Master equations for perturbation Einstein equations with matter

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Plan

Background

Three static fields: metric, electromagnetic field and scalar field.
In Schwarzschild–like coordinates (n+2 dimensions):

$$\bar{g} = -f(r)dt^2 + \frac{\zeta(r)}{f(r)^2}dr^2 + S(r)^2 dX_{(n)}^2\,, \tag{1}$$

$$\bar{A} = a(r)dt\,, \tag{2}$$

$$\bar{\varphi} = \phi(r)\,. \tag{3}$$

$$dX_{(n)}^2 = \begin{cases} dx_1^2 + \ldots + dx_{(n)}^2, & K = \ \ 0\,, \ \text{planar} \\ d\Omega_{(n)}, & K = +1\,, \ \text{spherical} \\ dH_{(n)}, & K = -1\,, \ \text{hyperbolic} \end{cases} . \tag{4}$$

Action:

$$S = \int d^{n+2}x\sqrt{-g}\left(R - 2\Lambda - \eta(\partial\phi)^2 - \frac{1}{4}Z(\phi)F^2 - V(\phi)\right) \tag{5}$$

Background

Einstein equations:

$$\phi'' = \phi'\left(\frac{\zeta'}{\zeta} - n\frac{S'}{S}\right) - \frac{a'^2 Z' + 4\eta f'\phi' - 2\zeta^2 V'}{4\eta f}, \tag{6}$$

$$a'' = a'\left(\frac{\zeta'}{\zeta} - n\frac{S'}{S} - \frac{Z'\phi'}{Z}\right), \tag{7}$$

$$S'' = \frac{\zeta' S'}{\zeta} - \frac{\eta}{n} S\phi'^2, \tag{8}$$

$$\begin{aligned} 0 =& S^2\left(2\eta f\phi'^2 - Za'^2\right) - 2nSf'S' - 2n(n-1)fS'^2 + \\ &+ 2\zeta^2\left(n(n-1)K - S^2(V+\Lambda)\right), \end{aligned} \tag{9}$$

$$f'' = Za'^2 + \frac{f'\zeta'}{\zeta} - (n-2)\frac{f'S'}{S} - \frac{2(n-1)}{S^2}\left(\zeta^2 K - fS'^2\right) - \frac{2\eta}{n} f\phi'^2. \tag{10}$$

Gravitational, electromagnetic and scalar perturbations

We seek for a new solution (for simplicity, we restrict to planar symmetry):

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\,\delta g_{\mu\nu} + \ldots \tag{11}$$
$$A_{\mu} = \bar{A}_{\mu} + \epsilon\,\delta a_{\mu} + \ldots \tag{12}$$
$$\varphi = \bar{\varphi} + \epsilon\,\delta\varphi + \ldots \tag{13}$$

Plug into Einstein equations, expand in ϵ.
System of nonlinear Einstein equations $\rightarrow$ infinite system of linearized equations.

Gravitational, electromagnetic and scalar perturbations

Form of perturbations (tensor, vector and scalar sectors):

$$\delta g_{\mu\nu} = \begin{pmatrix} h_{tt} & \frac{1}{2}h_{tr} & ik\,h_{tx} & 0 & \dots & 0 & h_{tz} \\ \frac{1}{2}h_{tr} & h_{rr} & ik\,h_{rt} & 0 & \dots & 0 & h_{rz} \\ ik\,h_{tx} & ik\,h_{rx} & -k^2 h_{xx} & 0 & \dots & 0 & ik\,h_{xz} \\ 0 & 0 & 0 & -k^2 h_{yy} & 0 & 0 & h_{yz} \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ h_{tz} & h_{rz} & ik\,h_{xz} & h_{yz} & 0 & 0 & -k^2 h_{zz} \end{pmatrix} e^{ikx} ,$$

$$\delta A_\mu = \left(a_t\,,\quad a_r\,,\quad ik a_x\,,\quad 0\,,\quad \dots\,,\quad 0\,,\quad a_z\right) e^{ikx}\,,$$

$$\delta\varphi = \delta\phi e^{ikx}\,, \tag{14}$$

also introduce $h_{xx} = h_+ + \frac{n}{n-1}h_-$, $h_{yy} = \dots h_{zz} = h_+ - \frac{1}{n}h_-$.

