Parton-level Monte Carlo for EIC

Andreas van Hameren
Institute of Nuclear Physics
Polish Academy of Sciences
Kraków

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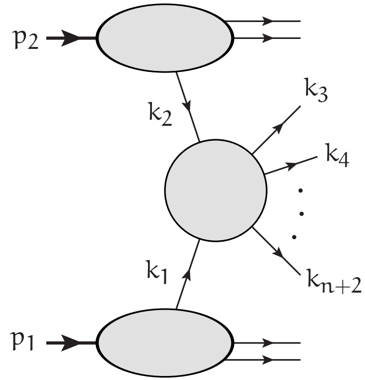
Outline

- Monte Carlo integration
- amplitude calculation
- phase space generation
- Improved Transverse Momentum Factorization
- phenomenology

Parton-level cross sections

Hadron-scattering process Y with partonic processes y contributing to multi-jet final state

$$d\sigma_Y(p_1,p_2;k_3,\dots,k_{2+n}) = \sum_{y\in Y}\int d^4k_1\,\mathcal{P}_{y_1}(k_1)\int d^4k_2\,\mathcal{P}_{y_2}(k_2)\,d\hat{\sigma}_y(k_1,k_2;k_3,\dots,k_{2+n})$$



Collinear factorization:

$$\mathcal{P}_{y_i}(k_i) = \int \frac{dx_i}{x_i}\,f_{y_i}(x_i,\mu)\,\delta^4(k_i - x_i p_i)$$

k_T-dependent factorization factorization:

$$\mathcal{P}_{y_i}(k_i) = \int \frac{d^2\mathbf{k}_{iT}}{\pi}\int \frac{dx_i}{x_i}\,\mathcal{F}_{y_i}(x_i,|\mathbf{k}_{iT}|,\mu)\,\delta^4(k_i - x_i p_i - k_{iT})$$

Differential partonic cross section:

$$\begin{aligned} d\hat{\sigma}_y(k_1,k_2;k_3,\dots,k_{2+n}) &= d\Phi_Y(k_1,k_2;k_3,\dots,k_{2+n})\,\Theta_Y(k_3,\dots,k_{2+n}) \\ &\quad \times \mathrm{flux}(k_1,k_2) \times \mathcal{S}_y\,|\mathcal{M}_y(k_1,\dots,k_{2+n})|^2 \end{aligned}$$

Parton-level phase space:

$$d\Phi_Y(k_1,k_2;k_3,\dots,k_{2+n}) = \left(\prod_{i=3}^{n+2} d^4k_i\delta_+(k_i^2 - m_i^2)\right)\delta^4(k_1+k_2-k_3-\cdots-k_{n+2})$$

Parton-level cross sections

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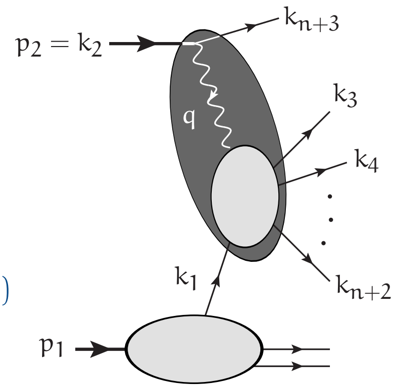

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KaTie https://bitbucket.org/hameren/katie

- parton level event generator, like Alpgen, Helac, MadGraph, etc.
- arbitrary hadron-hadron or hadron-lepton processes within the standard model (including effective Higgs-gluon coupling) with several final-state particles.
- **0, 1, or 2 space-like initial states.**
- produces (partially un)weighted event files, for example in the LHEF format.
- requires LHAPDF. TMD PDFs can be provided as files containing rectangular grids, or with TMDlib (Hautmann, Jung, Krämer, Mulders, Nocera, Rogers, Signori 2014).
- a calculation is steered by a single input file.
- employs an optimization stage in which the pre-samplers for all channels are optimized.
- during the generation stage several event files can be created in parallel.
- event files can be processed further by parton-shower program like CASCADE.
- (evaluation of) matrix elements separately available.

Monte Carlo integration

to calculate $\int_{\mathbf{M}} \mathrm{d}^{\omega}x\, f(x)$

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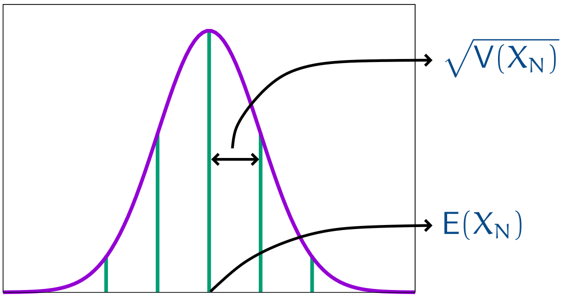
Let g be a *probability density* on $\mathbf{M}$ of dimension ω such that if $f(x) \neq 0$ then $g(x) \neq 0$.
Let $\{x_i\}$ be a sequence of points in $\mathbf{M}$ independently drawn at random from g.
Then, for $N \to \infty$, the probability distribution of the random variable

$$X_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{g(x_i)}$$

becomes Gausian, with expectation value and variance

$$E(X_N) = \int_{\mathbf{M}} d^\omega x\, f(x) \quad , \quad V(X_N) = \frac{1}{N}\left[\int_{\mathbf{M}} d^\omega x\, \frac{f(x)^2}{g(x)} - \left(\int_{\mathbf{M}} d^\omega x\, f(x)\right)^2\right]$$



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- X_N is an estimate of the integral of f with error estimate $\sqrt{V(X_N)}$
- $V(X_N)$ can be estimated itself with $\left[N^{-1} \sum_{i=1}^{N} f(x_i)^2/g(x_i)^2 - X_N^2\right]/(N-1)$
- the error decreases as $N^{-1/2}$, independently of $\mathbf{M}$
- *importance sampling*: convergence can be improved by choosing g such that it has the same shape as f. If you can construct $g(x) = f(x)/\int_M d^\omega y\, f(y)$, then you actually solved the integration problem without the need of Monte Carlo.

Monte Carlo integration

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- we are considering Monte Carlo *integration*: $f(x)$ can be evaluated for any x.
  The integral can, in principle, be calculated to arbitrary precision.

- Monte Carlo simulation: evaluating $f(x)$ is essentially impossible, and one can only try to approximate it as much as possible with $g(x)$.

Weighted event generation

$$\int_M d^\omega x \, f(x) \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{g(x_i)}$$

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Let φ be also be a function on $\mathbf{M}$ (but more like a coordinate function). Then we define a *bin* of φ as

$$\mathrm{bin}(\varphi; a, b) = \int_M d^\omega x\, f(x)\, \theta(a < \varphi(x) < b) \quad , \quad \theta(\Pi) = \begin{cases} 1 & \text{if } \Pi \text{ is true} \\ 0 & \text{if } \Pi \text{ is false} \end{cases}$$

From the Monte Carlo point of view, we only changed the integrand $f(x)$ to $f(x)\theta(a < \varphi(x) < b)$, and we can use the same density $g(x)$ to calculate the bin.

$$\mathrm{bin}(\varphi; a, b) \approx \frac{1}{N}\sum_{i=1}^{N} \frac{f(x_i)}{g(x_i)}\, \theta(a < \varphi(x_i) < b)$$

In case there are more than one, but non-overlapping, bins, then each x_i can only contribute to one of those, and we can make an unbiased estimate for all bin using the same set of random points

$$\mathrm{bin}(\varphi; a_j, b_j) \approx \frac{1}{N}\sum_{i=1}^{N} \frac{f(x_i)}{g(x_i)}\, \theta(a_j < \varphi(x_i) < b_j)$$

Despite that we are calculating “exact” integrals, we call this *making a histogram by weighted event generation*.

Zero-dimensional QFT

Consider ϕ^3-theory on a single space-time point

$$Z[J] = \int_{-\infty}^{\infty} d\phi \, \exp\left\{ \frac{i}{\hbar} [J\phi + S(\phi)] \right\} \quad , \quad S(\phi) = -\frac{m^2}{2}\phi^2 - \frac{g}{6}\phi^3 \ , \ \mathrm{Im}(m^2 < 0)$$

We trivially have the linear Dyson-Schwinger equation

$$0 = \int_{-\infty}^{\infty} d\phi \, \frac{\hbar}{i} \frac{d}{d\phi} \exp\left\{ \frac{i}{\hbar} [J\phi + S(\phi)] \right\} = \left(J - \frac{\hbar}{i} m^2 \frac{d}{dJ} + \frac{\hbar^2 g}{2} \frac{d^2}{dJ^2} \right) Z[J]$$

$Z[J]$ generates zero-dimensional "Green functions", connected "Green functions" generated by

$$W[J] = \ln Z[J]$$

Non-linear Dyson-Schwinger equation

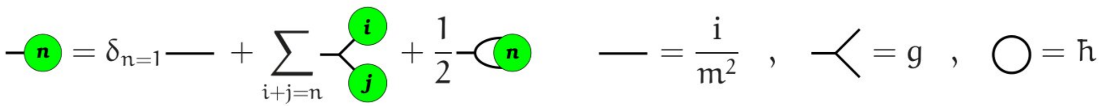
$$0 = J + i m^2 \frac{dW[J]}{dJ} + \frac{g}{2} \left[\hbar \frac{d^2 W[J]}{dJ^2} + \left(\frac{dW[J]}{dJ} \right)^2 \right]$$

Zero-dimensional QFT

Dyson-Schwinger equation for Green functions from $\frac{dW[J]}{dJ} = \sum_{n=0}^{\infty} \frac{C_{n+1} J^n}{n!}$

$$\frac{C_{n+1}}{n!} = \frac{i}{m^2} \left(\delta_{n=1} + g \sum_{i+j=n} \frac{C_{i+1}}{i!} \frac{C_{j+1}}{j!} + \frac{\hbar g}{2} \frac{C_{n+2}}{n!} \right)$$

We may cast the equation into a graphical form



Zero-dimensional QFT

Introduce more zero-dimensional points

$$S(\phi) = -\sum_{k,l} \frac{1}{2} A_{k,l}\phi_k\phi_l - \sum \frac{g}{6}\phi_l^3 \quad , \quad \mathrm{Im}(A_{k,k} < 0)$$

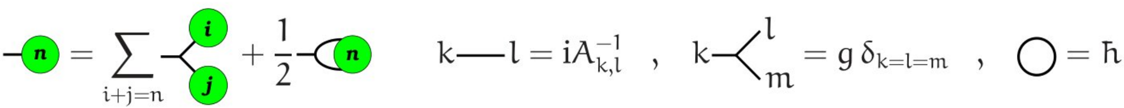
Dyson-Schwinger equation

$$0 = J_k + i\sum_l A_{k,l}\frac{\partial W[J]}{\partial J_l} + \frac{g}{2}\left[\hbar\frac{\partial^2 W[J]}{\partial J_k^2} + \left(\frac{\partial W[J]}{\partial J_k}\right)^2\right]$$

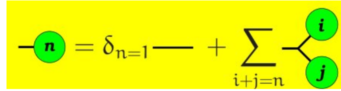
Expand generating function in terms of Green functions

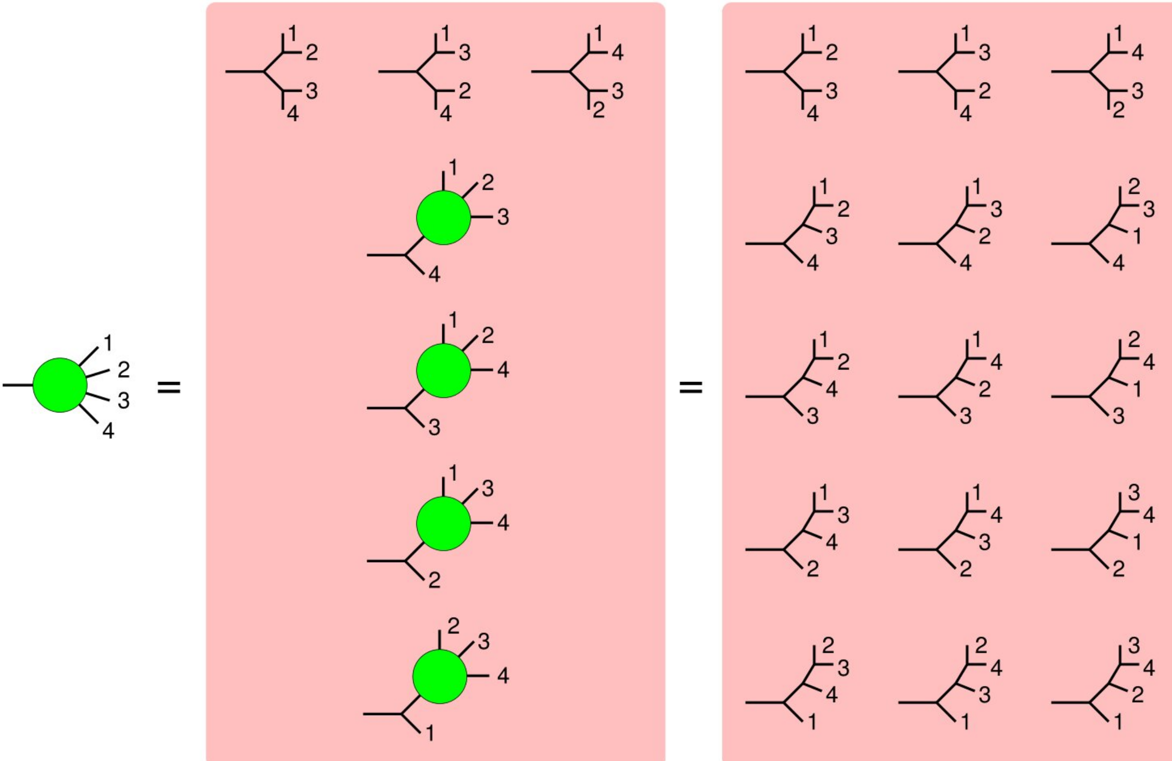
$$\frac{\partial W[J]}{\partial J_l} = \sum_{i_1+i_2+\cdots+i_k=n} C_{l;i_1 i_2\cdots i_k}\frac{J_1^{i_1}}{i_1!}\frac{J_2^{i_2}}{i_2!}\cdots\frac{J_k^{i_k}}{i_k!}$$

