Relativistic Quantum Mechanics of the Majorana Particle

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PLAN

Introduction

1. The Dirac equation

$$i\gamma^{\mu}\partial_{\mu}\psi(x) - m\psi(x) = 0, \qquad \psi(x) = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix}, \quad \psi^{\alpha}(x) \in \mathbb{C}$$

Scalar product: $\langle\psi_1|\psi_2\rangle = \int d^3x\, \psi_1(\mathbf{x}, t)^{\dagger}\, \psi_2(\mathbf{x}, t)$ with arbitrary t.
The Hamiltonian form: $i\partial_t\psi = \hat{H}\psi$, Hermitian $\hat{H} = -i\gamma^0\gamma^k\,\partial_k + m\gamma^0$.
General solution of the equation

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3p\, e^{i\mathbf{px}} \left(e^{-iE_p t}\, v^{(+)}(\mathbf{p}) + e^{iE_p t}\, v^{(-)}(\mathbf{p}) \right), \qquad (1)$$

where

$$(\gamma^0\gamma^l p^l + m\gamma^0)v^{(\pm)}(\mathbf{p}) = \pm E_p v^{(\pm)}(\mathbf{p}), \qquad E_p = +\sqrt{m^2 + \mathbf{p}^2}.$$

(1) is important as the starting point for QFT of the Dirac field.
Two views on (1): expansion in the basis of common eigenvectors of commuting observables $\hat{\mathbf{p}} = -i\nabla$ and $\hat{H}$ (physics); or merely the Fourier transformation (mathematics).

2. There exist Majorana representations for γ^μ matrices in which they are purely imaginary. For example,

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \;\; \gamma^1 = i \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \;\; \gamma^2 = i \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

$$\gamma^3 = -i \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}.$$

σ_k – the Pauli matrices, σ_0 – the 2×2 unit matrix. In the Majorana representations charge conjugation C is represented simply by the complex conjugation.

By definition, the Majorana bispinors are invariant under C. Thus, they have real components in the Majorana repr. The Hilbert space of the Majorana bispinors is over $\mathbb{R}$, not $\mathbb{C}$. The scalar product has the form $\langle\psi_1|\psi_2\rangle = \int d^3x \, \psi_1(\mathbf{x}, t)^T \psi_2(\mathbf{x}, t)$. QM without complex numbers.

The Dirac bispinor is a composite object:

- $\psi = \psi_1 + i\psi_2$ with real $\psi_{1,2}$, $C\psi_1 = \psi_1$, $C(i\psi_2) = -i\psi_2$;
- the decomposition $\psi = \psi_R + \psi_L$ into the Weyl bispinors;
- two irreducible unitary $s = 1/2$ reprs of the Poincaré group.

Axial momentum

Questions:
- What is the 'eigenvector' version of expansion (1) for the Majorana bispinors?
- $\hat{\mathbf{p}} = -i\nabla \to ?$

1. There exists 1:1 mapping M between linear spaces of the Majorana and right-handed (or left-handed) Weyl bispinors. Take arbitrary right-handed Weyl bispinor ϕ, $\gamma_5\phi = \phi$, and form $\psi = \phi + \phi^* \equiv M(\phi)$. ψ is a Majorana bispinor. M is invertible: $\phi = (I + \gamma_5)\psi/2 \equiv \psi_R$. M preserves linear combinations only if their coefficients are real.

The Weyl bispinors are complex, hence the standard momentum operator $\hat{\mathbf{p}} = -i\nabla$ is well-defined for them. It commutes with γ_5, therefore also $\hat{\mathbf{p}}\phi$ is right-handed Weyl bispinor. Let us find the Majorana bispinor that corresponds to $\hat{\mathbf{p}}\phi$:

$$M(\hat{\mathbf{p}}\phi) = \hat{\mathbf{p}}\phi + (\hat{\mathbf{p}}\phi)^* = \hat{\mathbf{p}}(\psi_R - \psi_L) = -i\gamma_5\nabla\psi \equiv \hat{\mathbf{p}}_5\psi.$$

Thus the standard momentum operator in the space of right-handed Weyl bispinors gives rise to $\hat{\mathbf{p}}_5 = -i\gamma_5\nabla$ – the axial momentum operator – in the space of Majorana bispinors.

Axial momentum

$\hat{\mathbf{p}}_5$ commutes with γ_5, therefore it can be used also in the space of right-handed Weyl bispinors. It is invariant under the mapping M: if $\psi = M(\phi)$ then $\hat{\mathbf{p}}_5\psi = M(\hat{\mathbf{p}}_5\phi)$ because the matrix $-i\gamma_5$ is real.

