Quantum Link Models and Quantum Simulation of Gauge Theories

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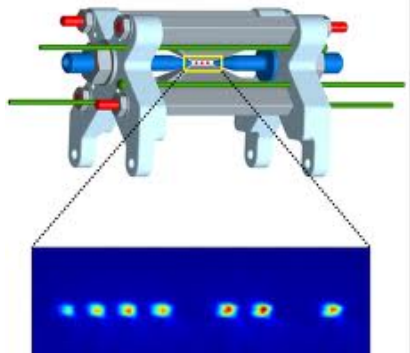
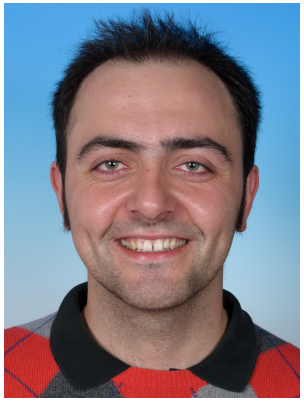
Summer School:
57. Crakow School
of Theoretical Physics
Zakopane, Poland, 2017



Bern-Innsbruck Particle Physics AMO Collaboration



Innsbruck: Marcello Dalmonte, Catherine Laflamme, Peter Zoller

Bilbao: Enrique Rico Ortega

Bern: Michael Bögli, Pascal Stebler, Philippe Widmer

Berlin: Debasish Banerjee

Outline

PART 1: FROM CLASSICAL TO QUANTUM SIMULATION

Pioneers of Quantum Computing and Quantum Simulation

Classical and Quantum Simulation of Quantum Spin Systems

The Sign Problem in Classical Simulations of Fermions

PART 2: QUANTUM LINK MODELS
AND QUANTUM SIMULATORS FOR
ABELIAN AND NON-ABELIAN GAUGE THEORIES

From Wilson's Lattice QCD to Quantum Link Models

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PART 3: CONTINUUM LIMIT OF QUANTUM SPIN
AND QUANTUM LINK MODELS

From $SU(N)$ Quantum Spin Ladders to $\mathbb{C}P(N-1)$ Models

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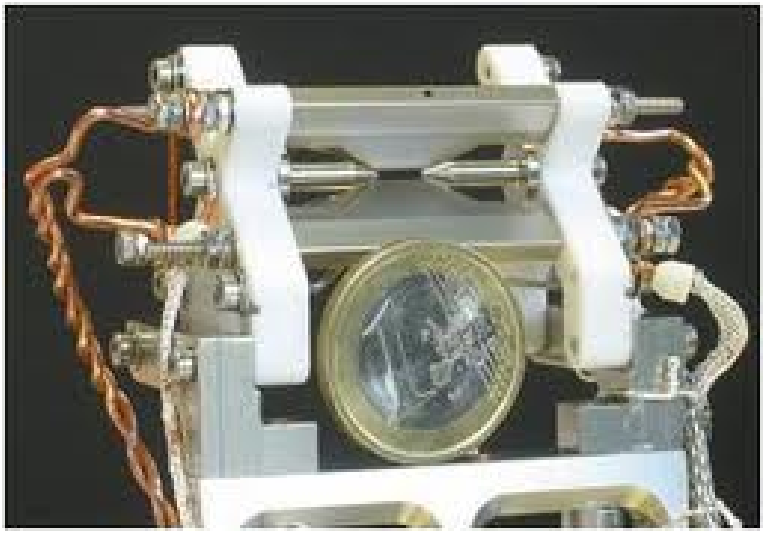
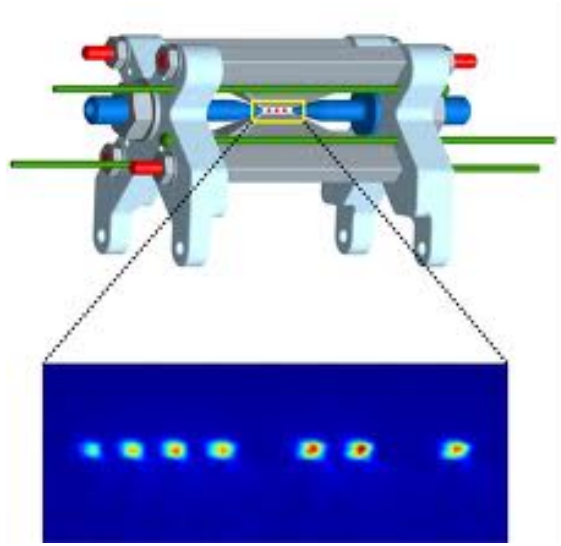
Richard Feynman's vision of 1982

"I'm not happy with all the analyses that go with just the classical theory, because nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy."

