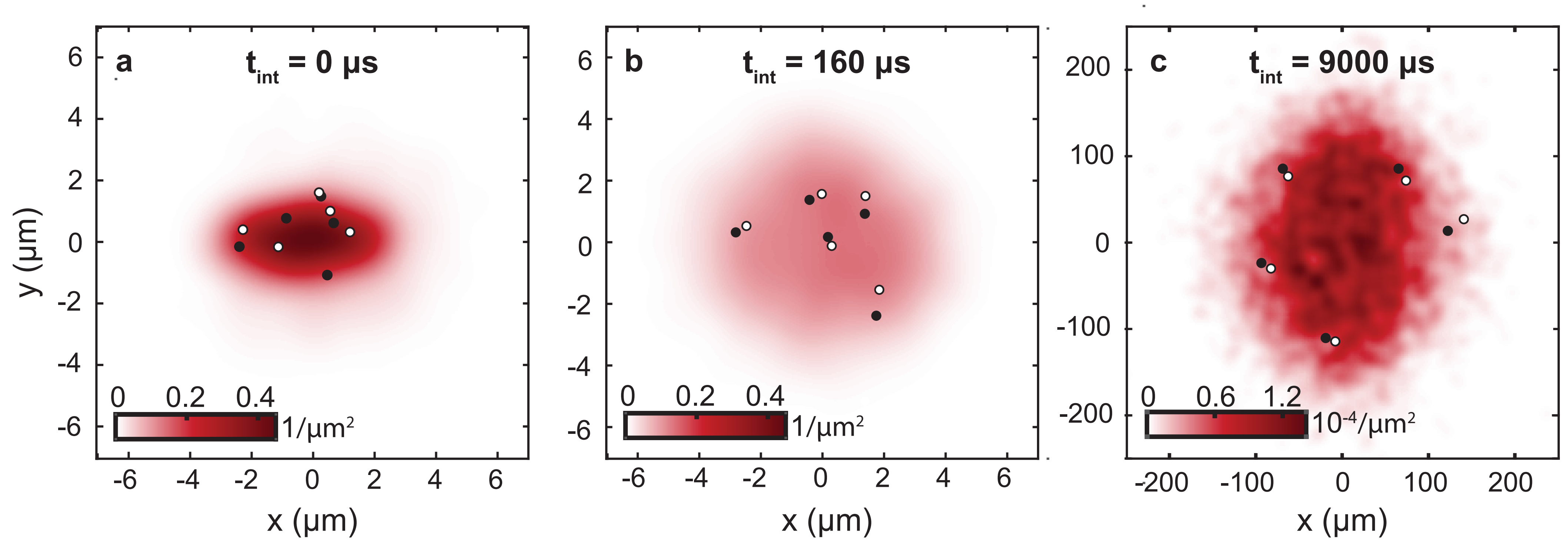


Gale et al, 1301.5893



Hydrodynamics is believed to be applicable near equilibrium

- And even more:



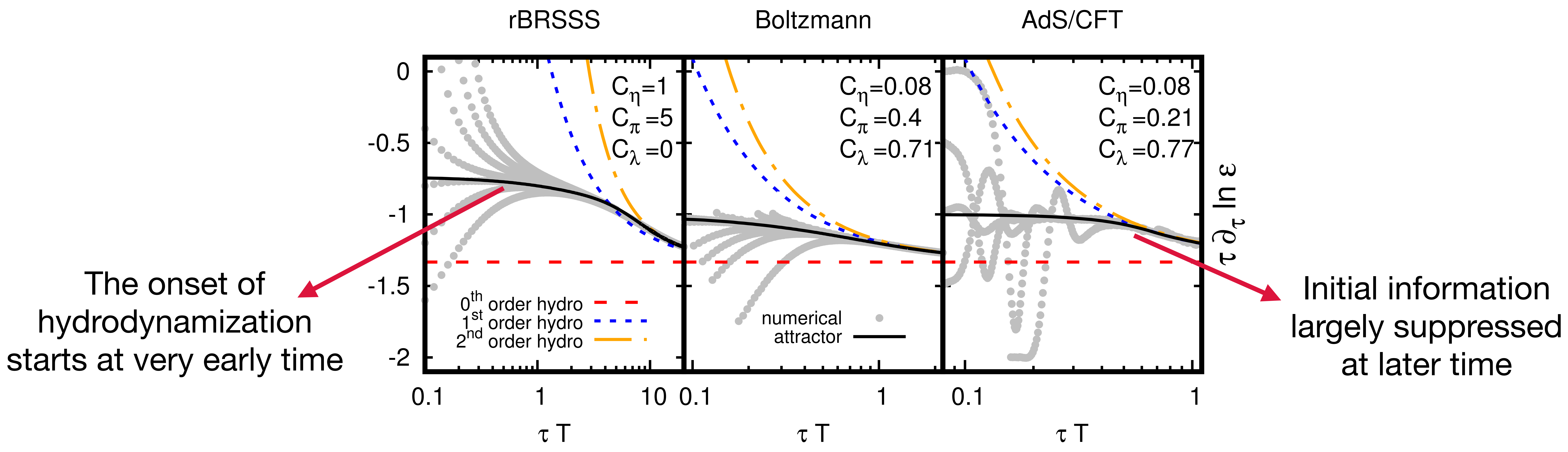
Density distribution in position space

5-particle hydrodynamics

Brandstetter et al, 2308.09699

Hydrodynamic attractor

- *Attractor* plays an important role to explain the success of hydrodynamics even far from equilibrium.



Florkowski et al, 1707.02282, Romatschke, 1712.05815

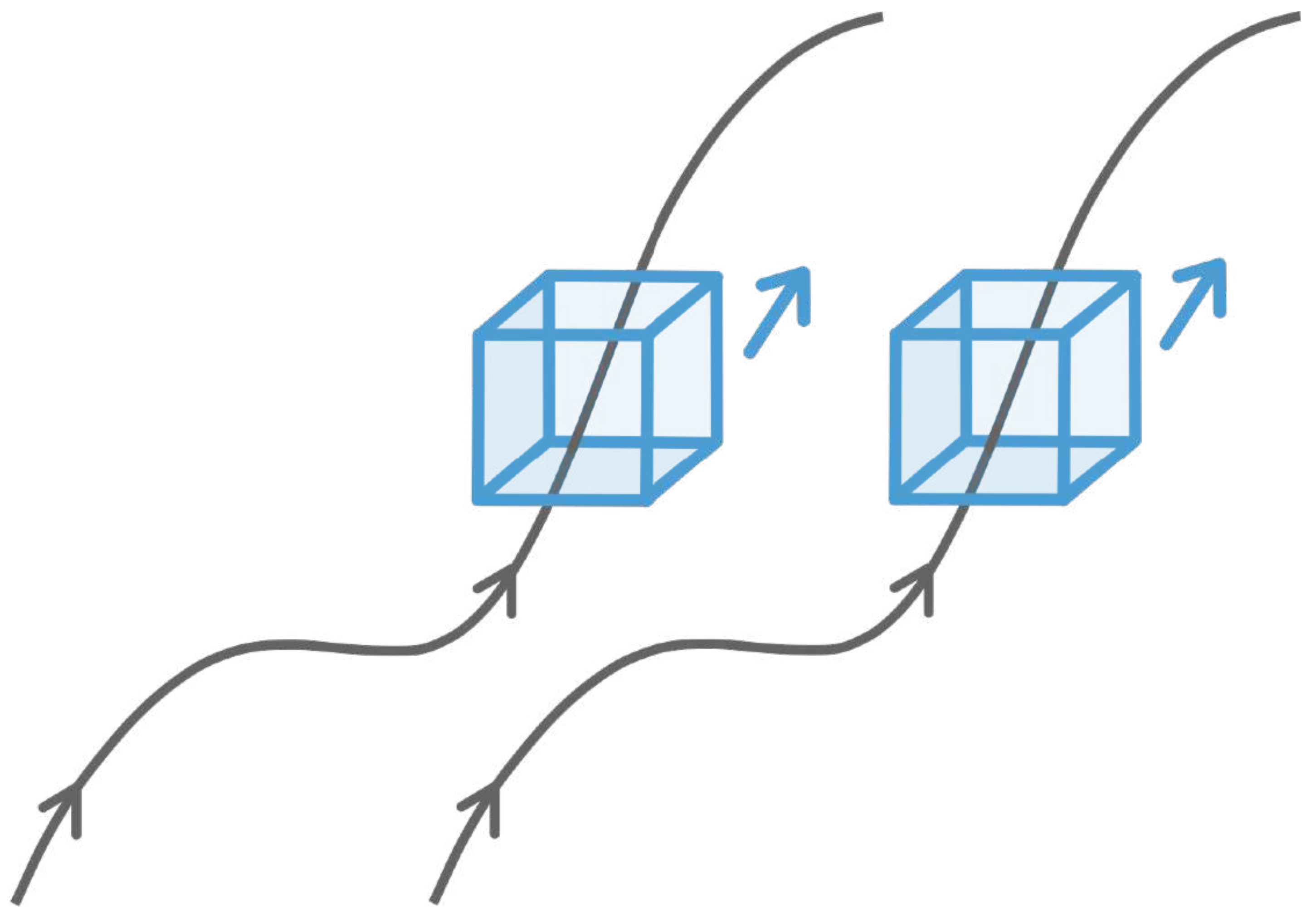
- Does attractor wash out everything? Does attractor exist with less symmetries? Can we understand it better analytically?

