Generic cosmological solution without singularity from bifurcation analysis

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- In general, **requires implementation of the potential** $U(\phi)$, affecting the scalar field ϕ.

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- A study of bifurcations within given scenarios—how many **qualitatively different behaviours** occur?

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$$S = S_g + S_\phi,$$

where

$$S_g = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}R \quad \text{(Einstein-Hilbert action)},$$

$$S_\phi = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[\underbrace{\varepsilon\nabla^\alpha\phi\nabla_\alpha\phi}_{\text{kinetic energy}}\ \underbrace{+2U(\phi)}_{\text{potential energy}}\ \underbrace{+\varepsilon\xi R\phi^2}_{R-\phi\text{ coupling}}\right],$$

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- The value of ξ is not arbitrary but is determined by the underlying physics (allowed ranges for ξ)

Field equations

- Variation of the action w.r.t. $g^{\mu\nu}$ and ϕ produces Einstein and Klein-Gordon equations, respectively:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T^{(\phi)}_{\mu\nu},$$

$$\Box\phi - \xi R\phi - \varepsilon U_{,\phi} = 0,$$

where the energy-momentum tensor

$$\begin{aligned} T^{(\phi)}_{\mu\nu} =& \varepsilon \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\varepsilon g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi - g_{\mu\nu}\, U(\phi) + \\ & + \varepsilon\xi\phi^2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \varepsilon\xi \left(g_{\mu\nu}\Box\phi^2 - \nabla_\mu\nabla_\nu\phi^2 \right), \end{aligned}$$

and

$$U_{,\phi} := \mathrm{d}U/\mathrm{d}\phi$$

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$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right),$$

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- In new variables, cosmological equations produce the **dynamical system**

$$\begin{aligned}
u' = & \left[-\frac{1}{2}u^2(2+n) - \frac{3}{2}u(1+4\xi n) + \frac{1}{2}\varepsilon n v^2 - 3\xi(1+n)\right]\left[\frac{1}{3}\varepsilon v^2 - 2\xi(1-6\xi)\right]^2 + \\
& + (6\xi + u)\left[\frac{1}{3}\varepsilon v^2 - 2\xi(1-6\xi)\right] \cdot \\
& \cdot \left(u^2\left[1-\xi(2-n)\right] + 4\xi u(2+3\xi n) - \frac{1}{2}\varepsilon v^2(1+2\xi n) + 3\xi\left[1+2\xi(1+n)\right]\right), \\
v' = & -uv\left[\frac{1}{3}\varepsilon v^2 - 2\xi(1-6\xi)\right]^2,
\end{aligned}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\ln a} = H^{-1}\dot{f}$.

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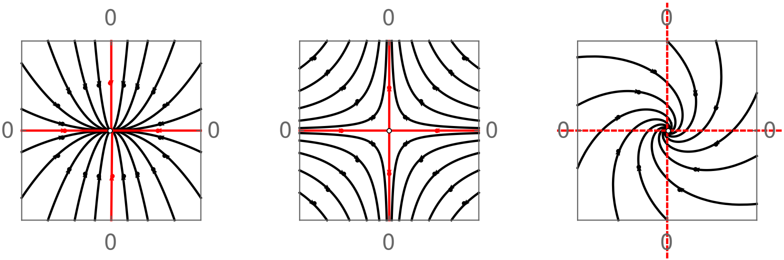

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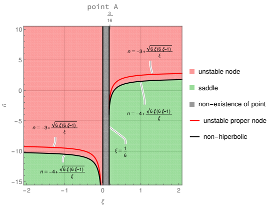
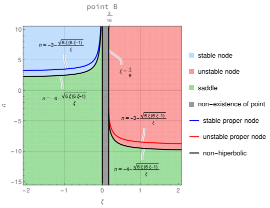
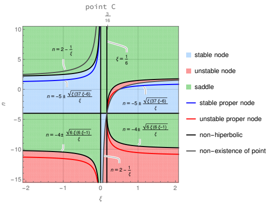
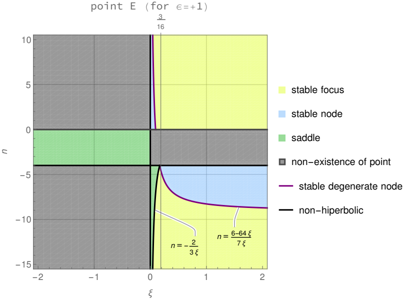
- **Definition:** The appearance of topologically nonequivalent phase portrait under variation of parameters is called a **bifurcation**.