Ion traps as a digital quantum computer





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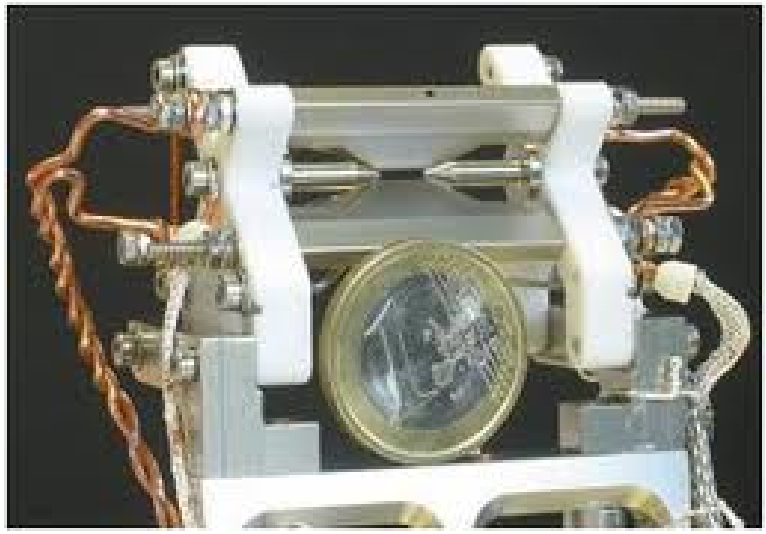
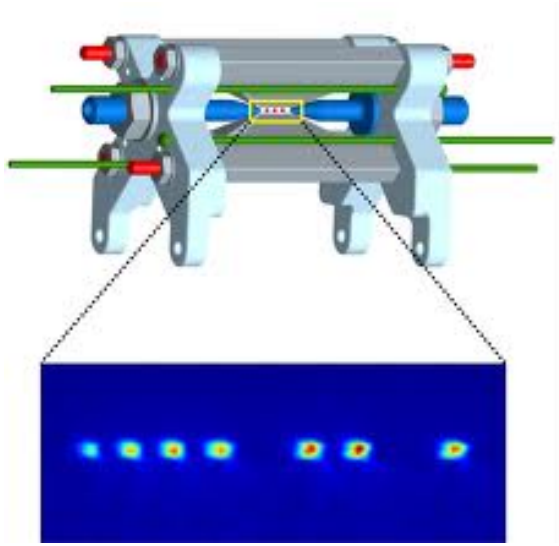
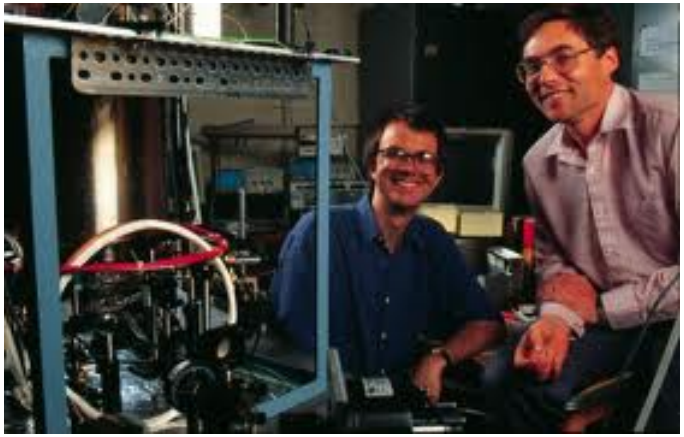