Attractors

Fluids in equilibrium: Euler equation

- Stress tensor is homogeneous in LRF.

$$T^{\mu\nu}_{(0)\mathrm{LRF}} = \begin{pmatrix} \varepsilon & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \xrightarrow{\text{boost}} T^{\mu\nu}_{(0)} = \varepsilon u^\mu u^\nu + p\Delta^{\mu\nu}$$



- Euler equation:

$$\partial_\mu T^{\mu\nu}_{(0)} = 0 \quad \Longrightarrow \quad \partial_t \psi = \nabla \cdot J_{(0)}[\psi] \quad \text{where} \quad \psi = (\,\mathrm{n}, \varepsilon, \pi, \ldots)$$

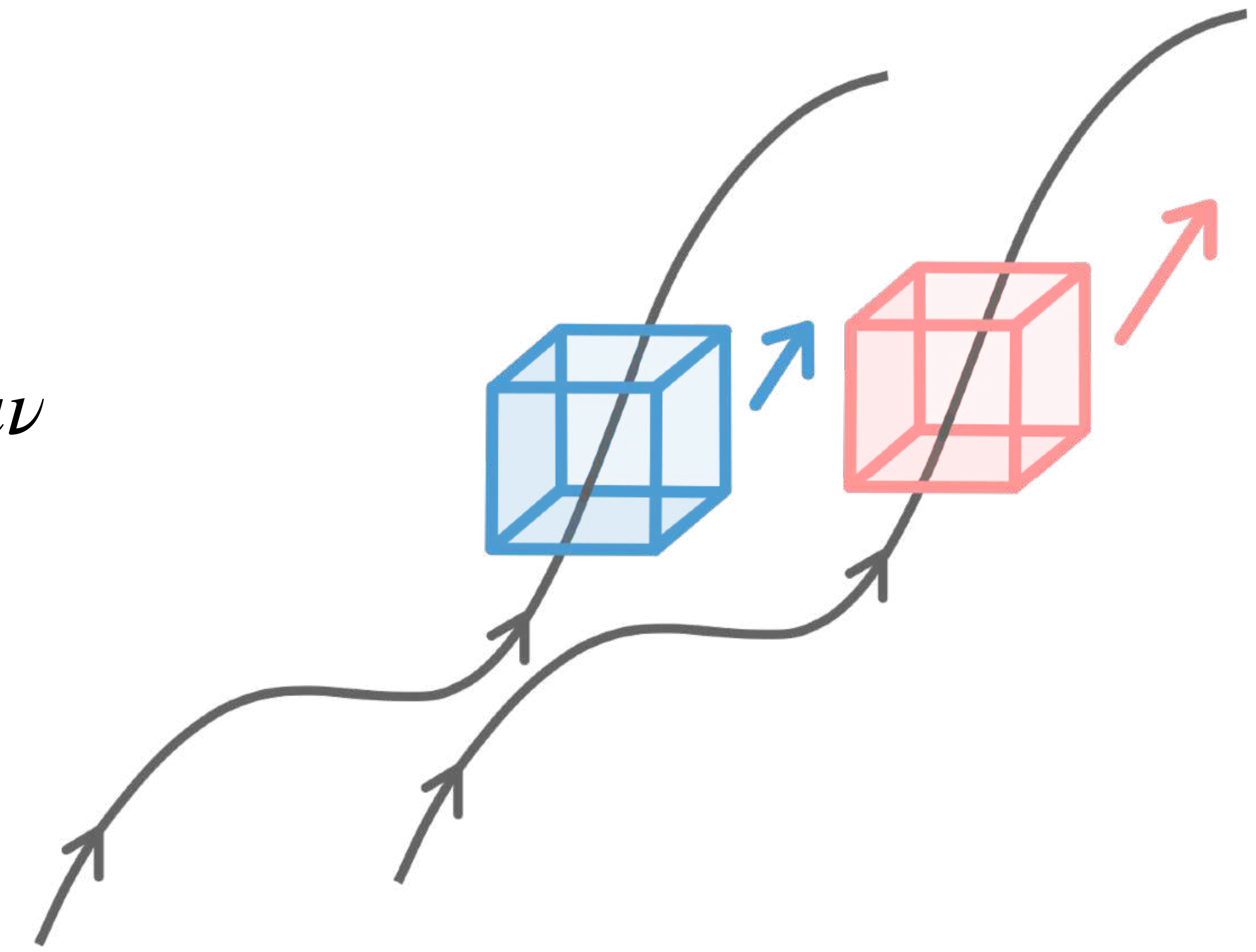
Conserved quantities evolve via advection & expansion.

Fluids near equilibrium: NS-like equations

- Stress tensor approximated by gradient expansion.

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + \dots$$

$$T^{\mu\nu}_{(1)} = -2\eta\sigma^{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \frac{1}{3}\partial \cdot u \Delta^{\mu\nu}$$



NB: there are infinite many equilibrium proxies for a non-equilibrium state.

- Navier-Stokes(NS)-like (e.g., Burnett, BRSSS, etc.) equations:

$$\partial_\mu T^{\mu\nu} = 0 \quad \Longrightarrow \quad \partial_t \psi = \nabla \cdot J[\psi, \nabla\psi, \dots] \quad \text{where} \quad \psi = (\mathrm{n}, \varepsilon, \pi, \dots)$$

Conserved quantities evolve via advection & expansion, as well as dissipation & diffusion.