Graphical interpretation

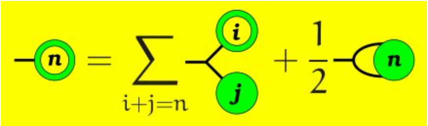


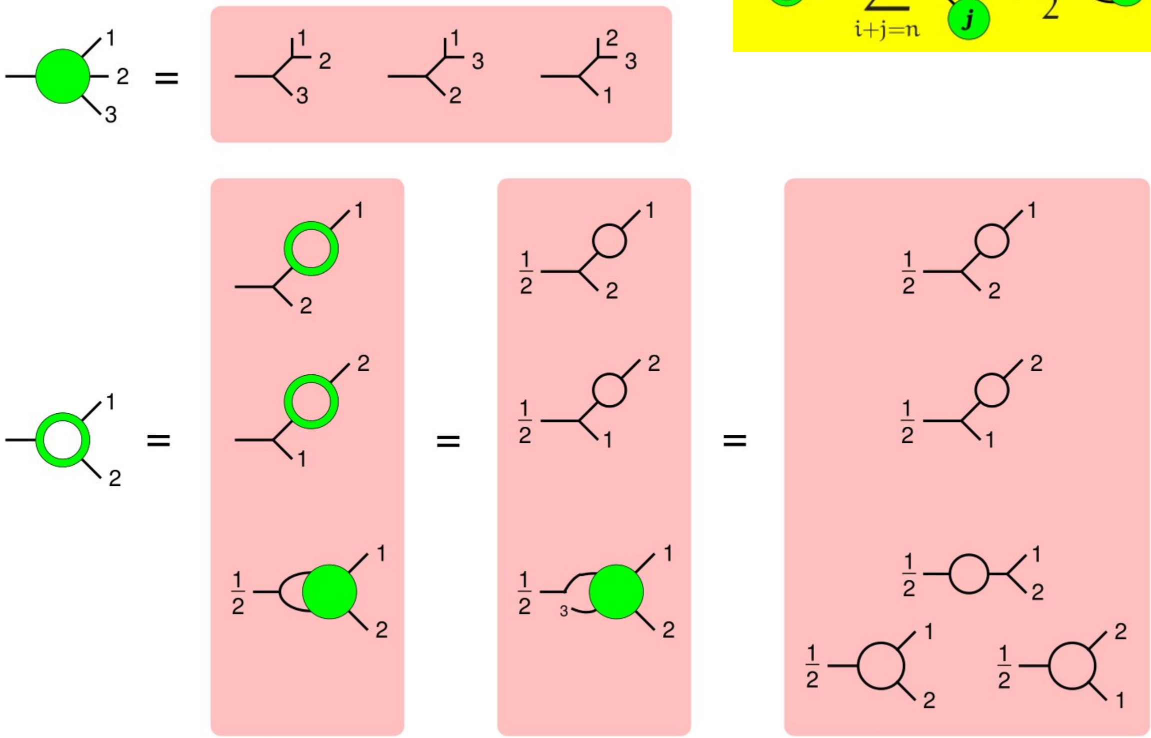
Tree-level recursion



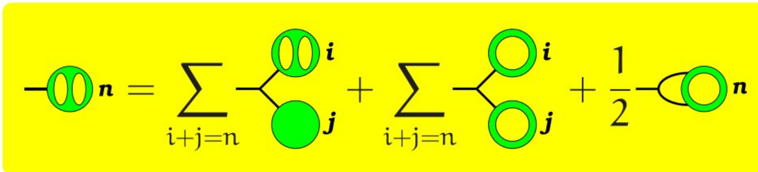


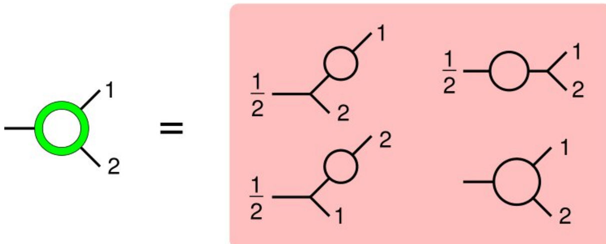
One-loop recursion

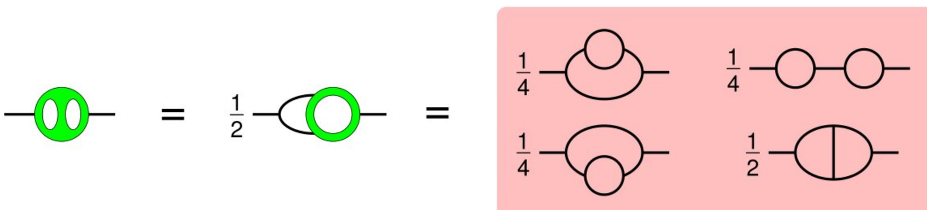




Two-loop recursion

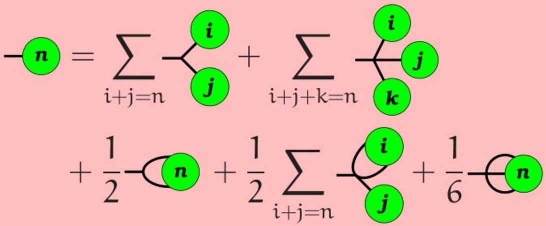




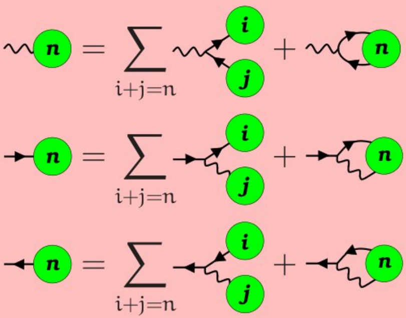


Generalization to realistic QFT

Theories with four-point vertices:



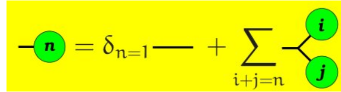
Theories with more types of currents:

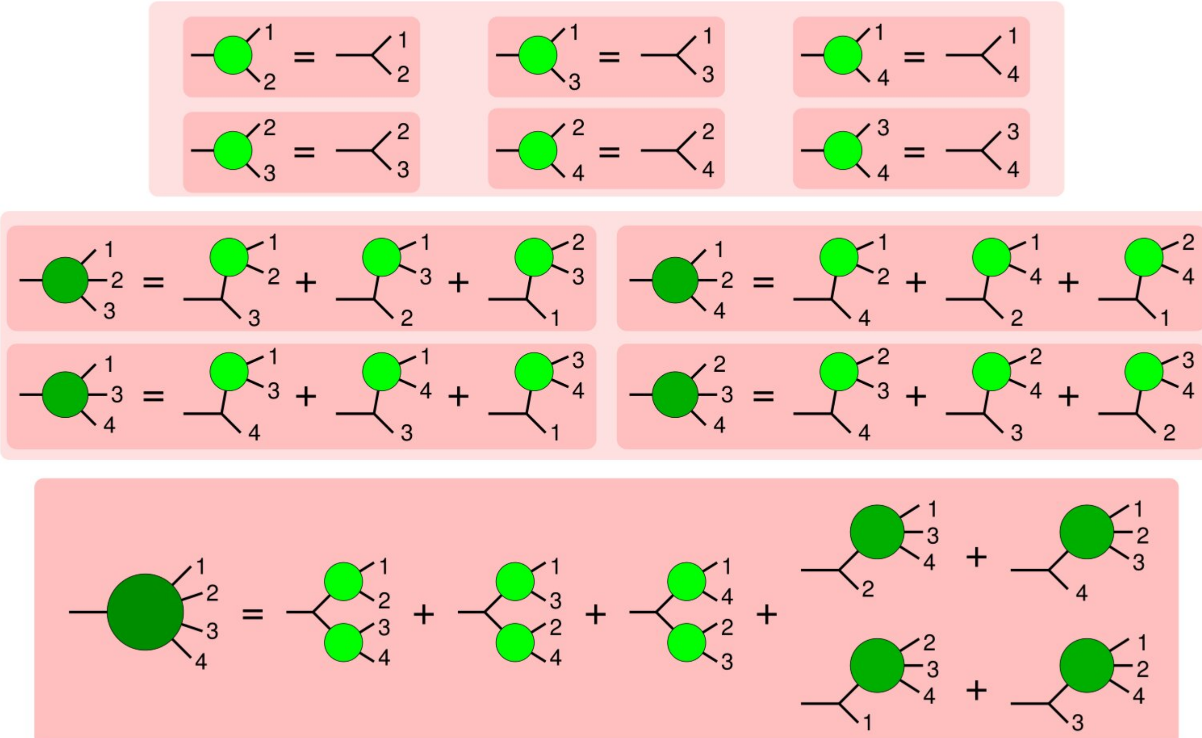


Currents may have several components.

- distinguishable external lines correspond to on-shell particles
  $\Longrightarrow$ polarization vectors, spinors, 1
- sum of momenta of on-shell lines is equal to momentum of off-shell line
- vertices directly from Feynman rules in momentum space
- off-shell line carries propagator from Feynman rules, in any gauge
- on-shell $(n+1)$-leg amplitude
  - from current with n on-shell legs
  - by omitting the final propagator
  - and contracting with pol.vec. or spinor instead

Recursive computation

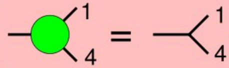


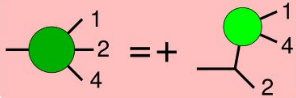


Caravaglios, Moretti 1995

DS skeleton for $\emptyset \to \mathbf{hhhhh}$

```
 1:   5[  3 h ] <--   2[  2 h ]   1[  1 h ]
 2:   6[  5 h ] <--   3[  4 h ]   1[  1 h ]
 3:   7[  9 h ] <--   4[  8 h ]   1[  1 h ]
 4:   8[  6 h ] <--   3[  4 h ]   2[  2 h ]
 5:   9[ 10 h ] <--   4[  8 h ]   2[  2 h ]
 6:  10[ 12 h ] <--   4[  8 h ]   3[  4 h ]
 7:  11[  7 h ] <--   3[  4 h ]   5[  3 h ]
 8:  11[  7 h ] <--   2[  2 h ]   6[  5 h ]
 9:  11[  7 h ] <--   1[  1 h ]   8[  6 h ]
10:  12[ 11 h ] <--   4[  8 h ]   5[  3 h ]
11:  12[ 11 h ] <--   2[  2 h ]   7[  9 h ]
12:  12[ 11 h ] <--   1[  1 h ]   9[ 10 h ]
13:  13[ 13 h ] <--   4[  8 h ]   6[  5 h ]
14:  13[ 13 h ] <--   3[  4 h ]   7[  9 h ]
15:  13[ 13 h ] <--   1[  1 h ]  10[ 12 h ]
16:  14[ 14 h ] <--   4[  8 h ]   8[  6 h ]
17:  14[ 14 h ] <--   3[  4 h ]   9[ 10 h ]
18:  14[ 14 h ] <--   2[  2 h ]  10[ 12 h ]
19:  15[ 15 h ] <--  10[ 12 h ]   5[  3 h ]
20:  15[ 15 h ] <--   9[ 10 h ]   6[  5 h ]
21:  15[ 15 h ] <--   8[  6 h ]   7[  9 h ]
22:  15[ 15 h ] <--   4[  8 h ]  11[  7 h ]
23:  15[ 15 h ] <--   3[  4 h ]  12[ 11 h ]
24:  15[ 15 h ] <--   2[  2 h ]  13[ 13 h ]
25:  15[ 15 h ] <--   1[  1 h ]  14[ 14 h ]
```





particle identifier for off-shell leg

$p_{13} = p_4 + p_9$

Binary representation of momenta: external momenta are labeled by powers of 2, and

$$p_{2^{n-1}-1} = p_1 + p_2 + p_4 + \cdots + p_{2^{n-2}} = -p_{2^{n-1}}$$

eg. for $n = 5$ we have $p_{15} = -p_{16}$

DS skeleton for $\emptyset \to e^+e^- \, e^+e^- \, \gamma$

```
 1: -1,   5[  5 A ] <--   3[  4 E-]   1[  1 E+]
 2: -1,   6[  5 Z ] <--   3[  4 E-]   1[  1 E+]
 3:  1,   7[  9 E+] <--   4[  8 A ]   1[  1 E+]
 4: -1,   8[  6 A ] <--   3[  4 E-]   2[  2 E+]
 5: -1,   9[  6 Z ] <--   3[  4 E-]   2[  2 E+]
 6:  1,  10[ 10 E+] <--   4[  8 A ]   2[  2 E+]
 7:  1,  11[ 12 E-] <--   4[  8 A ]   3[  4 E-]
 8: -1,  12[  7 E+] <--   2[  2 E+]   5[  5 A ]
 9: -1,  12[  7 E+] <--   2[  2 E+]   6[  5 Z ]
10:  1,  12[  7 E+] <--   1[  1 E+]   8[  6 A ]
11:  1,  12[  7 E+] <--   1[  1 E+]   9[  6 Z ]
12: -1,  13[ 13 A ] <--   3[  4 E-]   7[  9 E+]
13: -1,  14[ 13 Z ] <--   3[  4 E-]   7[  9 E+]
14: -1,  13[ 13 A ] <--   1[  1 E+]  11[ 12 E-]
15: -1,  14[ 13 Z ] <--   1[  1 E+]  11[ 12 E-]
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26: -1,  17[ 15 E+] <--   2[  2 E+]  14[ 13 Z ]
27:  1,  17[ 15 E+] <--   1[  1 E+]  15[ 14 A ]
28:  1,  17[ 15 E+] <--   1[  1 E+]  16[ 14 Z ].
```

same momentum, different particle

$$\overline{\Psi}_{11} =+ \overline{\Psi}_3 \not{A}_4(-ie)\frac{i}{\not{p}_{12} - m}$$

$$\Psi_{12} =+ \frac{i}{-\not{p}_7 - m}(-ie)\not{A}_8 \Psi_1$$

$$A^{\mu}_{13} =+ \frac{-i}{p^2_{13}}(-ie)\overline{\Psi}_{11}\gamma^{\mu}\Psi_1$$

fermi sign

$$(-1)^{\chi(p,q)} \quad , \quad \chi(p,q) = \sum_{i=n}^{2} \hat{p}_i \sum_{j=1}^{i-1} \hat{q}_j$$