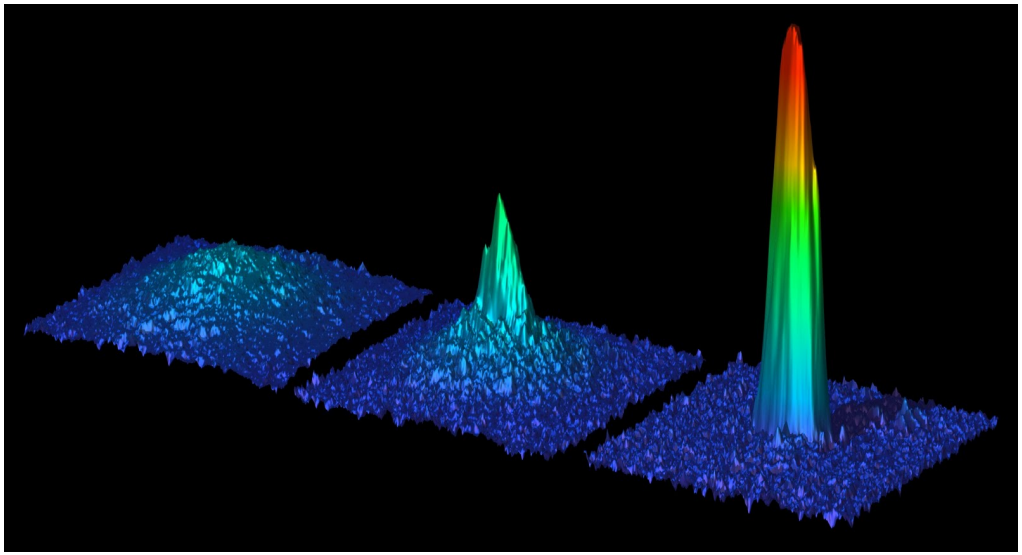


Franklin Medal 2010: I. Cirac, D. Wineland, P. Zoller

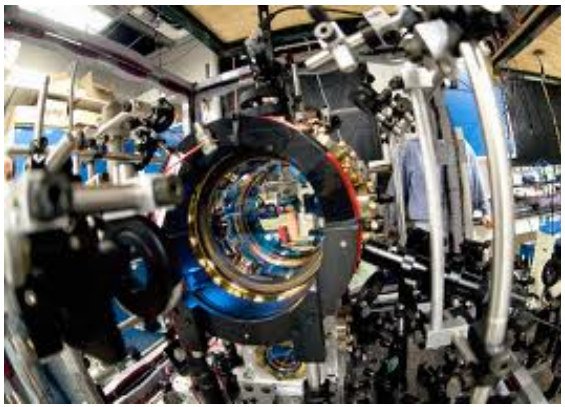
Bose-Einstein condensation in ultra-cold atomic gases

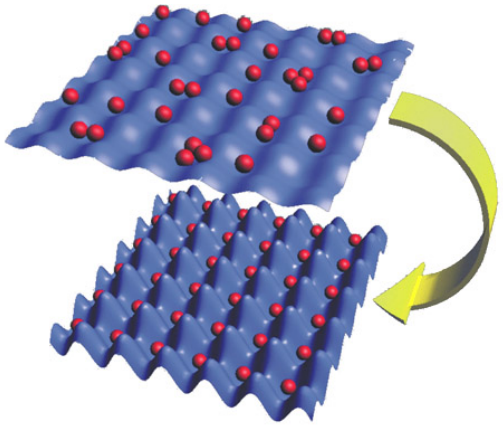


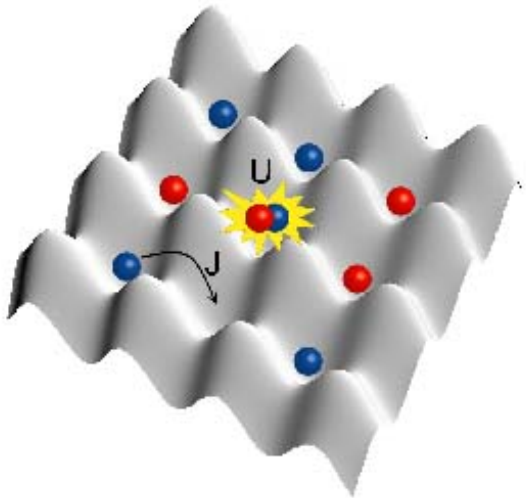
Eric Cornell, Carl Wieman, Wolfgang Ketterle, 1995