Fluids far from equilibrium: MIS-like equations

- Stress tensor involves non-hydrodynamic DOFs for UV completion.
  E.g., 0+1D boost-invariant conformal fluids:

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + \pi^{\mu\nu} + \ldots = \begin{pmatrix} \varepsilon & & & \\ & p_T & & \\ & & p_T & \\ & & & p_L \end{pmatrix}$$

$$p_T = p + \pi_T = p - \pi^\eta_\eta/2,$$
$$p_L = p + \pi^\eta_\eta$$

NB: π^η_η vanishes in equilibrium

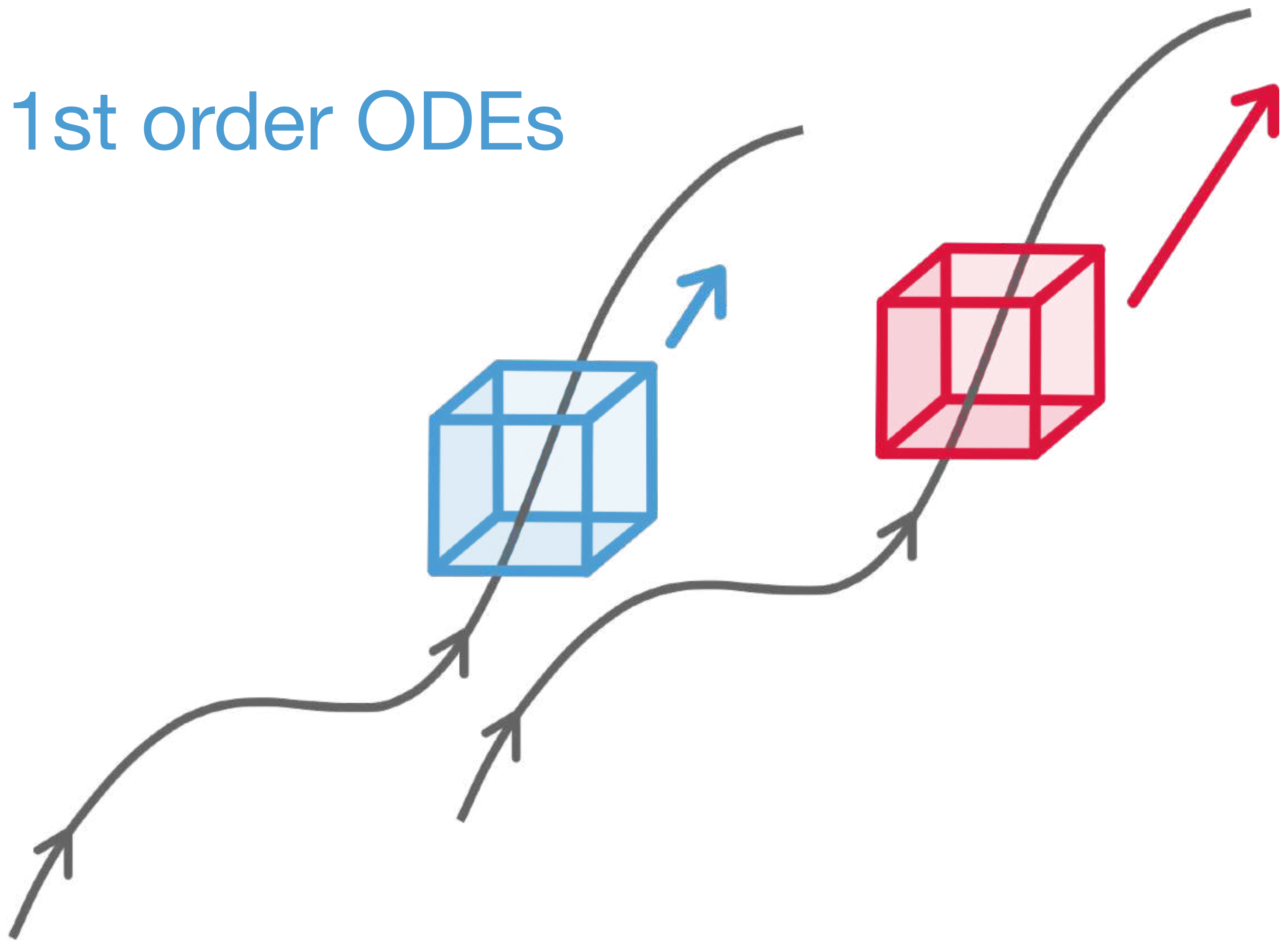
- Muller-Israel-Stewart(MIS)-like (e.g., Maxwell-Cattaneo, DNMR, BDNK etc.) equations:

$$\partial_\mu T^{\mu\nu} = 0 \quad \Longrightarrow \quad (\tau\partial_\tau + 1)\varepsilon + p + \pi^\eta_\eta = 0$$

coupled 1st order ODEs

MIS $$\left(\tau_\pi \partial_\tau + 1 + \frac{4\tau_\pi}{3\tau}\right)\pi^\eta_\eta + \frac{4\eta}{3\tau} = 0$$

where $\varepsilon = 3p = C_e T^4, \eta = \frac{4}{3} C_e C_\eta T^3, \tau_\pi = C_\tau T^{-1}$.

