Quantum fluctuation of energy and its pseudo-gauge dependence in subsystems of hot relativistic gas

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Refs: arXiv:2012.05662; Phys.Rev.D 103 (2021) 9, L091502

Work of AD is funded by the Polish National Science Center Grant No:2018/30/E/ST2/00432

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Motivation

- To model the bulk evolution of the strongly interacting matter produced in relativistic heavy-ion collisions, relativistic viscous hydrodynamics has become the basic theoretical tool. [1, 2].
- Concepts used in hydrodynamics: energy density and pressure, both are defined locally – formally, the fluid cell has zero size.
- Interestingly, hydro models which are successful in explaining the experimental data can be used to conclude about the energy density attained in the collision processes.
- Is the energy density is a well defined concept for fluid cell of arbitrary size?
- Does quantum fluctuation play any role?
- Noether's theorem does not give a unique choice for the energy-monetum tensor $\rightarrow$ pseudo gauge choices.
- Possible pseudo gauge dependence?

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[1] C. Gale, S. Jeon, B. Schenke, Int.J.Mod.Phys.A 28 (2013) 1340011

[2] S. Jeon, U. Heinz, Int.J.Mod.Phys.E 24 (2015) 10, 1530010.

Scale dependence of quantum fluctuation

- For a real scalar field the canonical energy-momentum tensor is :

$$\hat{T}^{00} = \pi\dot{\phi} - \mathcal{L} \equiv \mathcal{H}. \tag{1}$$

In the Natural units, $\hbar = c = 1$, $[\hat{T}^{00}] = [\mathcal{L}] = [M^d] = [\mathcal{H}]$.

- We define a smeared operator,

$$A(a) = (a\sqrt{\pi})^{1-d} \int d^{d-1}\mathbf{x}\, \mathcal{H}(0,\mathbf{x}) e^{-\mathbf{x}^2/a^2}. \tag{2}$$

- Variance of the operator $A(a)$,

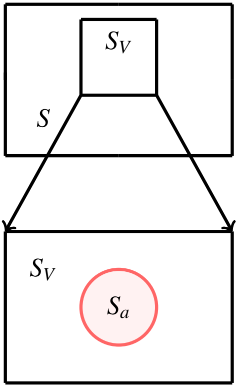
$$\mathrm{var}\, A = \langle A^2 \rangle - \langle A \rangle^2 \implies [\mathrm{var}\, A] = [A]^2 = [\mathcal{H}]^2 = [M]^{2d}. \tag{3}$$

- If we set,

$$\mathrm{var}\, A(a) \sim a^{\beta} \implies \beta = -2d \tag{4}$$

- Therefore the fluctuations of the energy density grow rapidly at small distances [3].

[3] Quantum Field Theory: Lectures of Sidney Coleman



- S is closed/isolated system described by microcanonical ensemble.
- S_V is a sub system of the closed system in equilibrium, described by the canonical ensemble. Fluctuation in energy in S_V (large volume limit):

$$\sigma_H^2 = \frac{\langle H^2 \rangle - \langle H \rangle^2}{\langle H \rangle^2} = \frac{T^2 C_V}{V \varepsilon^2} \to 0. \tag{5}$$

  H is the Hamiltonian, T is temperature, ε is energy density, C_V is specific heat.
- S_a is a subsystem of S_V which is described by the "Gaussian box": $(a\sqrt{\pi})^3 \exp(-\mathbf{x^2}/a^2)$.

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[4]Talk given by W. Florkowski at Workshop on QGP Phenomenology, Sharif University of Technology, Teheran, Iran May 28,2021

Quantum scalar field

- We describe our system by a quantum scalar field in thermal equilibrium[5].

$$\phi(t,\boldsymbol{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}} \left(a_{\boldsymbol{k}} e^{-ik\cdot x} + a^\dagger_{\boldsymbol{k}} e^{ik\cdot x} \right);\ \ [a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}] = \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}') \tag{6}$$

Single particle energy: $\omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m^2}$, $a^\mu b_\mu = a \cdot b = a^0 b^0 - \boldsymbol{a}\cdot\boldsymbol{b}$.