Ultra-cold atoms in optical lattices as analog quantum simulators







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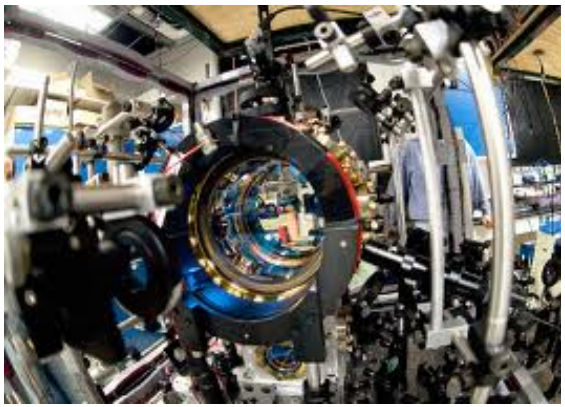
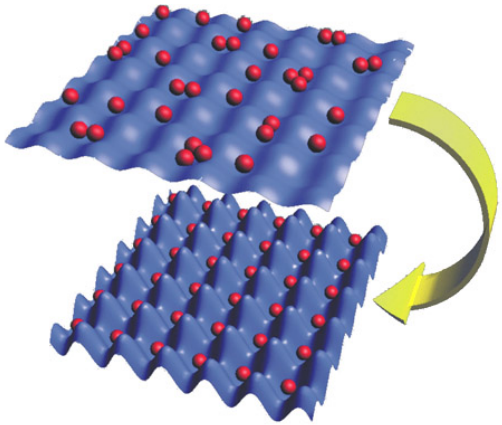
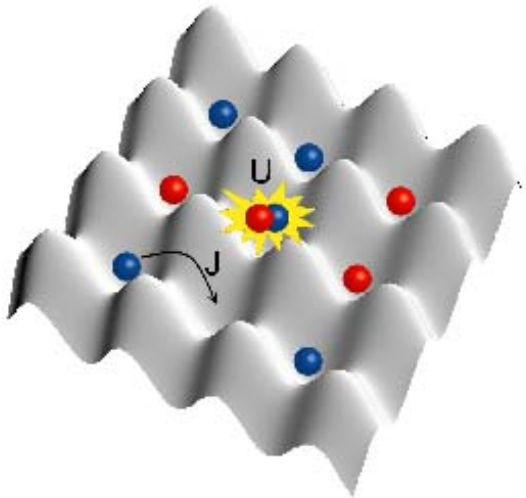
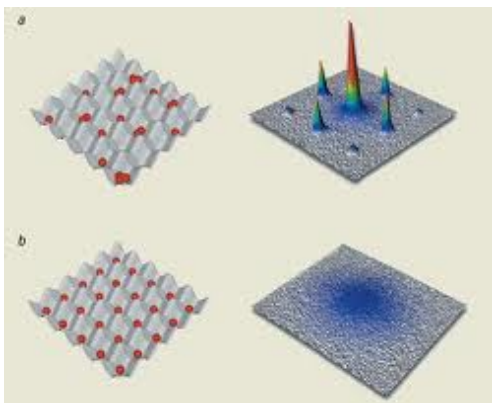






Transition from a superfluid to a Mott insulator



Theodor Hänsch

Immanuel Bloch

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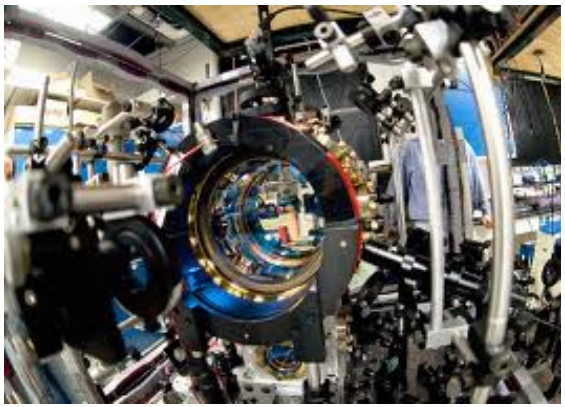