$\hat{p}_i = 1$ if external particle i is a fermion and is present in p, else $\hat{p}_i = 0$

Cross sections from Monte Carlo

Calculation of a cross section requires phase space integration and summation over spins and colors.

$$\sigma = \int d\Phi \sum_{\text{spin}} \sum_{\text{color}} |\mathcal{M}(\Phi, \text{spin}, \text{color})|^2 \, \mathcal{O}(\Phi)$$

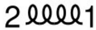
- Phase space must we dealt with within a Monte Carlo approach (that's why we need to be able to evaluate scattering amplitudes numerically efficiently)
- Spin may be dealt with within a Monte Carlo approach:

$$\sum_{+,-} \Rightarrow \int_0^1 d\rho \quad , \quad \varepsilon^\mu(\rho) = u_+(\rho)\varepsilon_+^\mu + u_-(\rho)\varepsilon_-^\mu \quad , \quad \int_0^1 u_i(\rho)u_j(\rho)^* = \delta_{i,j}$$

  - random helicities: $u_\pm(\rho) = \sqrt{2}\,\theta(\pm(\frac{1}{2} - \rho))$
  - random polarizations: $u_\pm(\rho) = e^{\pm i\pi\rho}$
- Color may be dealt with also within a Monte Carlo approach

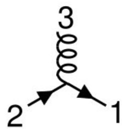
What color representation to use?

QCD Feynman rules



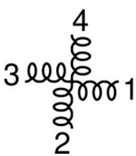
$$= \frac{-i}{p^2}\eta^{\mu_1\mu_2}\,\delta^{a_1 a_2}$$

$$= \frac{i}{\not{p} - m}\,\delta_{i_1 i_2}$$



$$= -ig\,T^{a_3}_{i_1 i_2}\,\gamma^{\mu_3}$$

$$= g\,f^{a_1 a_2 a_3}\left[(p_1 - p_2)^{\mu_3}\eta^{\mu_1\mu_2} + (p_2 - p_3)^{\mu_1}\eta^{\mu_2\mu_3} + (p_3 - p_1)^{\mu_2}\eta^{\mu_3\mu_1}\right]$$



$$\begin{aligned} = ig^2\Big[& \left(f^{a_1 a_3 b} f^{a_2 a_4 b} - f^{a_1 a_4 b} f^{a_3 a_2 b}\right)\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} \\ & + \left(f^{a_1 a_2 b} f^{a_3 a_4 b} - f^{a_1 a_4 b} f^{a_2 a_3 b}\right)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} \\ & + \left(f^{a_1 a_3 b} f^{a_4 a_2 b} - f^{a_1 a_2 b} f^{a_3 a_4 b}\right)\eta^{\mu_1\mu_4}\eta^{\mu_3\mu_2}\Big] \end{aligned}$$

Color representation

- Represent gluons as 8-times higher-dim vectors A_μ^a
  increases the number of operations per vertex unacceptably

- Treat gluons with different color as different particles

$$f^{abc} \neq 0 \quad \Rightarrow \quad abc \in \{123, 147, 156, 246, 257, 345, 367, 458, 678\}$$

all possible fusions unique, except $(4,5) \to \{3,8\}$ and $(6,7) \to \{3,8\}$

```
1:     5[   3 g 3 ] <--    2[   2 g 2 ]    1[   1 g 1 ]
2:     6[   5 g 7 ] <--    3[   4 g 4 ]    1[   1 g 1 ]
3:     7[   9 g 6 ] <--    4[   8 g 5 ]    1[   1 g 1 ]
4:     8[   6 g 6 ] <--    3[   4 g 4 ]    2[   2 g 2 ]
5:     9[  10 g 7 ] <--    4[   8 g 5 ]    2[   2 g 2 ]
6:    10[  12 g 3 ] <--    4[   8 g 5 ]    3[   4 g 4 ]
7:    10[  12 g 8 ] <--    4[   8 g 5 ]    3[   4 g 4 ]
```

skeleton depends on external color configuration

Color connection (flow) representation

$$\sum_a |\mathcal{A}^a|^2 = \sum_{a,b} \delta^{ab} \mathcal{A}^a \mathcal{A}^{b*} = \sum_{a,b} 2\mathrm{Tr}\{T^a T^b\} \mathcal{A}^a \mathcal{A}^{b*} = \sum_{i,j} |\mathcal{A}^i_j|^2 \ , \ \ \mathcal{A}^i_j = \sqrt{2}(T^a)^i_j \mathcal{A}^a$$

Contract all external gluons with $\sqrt{2}(T^a)^i_j$
and replace in all gluon propagators $\delta^{ab} = 2\mathrm{Tr}\{T^a T^b\}$
Color structure of the vertices become

$$\text{3-gluon:} \quad 2^{3/2} f^{abc} (T^a)^{i_1}_{j_1} (T^b)^{i_2}_{j_2} (T^c)^{i_3}_{j_3} = \frac{-i}{\sqrt{2}} \left(\delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1} - \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_2} \right)$$

$$\begin{aligned} \text{4-gluon:} \quad & 4(f^{abe}f^{cde} - f^{ade}f^{bce})(T^a)^{i_1}_{j_1} (T^b)^{i_2}_{j_2} (T^c)^{i_3}_{j_3} (T^d)^{i_4}_{j_4} \\ &= \frac{-1}{2} \Big(2\delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_4} \delta^{i_4}_{j_1} + 2\delta^{i_1}_{j_4} \delta^{i_2}_{j_1} \delta^{i_3}_{j_2} \delta^{i_4}_{j_3} \\ &\quad - \delta^{i_1}_{j_2} \delta^{i_2}_{j_4} \delta^{i_3}_{j_1} \delta^{i_4}_{j_3} - \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_4} \delta^{i_4}_{j_2} - \delta^{i_1}_{j_3} \delta^{i_2}_{j_4} \delta^{i_3}_{j_2} \delta^{i_4}_{j_1} - \delta^{i_1}_{j_4} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1} \delta^{i_4}_{j_2} \Big) \end{aligned}$$

$$\text{quark-gluon:} \quad \sqrt{2}\,(T^a)^{i_1}_{j_1} (T^b)^{i_2}_{j_2} = \frac{1}{\sqrt{2}} \left(\delta^{i_1}_{j_2} \delta^{i_2}_{j_1} - \frac{1}{N_c} \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \right)$$

$1/N_c$ contribution in quark-gluon vertex, but trivial gluon propagator: $\delta^{i_1}_{j_2} \delta^{i_2}_{j_1}$

Decomposition into partial amplitudes

Kanaki, Papadopoulos 2000; Maltoni, Paul, Stelzer, Willenbrock 2003

Scattering amplitude with n color pairs can be expressed as

$$\mathcal{M}^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_n} = \sum_{\text{all perm.}} \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \ldots \delta^{i_n}_{j_{\sigma(n)}} \mathcal{A}_\sigma(1, 2, \ldots, n)$$

where $\mathcal{A}_\sigma(1, 2, \ldots, n)$ does not depend on the external color, but may depend on N_c. For small n, the explicit color sum is more efficient than color sampling

$$\sum_{\text{color}} |\mathcal{M}|^2 = \sum_{\sigma, \sigma'} N_c^{y(\sigma, \sigma')} \mathcal{A}_\sigma \mathcal{A}^*_{\sigma'}$$

where $y(\sigma, \sigma')$ is the number of common cycles in σ and σ'.

The DS skeleton for $\mathcal{A}_\sigma$ can be found from $\mathcal{M}$, by imagining that $N_c = n$, and assigning the external color configuration

$$(1, \sigma(1)) \;\; (2, \sigma(2)) \;\; \cdots \;\; (n, \sigma(n))$$

and multiplying quark-gluon vertices by $-i\sqrt{N_c}$ if they involve an internal gluon with $i = j$.

