Neutron stars in the Skyrme model

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The action for the gravitating gauged BPS baby Skyrme model is given by

$$S = \int d^3x|g|^{\frac{1}{2}} \left(-\lambda^2\pi^2|g|^{-1}g_{\alpha\beta}\tilde{\mathcal{B}}^{\alpha}\tilde{\mathcal{B}}^{\beta} - \mu^2\mathcal{U} - \frac{1}{4e^2}F^2_{\mu\nu} \right) + S_{EH}, \tag{1}$$

where S_{EH} is the Einstein-Hilbert action in (2+1) dimensional space-time and $\tilde{\mathcal{B}}^{\mu}$ is a gauge invariant version of the topological current

$$\tilde{\mathcal{B}}^{\mu} = \frac{1}{8\pi}\epsilon^{\mu\nu\rho}\vec{\phi}\cdot\left(D_{\nu}\vec{\phi}\times D_{\nu}\vec{\phi}\right). \tag{2}$$

The covariant derivatives read

$$D_{\mu}\vec{\phi} = \vec{\phi}_{\mu} + A_{\mu}\vec{n}\times\vec{\phi},\ \ \vec{n} = (0,0,1). \tag{3}$$

The corresponding Einstein equations are

$$G_{\alpha\beta} = \frac{\kappa^2}{2} T_{\alpha\beta}, \tag{4}$$

where $\kappa^2 = 16\pi G$.
In the next step we assume the axial symmetry for the metric

$$ds^2 = \mathbf{A}(r)dt^2 - \mathbf{B}(r)dr^2 - r^2 d\phi^2 \tag{5}$$

and consider only the static case with no electric field, so the gauge field reads

$$A_\mu = (0, A_1(\vec{x}), A_2(\vec{x})). \tag{6}$$

Due to axially symmetric metric we can restrict ourselves to an axially symmetric matter and gauge field

$$A_t = A_r = 0, \; A_\phi = na(r). \tag{7}$$

The Skyrme field $\vec{\phi} \in \mathcal{S}^2$, so we can express it by the stereographic projection

$$\vec{\phi} = \frac{1}{1+|u|^2}\left(u+\bar{u}, -i(u-\bar{u})1-|u|^2\right) \tag{8}$$

and applay for it the ansatz

$$u = f(r)e^{in\varphi}, \quad h = 1 - \frac{1}{1+f^2}. \tag{9}$$

Now we have everything to write down the Einstein equations together with the Maxwell equations.

Einstein equations:

$$\frac{\mathbf{B}_r}{\mathbf{B}} = \kappa^2 r \mathbf{B}\tilde{\rho},$$
$$\frac{\mathbf{A}_r}{\mathbf{A}} = \kappa^2 r \mathbf{B}\tilde{p}, \tag{10}$$
$$(\tilde{p}\mathbf{B})_r = \kappa^2 r \mu^2 \mathbf{B}^2 \mathcal{U} \tilde{p},$$

where

$$\tilde{\rho} = \frac{n^2}{2e^2 r^2 \mathbf{B}} a_r^2 + \frac{\lambda^2 n^2}{4 r^2 \mathbf{B}} (1+a)^2 h_r^2 + \mu^2 \mathcal{U}, \tag{11}$$

$$\tilde{p} = \frac{n^2}{2e^2 r^2 \mathbf{B}} a_r^2 + \frac{\lambda^2 n^2}{4 r^2 \mathbf{B}} (1+a)^2 h_r^2 - \mu^2 \mathcal{U}. \tag{12}$$

Maxwell equation:

$$\frac{n}{e^2} \partial_r \left(\sqrt{\frac{\mathbf{A}}{\mathbf{B}}} \frac{a_r}{r} \right) = \lambda^2 \sqrt{\frac{\mathbf{A}}{\mathbf{B}}} \frac{n}{2} (1+a) \frac{h_r^2}{r}. \tag{13}$$

Is the model truly BPS?