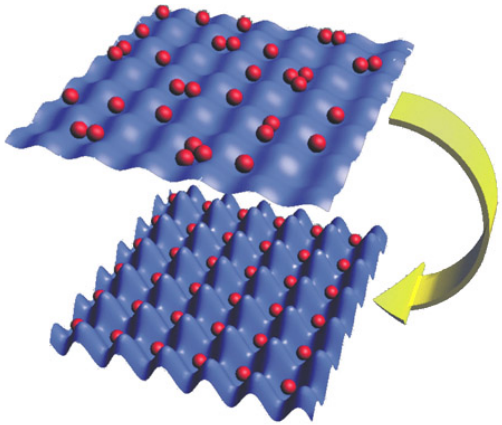

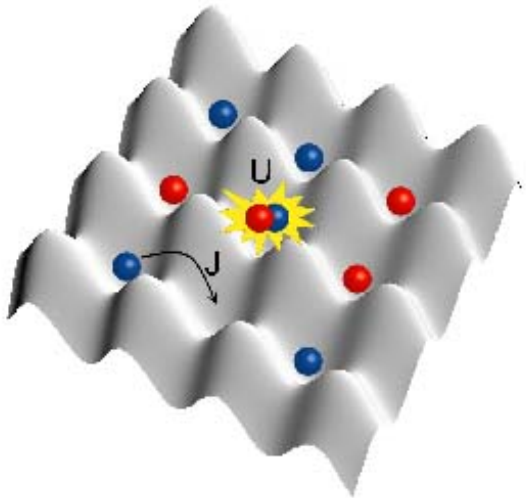

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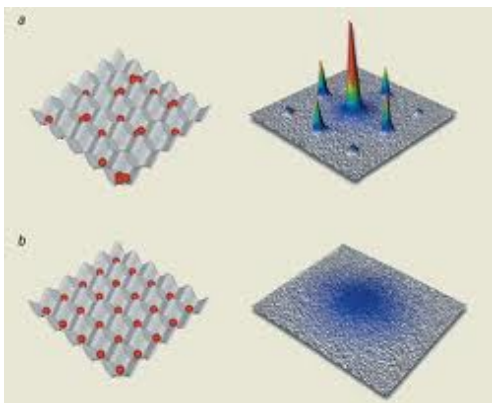

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Can one understand high-T_c superconductivity in this way?

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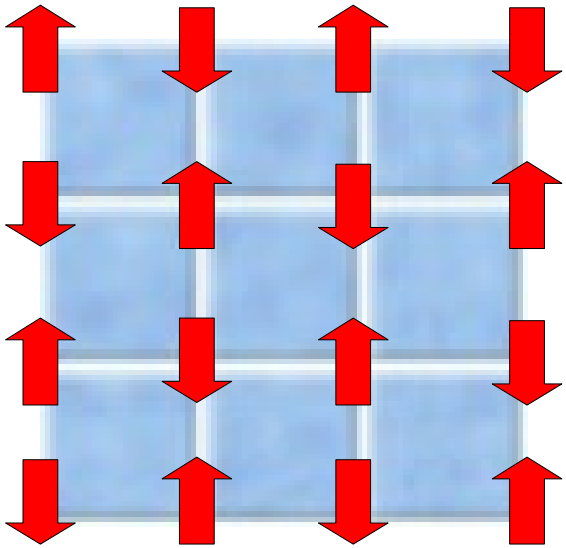
The spin $\frac{1}{2}$ quantum Heisenberg model



Quantum spins $[S_x^a, S_y^b] = i\delta_{xy}\varepsilon_{abc}S_x^c$ and their Hamiltonian

$$H = J\sum_{\langle xy\rangle} \vec{S}_x \cdot \vec{S}_y$$

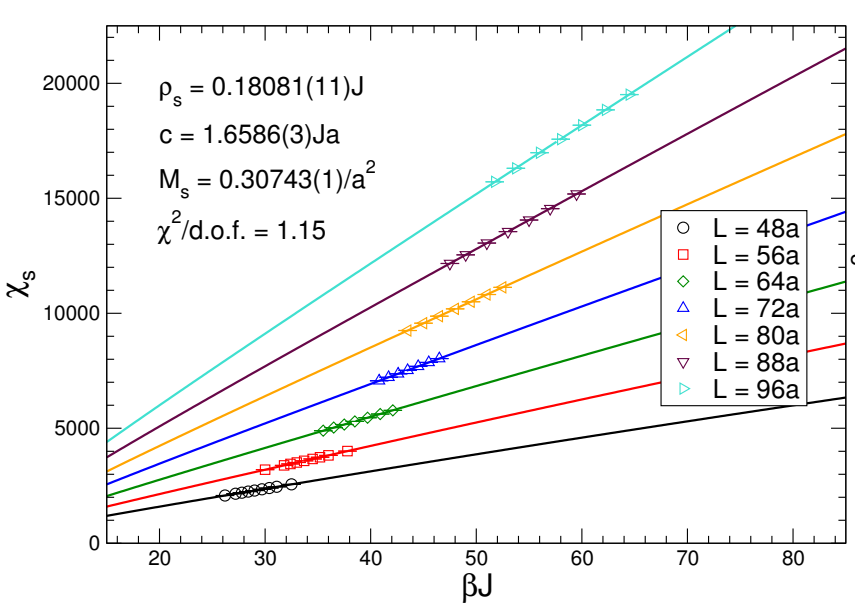
Partition function at inverse temperature $\beta = 1/T$