Partial amplitudes for $\emptyset \to g\, g\, \bar{u}\, d\, \mu^+ \nu_\mu$

```
Tree:  1, Label:1
  1:  1   6[  3 u~] <--  2[  2 u~]    1[  1 g ]
  2:  1   7[  5 d ] <--  3[  4 d ]    1[  1 g ]
  3: -1   9[ 24 W+] <--  5[ 16 Mn]    4[  8 M+]
  4:  1  13[ 26 d~] <--  2[  2 u~]    9[ 24 W+]
  5:  1  14[ 28 u ] <--  3[  4 d ]    9[ 24 W+]
  6:  1  17[ 27 d~] <--  9[ 24 W+]    6[  3 u~]
  7:  1  17[ 27 d~] <--  1[  1 g ]   13[ 26 d~]
  8:  1  18[ 29 u ] <--  9[ 24 W+]    7[  5 d ]
  9:  1  18[ 29 u ] <--  1[  1 g ]   14[ 28 u ]
 10: -1  21[ 31 g ] <--  7[  5 d ]   13[ 26 d~]
 11: -1  21[ 31 g ] <--  6[  3 u~]   14[ 28 u ]
 12: -1  21[ 31 g ] <--  3[  4 d ]   17[ 27 d~]
 13: -1  21[ 31 g ] <--  2[  2 u~]   18[ 29 u ]

Tree:  2, Label:2
  1:  1   6[  3 u~] <--  2[  2 u~]    1[  1 g ]
  2:  1   7[  5 d ] <--  3[  4 d ]    1[  1 g ]
  3: -1   9[ 24 W+] <--  5[ 16 Mn]    4[  8 M+]
  4:  1  13[ 26 d~] <--  2[  2 u~]    9[ 24 W+]
  5:  1  14[ 28 u ] <--  3[  4 d ]    9[ 24 W+]
  6:  1  17[ 27 d~] <--  9[ 24 W+]    6[  3 u~]
  7:  1  17[ 27 d~] <--  1[  1 g ]   13[ 26 d~]
  8:  1  18[ 29 u ] <--  9[ 24 W+]    7[  5 d ]
  9:  1  18[ 29 u ] <--  1[  1 g ]   14[ 28 u ]
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 12: -1  21[ 31 g ] <--  3[  4 d ]   17[ 27 d~]
 13: -1  21[ 31 g ] <--  2[  2 u~]   18[ 29 u ].

Tree:  3, Label:3
  1:  1   7[  5 d ] <--  3[  4 d ]    1[  1 g ]
  2: -1   9[ 24 W+] <--  5[ 16 Mn]    4[  8 M+]
  3:  1  13[ 26 d~] <--  2[  2 u~]    9[ 24 W+]
  4:  1  14[ 28 u ] <--  3[  4 d ]    9[ 24 W+]
  5:  1  18[ 29 u ] <--  9[ 24 W+]    7[  5 d ]
  6:  1  18[ 29 u ] <--  1[  1 g ]   14[ 28 u ]
  7: -1  20[ 30 g ] <--  3[  4 d ]   13[ 26 d~]
  8: -1  20[ 30 g ] <--  2[  2 u~]   14[ 28 u ]
  9: -1  21[ 31 g ] <--  7[  5 d ]   13[ 26 d~]
 10: -1  21[ 31 g ] <--  2[  2 u~]   18[ 29 u ]
 11:  1  21[ 31 g ] <--  1[  1 g ]   20[ 30 g ]

Tree:  4, Label:5
  1:  1   6[  3 u~] <--  2[  2 u~]    1[  1 g ]
  2: -1   9[ 24 W+] <--  5[ 16 Mn]    4[  8 M+]
  3:  1  13[ 26 d~] <--  2[  2 u~]    9[ 24 W+]
  4:  1  14[ 28 u ] <--  3[  4 d ]    9[ 24 W+]
  5:  1  17[ 27 d~] <--  9[ 24 W+]    6[  3 u~]
  6:  1  17[ 27 d~] <--  1[  1 g ]   13[ 26 d~]
  7: -1  20[ 30 g ] <--  3[  4 d ]   13[ 26 d~]
  8: -1  20[ 30 g ] <--  2[  2 u~]   14[ 28 u ]
  9: -1  21[ 31 g ] <--  6[  3 u~]   14[ 28 u ]
 10: -1  21[ 31 g ] <--  3[  4 d ]   17[ 27 d~]
 11:  1  21[ 31 g ] <--  1[  1 g ]   20[ 30 g ].

Tree:  5, Label:6
  1:  1   6[  3 u~] <--  2[  2 u~]    1[  1 g ]
  2:  1   7[  5 d ] <--  3[  4 d ]    1[  1 g ]
  3: -1   9[ 24 W+] <--  5[ 16 Mn]    4[  8 M+]
  4:  1  13[ 26 d~] <--  2[  2 u~]    9[ 24 W+]
  5:  1  14[ 28 u ] <--  3[  4 d ]    9[ 24 W+]
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  8:  1  18[ 29 u ] <--  9[ 24 W+]    7[  5 d ]
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```
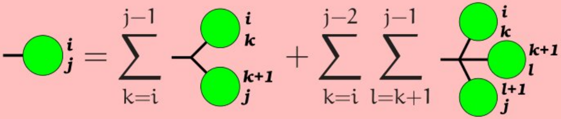

Planar recursion (Berends-Giele)

For planar multi-gluon tree-amplitudes:

$$p_{i,j} = p_i + p_{i+1} + \cdots + p_j$$



$$A^{\mu}_{i,j} = \frac{-i}{p^2_{i,j}} \left[\sum_{k=i}^{j-1} V^{\mu}_{\nu\rho}(p_{i,k}, p_{k+1,j})\, A^{\nu}_{i,k} A^{\rho}_{k+1,j} + \sum_{k=i}^{j-2} \sum_{l=k+1}^{j-1} W^{\mu}_{\nu\rho\sigma}\, A^{\nu}_{i,k} A^{\rho}_{k+1,l} A^{\sigma}_{l+1,j} \right]$$

$$V^{\mu}_{\nu\rho}(p, q) = \frac{i}{\sqrt{2}} \left[(p-q)^{\mu} g_{\nu\rho} + 2 g^{\mu}_{\rho}\, q_{\nu} - 2 g^{\mu}_{\nu}\, p_{\rho} \right]$$

$$W^{\mu}_{\nu\rho\sigma} = \frac{i}{2} \left[2 g^{\mu}_{\rho} g_{\nu\sigma} - g^{\mu}_{\nu} g_{\rho\sigma} - g^{\mu}_{\sigma} g_{\rho\nu} \right]$$

$$A^{\mu}_{i,i} = \varepsilon^{\mu}(p_i)$$

Explicit k_T-employing factorization

TMD factorization

- holds at leading power in k_T/μ
- on-shell parton-level matrix elements
- Transverse Momentum Dependent PDFs,
  evolve via the Collins-Soper-Sterman equations,
  re-sum large logs of k_T/μ

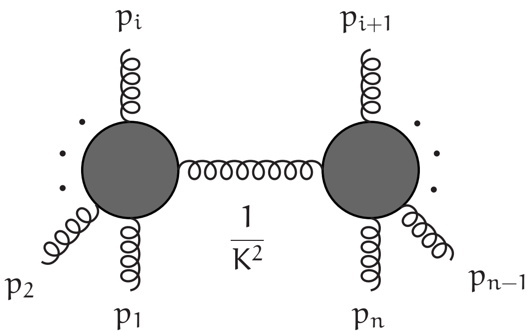
The following is in the context of **High energy factorization**

$$d\sigma_{hh} = \sum_{a,b} \int dx_1 \frac{d^2k_{T1}}{\pi} \int dx_2 \frac{d^2k_{T2}}{\pi} \mathcal{F}_a(x_1, k_{T1})\, \mathcal{F}_b(x_2, k_{T2})\, d\sigma_{ab}(x_1, k_{T1}, x_2, k_{T2})$$

- focus on small-x, not neglecting powers of k_T/μ
- **off-shell parton-level matrix elements**
- Transvers Momentum Dependent, or un-integrated, PDFs,
  evolve to resum logs of $1/x$,
  e.g. with BFKL or CCFM equations, or their non-linear extensions,

BCFW recursion for on-shell amplitudes

Amplitudes have poles at kinematical channels, and the residues factorize into amplitudes.



$$K^\mu = p_1^\mu + p_2^\mu + \cdots + p_i^\mu$$
$$= -p_{i+1}^\mu - \cdots - p_{n-1}^\mu - p_n^\mu$$

BCFW recursion for on-shell amplitudes

Britto, Cachazo, Feng, Witten 2005

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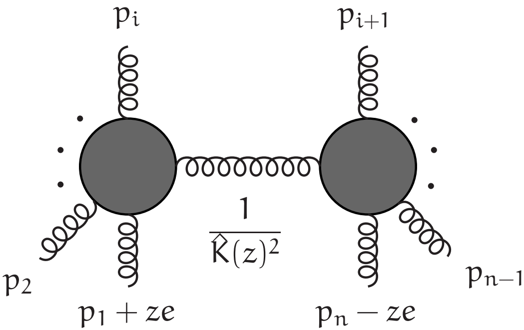

$$\hat{K}^\mu(z) = p_1^\mu + p_2^\mu + \cdots + p_i^\mu + ze^\mu$$
$$= -p_{i+1}^\mu - \cdots - p_{n-1}^\mu - p_n^\mu + ze^\mu$$

$$e^\mu = \tfrac{1}{2}\langle p_1|\gamma^\mu|p_n]$$
$$e\cdot e = e\cdot p_1 = e\cdot p_n = 0$$

$$\hat{K}(z)^2 = 0 \quad \Leftrightarrow \quad z = -\frac{(p_1 + \cdots + p_i)^2}{2(p_2 + \cdots + p_i)\cdot e}$$

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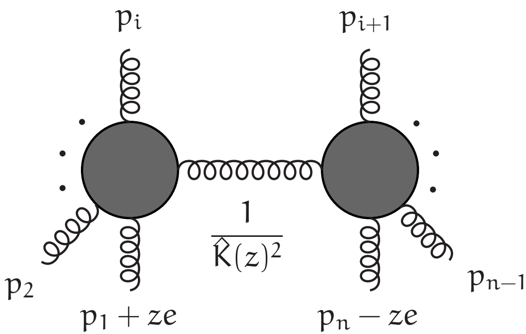

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$$\mathcal{A}(1^+, 2, \ldots, n{-}1, n^-) = \sum_{i=2}^{n-1} \sum_{h=+,-} \mathcal{A}(\hat{1}^+, 2, \ldots, i, -\hat{K}_{1,i}^h)\, \frac{1}{K_{1,i}^2}\, \mathcal{A}(\hat{K}_{1,i}^{-h}, i{+}1, \ldots, n{-}1, \hat{n}^-)$$

$$\mathcal{A}(1^+, 2^-, 3^-) = \frac{\langle 23\rangle^3}{\langle 31\rangle\langle 12\rangle} \quad , \quad \mathcal{A}(1^-, 2^+, 3^+) = \frac{[32]^3}{[21][13]}$$

Amplitudes with off-shell gluons

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n-parton amplitude is a function of n momenta $k_1, k_2, \ldots, k_n$
and n *directions* $p_1, p_2, \ldots, p_n$

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$$k_1^\mu + k_2^\mu + \cdots + k_n^\mu = 0 \qquad \text{momentum conservation}$$

$$p_1^2 = p_2^2 = \cdots = p_n^2 = 0 \qquad \text{light-likeness}$$

$$p_1 \cdot k_1 = p_2 \cdot k_2 = \cdots = p_n \cdot k_n = 0 \qquad \text{eikonal condition}$$

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$$\begin{aligned} k_1^\mu + k_2^\mu + \cdots + k_n^\mu &= 0 && \text{momentum conservation} \\ p_1^2 = p_2^2 = \cdots = p_n^2 &= 0 && \text{light-likeness} \\ p_1\cdot k_1 = p_2\cdot k_2 = \cdots = p_n\cdot k_n &= 0 && \text{eikonal condition} \end{aligned}$$

With the help of an auxiliary four-vector q^μ with $q^2 = 0$, we define

$$k_T^\mu(q) = k^\mu - x(q)p^\mu \quad \text{with} \quad x(q) \equiv \frac{q\cdot k}{q\cdot p}$$