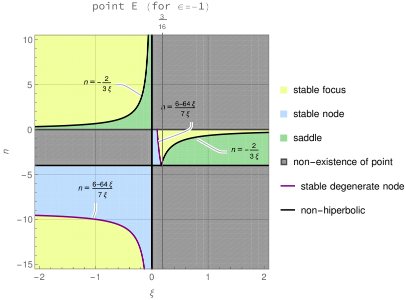
Bifurcation diagrams of local stability in equilibria











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- **De Sitter state describes dynamics of cosmic inflation.**

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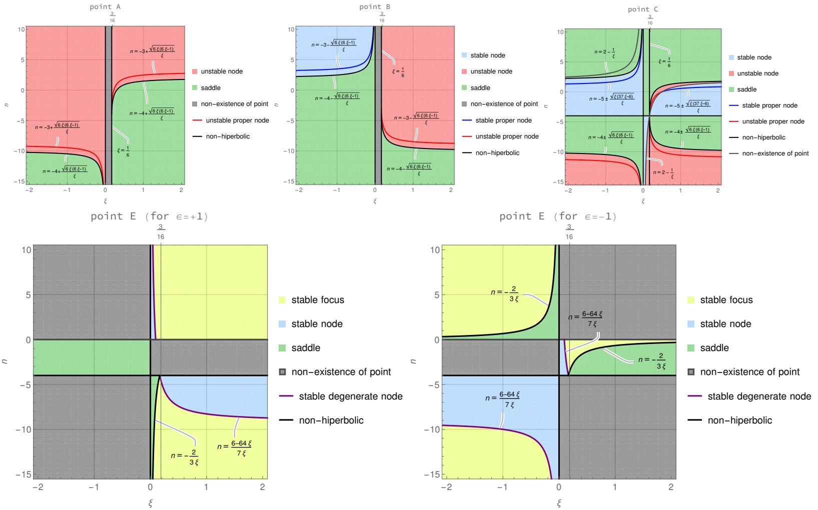


Cases of de Sitter–de Sitter evolution of the universe

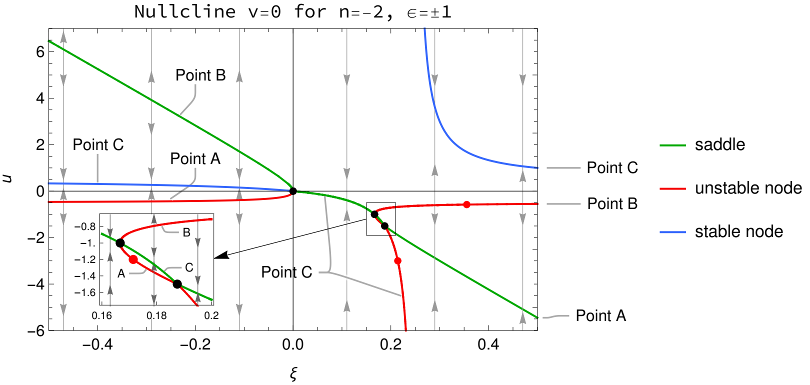
- Looking at bifurcations diagrams, we could extract ranges of parameters for which evolution of the universe **starts in de-Sitter state (inflation) and finishes in another de Sitter state (late time acceleration after the inclusion of matter)**.

Table: Sets of parameters for which the universe undergoes the evolution starting from the de Sitter state and finishing in the de Sitter state.

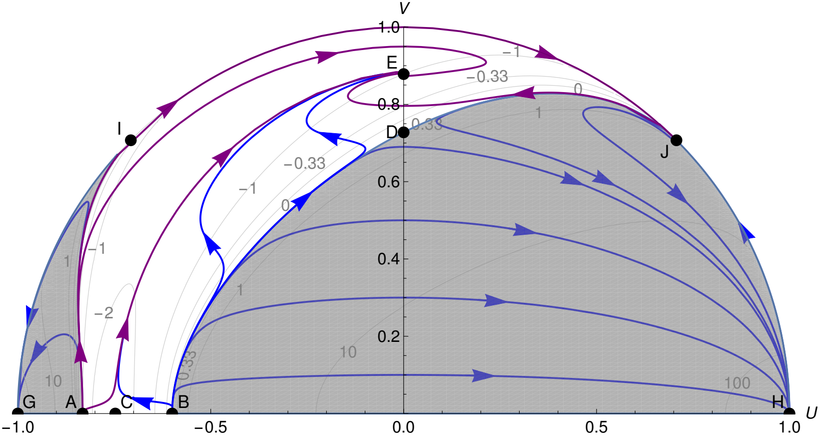
no.	ξ	n	ε	starting point	final point
1. Generic de Sitter–de Sitter evolution					
(a)	$\frac{3}{16}$	$(0,+\infty)$	$+1$	unstable node A	stable focus E
(b)	$\frac{3}{16}$	$(-2,0)$	-1	unstable node A	stable focus E
(c)	$\left(\frac{3}{16},\frac{1}{4}\right)$	-2	-1	unstable node C	stable focus E
2. Non-generic de Sitter–de Sitter evolution					
(d)	$(-\infty,0]$	-2	$+1$	saddle E	stable node C
(e)	$\frac{3}{16}$	$\left(-3\frac{5}{9},-2\right)$	-1	saddle A	stable focus E
(f)	$\frac{3}{16}$	$\left[-4,-3\frac{5}{9}\right)$	-1	saddle A	saddle E
(g)	$\left[0,\frac{3}{16}\right)$	-2	-1	saddle C	stable node/focus E
(h)	$\left(\frac{1}{3},+\infty\right)$	-2	-1	saddle E	stable node C