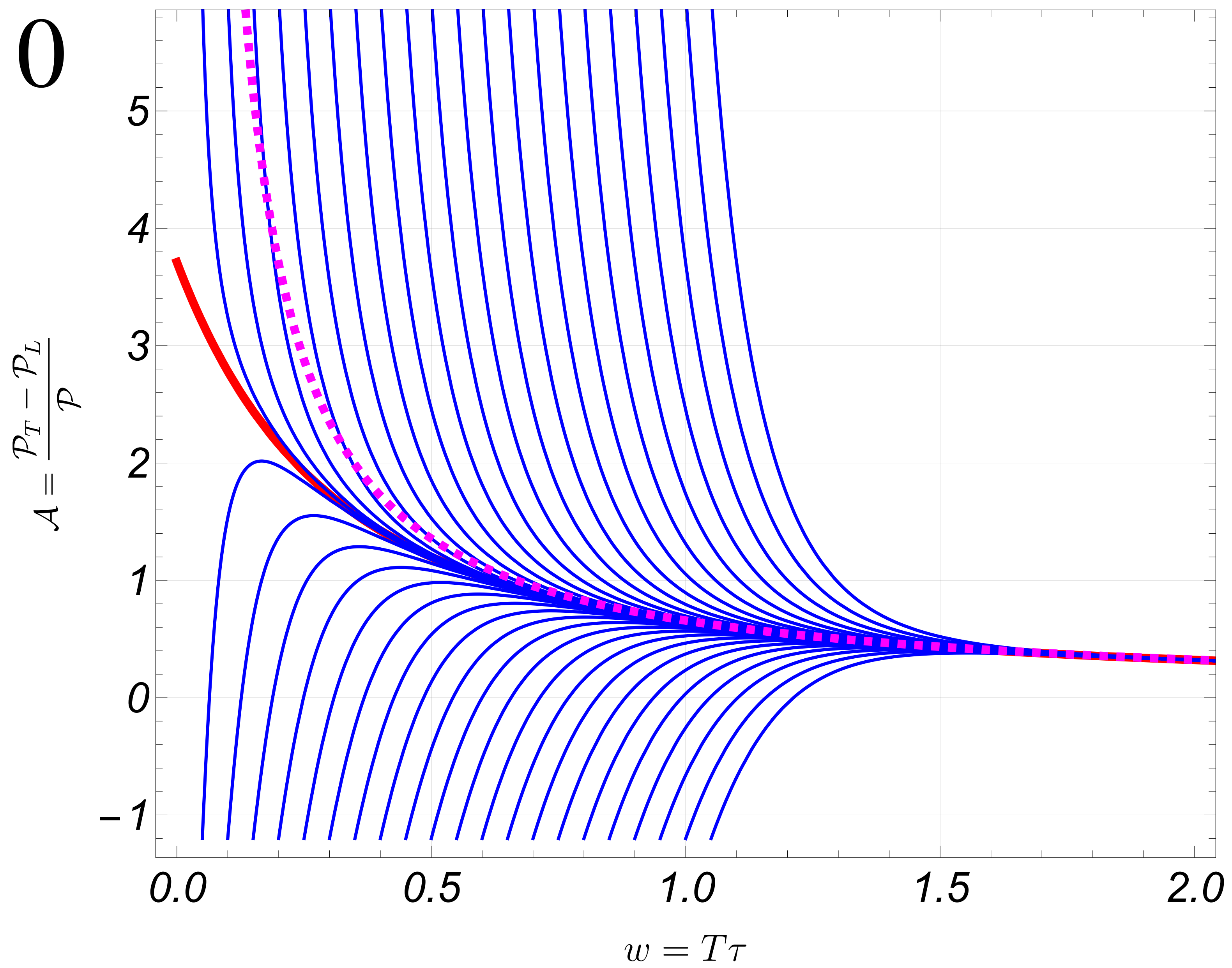


Hydrodynamic attractors

- In terms of $w = \tau T$, equation for pressure anisotropy $A(w) \equiv (P_T - P_L)/P$ decouples:

$$C_\tau \left(1 + \frac{A(w)}{12}\right) wA'(w) + \frac{1}{3} C_\tau A(w)^2 + \frac{3}{2} wA(w) - 12C_\eta = 0$$

decoupled 1st order ODE



with asymptotic solutions

$$A(w) = \frac{C_0}{w^4}\left(1 + \mathcal{O}(w)\right) + 6\sqrt{C_\eta / C_\tau} + \mathcal{O}(w), \quad w \to 0$$

longitudinal expansion dominates + early time attractor

Heller et al, 1503.07514; Jankowski et al, 2303.09414

$$A(w) = \frac{8C_\eta}{w}\left(1 + \frac{2C_\tau}{3w} + \mathcal{O}(w^{-2})\right) + C_\infty e^{-\frac{3w}{2C_\tau}} w^{\frac{C_\eta}{C_\tau}} \left(1 + \mathcal{O}(w^{-1})\right) + \ldots, \quad w \to \infty$$

hydrodynamic attractor + non-hydrodynamic (transseries) modes.

Alternative formulation of attractors

- In the presence of additional scales other than T, τ is more convenient as dynamic variable than $w = \tau T$.

two coupled 1st order ODEs
(one 2nd order ODE)

$$\tau T'(\tau) + T(\tau)\left(\frac{1}{3} - \frac{A(\tau)}{18}\right) = 0$$

$$C_\tau \tau A'(\tau) + \frac{2}{9} C_\tau A(\tau)^2 + \tau T(\tau) A(\tau) - 8 C_\eta = 0$$

- System of n coupled linear ODEs ⟶ one nth order ODE:

$$\tau T''(\tau) + \frac{3\tau T'(\tau)^2}{T(\tau)} + \left(\frac{11}{3} + \frac{\tau T(\tau)}{C_\tau}\right) T'(\tau) + \frac{T(\tau)^2}{3C_\tau} + \frac{4}{9\tau}\left(1 - \frac{C_\eta}{C_\tau}\right) T(\tau) = 0$$