Linear gauge invariants

Gauge transformations induced by linear gauge vector $\epsilon\,\zeta^\mu$:

$$h_{\mu\nu} \to h_{\mu\nu} + \mathcal{L}_\zeta \bar{g}_{\mu\nu}\,, \tag{15}$$
$$a_\mu \to a_\mu + \mathcal{L}_\zeta \bar{A}_\mu\,, \tag{16}$$
$$\delta\phi \to \delta\phi + \mathcal{L}_\zeta \bar{\varphi}\,. \tag{17}$$

Two ways: choose a certain gauge (e.g. Regge–Wheeler ($h_{rx} = h_{tx} = h_- = 0$), Detweiller ($h_+ = h_{tx} = h_- = 0$)) or construct gauge invariants.

Linear gauge invariants

Tensor sector:

$$H_{yz} \equiv h_{yz}\,, \tag{18}$$

Vector sector:

$$H_{tz} \equiv h_{tz} - \partial_t h_{xz}\,, \tag{19}$$

$$H_{rz} \equiv h_{rz} - \partial_r h_{xz} + 2\frac{S'}{S} h_{xz}\,, \tag{20}$$

$$A_z \equiv \delta a_z\,, \tag{21}$$

$$\tag{22}$$

Scalar sector:

$$H_{tt} \equiv h_{tt} - 2\partial_t h_{tx} + \partial_t^2 h_- + k^2 \frac{f'}{2nSS'}(h_- - n h_+)\,, \tag{23}$$

$$H_{tr}\,, H_{rr}\,, H_{rx}\,, A_t\,, A_r\,, \varphi = \dots\,. \tag{24}$$

Einstein Equations can be all fulfilled introducing master scalar functions:

Tensor sector	Vector sector	Scalar sector
$\Phi_G^{(T)}$	$\boldsymbol{\Phi}^{(V)} = \begin{pmatrix} \Phi_G^{(V)} \\ \Phi_E^{(V)} \end{pmatrix}$	$\boldsymbol{\Phi}^{(S)} = \begin{pmatrix} \Phi_G^{(S)} \\ \Phi_E^{(S)} \\ \Phi_S^{(S)} \end{pmatrix}$

Master scalars fulfil coupled wave equations:

$$\left(\bar{\Box} - W^{(T)}\right) \Phi_G^{(T)} = 0 \tag{25}$$

$$\left(\bar{\Box} - \mathbf{W}^{(V)}\right) \boldsymbol{\Phi}^{(V)} = 0 \tag{26}$$

$$\left(\bar{\Box} - \mathbf{W}^{(S)}\right) \boldsymbol{\Phi}^{(S)} = 0 \tag{27}$$

$\mathbf{W}^{(V)}$ and $\mathbf{W}^{(S)}$ are 2x2 and 3x3 symmetric matrices.
Express all variables in terms of Φ's and their derivatives.

Potentials

Tensor sector:

$$W^{(T)} = 0 \tag{28}$$

Vector sector:

$$\mathbf{W}^{(V)} = \begin{pmatrix} W_G^{(V)} & W_{G,E}^{(V)} \\ W_{G,E}^{(V)} & W_E^{(V)} \end{pmatrix} , \tag{29}$$

where

$$W_G^{(V)}(r) = -n\left(\frac{f'S'}{\zeta^2 S} - \frac{fS'^2}{\zeta^2 S^2} + \frac{K}{S^2}\right) + \eta\frac{f\phi'^2}{\zeta^2} .$$

$$W_{G,E}^{(V)}(r) = -\sqrt{k^2 - nK}\,\frac{\sqrt{Z}a'}{\zeta S} ,$$

$$W_E^{(V)}(r) = -\frac{f'S'}{\zeta^2 S} + (n-2)\left(\frac{K}{S^2} - \frac{fS'^2}{\zeta^2 S^2}\right) + \frac{Za'^2}{\zeta^2} + \frac{f\eta\phi'^2}{n\zeta^2} -$$
$$\frac{1}{8\eta\zeta^2}\frac{Z'}{Z}\left(-2\zeta^2 V' + a'^2 Z'\right) - \frac{Z'^2}{Z^2}\frac{f\phi'^2}{4\zeta^2} - \frac{Z'}{Z}\frac{fS'\phi'}{\zeta^2 S} + \frac{f\phi'^2 Z''}{2\zeta^2 Z} ,$$