We can notice that there is a formal solution to the zero pressure condition

$$\mathbf{A} = 1 \quad \text{and} \quad \tilde{p} = 0. \tag{14}$$

Now if we perform a change of the radial variable and introduce

$$\frac{dz}{dr} = r\sqrt{\mathbf{B}} \tag{15}$$

then we can show that our model will reduce from the gravitating model to a non-gravitating one. We already know that in the gauged BPS baby Skyrme model the Bogomolny equations have a form

$$\begin{aligned} na_z &= -e^2\lambda^2 W(h), \\ \frac{n}{2}(1+a)h_z &= -W_h(h), \end{aligned} \tag{16}$$

where $W(h)$ satisfies

$$\frac{e^2\lambda^4}{2}W^2 + \lambda^2 W_h^2 = \mu^2 \mathcal{U}(h). \tag{17}$$

If we introduce a new superpotential $\omega(h)$

$$\omega(h) = \frac{\lambda}{\mu} W(h) \tag{18}$$

we get

$$\omega_h^2 + \beta^2\omega^2 = \mathcal{U}, \tag{19}$$

where the new dimensionless parameter is

$$\beta^2 = \frac{e^2\lambda^2}{2}. \tag{20}$$

Observables

- The proper mass M- the energy of the soliton

$$M = \int d^2x |g|^{\frac{1}{2}} \tilde{\rho} = 2\pi |n| \lambda\mu |\omega(h=1)|. \tag{21}$$

- The total magnetic flux Φ (Magnetic field $H = \epsilon^{12} F_{12} = n a_z$)

$$\Phi = \int d^2x |g|^{\frac{1}{2}} H = 2\pi n a(z_0) = 2\pi n a_\infty, \tag{22}$$

where

$$\begin{aligned} a_\infty &= -1 + \exp\left(-\frac{F(1)}{4}\beta^2\right), \\ F(h) &= 4\int_0^h \frac{\omega(h')}{\omega_{h'}(h')} dh'. \end{aligned} \tag{23}$$

- The geometric volume of the soliton V

$$V = \int d^2x|g|^{\frac{1}{2}} = \pi\frac{\lambda}{\mu}|n|\exp\left(-\frac{F(1)}{4}\beta^2\right)\int_0^1 \frac{\exp\left(\frac{F(h)}{4}\beta^2\right)}{\omega_h}dh. \tag{24}$$

- The M_{ADM} mass

$$M_{ADM} = 2\pi\int_0^R rdr\tilde{\rho}(r) =$$
$$= 2\pi|n|\lambda\mu|\omega(h=1)|\left(1-\frac{\kappa^2\lambda\mu}{4}|n||\omega(h=1)|\right). \tag{25}$$

Due to regularity of the metric function there exist maximal value of the topological charge n_{max}, which implies existence of the maximal mass

$$n_{max} = \left\lfloor \frac{2}{\lambda\mu\kappa^2|\omega(1)|}\right\rfloor, \quad M_{ADM}^{max} = M_{ADM}(n^{max}) = \frac{M^{max}}{2} = \frac{2\pi}{\kappa^2}. \tag{26}$$

- The radius R of the baby Skyrmion

$$\frac{R^2}{2} = \int_0^R r dr = \int_0^{z_0} dz \left(1 - \frac{\kappa^2}{2} \int_0^z \tilde{\rho}(z') dz'\right). \tag{27}$$

It can be shown that this is equal to

$$\frac{R^2}{2} = \frac{V}{2\pi} - \frac{n^2 \lambda^2 \kappa^2}{2} \mathcal{A}(\beta), \tag{28}$$

where

$$\mathcal{A}(\beta) = \int_0^1 \frac{\exp\left(\frac{F(h)-F(1)}{4}\beta^2\right)}{\omega_h} \left[\omega(1) - \right. \\ \left. - \exp\left(\frac{F(h) - F(1)}{4}\beta^2\right) \omega(h)\right] dh. \tag{29}$$