similar equation can be obtained for $A(\tau)$

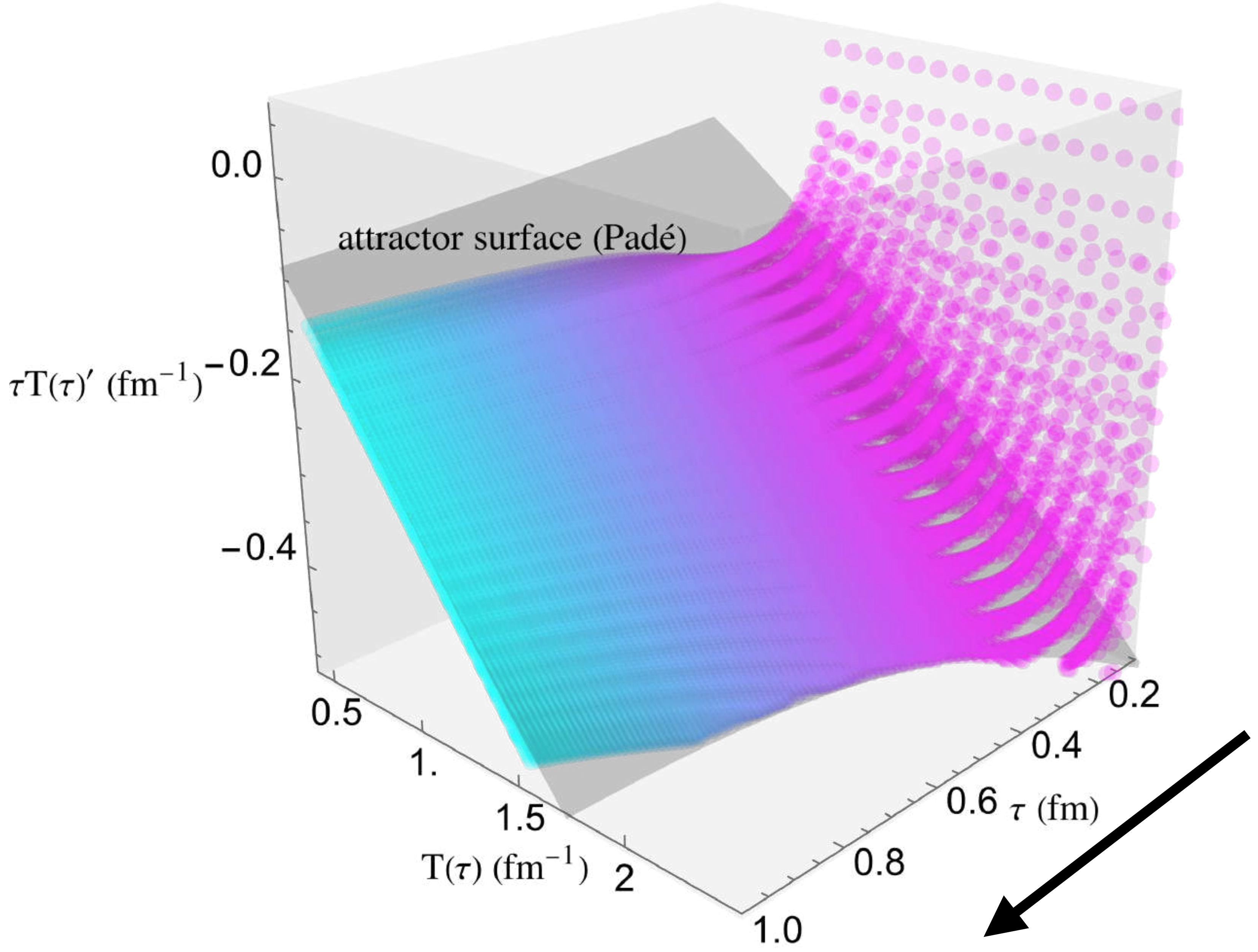
Early-time attractor

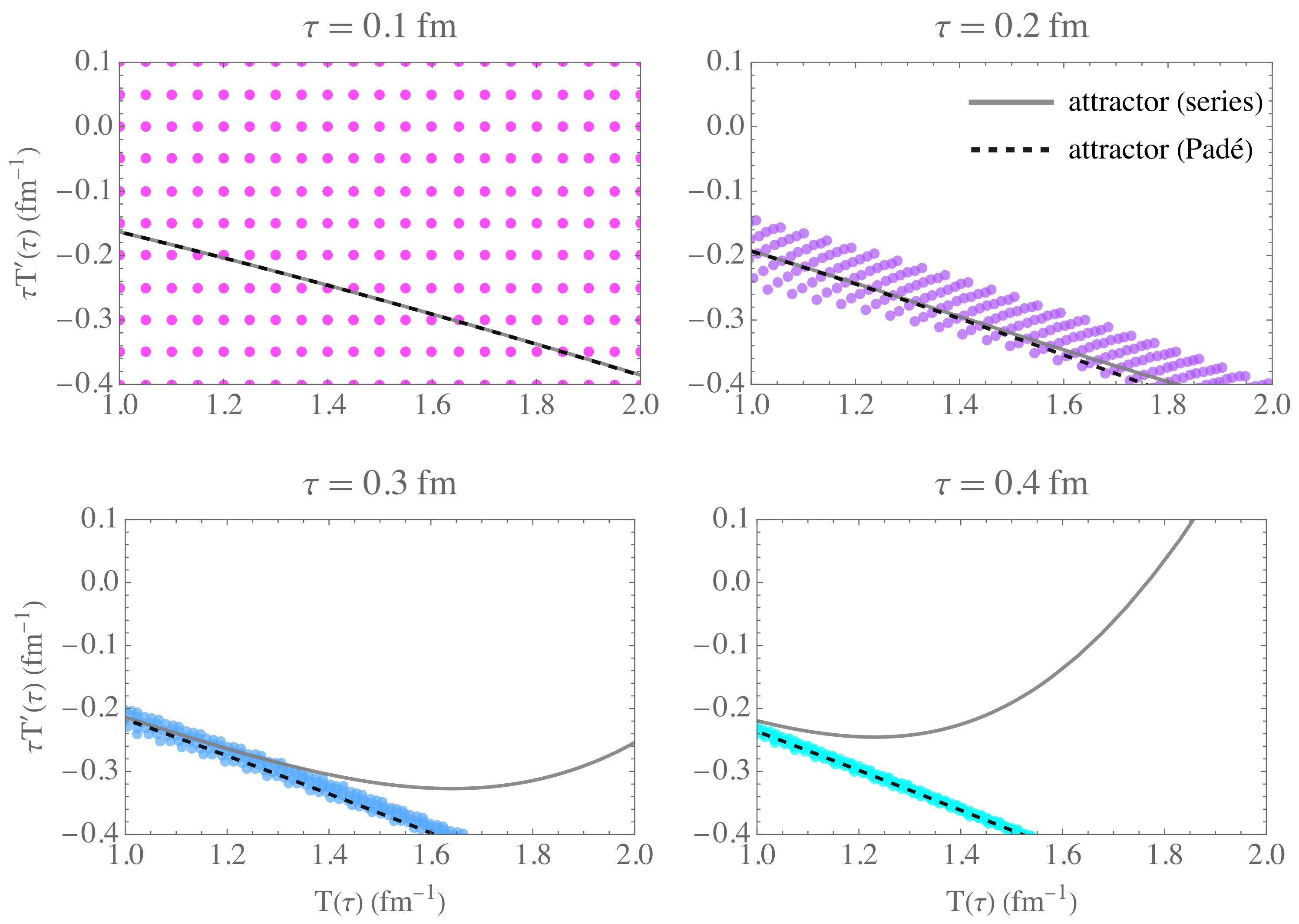
- Early-time *attractor* solutions:

μ: integration constant; $\alpha = \sqrt{C_\eta / C_\tau}$

$$T(\tau) \sim \mu(\mu\tau)^{-\frac{1-\alpha}{3}}\left(1 + \sum_{n=1}^{\infty} t_n (\mu\tau)^{\frac{n}{3}(2+\alpha)}\right), \qquad A(\tau) \sim 6\alpha\left(1 + \sum_{n=1}^{\infty} a_n (\mu\tau)^{\frac{n}{3}(2+\alpha)}\right)$$

- Generic solutions rapidly approach the *attractor surface* in phase space $(\tau T', T, \tau)$ at early time.





snapshot of $(\tau T', T)$ plane at different τ

Later-time attractor

- Later-time asymptotic solutions

$$T(\tau) \sim \Lambda(\Lambda\tau)^{-\frac{1}{3}}\left(1+\sum_{n=1}^{\infty} t_n(\Lambda\tau)^{-\frac{2}{3}n}\right) + C_\infty(\Lambda\tau)^{-\frac{2}{3}(1-\alpha^2)} e^{-\frac{3}{2C_\tau}(\Lambda\tau)^{2/3}}\left(1+\mathcal{O}((\Lambda\tau)^{-2/3})\right) + \dots$$

$$A(\tau) \sim 8C_\eta(\Lambda\tau)^{-\frac{2}{3}}\left(1+\sum_{n=1}^{\infty} a_n(\Lambda\tau)^{-\frac{2}{3}n}\right) + C'_\infty(\Lambda\tau)^{-\frac{1}{3}+\alpha^2} e^{-\frac{3}{2C_\tau}(\Lambda\tau)^{2/3}}\left(1+\mathcal{O}((\Lambda\tau)^{-2/3})\right) + \dots$$

hydrodynamic attractor + non-hydrodynamic (transseries) modes.

Λ, C_∞: independent integration constant

The suppression is mild since the typical $\tau \sim 10$ fm in HIC is not large.

Perturbations

Linearized modes

- Linearization of MIS theory around the attractor for 6 independent fields:

$$(\delta T, \delta\theta, \delta\omega, \delta\pi_{11}, \delta\pi_{22}, \delta\pi_{12})(\tau, \mathbf{x})$$

where $\delta\theta \equiv \partial_i \delta u_i$ and $\delta\omega \equiv \epsilon_{ij}\partial_i \delta u_j$, $i = 1,2$.

The translation invariance symmetry in transverse plane is broken.

- The dynamic system is reduced to a set of linear 2nd order ODEs for $\phi(\tau, \mathbf{x}) = (\delta T, \delta\theta, \delta\omega)(\tau, \mathbf{x}) \longrightarrow \hat{\phi}(\tau, \mathbf{k}) = (\delta\hat{T}, \delta\hat{\theta}, \delta\hat{\omega}) \equiv (\delta T/T, \delta\theta/k, \delta\omega/k)(\tau, \mathbf{k})$:

$$\hat{\phi}''(\tau, \mathbf{k}) + P_1(\tau, \mathbf{k})\hat{\phi}'(\tau, \mathbf{k}) + P_0(\tau, \mathbf{k})\hat{\phi}(\tau, \mathbf{k}) = 0$$