Tensor sector:

$$H_{yz} \equiv S^2 \Phi_G^{(T)}\,, \tag{30}$$

Vector sector:

$$\begin{aligned} H_{tz} &\equiv n\frac{fSS'}{\zeta}\Phi_G^{(V)} + \frac{fS^2}{\zeta}\partial_r \Phi_G^{(V)}\,,\\ H_{rz} &\equiv \frac{\zeta S^2}{f}\partial_t \Phi_G^{(V)}\,,\\ A_z &\equiv \sqrt{k^2 - nK}\frac{S}{\sqrt{Z}}\Phi_E^{(V)}\,. \end{aligned} \tag{31}$$

Scalar sector:

$$\begin{aligned} H_{rx} =& \sqrt{\tfrac{n-1}{n}}\sqrt{k^2-nK}\,\tfrac{\zeta S^2}{\mathcal{D}}\left(2\sqrt{\eta}k\zeta\phi'\Phi_S^{(S)} - \sqrt{2}nS'a'Z\Phi_E^{(S)}\right) + S^2\partial_r\Phi_G^{(S)}\\ &+ \tfrac{\zeta^2 S}{nS' f\mathcal{D}}\Big(nS'\big(k^2Sf' + 2fS'\big((n-2)k^2 - n(n-1)K\big)\big) + 2\eta k^2 S^2 f(\phi')^2 + 2k^4\zeta^2\Big)\Phi_G^{(S)} \end{aligned} \tag{32}$$

...

Application: quasinormal spectrum

Calculating quasinormal modes more efficient:

QNM from master scalar equation
$(\Phi(t, r) = e^{-i\omega t}\Psi(r))$:

n	$Re(\omega_n)$	$Im(\omega_n)$
1	± 0.7414299655	-0.2862800072
2	± 1.733511095	-1.343007549
3	± 2.705539866	-2.357061908
4	± 3.689391462	-3.363863379
5	± 4.678735426	-4.367980846
6	± 5.671090621	-5.370783926
7	± 6.665291123	-6.372835299
8	± 7.660712908	-7.374412239
9	± 8.656989607	-8.375668528
10	± 9.653890825	-9.376696881
11	± 10.65126380	-10.37755679
12	± 11.649003	-11.378288
13	± 12.6470	-12.3789
14	± 13.65	-13.38
15	$\pm 15.$	$-14.$

QNM (Kovtun, Starinets):

n	$Re(\omega_n)$	$Im(\omega_n)$
1	± 0.7414299655	-0.2862800072
2	± 1.733511095	-1.343007549
3	± 2.705539866	-2.357061908
4	± 3.689391462	-3.363863379
5	± 4.678735426	-4.367980846
6	± 5.671090621	-5.370783926
7	± 6.66529112	-6.37283530
8	± 7.66071	-7.37441
9	± 8.65	-8.37

Spectrum was found using A. Jansen, Eur. Phys. J. Plus 132:546 (2017).

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Perturbation analysis for gravitational and electromagnetic radiation in a Reissner-Nordström geometry

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We consider the gravitational and electromagnetic fields produced by a charged (or uncharged) test particle moving in a Reissner-Nordström geometry as perturbations on the background Reissner-Nordström geometry and its associated electric field, respectively. The gravitational perturbations are expanded in tensor harmonics in the manner of Regge and Wheeler, while the electromagnetic field is expanded in vector harmonics. Following a previously proposed convention, we find that in the Einstein-Maxwell system of equations, electric gravitational multipoles couple only to electric (TM) electromagnetic multipoles and similarly for magnetic multipoles. It is possible to reduce the entire Einstein-Maxwell system for each type of multipole to two second-order Schrödinger-type equations.

$$\frac{d^2R^{(e)}_{LM}}{dr^{*2}}+(\omega^2-V^{(e)}_L)R^{(e)}_{LM}=-e^{2\nu}(\lambda r^2+3mr-2e^2)^{-1}[e^{-\nu}r^{-2}(\lambda r+m)+(2\lambda r+3m)(\lambda r^2+3mr-2e^2)^{-1}](8e/i\omega)f^{(e)}_{LM}$$
$$+S^{(\mathrm{matter})}_{LM}, \tag{48}$$

$$\frac{d^2f^{(e)}_{LM}}{dr^{*2}}+\left\{\omega^2-e^{\nu}\left[\frac{L(L+1)}{r^2}+e^{\nu}\frac{4e^2}{r^2(\lambda r^2+3mr-2e^2)}\right]\right\}f^{(e)}_{LM}=-\frac{i\omega e}{r^2}e^{\nu}\left[f(r)R^{(e)}_{LM}+\frac{dR^{(e)}_{LM}}{dr^*}\right]+\frac{i\omega e}{2r^2}e^{\nu}A_{23}$$
$$+\frac{i\omega e}{r^2}\hat{S}_1+4\pi e^{\nu}\left[e^{\nu}u-\frac{d}{dr^*}(e^{\nu}w)\right], \tag{49}$$