If we introduce a new variable $x = |n|/n_{max} \in [0, 1]$ then we can study the mass-radius relation in a parametric fashion

$$\begin{cases} \dfrac{\kappa^2 M_{ADM}}{2\pi} = x\,(2 - x) \\[2ex] \dfrac{\kappa^2 \mu^2 R^2}{2} = \dfrac{\mathcal{A}(\beta)}{|\omega(1)|^2} x \left(\dfrac{\mathcal{C}(\beta)|\omega(1)|}{\mathcal{A}(\beta)} - x \right) \end{cases} \tag{30}$$

where

$$\mathcal{C}(\beta) = \exp\left(-\frac{F(1)}{4}\beta^2 \right) \int_0^1 \frac{\exp\left(\frac{F(h)}{4}\beta^2 \right)}{\omega_h} dh. \tag{31}$$

Let's define a new parameter $\Omega(\beta)$

$$\Omega(\beta) = \frac{\mathcal{C}(\beta)|\omega(1)|}{\mathcal{A}(\beta)}. \tag{32}$$

Now we have three cases

- For $\Omega = 2$, M_{ADM} is a linear function of R^2.
- For $\Omega < 2$ the $M_{ADM} - R$ curve turns left at some value of the topological charge (or x) which means that the maximal radius does not coincide with the maximal mass.
- For $\Omega > 2$, where the curve bends right.

Example- Old baby potential

As an example the old baby potential was considered

$$\mathcal{U} = \frac{h}{4}. \tag{33}$$

The super-potential equation is

$$\omega_h^2 + \beta^2\omega^2 = \frac{h}{4}, \quad \omega(0) = 0. \tag{34}$$

For this equation one can find the approximated solutions by applying the perturbative expansion

$$\omega_{small} = h^{3/2}\left(\frac{1}{3} - \frac{2}{63}(\beta h)^2 + \frac{10}{6237}(\beta h)^4 - \frac{92}{5893965}(\beta h)^6\right). \tag{35}$$

$$\omega_{large} = h^{3/2}\left(\frac{1}{2}(\beta h)^{-1} - \frac{1}{16}(\beta h)^{-3} - \frac{13}{256}(\beta h)^{-5} - \frac{213}{2048}(\beta h)^{-7}\right). \tag{36}$$

The approximated solution then reads

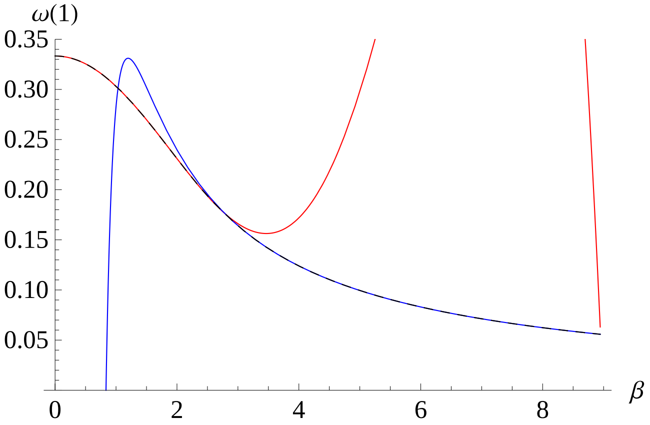
$$\omega_{approx} = \begin{cases} \omega_{small} & h \in [0, h_0] \\ \omega_{large} & h \in [h_0, 1] \end{cases} \tag{37}$$

where the gluing point h_0 is defined as

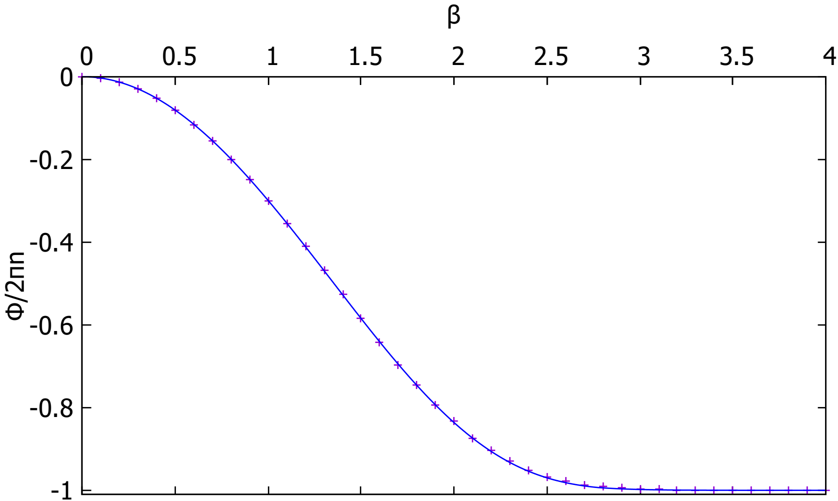
$$\omega_{small}(h_0) = \omega_{large}(h_0) \tag{38}$$