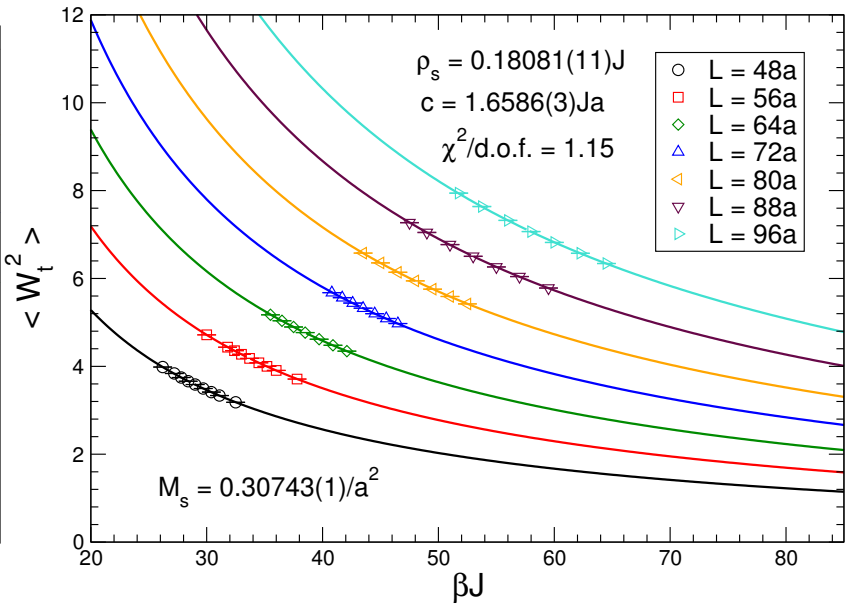
$$Z = \mathrm{Tr}\exp(-\beta H)$$

Low-energy effective action for antiferromagnetic magnons

$$S[\vec{e}] = \int_0^\beta dt \int d^2x \, \frac{\rho_s}{2} \left(\partial_i \vec{e} \cdot \partial_i \vec{e} + \frac{1}{c^2} \partial_t \vec{e} \cdot \partial_t \vec{e} \right)$$