2. Are the two quantum mechanics, Majorana and Weyl, equivalent? No, because the mapping M does not preserve scalar product. Take $\psi_1 = M(\phi_1),\ \psi_2 = M(\phi_2)$,

$$\int d^3x\, \psi_1^T \psi_2 = \int d^3x\, (\phi_1^\dagger \phi_2 + (\phi_1^\dagger \phi_2)^*).$$

Moreover, there are differences in evolution equations. In the Weyl case, evolution equation has the form (1) with $m = 0$; in the Majorana case $m \neq 0$ is allowed. Using M^{-1} one can transform Eq. (1) for ψ to the space of right-handed Weyl bispinors:

$$i\gamma^\mu \partial_\mu \phi - m\phi^* = 0.$$

This equation is known as the Majorana equation for ϕ (ϕ^* is charge conjugation of ϕ). It can not be accepted as quantum evolution equation for the Weyl ϕ because it is not linear over $\mathbb{C}$. The Hilbert space of the Weyl bispinors is linear over $\mathbb{C}$ – it includes all bispinors such that $\gamma_5\phi = \phi$.

Axial momentum

3. The normalized eigenvectors of the axial momentum obey the conditions

$$\hat{\mathbf{p}}_5\psi_{\mathbf{p}}(\mathbf{x}) = \mathbf{p}\,\psi_{\mathbf{p}}(\mathbf{x}), \quad \int d^3x\,\psi^T_{\mathbf{p}}(\mathbf{x})\,\psi_{\mathbf{q}}(\mathbf{x}) = \delta(\mathbf{p}-\mathbf{q}).$$

They have the form

$$\psi_{\mathbf{p}}(\mathbf{x}) = (2\pi)^{-3/2}\exp(i\gamma_5\mathbf{px})\,v,$$

where v an arbitrary real, constant, normalized ($v^T v = 1$) bispinor, and

$$\exp(i\gamma_5\mathbf{px}) = \cos(\mathbf{px})I + i\gamma_5\sin(\mathbf{px}).$$

We call them the axial plane waves.

4. Commutator of the axial momentum with position operator $\hat{\mathbf{x}}$ has the form

$$[\hat{x}^j,\,\hat{p}_5^k] = i\delta_{jk}\gamma_5.$$

The implied uncertainty relation is the well-known one

$$\langle\psi|(\Delta\hat{x}^j)^2|\psi\rangle\langle\psi|(\Delta\hat{p}_5^k)^2|\psi\rangle \geq \frac{1}{4}\delta_{jk},$$

where $\Delta\hat{x}^j = \hat{x}^j - \langle\psi|\hat{x}^j|\psi\rangle,\ \Delta\hat{p}_5^k = \hat{p}_5^k - \langle\psi|\hat{p}_5^k|\psi\rangle$.

Axial momentum

5. The axial momentum commutes with the Hamiltonian
$\hat{h} = -\gamma^0\gamma^k\partial_k - im\gamma^0$ only in the massless case:

$$[\hat{h}, \hat{\mathbf{p}}_5] = -2im\gamma^0\hat{\mathbf{p}}_5.$$

The operator ∇ commutes with $\hat{h}$. Therefore, in the Heisenberg picture, $\hat{\mathbf{p}}_5(t) = -i\hat{\gamma_5}(t)\nabla$. It turns out that the operator $\hat{\gamma_5}(t)$ has the following form

$$\hat{\gamma_5}(t) = \gamma_5 + i\,\frac{m}{\hat{\omega}}\,\gamma^0\gamma_5\,[\sin(2\hat{\omega}t) + \hat{J}\,(1 - \cos(2\hat{\omega}t))].$$

Here $\hat{J} = \hat{h}/\hat{\omega},\ \hat{\omega} = \sqrt{m^2 - \Delta} \to E_p = \sqrt{m^2 + \vec{p}^{\,2}}.$ Note that $\hat{J}^2 = -I$.