- Hamiltonian density:

$$\mathcal{H} = \frac{1}{2}\left(\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right). \tag{7}$$

- We define an operator $\mathcal{H}_a$ for a *finite* subsystem S_a placed at the origin of the coordinate system,

$$\mathcal{H}_a = \frac{1}{(a\sqrt{\pi})^3} \int d^3\boldsymbol{x}\, \mathcal{H}(x)\ \exp\left(-\frac{\boldsymbol{x}^2}{a^2}\right). \tag{8}$$

- Our objective:

$$\sigma^2(a,m,T) = \langle \mathcal{H}_a \mathcal{H}_a \rangle - \langle \mathcal{H}_a \rangle^2,\ \ \sigma_n(a,m,T) = \frac{(\langle \mathcal{H}_a \mathcal{H}_a \rangle - \langle \mathcal{H}_a \rangle^2)^{1/2}}{\langle \mathcal{H}_a \rangle}. \tag{9}$$

[5] S. Coleman, Lectures of Sidney Coleman on Quantum Field Theory. WSP, Hackensack, 12, 2018.

- Normal ordering: Composite QFT operators $\rightarrow$ Some "Normal ordering" prescription required.

$$\mathcal{H}_a \rightarrow : \mathcal{H}_a : \tag{10}$$

$$\mathcal{H}_a \mathcal{H}_a \rightarrow : \mathcal{H}_a :: \mathcal{H}_a : \tag{11}$$

- To perform thermal averaging, it is sufficient to know the thermal expectation values of the products of two and four creation and/or annihilation operators [6,7,8]

$$\langle a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}'} \rangle = \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}') f(\omega_{\boldsymbol{k}}), \tag{12}$$

$$\langle a^\dagger_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}'} a_{\boldsymbol{p}} a_{\boldsymbol{p}'} \rangle = \Big(\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p})\, \delta^{(3)}(\boldsymbol{k}'-\boldsymbol{p}') + \delta^{(3)}(\boldsymbol{k}-\boldsymbol{p}')\, \delta^{(3)}(\boldsymbol{k}'-\boldsymbol{p})\Big) f(\omega_{\boldsymbol{k}}) f(\omega_{\boldsymbol{k}'})$$

The Bose–Einstein distribution function, $f(\omega_{\boldsymbol{k}}) = 1/(\exp[\beta\, \omega_{\boldsymbol{k}}] - 1)$.

- Any other combinations of two and four creation and/or annihilation operators can be obtained through the commutation relation between $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}}$.

[6] C. Cohen-Tannoudji, B. Diu, F. Lalo e, and S. R. Hemley, Quantum mechanics: Vol. 3, Wiley, New York, 1977.

[7] T. Evans and D. A. Steer, Nucl. Phys. B 474 (1996) 481.

[8] C. Itzykson and J. Zuber, Quantum Field Theory. International Series In Pure and Applied Physics. McGraw-Hill, New York, 1980.

- The thermal expectation value of the operator $\mathcal{H}_a$ is

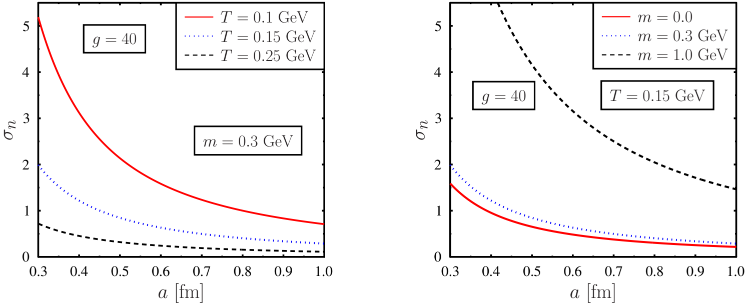
$$\langle : \mathcal{H}_a : \rangle = \int \frac{d^3k}{(2\pi)^3}\, \omega_{\boldsymbol{k}}\, f(\omega_{\boldsymbol{k}}) \equiv \varepsilon(T) \quad \text{well known result} \tag{13}$$