Nonlinear Reissner–Nordström perturbations

- for every sector there exist two master scalar variables $\Phi_G^{(S)}(t,r)$ and $\Phi_E^{(S)}(t,r)$ fulfilling a coupled system of wave equations:

$$r(-\bar{\Box} + V_G^{(S)})\frac{{}^{(i)}\Phi_G^{(S)}}{r} + V_{EG}^{(S)}\ {}^{(i)}\Phi_E^{(S)} = {}^{(i)}\tilde{S}_G^{(S)}\,, \tag{33}$$

$$r(-\bar{\Box} + V_E^{(S)})\frac{{}^{(i)}\Phi_E^{(S)}}{r} + V_{EG}^{(S)}\ {}^{(i)}\Phi_G^{(S)} = {}^{(i)}\tilde{S}_E^{(S)}\,. \tag{34}$$

- ${}^{(i)}H_{\mu\nu}$ and ${}^{(i)}A_{\mu}$ a are sums of solutions to the homogeneous equations and functions responsible for inhomogeneities:

$${}^{(i)}H_{\ell\ tr} = \ldots + {}^{(i)}\alpha\,, \tag{35}$$

$${}^{(i)}H_{\ell\ rr} = \ldots + {}^{(i)}\beta\,, \tag{36}$$

$${}^{(i)}H_{\ell\ +} = \ldots + {}^{(i)}\gamma\,, \tag{37}$$

$${}^{(i)}A_{\ell\ t} = \ldots + {}^{(i)}\lambda\,, \tag{38}$$

$${}^{(i)}A_{\ell\ r} = \ldots + {}^{(i)}\kappa\,. \tag{39}$$

$$
{}^{(i)}\alpha = -\frac{2r^2\left(r^2 A^{2(i)}S^G_{\ell\ rr} + r^{2(i)}S^G_{\ell\ tt} + 2A^{(i)}S^G_{\ell\ +}\right)}{\ell(\ell+1)r^2\left(rA' - 2A + \ell(\ell+1)\right)} + \\
- \frac{16Q^2 A^{(i)}S^G_{\ell\ -}}{\ell(\ell+1)r^2\left(rA' - 2A + \ell(\ell+1)\right)}, \tag{40}
$$

$$
{}^{(i)}\beta = r\left(\frac{2r^{(i)}S^G_{\ell\ tr}}{\ell(\ell+1)} + \frac{\partial_t{}^{(i)}\alpha}{A}\right), \tag{41}
$$

$$
{}^{(i)}\gamma = \frac{r\partial_r{}^{(i)}\alpha + {}^{(i)}\alpha}{A} - \frac{{}^{(i)}\alpha\left(rA' + \ell(\ell+1)\right)}{2A^2}, \tag{42}
$$

$$
{}^{(i)}\kappa = \frac{r^{2(i)}S^M_{\ell\ r}}{\ell(\ell+1)} + \frac{2Q\partial_t{}^{(i)}S^G_{\ell\ -}}{A\ell(\ell+1)}, \tag{43}
$$

$$
{}^{(i)}\lambda = \frac{r^{2(i)}S^M_{\ell\ t}}{\ell(\ell+1)} + \frac{2QA\partial_r{}^{(i)}S^G_{\ell\ -}}{\ell(\ell+1)}, \tag{44}
$$

$$
{}^{(i)}\tilde{S}^{(S)}_E = \ldots \quad , \quad {}^{(i)}\tilde{S}^{(S)}_G = \ldots . \tag{45}
$$

(46)

Summary:

- We can make the system of linearised Einstein equations with fundamental fields surprisingly simple: coupled wave equations (no derivative couplings, symmetric potential matrices)
- Form is useful for further generalisations to higher perturbation orders.
- Usefulness for numerics (1st order equations instead of 4th w.r.t. time), linear stability
- Conceptual simplicity means possible use to other models (e.g, cosmological perturbations, A. Rostworowski arXiv:1902.05090)

Thank you