Oscillating matrix elements of the axial momentum:

$$\int d^3x\, \psi_p^T(\vec{x})\hat{\mathbf{p}}_5(t)\psi_q(\vec{x}) = \mathbf{p}\left[1 + \frac{m^2}{E_p^2}(\cos(2E_pt) - 1)\right](v^Tw)\,\delta(\vec{p} - \vec{q})$$

$$-\mathbf{p}\frac{m}{E_p}\left[i\sin(2E_pt)(v^T\gamma^0w) + (1 - \cos(2E_pt))(v^T\gamma_5\frac{\gamma^jp^j}{E_p}w)\right]\delta(\vec{p} + \vec{q}).$$

General solution of the evolution equation

1. The expansion of $\psi(\mathbf{x}, t)$ into the axial plane waves

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \sum_{\alpha=1}^{2} \int d^3p \, e^{i\gamma_5 \mathbf{px}} \left(v_\alpha^{(+)}(\mathbf{p}) c_\alpha(\mathbf{p}, t) + v_\alpha^{(-)}(\mathbf{p}) d_\alpha(\mathbf{p}, t) \right).$$

As the orthonormal basis of the real bispinors we take the eigenvectors of $\gamma^0 \gamma^k p^k$ (this matrix commutes with $\mathbf{p}_5$):

$$v_1^{(+)}(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| - p^2)}} \begin{pmatrix} -p^3 \\ p^2 - |\mathbf{p}| \\ p^1 \\ 0 \end{pmatrix}, \quad v_2^{(+)}(\mathbf{p}) = i\gamma_5 \, v_1^{(+)}(\mathbf{p}),$$

$$v_1^{(-)}(\mathbf{p}) = i\gamma^0 \, v_1^{(+)}(\mathbf{p}), \quad v_2^{(-)}(\mathbf{p}) = i\gamma_5 \, v_1^{(-)}(\mathbf{p}) = -\gamma_5 \gamma^0 v_1^{(+)}(\mathbf{p}).$$

This is rather interesting basis: scale invariant, and real. Can be used also in the Dirac quantum mechanics.

Another expansion (without $\hat{\mathbf{p}}_5$): L. Pedro, arXiv:1212.5465 (2012).

General solution of the evolution equation

2. Time dependence of the real amplitudes $c_\alpha(\mathbf{p},t),\ d_\alpha(\mathbf{p},t)$ is found from the Dirac equation. The amplitudes are split into the even and odd parts,

$$c_\alpha(\mathbf{p},t) = c_\alpha^{'}(\mathbf{p},t) + c_\alpha^{''}(\mathbf{p},t), \quad d_\alpha(\mathbf{p},t) = d_\alpha^{'}(\mathbf{p},t) + d_\alpha^{''}(\mathbf{p},t),$$

where $c_\alpha^{'}(-\mathbf{p},t) = c_\alpha^{'}(\mathbf{p},t),\ c_\alpha^{''}(-\mathbf{p},t) = -c_\alpha^{''}(\mathbf{p},t)$, similarly for d', d''.
Furthermore, we introduce

$$\vec{c}(\mathbf{p},t) = \begin{pmatrix} c_1' \\ c_1'' \\ c_2' \\ c_2'' \end{pmatrix},\ \vec{d}(\mathbf{p},t) = \begin{pmatrix} d_1' \\ d_1'' \\ d_2' \\ d_2'' \end{pmatrix},\ K_\pm(\mathbf{p}) = \begin{pmatrix} 0 & -n^1 & \pm n^2 & \pm n^3 \\ n^1 & 0 & \mp n^3 & \pm n^2 \\ \mp n^2 & \pm n^3 & 0 & n^1 \\ \mp n^3 & \mp n^2 & -n^1 & 0 \end{pmatrix},$$

where $E_p = \sqrt{\mathbf{p}^2 + m^2}$, and

$$n^1 = \frac{m\, p^1}{E_p \sqrt{(p^1)^2 + (p^3)^2}}, \quad n^2 = \frac{|\mathbf{p}|}{E_p}, \quad n^3 = \frac{m\, p^3}{E_p \sqrt{(p^1)^2 + (p^3)^2}}.$$

General solution of the evolution equation

The time dependence of the amplitudes is given by

$$\vec{c}(\mathbf{p}, t) = \exp(t\, E_p\, K_+(\mathbf{p}))\, \vec{c}(\mathbf{p}, 0), \quad \vec{d}(\mathbf{p}, t) = \exp(t\, E_p\, K_-(\mathbf{p}))\, \vec{d}(\mathbf{p}, 0).$$

The matrices $K_\pm$ are antisymmetric, the matrices $\exp(t\, E_p\, K_\pm(\mathbf{p}))$ belong to the $SO(4)$ group.