- Important new result: Fluctuation,

$$\sigma^2(a,m,T) = \int dK\, dK' f(\omega_{\boldsymbol{k}})(1+f(\omega_{\boldsymbol{k}'})) \times \left[(\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'} + \boldsymbol{k}\cdot\boldsymbol{k}' + m^2)^2 e^{-\frac{a^2}{2}(\boldsymbol{k}-\boldsymbol{k}')^2} + (\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'} + \boldsymbol{k}\cdot\boldsymbol{k}' - m^2)^2 e^{-\frac{a^2}{2}(\boldsymbol{k}+\boldsymbol{k}')^2}\right]. \tag{14}$$

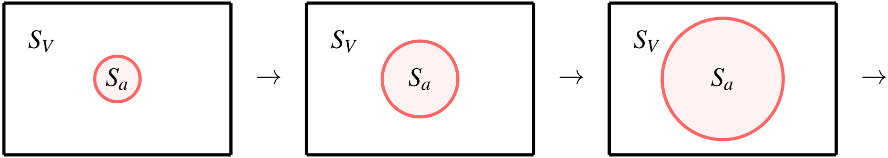
  here $dK = d^3k/((2\pi)^3 2\omega_{\boldsymbol{k}})$.
- All the vacuum energy term coming from the composite operator may not be removed by "Normal ordering".
- $\langle : \mathcal{H}_a : \rangle$ is independent of the scale a, but the fluctuation $\sigma^2(a,m,T)$ depends on the scale.
- Degeneracy factor: $\varepsilon \to g\varepsilon, \quad \sigma^2 \to g\sigma^2$.

Temperature and mass dependence of σ_n



- With a possible interpretation of heavy-ion data in mind, we consider temperatures in the range 0.1 GeV $< T < 0.25$ GeV, and particle masses: $m\ =\ 0,\ 0.3$ and 1.0 GeV.
- The value of g varies between 37 (for two quark flavor QGP) and 47.5 (for three quark flavor QGP). $g \sim 400$ for Hadron gas, but high mass hadronic contribution is thermally suppressed.
- For high T and m, σ_n is small, but the fluctuation σ is large.

Thermodynamic limit



- Gaussian representation of the three dimensional Dirac delta function

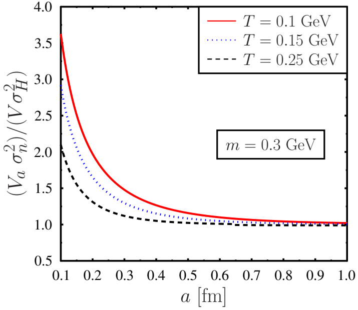
$$\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p}) = \lim_{a\to\infty} \frac{a^3}{(2\pi)^{3/2}} e^{-\frac{a^2}{2}(\boldsymbol{k}-\boldsymbol{p})^2}. \tag{15}$$

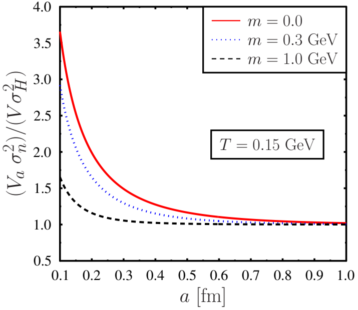
- One obtains,

$$V_a \sigma_n^2 = \frac{T^2 c_V}{\varepsilon^2} = V \frac{\langle H^2\rangle - \langle H\rangle^2}{\langle H\rangle^2} \equiv V\sigma_H^2, \tag{16}$$

- $V_a = a^3(2\pi)^{3/2}$ can be considered as the volume of the subsystem S_a — a nontrivial factor of $(2\pi)^{3/2}$ is an artifact of using the "Gaussian" box.
- $V\sigma_H^2$ can be identified as the normalized energy fluctuation in the system S_V.