'Collisions' of equilibria for generic dS–dS scenarios

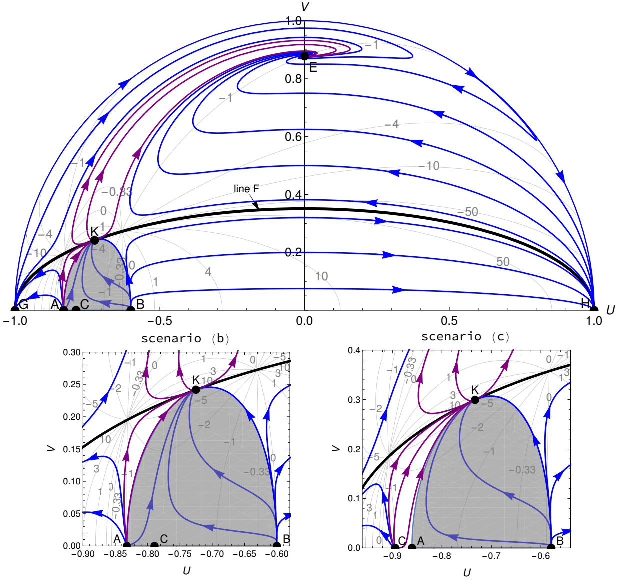
- Again, bifurcation diagrams (showing 'collisions' of equilibria) indicated that one phase portrait is fully representative for each of scenarios (a)–(c).



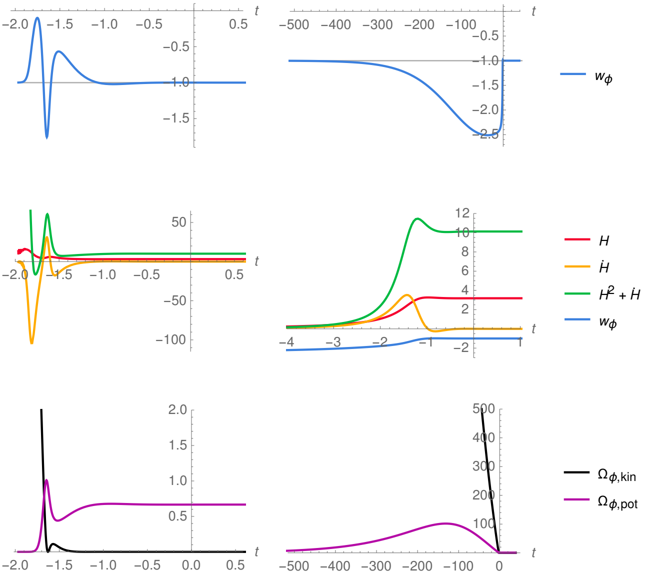
Phase portrait for scenario (a)



Phase portrait for scenarios (b) and (c)



Evolution of physical quantities for case (a)



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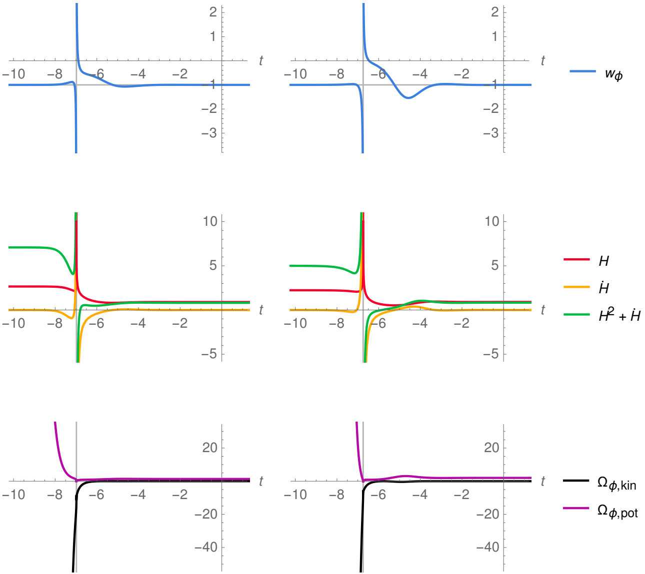

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- *THANK YOU FOR YOUR ATTENTION!*