3. This solution can be rewritten in terms of quaternions. The quaternionic units $\hat{i}, \hat{j}, \hat{k}$ are introduced as follows:

$$\hat{i} = i\gamma_5, \quad \hat{j} = i\gamma^0, \quad \hat{k} = -\gamma_5\gamma^0.$$

They obey the usual conditions

$$\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = -I, \quad \hat{i}\hat{j} = \hat{k}, \quad \hat{k}\hat{i} = \hat{j}, \quad \hat{j}\hat{k} = \hat{i}.$$

The bispinor basis $v_\alpha^{(\pm)}(\mathbf{p})$ is generated from $v_1^{(+)}(\mathbf{p})$ by acting with $\hat{i}, \hat{j}, \hat{k}$. Moreover, $K_\pm(\mathbf{p}) = -n^1\hat{k} \mp n^2\hat{i} \pm n^3\hat{j}$. Therefore, the time evolution of the amplitudes $\vec{c}$, $\vec{d}$ at each fixed value of the axial momentum $\mathbf{p}$ is given by a time dependent quaternion.

General solution of the evolution equation

4. The general solution can be rewritten in the form of superposition of traveling plane waves:

$$\psi(\mathbf{x},t)=\frac{1}{2(2\pi)^{3/2}}\int d^3p\,[\cos(\mathbf{px}-E_pt)\,A_+(\mathbf{p})+\cos(\mathbf{px}+E_pt)\,A_-(\mathbf{p})$$

$$+\sin(\mathbf{px}-E_pt)\,B_+(\mathbf{p})+\sin(\mathbf{px}+E_pt)\,B_-(\mathbf{p})]\,,$$

where

$$A_\pm(\mathbf{p})=v_1^{(+)}(\mathbf{p})A_\pm^1(\mathbf{p})+v_2^{(+)}(\mathbf{p})A_\pm^2(\mathbf{p})+v_1^{(-)}(\mathbf{p})A_\pm^3(\mathbf{p})+v_2^{(-)}(\mathbf{p})A_\pm^4(\mathbf{p}),$$

$$B_\pm(\mathbf{p})=v_1^{(+)}(\mathbf{p})B_\pm^1(\mathbf{p})+v_2^{(+)}(\mathbf{p})B_\pm^2(\mathbf{p})+v_1^{(-)}(\mathbf{p})B_\pm^3(\mathbf{p})+v_2^{(-)}(\mathbf{p})B_\pm^4(\mathbf{p}),$$

and

$$A_\pm^1=(1\pm\frac{p}{E_p})c_1\mp\frac{m}{E_p}d_2,\quad A_\pm^2=(1\pm\frac{p}{E_p})c_2\mp\frac{m}{E_p}d_1,$$

General solution of the evolution equation

$$A^3_\pm = (1 \mp \frac{p}{E_p})d_1 \pm \frac{m}{E_p}c_2, \quad A^4_\pm = (1 \mp \frac{p}{E_p})d_2 \pm \frac{m}{E_p}c_1,$$

$$B^1_\pm = -(1 \pm \frac{p}{E_p})c_2 \mp \frac{m}{E_p}d_1, \quad B^2_\pm = (1 \pm \frac{p}{E_p})c_1 \pm \frac{m}{E_p}d_2,$$

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Here $p \equiv |\mathbf{p}|$, $E_p = \sqrt{\mathbf{p}^2 + m^2}$, the amplitudes c_1, c_2, d_1, d_2 are the ones introduced earlier. $\mathbf{p}$ is the eigenvalue of the axial momentum.

In the massive case one can not have $A_- = 0 = B_-$ and $(A_+)^2 + (B_+)^2 > 0$, or vice versa. Always a component propagating in the opposite direction is present. It can be weak, $\sim m/E_p$.

Relativistic invariance

1. The Poincaré transformations of the real bispinor $\psi(x)$ have the standard form,

$$\psi'_{L,a}(x) = S(L)\psi(L^{-1}(x-a)),$$

where $S(L) = \exp(\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]/8)$, and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ parameterize the proper orthochronous Lorentz group, $L = \exp(\omega^\mu{}_\nu)$.

In order to identify the pertinent unitary irreducible representations of the Poincaré group we write

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3p \, e^{i\gamma_5 \mathbf{px}} v(\mathbf{p}, t),$$

where $v(\mathbf{p}, t)$ is a real bispinor. The Dirac equation $\Rightarrow$

$$\dot{v}(\mathbf{p}, t) = -i\gamma^0\gamma^k\gamma_5 p^k v(\mathbf{p}, t) - im\gamma^0 v(-\mathbf{p}, t).$$

In the last term we have $v(-\mathbf{p}, t)$ because γ^0 anticommutes with γ_5. It follows that $\ddot{v}(\mathbf{p}, t) = -E_p^2 v(\mathbf{p}, t)$.