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Construct k_T^μ explicitly in terms of p^μ and q^μ:

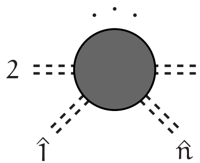
$$k_T^\mu(q) = -\frac{\kappa}{2}\,\varepsilon^\mu - \frac{\kappa^*}{2}\,\varepsilon^{*\mu} \quad \text{with} \quad \begin{cases} \varepsilon^\mu = \dfrac{\langle p|\gamma^\mu|q]}{[pq]} \quad , \quad \kappa = \dfrac{\langle q|\not{k}|p]}{\langle qp\rangle} \\ \varepsilon^{*\mu} = \dfrac{\langle q|\gamma^\mu|p]}{\langle qp\rangle} \quad , \quad \kappa^* = \dfrac{\langle p|\not{k}|q]}{[pq]} \end{cases}$$

$k^2 = -\kappa\kappa^*$ is independent of q^μ, but also individually κ and κ^* are independent of q^μ.

BCFW recursion for off-shell amplitudes

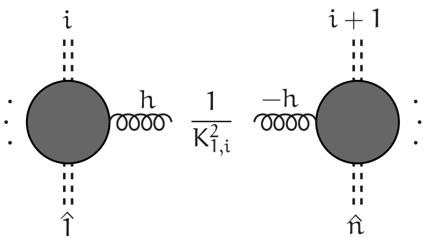
AvH 2014
AvH, Serino 2015

The BCFW recursion formula becomes



$$= \sum_{i=2}^{n-2} \sum_{h=+,-} A_{i,h}$$

$A_{i,h} =$



"On-shell condition" for "off-shell" gluons: $p_i \cdot k_i = 0$

BCFW recursion for off-shell amplitudes

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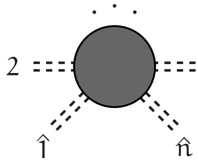
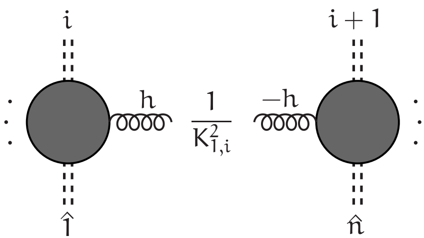
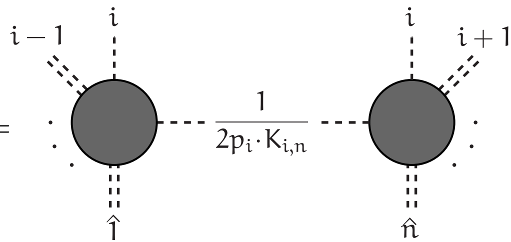


$$= \sum_{i=2}^{n-2} \sum_{h=+,-} A_{i,h} + \sum_{i=2}^{n-1} B_i$$

$A_{i,h} =$



$B_i =$



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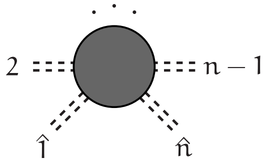
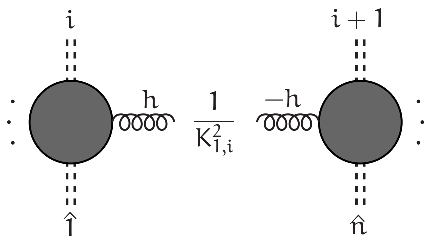
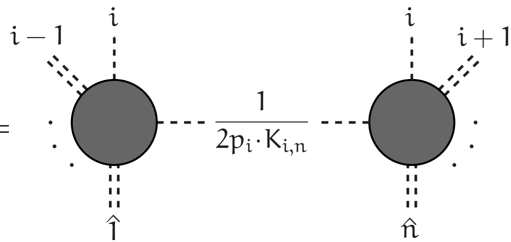


$$= \sum_{i=2}^{n-2} \sum_{h=+,-} A_{i,h} + \sum_{i=2}^{n-1} B_i + C + D \,,$$

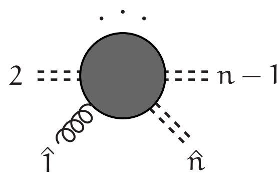
$A_{i,h} =$



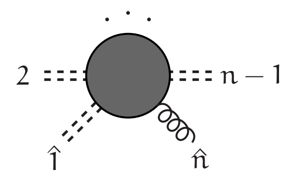
$B_i =$



$C = \dfrac{1}{\kappa_1}$



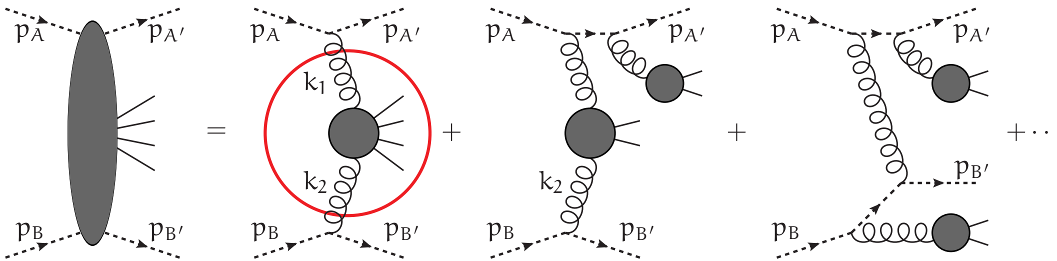
$D = \dfrac{1}{\kappa_1^*}$

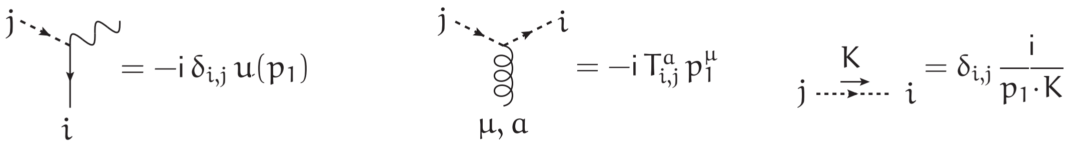


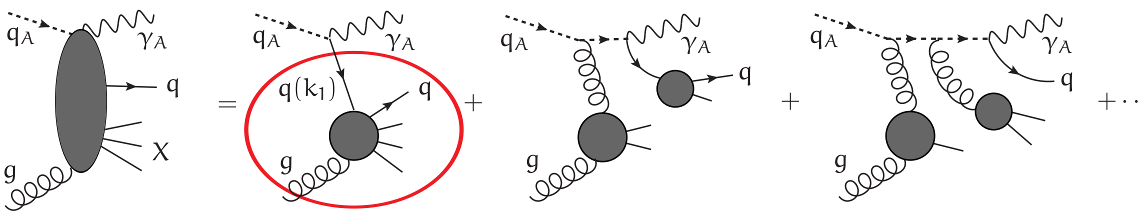
off-shell amplitude as embedding

AvH, Kutak, Kotko 2013
AvH, Kutak, Salwa 2013

Embed the process in an on-shell process with auxiliary partons and eikonal Feynman rules.







Parton-level event generation

- choose partonic subprocess $y = y_1, y_2 \to y_3, \dots y_{n+2}$ with probability $P(y)$
- generate initial-state variables x_1, x_2, k_{T1}, k_{T2} with probability $P(y; x_1, x_2, k_{T1}, k_{T2})$
- generate final-state momenta $k_3, \dots, k_{n+2}$ with differential probability

$$dF(y, k_1, k_2; k_3 \dots, k_{2+n}) = d\Phi_Y(k_1, k_2; k_3, \dots, k_{2+n})\, P(y, k_1, k_2; k_3, \dots, k_{2+n})$$

- assign weight $= 0$ to phase space point if it does not satisfy the inclusive cuts...
- ...else evaluate PDFs and matrix element and assign weight

$$\mathcal{W}_y(k_1, \dots, k_{2+n}) = \frac{\mathcal{F}_{y_1}(x_1, k_{T1})\, \mathcal{F}_{y_2}(x_2, k_{T2})\, |\mathcal{M}_y(k_1, \dots, k_{2+n})|^2\, \mathcal{S}_y\, \mathrm{flux}(k_1, k_2)}{P(y)\, P(y; x_1, x_2, k_{T1}, k_{T2})\, P(y, k_1, k_2; k_3, \dots, k_{2+n})}$$

- choose/create probabilities P wisely/adaptively in order to let $\mathcal{W}_y(k_1, \dots, k_{2+n})$ fluctuate as little as possible from event to event ...
- ...this requires an optimization stage for each subprocess y during which crude estimates of partonic cross sections are made
- there is a lot of engineering/parameters in P, but there is only QFT in $\mathcal{M}_y$

Phase Space

The differential volume of n-particle phase space is given by

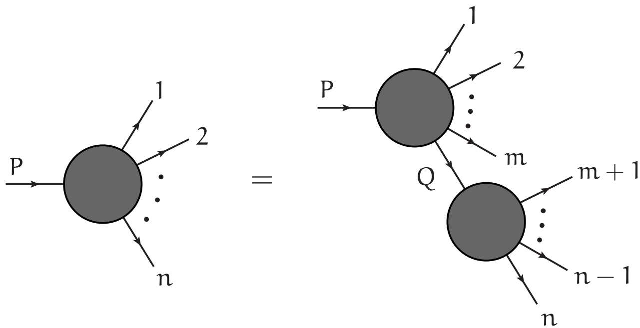
$$d\Phi_n(\,p_1, s_1\,,\, p_2, s_2\,\ldots p_n, s_n\,;\, P\,) = \\ d^4p_1\delta(p_1^2 - s_1)\, d^4p_2\delta(p_2^2 - s_2)\cdots d^4p_n\delta(p_n^2 - s_n)\,\delta^4(P - p_1 - p_2 - \cdots - p_n)$$

and satisfies the recursive relation

$$d\Phi_n(\,p_1, s_1\,,\, p_2, s_2\,\ldots p_n, s_n\,;\, P\,) = \\ dS\, d\Phi_{m+1}(\,p_1, s_1\,,\, p_2, s_2\,\ldots p_m, s_m\,,\, Q, S\,;\, P\,) \\ \times\, d\Phi_{n-m}(\,p_{m+1}, s_{m+1}\,,\, p_{m+2}, s_{m+2}\,\ldots p_n, s_n\,;\, Q\,)$$

with integration over S and Q.

So phase space can be completely decomposed into 2-particle phase spaces, and can be written in terms of invariants and angles.



2-particle phase space

We want to generate p_a, p_b in a 2-particle phase space $\Phi(\,p_a, s_a\,,\,p_b, s_b\,;\,P\,)$. This implies that P and also s_a, s_b are given (generated or squared external masses) and we can define

$$|\vec{q}| = \sqrt{\frac{\lambda(P^2, s_a, s_b)}{4P^2}} \quad \text{with} \quad \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$

Now, we can

1. generate $\varphi \in [0, 2\pi]$ and $z \in [-1, 1]$
2. construct $q^0 = \sqrt{s_a + |\vec{q}|^2}$ and $\vec{q} = |\vec{q}|\left(\sqrt{1-z^2}\cos\varphi\,,\,\sqrt{1-z^2}\sin\varphi\,,\,z\right)$
3. q is p_a in the center-off-mass frame of P, and needs to be boosted:

$$p_a^\mu = \left(E, \vec{q} + V\vec{P}\,\right) \quad \text{with} \quad E = \frac{q\cdot P}{\sqrt{P^2}} \quad \text{and} \quad V = \frac{q^0 + E}{P^0 + \sqrt{P^2}}$$

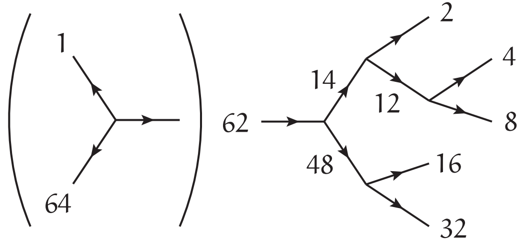
4. and finally $p_b = P - p_a$

This construction gives a Jacobian $\dfrac{\sqrt{P^2}}{\pi|\vec{q}|} = \dfrac{2P^2}{\pi\sqrt{\lambda(P^2, s_a, s_b)}}$

n-particle phase space

To generate, for example, 5-particle phase space, choose a decomposition into 2-particle phase spaces.

External momenta labelled by a power of 2 and $p_i + p_j = p_{i+j}$ and $p_{2^{n+3}-1} = 0$, so for $n = 5$: $p_{127} = 0$ and $p_i = -p_{127-i}$



The density factor of the example graph is

$$g(\{p\}) = g_{48}(s_{48})\, g_{14}(s_{14}) \frac{2s_{62}}{\pi\sqrt{\lambda(s_{64}, s_{48}, s_{14})}}\, g_{12}(s_{12}) \frac{2s_{14}}{\pi\sqrt{\lambda(s_{14}, s_{12}, s_2)}} \times \frac{2s_{12}}{\pi\sqrt{\lambda(s_{12}, s_8, s_4)}} \frac{2s_{48}}{\pi\sqrt{\lambda(s_{48}, s_{32}, s_{16})}}$$

The virtual invariants (s_{48}, s_{14}, s_{12}) need to be generated, and one can use densities anticipating the behavior of the integrand

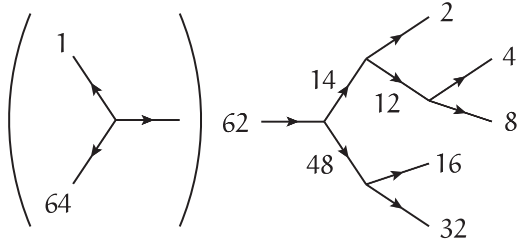
$$\text{e.g.} \quad g_{12}(s) \propto \frac{1}{(s - M_Z^2)^2 + \Gamma_Z^2 M_Z^2}$$

More graphs can be included via the multi-channel method. This way, the squared graphs in a squared amplitude can be matched, while interferences cannot.

n-particle phase space

To generate, for example, 5-particle phase space, choose a decomposition into 2-particle phase spaces.

External momenta labelled by a power of 2 and $p_i + p_j = p_{i+j}$ and $p_{2^{n+3}-1} = 0$, so for $n = 5$: $p_{127} = 0$ and $p_i = -p_{127-i}$



Most of the time, matching the behavior of the sum of squared graphs is enough to tame the phase space behavior of the squared amplitude.

$$\frac{|\text{exact amplitude}|^2}{\sum_i |\text{graph}_i|^2} \quad \text{behaves reasonably well}$$