Approach to the thermodynamic limit





- Variation of $V_a\sigma_n^2/V\sigma_H^2$ with the size of the subsystem S_a in the case where particles have a non vanishing mass and they obey Bose-Einstein statistics.
- One expects that in the thermodynamic limit $V_a\sigma_n^2/V\sigma_H^2$ should approach unity.
- Quantum fluctuations agree with the thermodynamic ones already for $a > 1$ fm.
- Quantum fluctuations become very important at the scale of 0.1 fm.

System of fermions

- We describe our system by a spin-$1/2$ field in thermal equilibrium. The field operator has the standard form [9]

$$\psi(t,\boldsymbol{x}) = \sum_r \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\boldsymbol{k}}}} \Big(U_r(\boldsymbol{k}) a_r(\boldsymbol{k}) e^{-ik\cdot x} + V_r(\boldsymbol{k}) b_r^\dagger(\boldsymbol{k}) e^{ik\cdot x} \Big), \tag{17}$$

- The canonical anti-commutation relations,

$$\{a_r(\boldsymbol{k}), a_s^\dagger(\boldsymbol{k}')\} = (2\pi)^3 \delta_{rs} \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}') \tag{18}$$

$$\{b_r(\boldsymbol{k}), b_s^\dagger(\boldsymbol{k}')\} = (2\pi)^3 \delta_{rs} \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}') \tag{19}$$

- Normalization of Dirac spinors,

$$\bar{U}_r(\boldsymbol{k}) U_s(\boldsymbol{k}) = 2m\delta_{rs} \tag{20}$$

$$\bar{V}_r(\boldsymbol{k}) V_s(\boldsymbol{k}) = -2m\delta_{rs} \tag{21}$$

[9] L. Tinti and W. Florkowski, arXiv:2007.04029

- To perform thermal averaging:

$$\langle a_r^\dagger(\boldsymbol{k})a_s(\boldsymbol{k}')\rangle = (2\pi)^3\delta_{rs}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')f(\omega_{\boldsymbol{k}}), \tag{22}$$

$$\begin{aligned}&\langle a_r^\dagger(\boldsymbol{k})a_s^\dagger(\boldsymbol{k}')a_{r'}(\boldsymbol{p})a_{s'}(\boldsymbol{p}')\rangle \\ &= (2\pi)^6\Big(\delta_{rs'}\delta_{r's}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p}')\ \delta^{(3)}(\boldsymbol{k}'-\boldsymbol{p}) \\ &- \delta_{rr'}\delta_{ss'}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p})\ \delta^{(3)}(\boldsymbol{k}'-\boldsymbol{p}')\Big)f(\omega_{\boldsymbol{k}})f(\omega_{\boldsymbol{k}'}).\end{aligned} \tag{23}$$

  Here $f(\omega_{\boldsymbol{k}}) = 1/(\exp[\beta\ (\omega_{\boldsymbol{k}}-\mu)]+1)$ is the Fermi–Dirac distribution function for particles.
- Anti particle operators also satisfies similar relation.
- For antiparticles, the Fermi–Dirac distribution function differs by the sign of the chemical potential μ, i.e. $\bar{f}(\omega_{\boldsymbol{k}}) = 1/(\exp[\beta\ (\omega_{\boldsymbol{k}}+\mu)]+1)$
- We consider a case with zero baryon chemical potential.