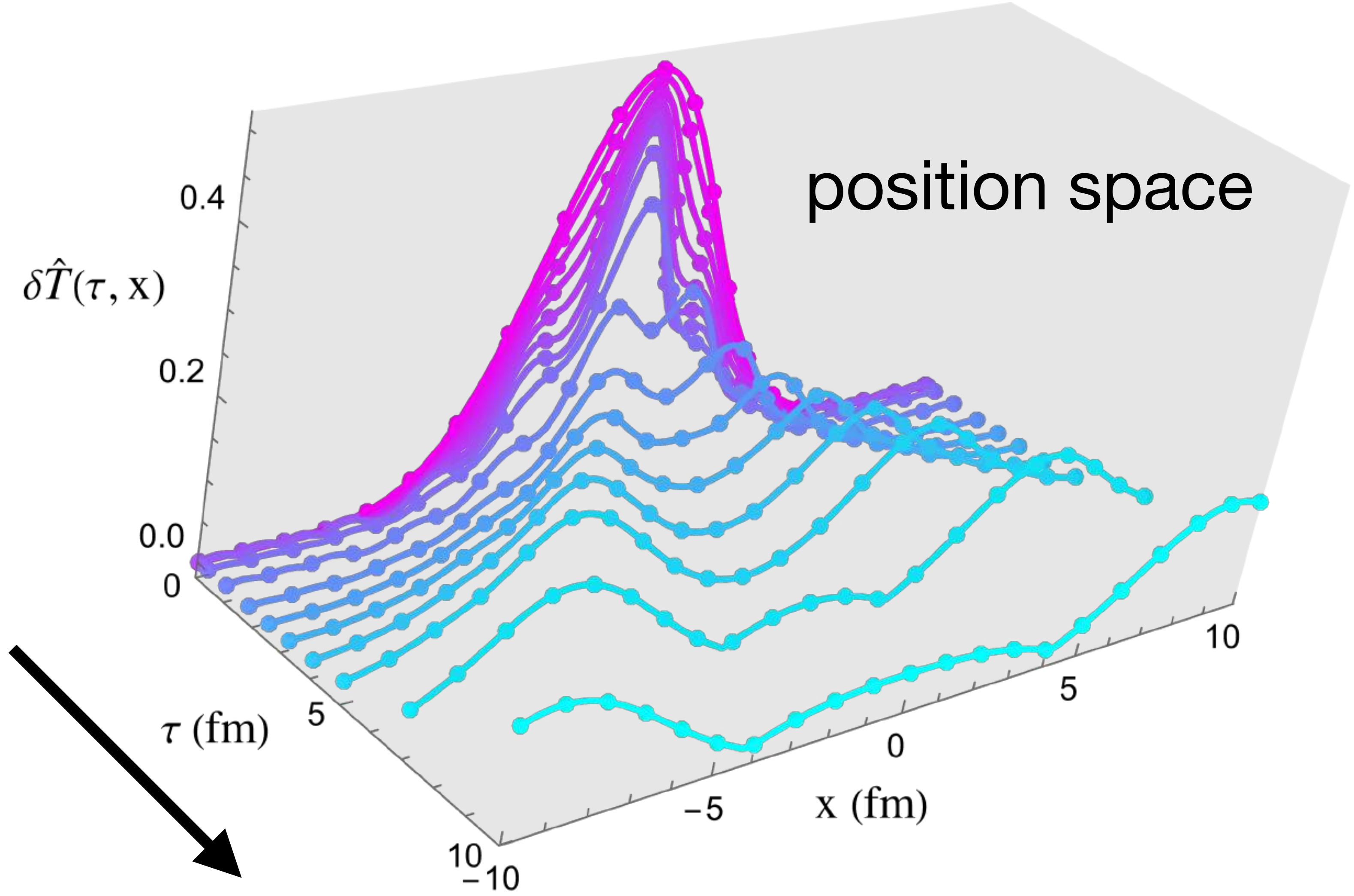
where P_1, P_0 are block-diagonal-matrix coefficients.

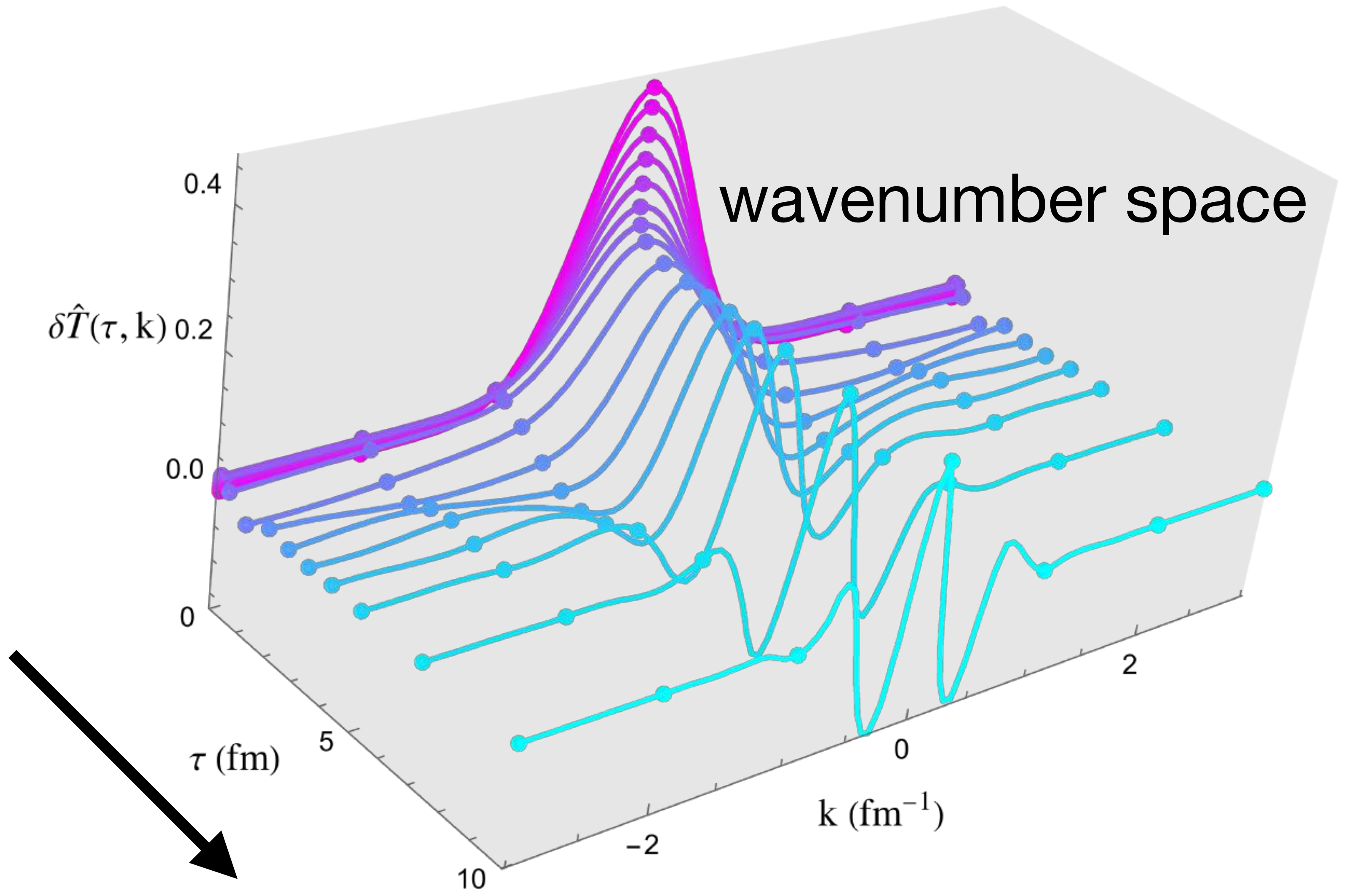
NB: the 2nd order ODE for $\delta\hat{\omega}$ decouples from that for $\delta\hat{T}$ and $\delta\hat{\theta}$, the latter can also be converted to a single 4th order ODE.

The transverse structure of initial states can be encoded in finite set of Fourier modes.