Fit to analytic predictions of effective theory

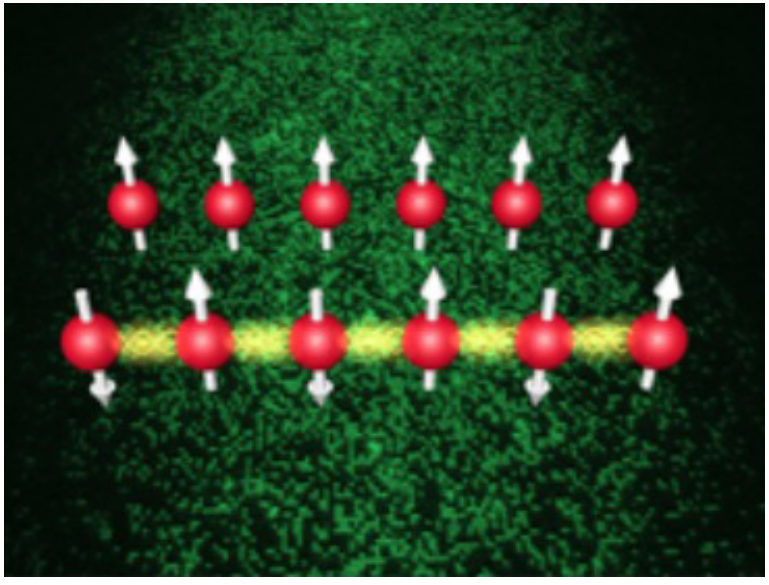


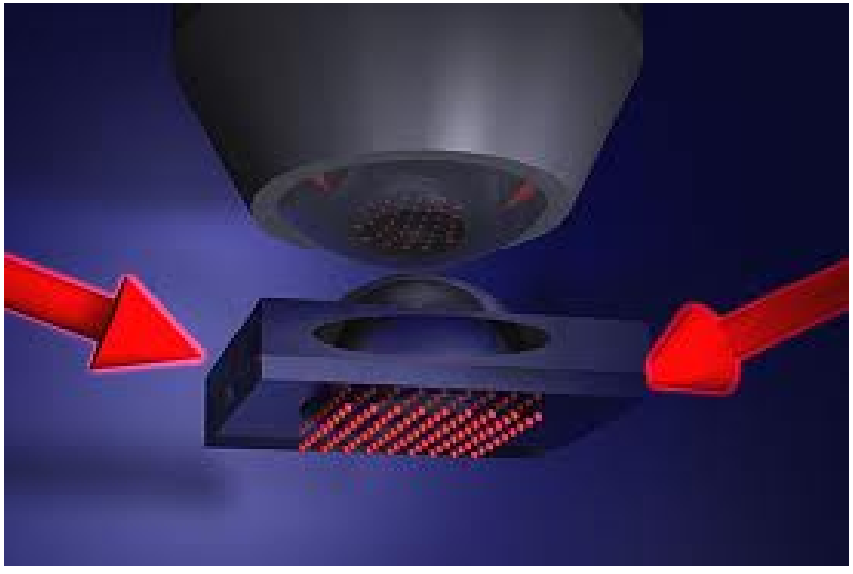


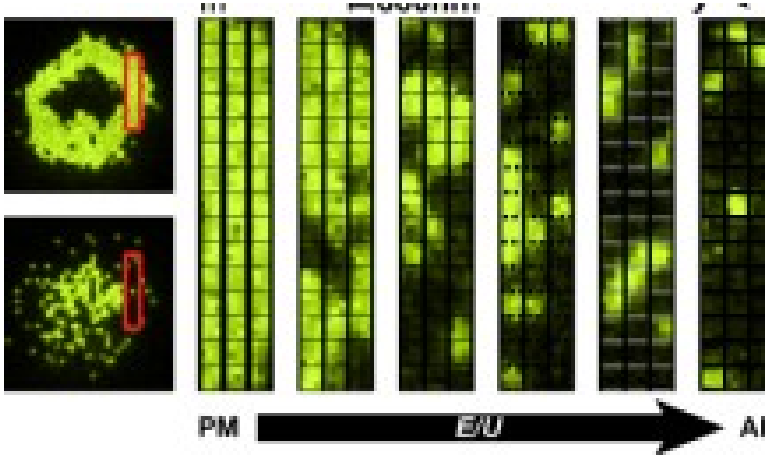
$$\mathcal{M}_s = 0.30743(1), \quad \rho_s = 0.18081(11)J, \quad c = 1.6586(3)Ja$$

UJW, H.-P. Ying (1994); F.-J. Jiang, UJW (2010)

Optical lattice quantum simulation of quantum spin systems







J. Simon, W. S. Bakir, R. Ma, M. E. Tal, P. M. Preis, M. Greiner, Nature 472 (2011) 307.

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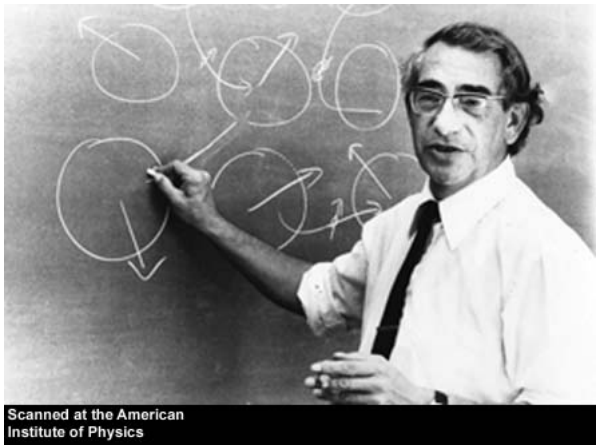
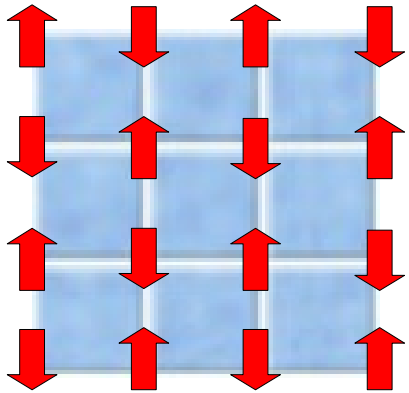
The Hubbard Model for doped antiferromagnets