- Contrary to the real scalar field, the canonical energy momentum tensor operator is not symmetric,

$$\hat{T}^{\mu\nu}_{\text{Can}} = \frac{i}{2}\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\psi - g^{\mu\nu}\mathcal{L}_{\text{D}} = \frac{i}{2}\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\psi, \quad \overleftrightarrow{\partial}^{\mu} \equiv \overrightarrow{\partial}^{\mu} - \overleftarrow{\partial}^{\mu} \tag{24}$$

Here $\mathcal{L}_{\text{D}}$ denotes the Lagrangian density of a spin-1/2 field, which can be expressed as

$$\mathcal{L}_{\text{D}} = \frac{i}{2}\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\psi - m\bar{\psi}\psi, \tag{25}$$

- Mathematically, for any original energy-momentum tensor $\hat{T}^{\mu\nu}$ satisfying $\partial_{\mu}\hat{T}^{\mu\nu} = 0$ we can construct a different one by adding the divergence of an antisymmetric object, namely [10]

$$\hat{T}'^{\,\mu\nu} = \hat{T}^{\mu\nu} + \partial_{\lambda}\hat{A}^{\nu\mu\lambda}; \;\; \hat{A}^{\nu\mu\lambda} = -\hat{A}^{\nu\lambda\mu} \tag{26}$$

- By construction, the new tensor is also conserved, i.e., $\partial_{\mu}\hat{T}'^{\,\mu\nu} = 0$.
- For spin $1/2$ field the energy momentum tensor is "Pseudo-gauge" dependent.

[10] E. Speranza and N. Weickgenannt, "Spin tensor and pseudo-gauges: from nuclear collisions to gravitational physics," arXiv:2007.00138 [nucl-th].

- Belinfante-Rosenfeld framework (BR):

$$\hat{T}^{\mu\nu}_{\rm BR} = \frac{i}{2}\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\psi - \frac{i}{16}\partial_{\lambda}\Big(\bar{\psi}\Big\{\gamma^{\lambda},\Big[\gamma^{\mu},\gamma^{\nu}\Big]\Big\}\psi\Big). \tag{27}$$

- de Groot-van Leeuwen-van Weert framework (GLW)[11]:

$$\begin{aligned}\hat{T}^{\mu\nu}_{\rm GLW} &= -\frac{1}{4m}\bar{\psi}\overleftrightarrow{\partial}^{\mu}\overleftrightarrow{\partial}^{\nu}\psi - g^{\mu\nu}\mathcal{L}_{\rm D}\\ &= \frac{1}{4m}\Big[-\bar{\psi}(\partial^{\mu}\partial^{\nu}\psi) + (\partial^{\mu}\bar{\psi})(\partial^{\nu}\psi) + (\partial^{\nu}\bar{\psi})(\partial^{\mu}\psi)\\ &\qquad - (\partial^{\mu}\partial^{\nu}\bar{\psi})\psi\Big].\end{aligned} \tag{28}$$

- Hilgevoord-Wouthuysen framework (HW)[12]:

$$\begin{aligned}\hat{T}^{\mu\nu}_{\rm HW} = \hat{T}^{\mu\nu}_{\rm Can} &+ \frac{i}{2m}\left(\partial^{\nu}\bar{\psi}\sigma^{\mu\beta}\partial_{\beta}\psi + \partial_{\alpha}\bar{\psi}\sigma^{\alpha\mu}\partial^{\nu}\psi\right)\\ &- \frac{i}{4m}g^{\mu\nu}\partial_{\lambda}\left(\bar{\psi}\sigma^{\lambda\alpha}\overleftrightarrow{\partial}_{\alpha}\psi\right),\end{aligned} \tag{29}$$

[11] S. R. De Groot, Relativistic Kinetic Theory. Principles and Applications. 1980.