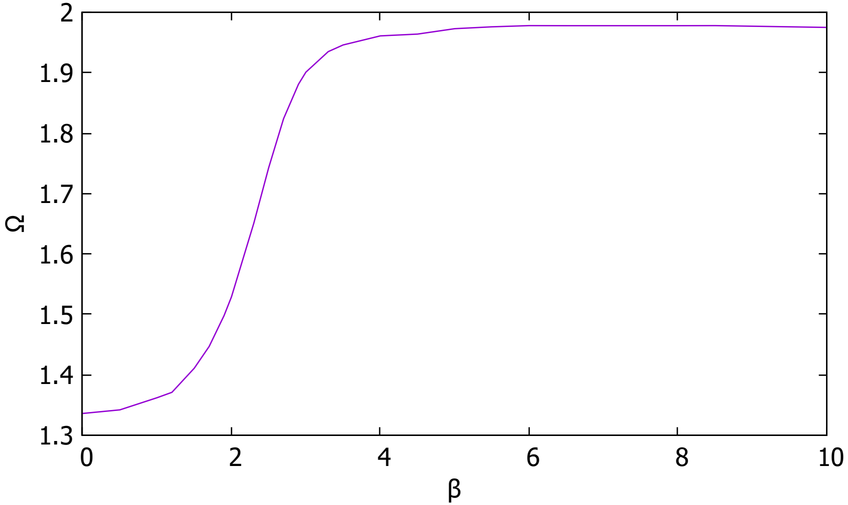
which, for the order of the expansion assumed above, is

$$h_0 = 2.7821 \frac{1}{\beta}. \tag{39}$$

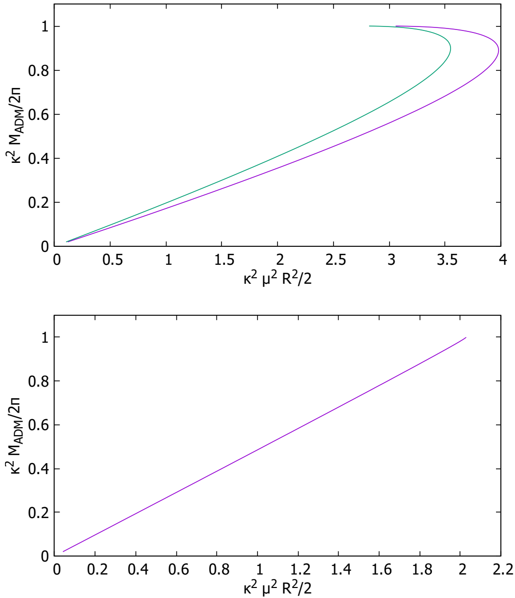


ω(1)
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
2
4
6
8
β


β
0
0.5
1
1.5
2
2.5
3
3.5
4
0
-0.2
-0.4
-0.6
-0.8
-1
Φ/2πn



2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
Ω
0
2
4
6
8
10
β



1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
4
κ2 MADM/2π
κ2 μ2 R2/2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
κ2 MADM/2π
κ2 μ2 R2/2

Summary

- Simultaneous inclusion of the gravity and magnetic field does not destroy BPS property of the BPS baby Skyrme model.
- All observables are given as some functions of the topological charge, with coefficients which are target space integrals depending on the coupling constant $\beta = e\lambda/\sqrt{2}$ and a particular model (particular potential).
- A non-zero value of the coupling constant β modifies entirely the constants in the parametric mass-radius formula leaving the functional form unchanged.

Thank you for your attention