Given a set of densities that can be generated, the sum of those densities can also be generarated (throw random number, choose graph, generate according to graph, etc.).

But how to deal with $\mathcal{O}(n!)$ graphs again, needed for total density?

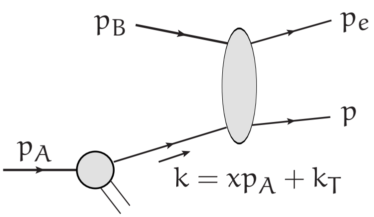
Answer: generate splittings instead of graphs, then the density can be calculated via a Dyson-Schwinger-type recursion (Gleisberg, Höche 2008).

Electron-hadron scattering

Electron-hadron scattering

$$\frac{d^2\sigma^{u-\text{quark}}_{e^-p\to e^-X}}{dx_{Bj}\, dQ^2} = \int dx \int \frac{d^2k_T}{\pi}\, \mathcal{F}_u(x, |\vec{k}_T|, Q) \int d\Phi\left(p_B + k \to \{p_e, p\}\right) \frac{1}{2xs}\, |\overline{\mathcal{M}}(e^-u^* \to e^-u)|^2$$

$$\times\, \delta\left(Q^2 + (p_B - p_e)^2\right) \delta\left(x_{Bj} - \frac{Q^2}{2p_A\cdot(p_B - p_e)}\right)$$



$$p_A^2 = p_B^2 = p_e^2 = p^2 = 0 \qquad s = 2p_A\cdot p_B \qquad y = \frac{Q^2}{x_{Bj}\, s}$$

collinear factorization: $\mathcal{F}_u(x, |\vec{k}_T|, Q) \to f_u(x, Q)\, \delta\left(|\vec{k}_T|^2\right)$

Electron-hadron scattering

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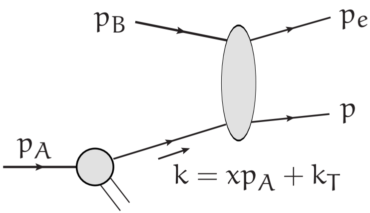

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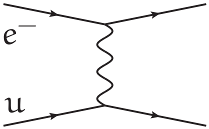
collinear factorization: $\mathcal{F}_u(x,|\vec{k}_T|,Q) \to f_u(x,Q)\,\delta\left(|\vec{k}_T|^2\right)$

$$\int d\Phi\left(p_B + k \to \{p_e, p\}\right)\delta\left(Q^2 + (p_B - p_e)^2\right)\delta\left(x_{Bj} - \frac{Q^2}{2p_A\cdot(p_B - p_e)}\right)$$

$$= \left\{\text{collinear}: \frac{\delta(x - x_{Bj})}{8\pi\,x_{Bj}\,s}\;,\;\; k_T\text{-factorization}: \frac{1}{16\pi^2\,x_{Bj}^2\,s\sqrt{\Delta(x,k_T)}}\right\}$$

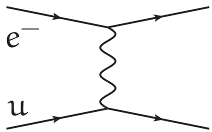
$$\Delta(x,|\vec{k}_T|) = -\prod_{i=1}^{4}\left(\frac{|\vec{k}_T|}{Q} \pm \sqrt{x/x_{Bj} - y} \pm \sqrt{1-y}\right)$$

Electron-hadron scattering



$$\frac{\left|\overline{\mathcal{M}}(e^- u \to e^- u)\right|^2}{(4\pi\alpha C_u)^2} = \frac{2xs}{Q^2}\frac{1+(1-\tilde{y})^2}{\tilde{y}} \quad , \quad \tilde{y} = \frac{x_{\mathrm{Bj}}}{x}y$$

Electron-hadron scattering



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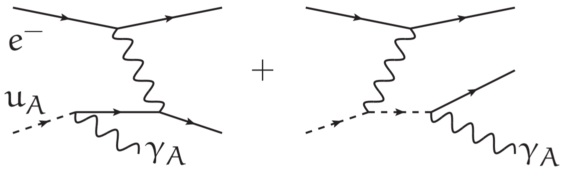
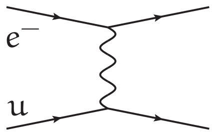


$$\frac{\left|\overline{\mathcal{M}}(e^- u^* \to e^- u)\right|^2}{(4\pi\alpha C_u)^2} = \frac{2xs}{Q^2}\frac{1+(1-y)^2}{y}$$

Electron-hadron scattering



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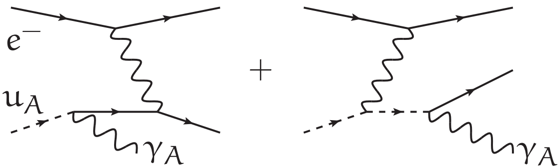
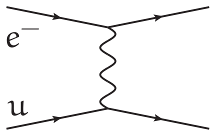


$$\frac{\left|\overline{\mathcal{M}}(e^- u^* \to e^- u)\right|^2}{(4\pi\alpha C_u)^2} = \frac{2xs}{Q^2}\frac{1+(1-y)^2}{y}$$

$$\frac{1}{2\pi\alpha^2}\,\frac{x_{Bj}\,Q^4}{1+(1-y)^2}\,\frac{d^2\sigma^{u-\text{quark}}_{e^-p\to e^-X}}{dx_{Bj}\,dQ^2} = \frac{C_u^2}{\pi}\int_{Q^2/s}^{1} dx \int_0^\infty dk_T^2\, \mathcal{F}_u(x, k_T^2, Q)\, \frac{\theta\big(\Delta(x,k_T)\big)}{\sqrt{\Delta(x,k_T)}}$$

Electron-hadron scattering



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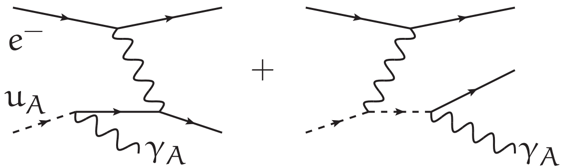

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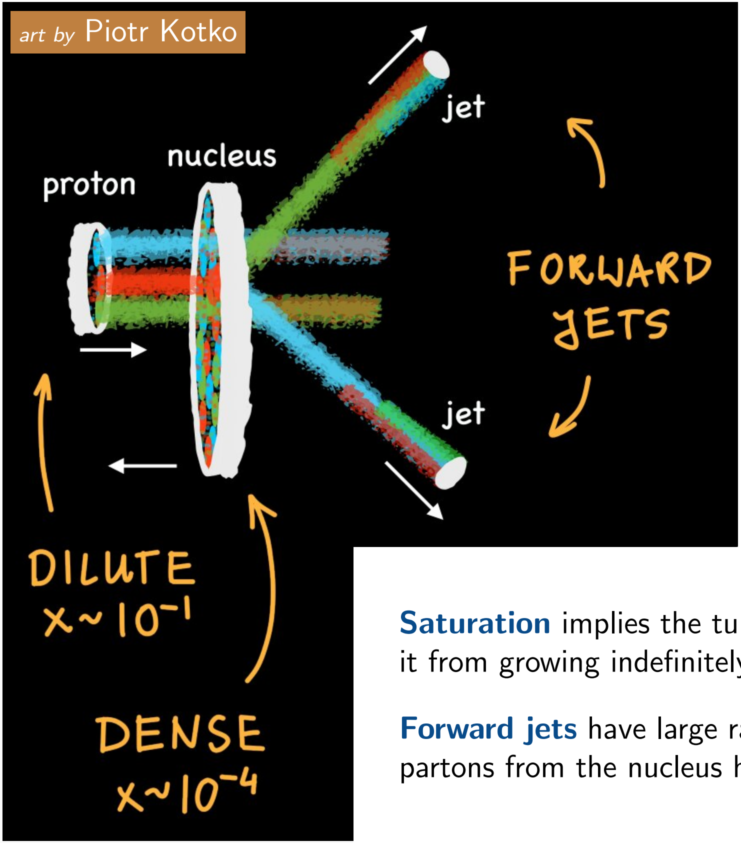
$$\left[\xi = 1 + \frac{k_T^2}{Q^2} - 2\cos(\pi\rho)\sqrt{1-y}\,\frac{k_T}{Q}\right] \qquad = C_u^2\, x_{Bj} \int_0^1 d\rho \int_0^{Q^2\kappa_+(1)} dk_T^2\, \mathcal{F}_u\left(x_{Bj}\xi(\rho,k_T)\,,\, k_T^2\,,\, Q\right)$$

QCD evolution, dilute vs. dense, forward jets



A dilute system carries a few high-x partons contributing to the hard scattering.

A dense system carries many low-x partons.

At high density, gluons are imagined to undergo recombination, and to saturate.

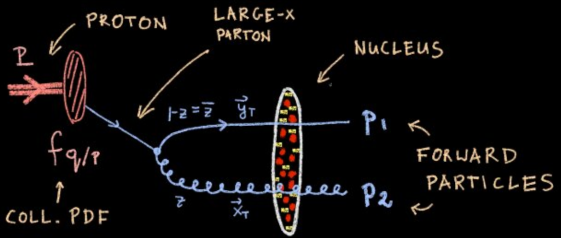
This is modeled with non-linear evolution equations, involving explicit non-vanishing k_T.

Saturation implies the turnover of the gluon density, stopping it from growing indefinitely for small x.

Forward jets have large rapidities, and trigger events in which partons from the nucleus have small x.

art by Piotr Kotko

pA (dilute-dense) collisions within CGC



$$\frac{d\sigma_{qA\to 2j}}{d^3p_1 d^3p_2} \sim \int \frac{d^2x}{(2\pi)^2}\frac{d^2x'}{(2\pi)^2}\frac{d^2y}{(2\pi)^2}\frac{d^2y'}{(2\pi)^2}\, e^{-i\vec{p}_{T1}\cdot(\vec{x}_T-\vec{x}'_T)}e^{-i\vec{p}_{T2}\cdot(\vec{y}_T-\vec{y}'_T)}$$

$$\times\,\psi_z^*\left(\vec{x}'_T-\vec{y}'_T\right)\psi_z\left(\vec{x}_T-\vec{y}_T\right)$$ ← QUARK WAVE FUNCTION

$$\times\Big\{S_x^{(6)}\left(\vec{y}_T,\vec{x}_T,\vec{y}'_T,\vec{x}'_T\right)-S_x^{(4)}\left(\vec{y}_T,\vec{x}_T,\bar{z}\vec{y}'_T+z\vec{x}'_T\right)$$

$$-S_x^{(4)}\left(\bar{z}\vec{y}_T+z\vec{x}_T,\vec{y}'_T,\vec{x}'_T\right)-S_x^{(2)}\left(\bar{z}\vec{y}_T+z\vec{x}_T,\bar{z}\vec{y}'_T+z\vec{x}'_T\right)\Big\}$$

CORRELATORS OF WILSON LINES

$$S_x^{(2)}\left(\vec{y}_T,\vec{x}_T\right)=\frac{1}{N_c}\left\langle \mathrm{Tr}\, U(\vec{y}_T)U^\dagger(\vec{x}_T)\right\rangle_x$$

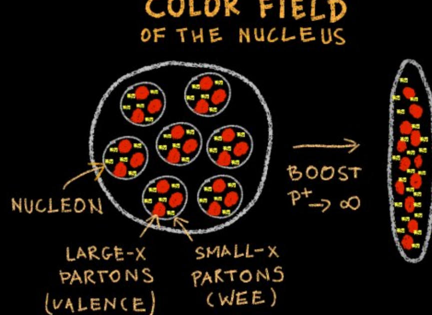
$$S_x^{(4)}\left(\vec{z}_T,\vec{y}_T,\vec{x}_T\right)=\frac{1}{2C_FN_c}\left\langle \mathrm{Tr}\left[U(\vec{z}_T)U^\dagger(\vec{y}_T)\right]\mathrm{Tr}\left[U(\vec{y}_T)U^\dagger(\vec{x}_T)\right]\right\rangle_x-S_x^{(2)}\left(\vec{z}_T,\vec{x}_T\right)$$

etc...

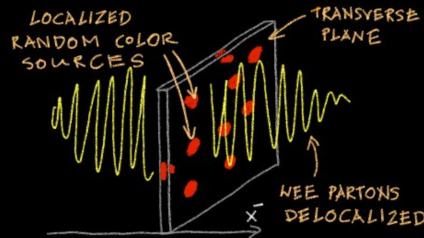
$$U(\vec{x}_T)=\mathscr{P}\exp\left\{ig\int_{-\infty}^{+\infty}dx^+\,A_a^-\left(x^+,\vec{x}_T\right)t^a\right\}$$

[C. Marquet, 2007]

COLOR FIELD OF THE NUCLEUS



[L. McLerran, R. Venugopalan, 1993]



Large-x partons — the color source for wee partons:

$$\left(D_\mu F^{\mu\nu}\right)_a\left(x^-,\vec{x}_T\right)=\delta^{\nu+}\rho_a(\vec{x}_T)\,\delta(x^-)$$

RANDOM DISTRIBUTION OF COLOR SOURCES

AVERAGE OVER COLOR SOURCES
GAUSSIAN FUNCTIONAL
B-JIMWLK EVOLUTION IN X → $\mathscr{W}_x[\rho]$

[Balitsky-Jalilian-Marian-Iancu-McLerran -Weigert-Leonidov-Kovner, 1996-2002]

ITMD Factorization

For forward dijet production
in dilute-dense hadronic collisions

Generalized TMD factorization (Dominguez, Marquet, Xiao, Yuan 2011)

$$d\sigma_{AB\to X} = \int dk_T^2 \int dx_A \sum_i \int dx_B \sum_b \phi_{gb}^{(i)}(x_A, k_T, \mu)\, f_{b/B}(x_B, \mu)\, d\hat{\sigma}_{gb\to X}^{(i)}(x_A, x_B, \mu)$$

For $x_A \ll 1$ and $P_T \gg k_T \sim Q_s$ (jets almost back-to-back).
TMD gluon distributions $\phi_{gb}^{(i)}(x_A, k_T, \mu)$ satisfy non-linear evolution equations.
Partonic cross section $d\hat{\sigma}_{gb}^{(i)}$ is on-shell, but depends on color-structure i.

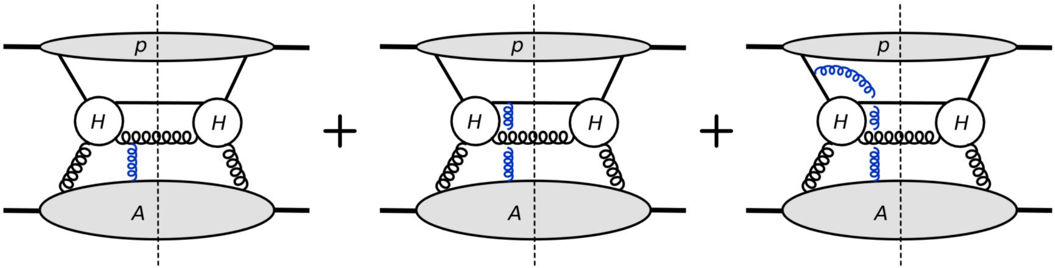
Improved TMD factorization (Kotko, Kutak, Marquet, Petreska, Sapeta, AvH 2015)

$$d\sigma_{AB\to X} = \int dk_T^2 \int dx_A \sum_i \int dx_B \sum_b \phi_{gb}^{(i)}(x_A, k_T, \mu)\, f_{b/B}(x_B, \mu)\, d\hat{\sigma}_{gb\to X}^{(i)}(x_A, x_B, k_T, \mu)$$