[12]. Hilgevoord and S. Wouthuysen, Nucl. Phys. 40 (1963) 1-12; J. Hilgevoord and E. De Kerf, Physica 31 No.7 (1965) 1002-1016

Energy density is pseudo-gauge independent

- We define an operator, $\hat{T}^{00}_a$:

$$\hat{T}^{00}_a = \frac{1}{(a\sqrt{\pi})^3} \int d^3\boldsymbol{x}\, \hat{T}^{00}(x) \; \exp\left(-\frac{\boldsymbol{x}^2}{a^2}\right). \tag{30}$$

- We consider the variance

$$\sigma^2(a,m,T) = \langle : \hat{T}^{00}_a :: \hat{T}^{00}_a : \rangle - \langle : \hat{T}^{00}_a : \rangle^2 \tag{31}$$

and the normalized standard deviation

$$\sigma_n(a,m,T) = \frac{\left(\langle : \hat{T}^{00}_a :: \hat{T}^{00}_a : \rangle - \langle : \hat{T}^{00}_a : \rangle^2\right)^{1/2}}{\langle : \hat{T}^{00}_a : \rangle}. \tag{32}$$

- Mean/thermal averaged $\hat{T}^{00}_a$:

$$\langle : \hat{T}^{00}_{\mathrm{Can},a} : \rangle = 4 \int \frac{d^3k}{(2\pi)^3}\, \omega_{\boldsymbol{k}}\, f(\omega_{\boldsymbol{k}}) \equiv \varepsilon_{\mathrm{Can}}(T) \tag{33}$$

$$= \langle : \hat{T}^{00}_{\mathrm{BR},a} : \rangle = \langle : \hat{T}^{00}_{\mathrm{GLW},a} : \rangle = \langle : \hat{T}^{00}_{\mathrm{HW},a} : \rangle \tag{34}$$

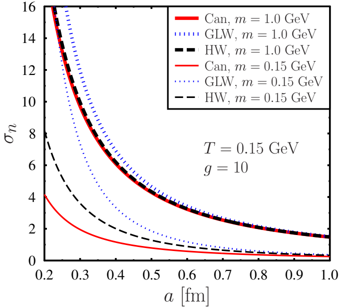
Energy density fluctuation– pseudo-gauge dependent

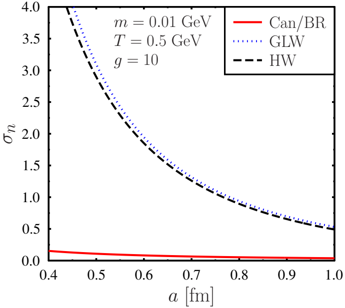
- Contrary to energy density the energy density fluctuation is pseudo-gauge dependent, e.g.
- For the Canonical framework:

$$\sigma^2_{\text{Can}}(a,m,T) = 2\int dK\, dK' f(\omega_{\boldsymbol{k}})(1-f(\omega_{\boldsymbol{k}'}))$$
$$\times\Bigg[(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})^2(\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}+\boldsymbol{k}\cdot\boldsymbol{k}'+m^2)e^{-\frac{a^2}{2}(\boldsymbol{k}-\boldsymbol{k}')^2}$$
$$-(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})^2(\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}+\boldsymbol{k}\cdot\boldsymbol{k}'-m^2)e^{-\frac{a^2}{2}(\boldsymbol{k}+\boldsymbol{k}')^2}\Bigg], \quad (35)$$

- For the de Groot-van Leeuwen-van Weert framework:

$$\sigma^2_{\text{GLW}}(a,m,T) = \frac{1}{2m^2}\int dK\, dK' f(\omega_{\boldsymbol{k}})(1-f(\omega_{\boldsymbol{k}'}))$$
$$\times\Bigg[(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})^4\left(\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}-\boldsymbol{k}\cdot\boldsymbol{k}'+m^2\right)e^{-\frac{a^2}{2}(\boldsymbol{k}-\boldsymbol{k}')^2}$$
$$-(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})^4\left(\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}-\boldsymbol{k}\cdot\boldsymbol{k}'-m^2\right)e^{-\frac{a^2}{2}(\boldsymbol{k}+\boldsymbol{k}')^2}\Bigg]. \quad (36)$$





- A comparison of the normalized standard deviation of fluctuations obtained for three different pseudo-gauges (Can=BR, GLW, HW).
- For $a < 0.5$ fm, we observe that the results obtained with various pseudo-gauges differ, with differences growing as a decreases.
- Irrespective of the choice of pseudo-gauges with growing system size the normalized standard deviation of fluctuations (σ_n) decreases.