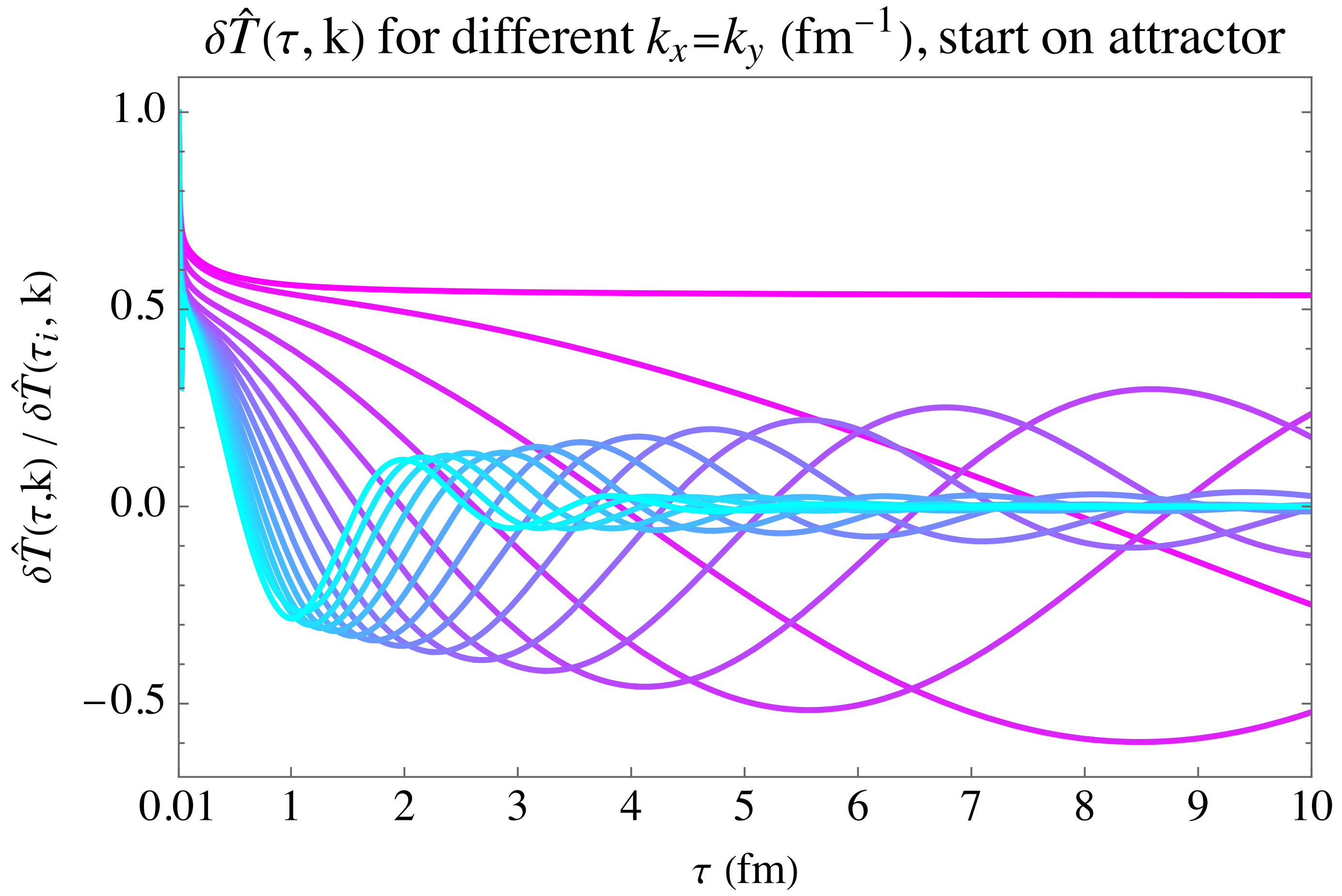
Transverse scale dependence

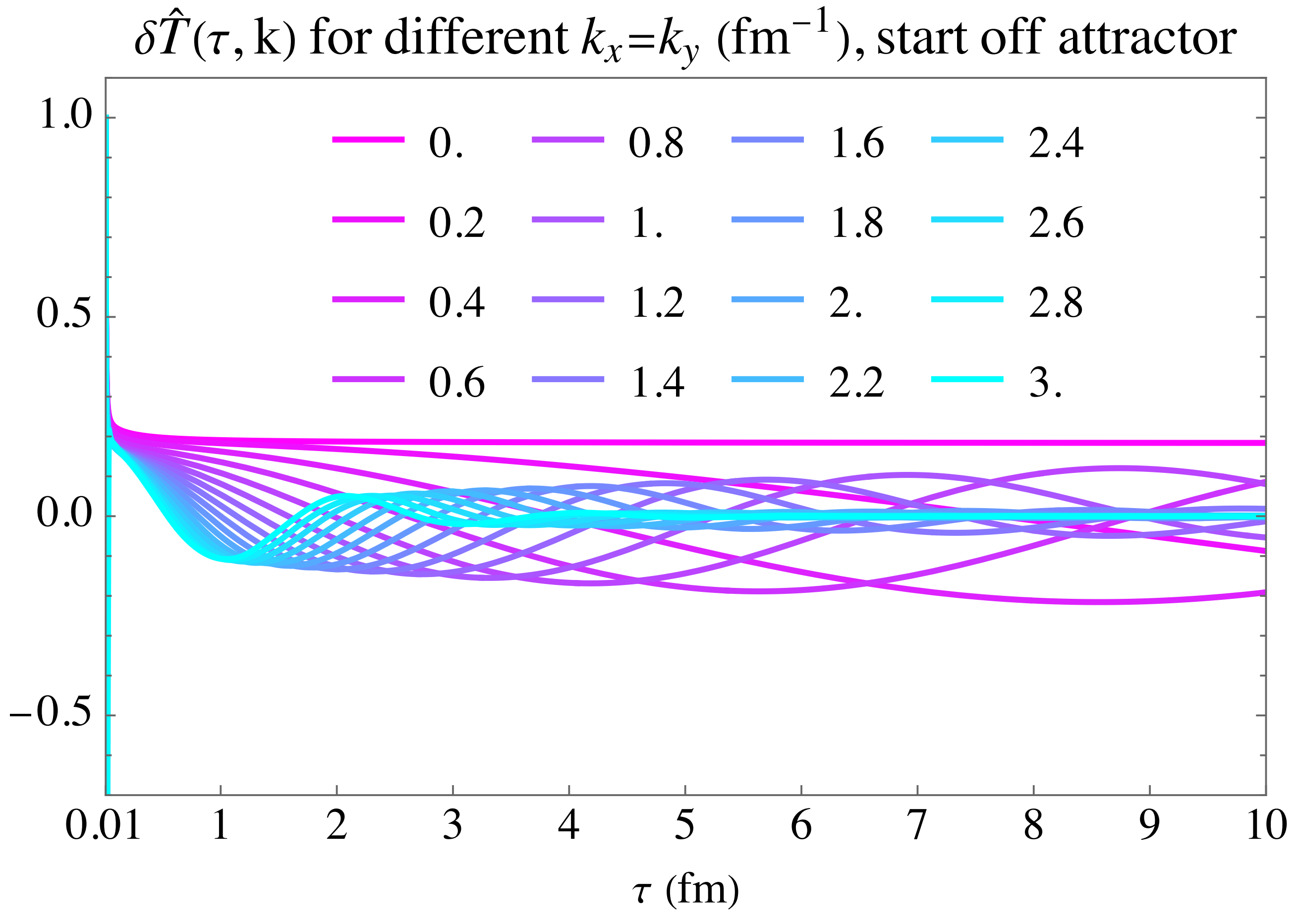
- Tomography in 2D transverse plane.





- Strong damping of large k modes and off-attractor perturbations.