Relativistic invariance

Solving the latter equation we obtain the explicit dependence on t

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 p}{E_p}\, e^{i\gamma_5 \mathbf{px}} \left(e^{-i\gamma_5 E_p t} v_+(\mathbf{p}) + e^{i\gamma_5 E_p t} v_-(-\mathbf{p}) \right),$$

where the amplitudes $v_\pm(\mathbf{p})$ are restricted by the conditions

$$E_p\, v_\pm(\mathbf{p}) = \gamma^0 \gamma^k p^k v_\pm(\mathbf{p}) \pm m\, \gamma_5 \gamma^0\, v_\mp(\mathbf{p}).$$

The Lorentz transformations of $\psi(x)$ are equivalent to the following transformation of the bispinor amplitudes,

$$v'_\pm(p) = S(L)\, v_\pm(L^{-1} p). \qquad (2)$$

The spatial translations $\mathbf{x}' = \mathbf{x} + \mathbf{a}$ are represented by

$$v'_\pm(\mathbf{p}, t) = e^{\mp i\gamma_5 \mathbf{pa}} v_\pm(\mathbf{p}, t).$$

Relativistic invariance

2. In the massive case, $v_-(\mathbf{p})$ can be expressed by $v_+(\mathbf{p})$. The scalar product acquires explicitly Lorentz invariant form

$$\langle\psi_1|\psi_2\rangle = \frac{1}{m^2}\int \frac{d^3p}{E_p}\,\overline{v_{1+}(\mathbf{p})}\,(\gamma^0 E_p - \gamma^k p^k)\,v_{2+}(\mathbf{p}),$$

where $\overline{v_{1+}(\mathbf{p})} = v_{1+}(\mathbf{p})^T\gamma^0$.

Representation (2) is unitary with respect to this scalar product. It is irreducible, and equivalent to a real version of the standard spin 1/2 unitary representation.

For comparison, in the case of massive Dirac particle there are two spin 1/2 representations.

3. In the massless case, the bispinors $v_+(\mathbf{p}),\ v_-(\mathbf{p})$ are independent. They are restricted by the conditions

$$E_p\, v_\pm(\mathbf{p}) = \gamma^0\gamma^k p^k v_\pm(\mathbf{p}), \qquad E_p = |\mathbf{p}|.$$

The resulting linear subspaces of real bispinors are two dimensional. Each spans the representation (2).

Relativistic invariance

These representations are irreducible, unitary, and characterized by helicities $\pm 1/2$. The Lorentz invariant form of the scalar product:

$$\langle\psi_1|\psi_2\rangle = -2\int\frac{d^3p}{|\mathbf{p}|}\;\Big[\overline{w_{1+}(\mathbf{p})}\,(\gamma^0|\mathbf{p}| - \gamma^k p^k)\,w_{2+}(\mathbf{p})$$

$$+\,\overline{w_{1-}(\mathbf{p})}\,(\gamma^0|\mathbf{p}| - \gamma^k p^k)\,w_{2-}(\mathbf{p})\Big]\,,$$

where $w_\pm(\mathbf{p})$ are related to $v_\pm(\mathbf{p})$ by

$$v_\pm(\mathbf{p}) = (\gamma^0|\mathbf{p}| - \gamma^k p^k)\,w_\pm(\mathbf{p}).$$

Interestingly, the last formula determines $w_\pm$ up to a gauge transformation of the form

$$w'_\pm(\mathbf{p}) = w_\pm(\mathbf{p}) + (\gamma^0|\mathbf{p}| - \gamma^k p^k)\,\chi_\pm(\mathbf{p})$$

with arbitrary $\chi_\pm(\mathbf{p})$, because $(\gamma^0|\mathbf{p}| - \gamma^k p^k)^2 = 0$ (nilpotency). The scalar product is invariant with respect to these transformations.

Summary

- The axial momentum $\mathbf{p}_5 = -i\gamma_5\nabla$ is a viable observable for the Majorana particle (to replace $\mathbf{p} = -i\nabla$). Sensitive to $m \neq 0$.
- The eigenvectors of $\mathbf{p}_5$ are not stationary states. The minimal stationary block in the Hilbert space is spanned by the two modes: $\mathbf{p}, -\mathbf{p}$.
- The main features of the expansion into the eigenvectors of $\mathbf{p}_5$ include:
  - the presence of quaternions
  - $SO(4)$ 'phase' factors
  - the presence of pairs of plane waves traveling in the opposite directions $(1 : m/E)$
- Relativistic invariance: as expected, and simple.
- Example of QM without complex numbers.

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H. A., Phys. Lett. A **383** (2019) 1242-1246 (arxiv:1805:03016);
H.A. and Z. Świerczyński, a forthcoming paper.

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THANK YOU!

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