Originally a model interpolating between High Energy Factorization and Generalized TMD factorization: $P_T \gtrsim k_T \gtrsim Q_s$.
Partonic cross section $d\hat{\sigma}_{gb}^{(i)}$ is off-shell and depends on color-structure i.

ITMD formalism is obtained from the CGC formalism, by including so-called kinematic twist corrections (Antinoluk, Boussarie, Kotko 2019).

Definition of gluon TMDs



$+$ similar diagrams with $2, 3, \ldots$ gluon exchanges

Resummation of gluon exchanges leads to Wilson line $U_\gamma = \mathcal{P}\exp\left\{-ig\int_\gamma dz\cdot A(z)\right\}$ acting as a gauge link for the gauge invariant definition of a TMD

$$\mathcal{F}_{g/A}(x, k_T) = 2\int \frac{d^4\xi\, \delta(\xi^+)}{(2\pi)^3\, p_A^+} \exp\{ixp_A^+\xi^- - i\vec{k}_T\cdot\vec{\xi}_T\} \left\langle A \middle| \mathrm{Tr}\{\hat{F}^{i+}(\xi)U_{\gamma(\xi,0)}\hat{F}^{i+}(0)\} \middle| A \right\rangle$$

ITMD* factorization for more than 2 jets

* only manifestly gauge invariant contribution included

Schematic hybrid (non-ITMD) factorization fomula

$$d\sigma = \sum_{y=g,u,d,\dots} \int dx_1 d^2k_T \int dx_2 \; d\Phi_{g^*y\to n} \, \frac{1}{\mathsf{flux}_{gy}} \, \mathcal{F}_g(x_1, k_T, \mu) \, f_y(x_2, \mu) \sum_{\mathsf{color}} \left| \mathcal{M}^{(\mathsf{color})}_{g^*y\to n} \right|^2$$

$$\mathcal{F}_g \sum_{\mathsf{color}} \left| \mathcal{M}^{(\mathsf{color})} \right|^2 = \mathcal{F}_g \sum_{i_1,i_2,\dots,i_{n+2}} \sum_{j_1,j_2,\dots,j_{n+2}} \left(\tilde{\mathcal{M}}^{i_1 i_2 \dots i_{n+2}}_{j_1 j_2 \dots j_{n+2}} \right)^* \left(\tilde{\mathcal{M}}^{i_1 i_2 \dots i_{n+2}}_{j_1 j_2 \dots j_{n+2}} \right)$$

ITMD* factorization for more than 2 jets

Schematic hybrid (non-ITMD) factorization fomula

$$d\sigma = \sum_{y=g,u,d,\dots} \int dx_1 d^2k_T \int dx_2\, d\Phi_{g^*y\to n} \frac{1}{\text{flux}_{gy}} \mathcal{F}_g(x_1, k_T, \mu)\, f_y(x_2,\mu) \sum_{\text{color}} \left|\mathcal{M}^{(\text{color})}_{g^*y\to n}\right|^2$$

ITMD* formula: replace

$$\mathcal{F}_g \sum_{\text{color}} \left|\mathcal{M}^{(\text{color})}\right|^2 = \mathcal{F}_g \sum_{i_1,i_2,\dots,i_{n+2}} \sum_{j_1,j_2,\dots,j_{n+2}} \left(\tilde{\mathcal{M}}^{i_1 i_2\dots i_{n+2}}_{j_1 j_2\dots j_{n+2}}\right)^* \left(\tilde{\mathcal{M}}^{i_1 i_2\dots i_{n+2}}_{j_1 j_2\dots j_{n+2}}\right)$$

with (Bomhof, Mulders, Pijlman 2006; Bury, Kotko, Kutak 2018)

$$(N_c^2-1) \sum_{i_1,\dots,i_n} \sum_{j_1,\dots,j_{n+2}} \sum_{\bar{\imath}_1,\dots,\bar{\imath}_{n+2}} \sum_{\bar{\jmath}_1,\dots,\bar{\jmath}_{n+2}} \left(\tilde{\mathcal{M}}^{i_1 i_2\cdots i_{n+2}}_{j_1 j_2\cdots j_{n+2}}\right)^* \left(\tilde{\mathcal{M}}^{\bar{\imath}_1 \bar{\imath}_2\cdots \bar{\imath}_{n+2}}_{\bar{\jmath}_1 \bar{\jmath}_2\cdots \bar{\jmath}_{n+2}}\right)$$

$$\times\, 2\int \frac{d^4\xi}{(2\pi)^3 P^+} \delta(\xi_+)\, e^{ik\cdot\xi} \left\langle P\right| \left(\hat{F}^+(\xi)\right)_{i_1}^{j_1} \left(\hat{F}^+(0)\right)_{\bar{\imath}_1}^{\bar{\jmath}_1} \left(\mathcal{U}^{[\lambda_2]}\right)_{i_2\bar{\imath}_2} \left(\mathcal{U}^{[\lambda_2]\dagger}\right)^{j_2\bar{\jmath}_2} \cdots$$

$$\cdots \left(\mathcal{U}^{[\lambda_{n+2}]}\right)_{i_{n+2}\bar{\imath}_{n+2}} \left(\mathcal{U}^{[\lambda_{n+2}]\dagger}\right)^{j_{n+2}\bar{\jmath}_{n+2}} \left|P\right\rangle$$

where P is the light-like momentum of the hadron (with $P^- = 0$), and $k^\mu = xP^\mu + k_T^\mu$, where $\hat{F}$ is the field strenght,
and $\mathcal{U}^\pm$ is a Wilson line from 0 to ξ via a "staple-like detour" to $\pm\infty$ depending on the type and state (initial/final) of parton.

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with (Bomhof, Mulders, Pijlman 2006; Bury, Kotko, Kutak 2018)

$$(N_c^2-1) \sum_{i_1,\ldots,i_n} \sum_{j_1,\ldots,j_{n+2}} \sum_{\bar{\imath}_1,\ldots,\bar{\imath}_{n+2}} \sum_{\bar{\jmath}_1,\ldots,\bar{\jmath}_{n+2}} \left(\tilde{\mathcal{M}}^{i_1 i_2\cdots i_{n+2}}_{j_1 j_2\cdots j_{n+2}}\right)^* \left(\tilde{\mathcal{M}}^{\bar{\imath}_1 \bar{\imath}_2\cdots \bar{\imath}_{n+2}}_{\bar{\jmath}_1 \bar{\jmath}_2\cdots \bar{\jmath}_{n+2}}\right)$$

$$\times 2 \int \frac{d^4\xi}{(2\pi)^3 P^+} \delta(\xi_+)\, e^{ik\cdot\xi} \left\langle P \middle| \left(\hat{F}^+(\xi)\right)^{j_1}_{i_1} \left(\hat{F}^+(0)\right)^{\bar{\jmath}_1}_{\bar{\imath}_1} \left(\mathcal{U}^{[\lambda_2]}\right)_{i_2\bar{\imath}_2} \left(\mathcal{U}^{[\lambda_2]\dagger}\right)^{j_2\bar{\jmath}_2} \cdots\right.$$

$$\left.\cdots \left(\mathcal{U}^{[\lambda_{n+2}]}\right)_{i_{n+2}\bar{\imath}_{n+2}} \left(\mathcal{U}^{[\lambda_{n+2}]\dagger}\right)^{j_{n+2}\bar{\jmath}_{n+2}} \middle| P \right\rangle$$

where P is … $P^\mu + k_T^\mu$,

where $\hat{F}$ is …

and $\mathcal{U}^\pm$ is … ling on the

type and st…

$$\tilde{\mathcal{M}}^{i_1 i_2\ldots i_{n+2}}_{j_1 j_2\ldots j_{n+2}} = \sum_{\sigma\in S_{n+2}} \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \cdots \delta^{i_{n+2}}_{j_{\sigma(n+2)}} \mathcal{A}_\sigma$$

ITMD* factorization for more than 2 jets

Schematic hybrid (non-ITMD) factorization fomula

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ITMD* formula: replace

$$\mathcal{F}_g \sum_{\mathrm{color}} \left|\mathcal{M}^{(\mathrm{color})}\right|^2 = \mathcal{F}_g \sum_{\sigma\in S_{n+2}} \sum_{\tau\in S_{n+2}} \mathcal{A}^*_\sigma\, \mathcal{C}_{\sigma\tau}\, \mathcal{A}_\tau \quad , \quad \mathcal{C}_{\sigma\tau} = N_c^{\lambda(\sigma,\tau)}$$

with "TMD-valued color matrix"

$$(N_c^2-1) \sum_{\sigma\in S_{n+2}} \sum_{\tau\in S_{n+2}} \mathcal{A}^*_\sigma\, \tilde{\mathcal{C}}_{\sigma\tau}(x,|k_T|)\, \mathcal{A}_\tau \quad , \quad \tilde{\mathcal{C}}_{\sigma\tau}(x,|k_T|) = N_c^{\bar\lambda(\sigma,\tau)} \tilde{\mathcal{F}}_{\sigma\tau}(x,|k_T|)$$

where each function $\tilde{\mathcal{F}}_{\sigma\tau}$ is one of 10 functions

$$\mathcal{F}^{(1)}_{qg} \;,\; \mathcal{F}^{(2)}_{qg} \;,\; \mathcal{F}^{(3)}_{qg}$$

$$\mathcal{F}^{(1)}_{gg} \;,\; \mathcal{F}^{(2)}_{gg} \;,\; \mathcal{F}^{(3)}_{gg} \;,\; \mathcal{F}^{(4)}_{gg} \;,\; \mathcal{F}^{(5)}_{gg} \;,\; \mathcal{F}^{(6)}_{gg} \;,\; \mathcal{F}^{(7)}_{gg}$$

ITMD* factorization for more than 2 jets

$$\mathcal{F}_{qg}^{(1)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle \quad , \quad \left\langle \cdots \right\rangle = 2\int \frac{d^4\xi\,\delta(\xi_+)}{(2\pi)^3 P^+}\, e^{ik\cdot\xi}\left\langle P\middle| \cdots \middle|P\right\rangle$$