Observables

Late-time asymptotics

- Late-time asymptotic solutions perturbed around attractor:

$$\delta\hat{T} = C_1(\Lambda\tau)^{a_1}\, e^{-\frac{3}{2C_\tau}(\Lambda\tau)^{2/3}} + C_2(\Lambda\tau)^{a_2}\, e^{-\frac{3}{2\gamma^2 C_\tau}(\Lambda\tau)^{2/3}} + e^{-\frac{3\alpha^2}{\gamma^2 C_\tau}(\Lambda\tau)^{2/3}}\left(C_3\, e^{i\sqrt{3}\gamma k\tau} + C_4\, e^{-i\sqrt{3}\gamma k\tau}\right)$$

$$\delta\hat{\theta} = C_1'(\Lambda\tau)^{a_1-1}\, e^{-\frac{3}{2C_\tau}(\Lambda\tau)^{2/3}} + C_2'(\Lambda\tau)^{a_2-\frac{1}{3}}\, e^{-\frac{3}{2\gamma^2 C_\tau}(\Lambda\tau)^{2/3}} + e^{-\frac{3\alpha^2}{\gamma^2 C_\tau}(\Lambda\tau)^{2/3}}\left(C_3'\, e^{i\sqrt{3}\gamma k\tau} + C_4'\, e^{-i\sqrt{3}\gamma k\tau}\right)$$

$$\delta\hat{\omega} = e^{-\frac{3}{4C_\tau}(\Lambda\tau)^{2/3}}\left(C_5\, e^{i\alpha k\tau} + C_6\, e^{-i\alpha k\tau}\right)$$

NB: the solutions for $\delta\hat{\pi}_{ij} \equiv \delta\pi_{ij}/T^4$ can be determined accordingly from $\delta\hat{T}$, $\delta\hat{\theta}$ and $\delta\hat{\omega}$.

$\gamma = \sqrt{1+\alpha^2}$; $\Lambda, C_1, \ldots, C_6$: independent integration constants

- The attractor is stable against transverse dynamics, and note again that the suppression is mild since the typical $\tau \sim 10$ fm in HIC is not large.

Observables

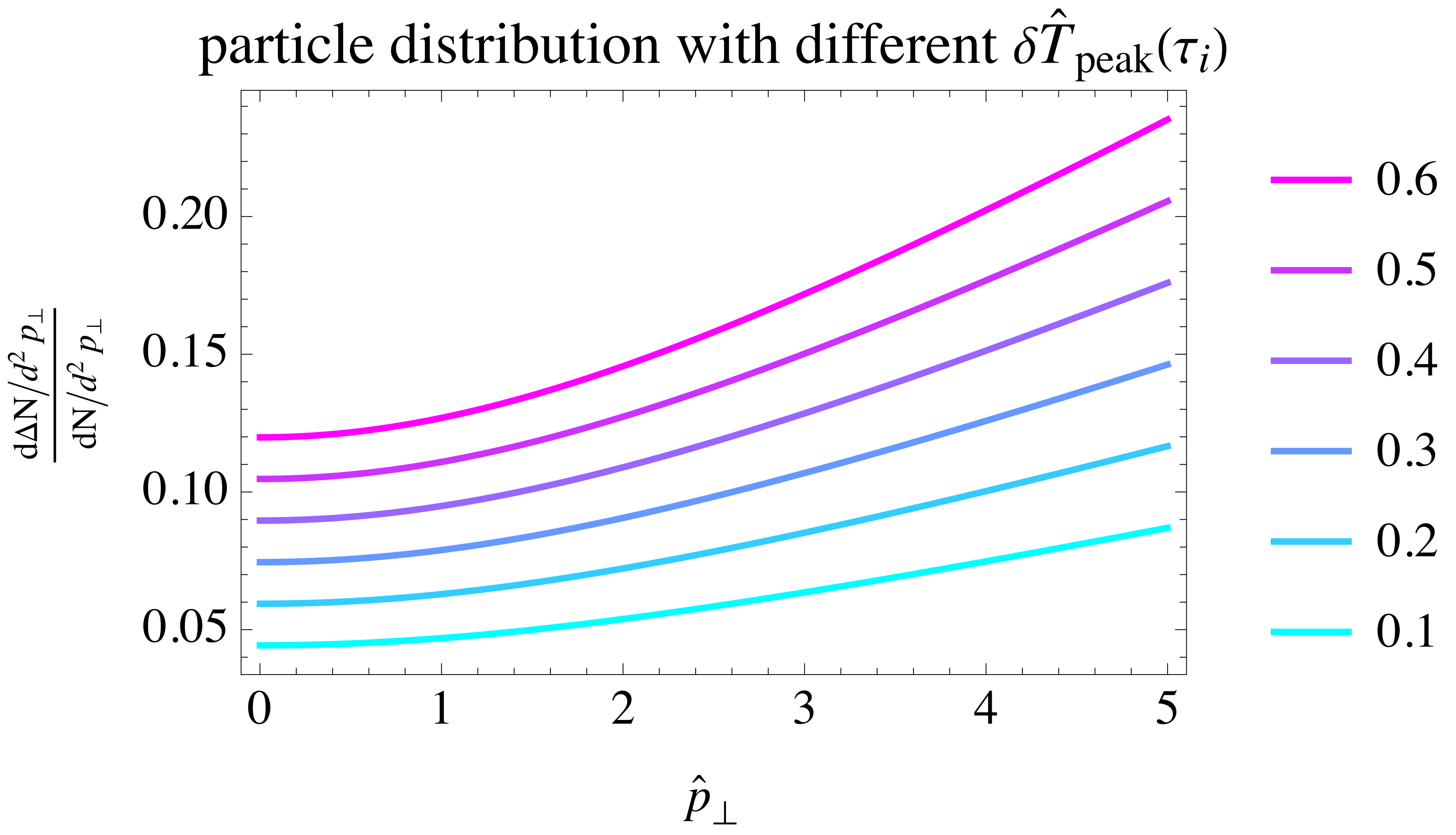
- Physical observables can be extracted from the asymptotic data of $(\delta\hat{T}, \delta\hat{\theta}, \delta\hat{\omega}, \delta\hat{\pi}_{ij})$ determined by $(C_1, \ldots, C_6)$.

- Linearized Cooper–Frye freezeout formula:

$$\frac{d\Delta N/d^2\mathbf{p}_\perp}{dN/d^2\mathbf{p}_\perp} = \left(1 + \hat{m}_\perp \frac{K_0(\hat{m}_\perp)}{K_1(\hat{m}_\perp)}\right)\langle\delta\hat{T}\rangle_\perp + \hat{\mathbf{p}}_\perp \cdot \langle\delta\mathbf{u}\rangle_\perp$$

$$\langle\delta\mathbf{u}\rangle_\perp \;||\; \delta\mathbf{u}(\delta\hat{\theta}, \delta\hat{\omega})$$

$$\hat{m}_\perp = \frac{\sqrt{m^2 + \mathbf{p}_\perp^2}}{T}, \quad \hat{\mathbf{p}}_\perp = \frac{\mathbf{p}_\perp}{T},$$ K_n : Bessel function



- Other observables (such as momentum anisotropy $A_T \sim v_2$)…

$$A_T \equiv \frac{\langle T_{11} - T_{22}\rangle_\perp}{\langle T_{11} + T_{22}\rangle_\perp} = \frac{9\langle\delta\hat{\pi}_{11} - \delta\hat{\pi}_{22}\rangle_\perp}{2C_e(3 + A)}$$

See XA and Spalinski, to appear for more details

Conclusion

Recap

- Transverse dynamics can be described by perturbations around the attractor background.

- The problem reduces to a set of linear ODEs which can be analyzed semi-analytically.

- Physics is captured by finite asymptotic data, mostly exponentially suppressed.

Outlook

- Systems with lesser symmetries.

- Implementation with jets or noises.

- More…