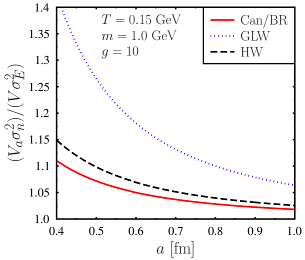
What about thermodynamic limit??

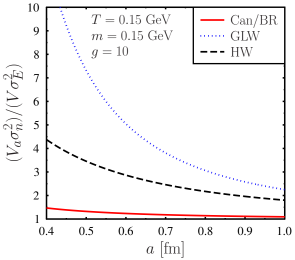
- Using the Gaussian representation of the Dirac delta function it can be shown that, in the large a limit,

$$\sigma^2_{\text{Can}} = \frac{4\,g}{(2\pi)^{3/2}a^3}\int \frac{d^3k}{(2\pi)^3}\,\omega_{\boldsymbol{k}}^2 f(\omega_{\boldsymbol{k}})(1-f(\omega_{\boldsymbol{k}})) = \sigma^2_{\text{BR}} = \sigma^2_{\text{GLW}} = \sigma^2_{\text{HW}}. \quad (37)$$

- In the large a limit we find,

$$V_a\sigma_n^2 = \frac{T^2c_V}{\varepsilon^2} = V\frac{\langle E^2\rangle - \langle E\rangle^2}{\langle E\rangle^2} \equiv V\sigma_E^2; \quad V_a = a^3(2\pi)^{3/2}. \quad (38)$$





Conclusions

- We have derived the formula characterizing the quantum fluctuation of energy in subsystems of a hot relativistic gas.
- It agrees with the expression for thermodynamic fluctuations, if the size of the subsystem is sufficiently large.
- For smaller sizes the effects of quantum fluctuations become relevant and the classical description with "well defined energy density" makes sense only after coarse graining over sufficiently large scale.
- For fermions quantum fluctuation of energy density does depend on the choice of the pseudo-gauge.
- On the practical side, the results of our calculations can be used to determine a scale of coarse graining for which the choice of the pseudo-gauge becomes irrelevant.
- This may be useful, in particular, in the context of hydrodynamic modeling of high-energy collisions.
- These results might be relevant for small systems.

Thank You!

Normal ordering: alternative approach

- For a composite operator we considered the following normal ordering method: $\mathcal{H}_a\mathcal{H}_a \rightarrow: \mathcal{H}_a :: \mathcal{H}_a :$.
- Therefore we are first normal ordering first then then multiplying to construct the composite operator.
- Alternatively one can also argue about different method of normal ordering:

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Semiclassical gravity theory and quantum fluctuations

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$$\Delta(x) \equiv \left| \frac{\langle : T_{00}{}^2(x) : \rangle - \langle : T_{00}(x) : \rangle^2}{\langle : T_{00}{}^2(x) : \rangle} \right| .$$

- If we consider such a normal ordering then:

$$\sigma^2(a,m,T) = \langle : \mathcal{H}_a\mathcal{H}_a : \rangle - \langle : \mathcal{H}_a : \rangle^2 = \int dK\, dK' f(\omega_{\boldsymbol{k}}) f(\omega_{\boldsymbol{k}'})$$
$$\times \left[(\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'} + \boldsymbol{k}\cdot\boldsymbol{k}' + m^2)^2 e^{-\frac{a^2}{2}(\boldsymbol{k}-\boldsymbol{k}')^2} + (\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'} + \boldsymbol{k}\cdot\boldsymbol{k}' - m^2)^2 e^{-\frac{a^2}{2}(\boldsymbol{k}+\boldsymbol{k}')^2} \right]. \tag{39}$$