$$\mathcal{F}_{qg}^{(2)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\square]}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{qg}^{(3)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[\square]}\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(1)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\square]\dagger}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(2)}(x,k_T) = \frac{1}{N_c}\left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[\square]\dagger}\right]\mathrm{Tr}\left[\hat{F}^{i+}(0)\,\mathcal{U}^{[\square]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(3)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(4)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[-]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(5)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[\square]\dagger}\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[\square]}\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(6)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\square]}\right]}{N_c}\frac{\mathrm{Tr}\left[\mathcal{U}^{[\square]\dagger}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(7)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\square]}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[\square]\dagger}\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

ITMD gluons

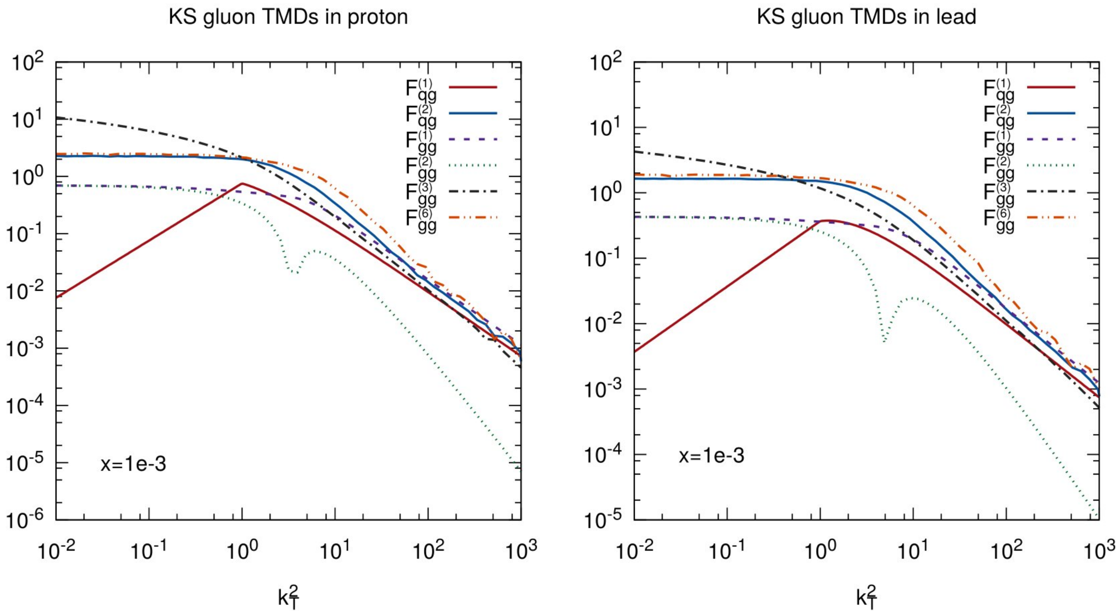
Start with dipole distribution $\mathcal{F}_{qg}^{(1)}(x, k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$ evolved via the BK equation formulated in momentum space supplemented with subleading corrections and fitted to F_2 data (Kutak, Sapeta 2012)

All other distribution appearing in dijet production, $\mathcal{F}_{qg}^{(2)}, \mathcal{F}_{gg}^{(1)}, \mathcal{F}_{gg}^{(2)}, \mathcal{F}_{gg}^{(6)}$, in the mean-field approximation (AvH, Marquet, Kotko, Kutak, Sapeta, Petreska 2016).

This is, at leading order in $1/N_c$. In this approximation, the same distributions suffice for trijets.

ITMD gluons

Bury, AvH, Kotko, Kutak 2020

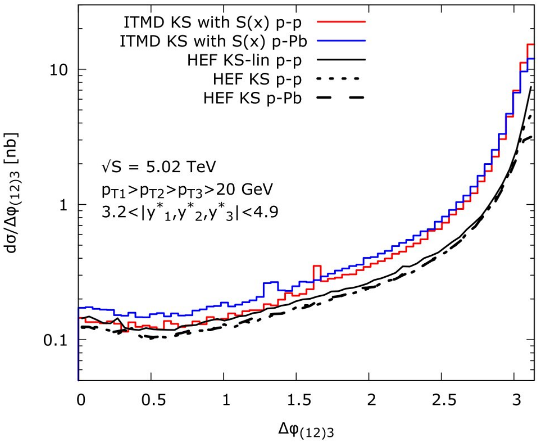


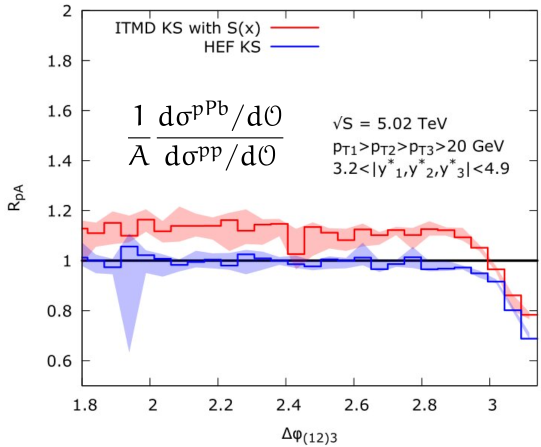
Dependence of $\mathcal{F}_{qg}^{(1)}$ on k_T below 1GeV approximated by power-like fall-off. For higher values of $|k_T|$ it is a solution to the BK equation.

TMDs decrease as $1/|k_T|$ for increasing $|k_T|$, except $\mathcal{F}_{gg}^{(2)}$, which decreases faster (even becomes negative, absolute value shown here).

pp and pPb to forward trijets

Bury, AvH, Kotko, Kutak 2020





$\Delta\varphi_{(12),3}$ is the angle between the sum of the two hardest jets, and the third jet. Is particularly sensitive to the final-state momentum inbalance.

ITMD* normalization significantly larger than HEF, due to different shape and normalization of the extra TMDs present in ITMD* but not in HEF.

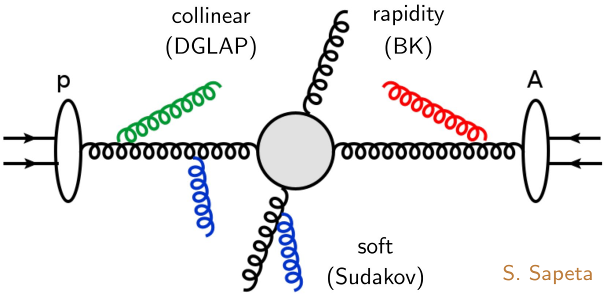
$S(x)$ refers to the x-dependent treatment of the nuclear target area, guaranteeing unitarity.

Saturation effect for $\Delta\varphi_{(12)3} \approx \pi$, enhancement of pPb result for $\Delta\varphi_{(12)3} < \pi$ due to broadening of the TMD distributions.

Calculations with LxJet (Kotko) and KaTie (AvH).

Sudakov resummation for dijets

Having hard jets in the final state, large logarithms associated with the hard scale have to be resummed. This resummation can be accounted for by inclusion of the Sudakov factor.



S. Sapeta

Within the small-x saturation formalism, Sudakov effects are most conveniently included in b-space (Mueller, Xiao, Yuan 2013; Staśto, Wei, Xiao, Yuan 2018)

$$\mathcal{F}_{g^*/B}^{ag\to cd}(x, q_T, \mu) = \frac{-N_c S_\perp}{2\pi\alpha_s}\int \frac{b_T db_T}{2\pi} J_0(b_T q_T)\, e^{-S_{\text{Sud}}^{ag\to cd}(\mu, b_T)}\, \nabla_{b_T}^2 S(x, b_T)$$

where $S_\perp$ is the transverse area of the target, and $S(x, b_T)$ the dipole scattering amplitude. This can be translated into a relation for momentum dependent distributions as

$$\mathcal{F}_{g^*/B}^{ag\to cd}(x, k_T, \mu) = \int db_T\, b_T\, J_0(b_T k_T)\, e^{-S_{\text{Sud}}^{ag\to cd}(\mu, b_T)} \int dk_T'\, k_T'\, J_0(b_T k_T')\, \mathcal{F}_{g^*/B}(x, k_T')$$

Sudakov resummation for dijets

The Sudakov receives perturbative and non-perturbative contributions for each cannel

$$S_{Sud}^{ab\to cd}(\mu, b_T) = \sum_{i=a,b,c,d} S_p^i(\mu, b_T) + \sum_{i=a,c,d} S_{np}^i(\mu, b_T)$$

Perturbative part Mueller, Xiao, Yuan 2013; Staśto, Wei, Xiao, Yuan 2018

$$S_p^i(Q, b_T) = \frac{\alpha_s}{2\pi}\int_{\mu_b^2}^{Q^2}\frac{d\mu^2}{\mu^2}\left[A^i \ln\frac{Q^2}{\mu^2} - B^i\right]$$

$$\{A, B\}^{qg\to qg} = \left\{2(C_A + C_B)\,,\, 3C_F + 2C_A\beta_0\right\} \;\;,\;\; \{A, B\}^{gg\to gg} = \left\{4C_A\,,\, 6C_A\beta_0\right\}$$

$$\mu_b = 2e^{-\gamma_E}/b_* \;\;,\;\; b_* = b_T/\sqrt{1 + b_T^2/b_{max}^2} \;\;,\;\; b_{max} = 0.5\text{GeV}^{-1}$$

Non-perturbative part Sun, Isaacson, Yuan, Yuan 2014; Prokudin, Sun, Yuan 2015

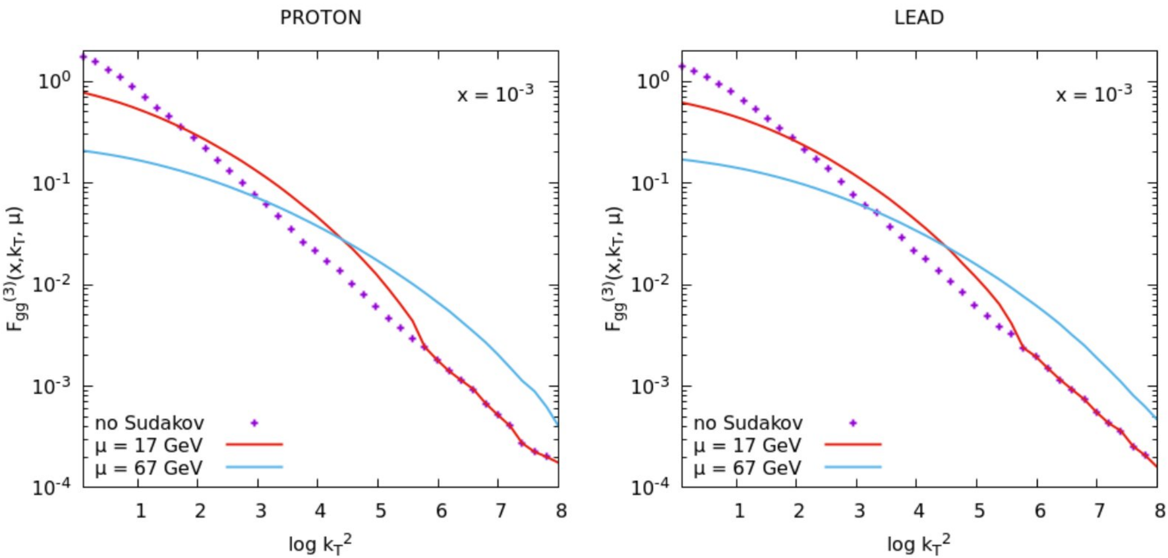
$$S_{np}^i(Q, b_T) = C^i\left[g_1 b_T^2 + g_2 \ln\frac{Q}{Q_0}\ln\frac{b_T}{b_*}\right] \;,\;\; C^{qg\to qg} = 1 + \frac{C_A}{2C_F} \;,\;\; C^{gg\to gg} = \frac{3C_A}{2C_F}$$

$$g_1 = 0.212\;,\;\; g_2 = 0.84\;,\;\; Q_0^2 = 2.4\text{GeV}^2$$

Non-perturbative contribution for small-x gluon already in TMD and omitted here.

$\mathcal{F}_{gg}(3)$ with Sudakov

AvH, Kotko, Kutak, Sapeta 2021



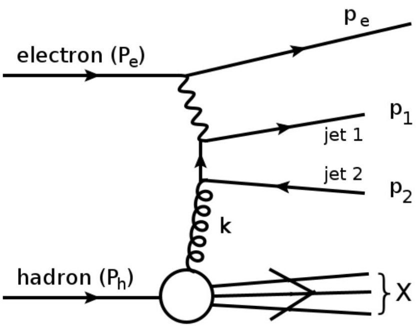
Within the Gaussian approximation, $\mathcal{F}_{gg}^{(3)}$ can be obtained from $\mathcal{F}_{qg}^{(1)}$ via

$$\mathcal{F}_{gg}^{(3)}(x,k_T) = \frac{\pi\alpha_s}{N_c k_T^2 S_\perp}\int_{k_T^2} dr_T^2 \,\ln\frac{r_T^2}{k_T^2}\int\frac{d^2q_T}{q_T^2}\,\mathcal{F}_{qg}^{(1)}(x,q_T)\,\mathcal{F}_{qg}^{(1)}(x,r_T-q_T)$$

where $S_\perp$ is the target's transverse area.

Dijets in DIS

AvH, Kotko, Kutak, Sapeta 2021

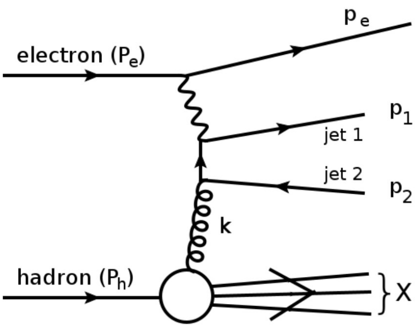


$$
d\sigma_{eh\to e'+2j+X} = \int \frac{dx}{x}\frac{d^2k_T}{\pi}\,\mathcal{F}_{gg}^{(3)}(x,k_T,\mu) \times \frac{1}{4xP_e\cdot P_h}\,d\Phi(P_e,k;p_e,p_1,p_2)\,|\overline{M}_{eg^*\to e'+2j}|^2
$$

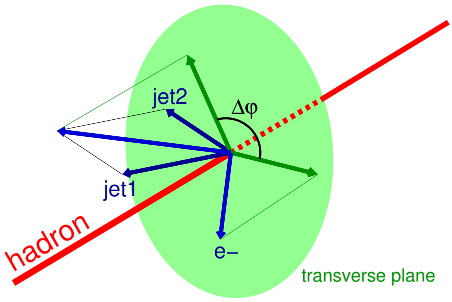
ITMD for DIS only requires $\mathcal{F}_{gg}^{(3)}$,
aka the Weizsäcker-Williams density

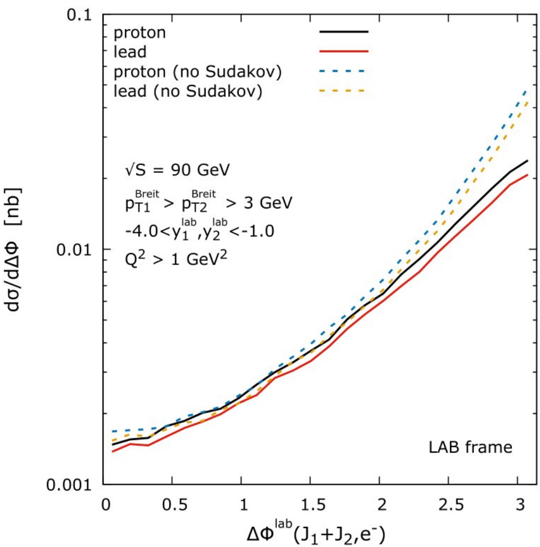
Dijets in DIS

AvH, Kotko, Kutak, Sapeta 2021



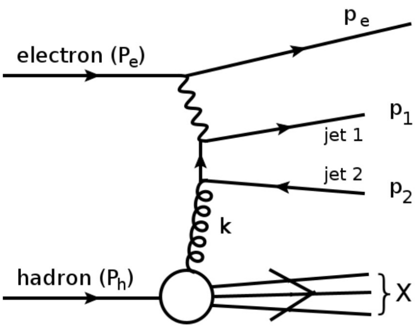
$$d\sigma_{eh\to e'+2j+X} = \int \frac{dx}{x}\frac{d^2k_T}{\pi}\,\mathcal{F}^{(3)}_{gg}(x,k_T,\mu) \times \frac{1}{4xP_e\cdot P_h}\,d\Phi(P_e,k;p_e,p_1,p_2)\,|\overline{M}_{eg^*\to e'+2j}|^2$$



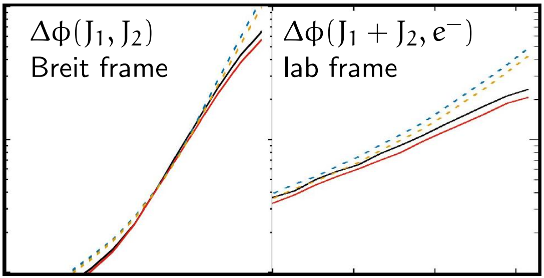


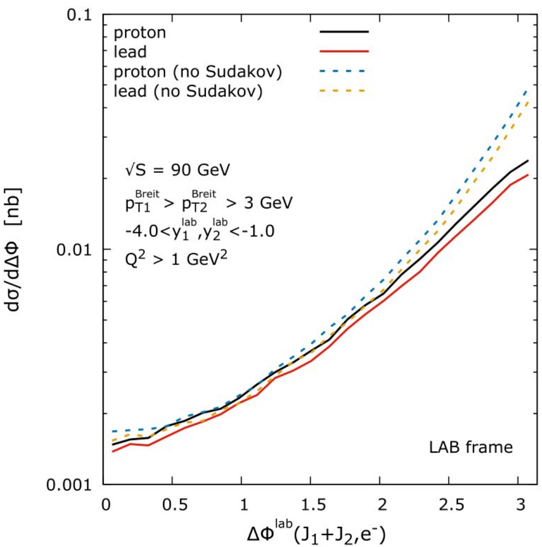
Dijets in DIS

AvH, Kotko, Kutak, Sapeta 2021



$$d\sigma_{eh\to e'+2j+X} = \int \frac{dx}{x}\frac{d^2k_T}{\pi}\,\mathcal{F}^{(3)}_{gg}(x,k_T,\mu) \times \frac{1}{4xP_e\cdot P_h}\,d\Phi(P_e,k;p_e,p_1,p_2)\,|\overline{M}_{eg^*\to e'+2j}|^2$$





Differences between curves slightly more pronounced for $\Delta\phi(J_1+J_2,e^-)$ in lab frame than for $\Delta\phi(J_1,J_2)$ in Breit frame.