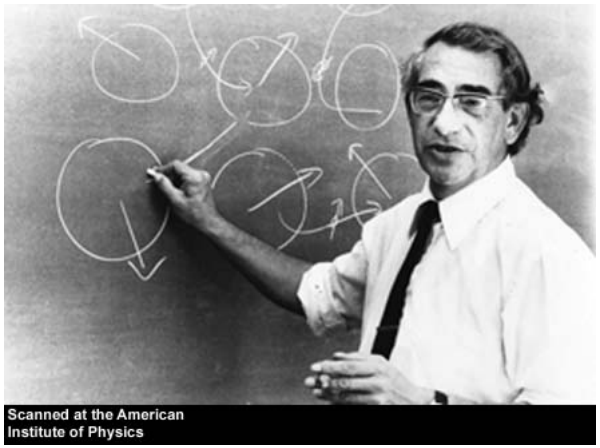
$$H = -t \sum_{\langle xy \rangle} (c_x^\dagger c_y + c_y^\dagger c_x) + U \sum_x (c_x^\dagger c_x - 1)^2, \quad c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix}$$

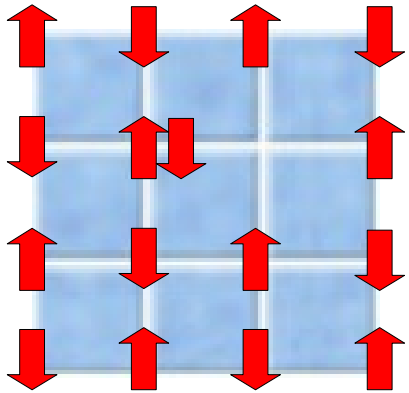
reduces to the Heisenberg model at half-filling for $U \gg t$

$$H = J \sum_{\langle xy \rangle} \vec{S}_x \cdot \vec{S}_y$$

Important open question:
Does the Hubbard model explain high-T_c superconductivity?

The Hubbard Model for doped antiferromagnets





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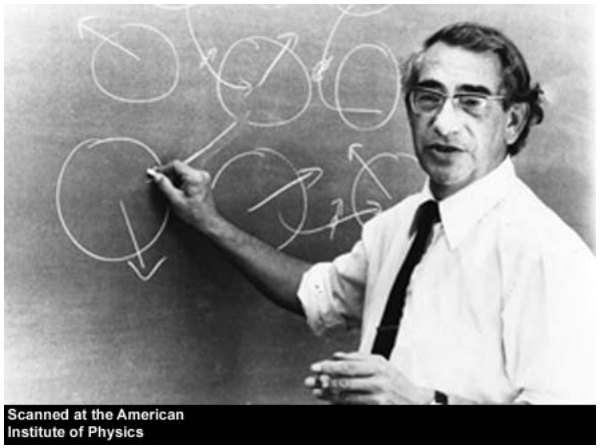

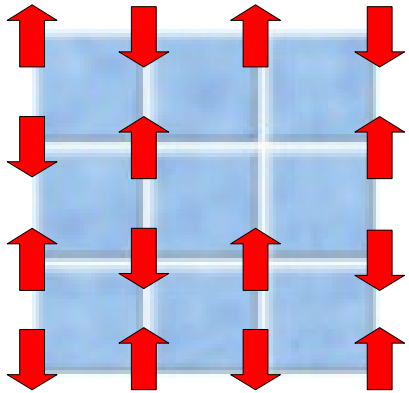

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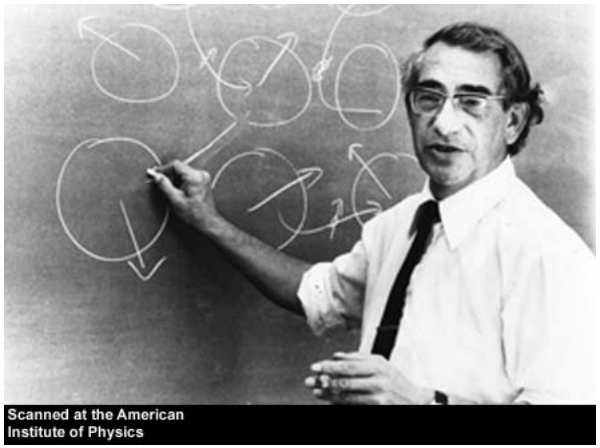

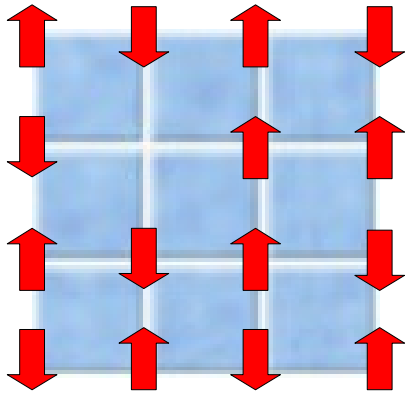

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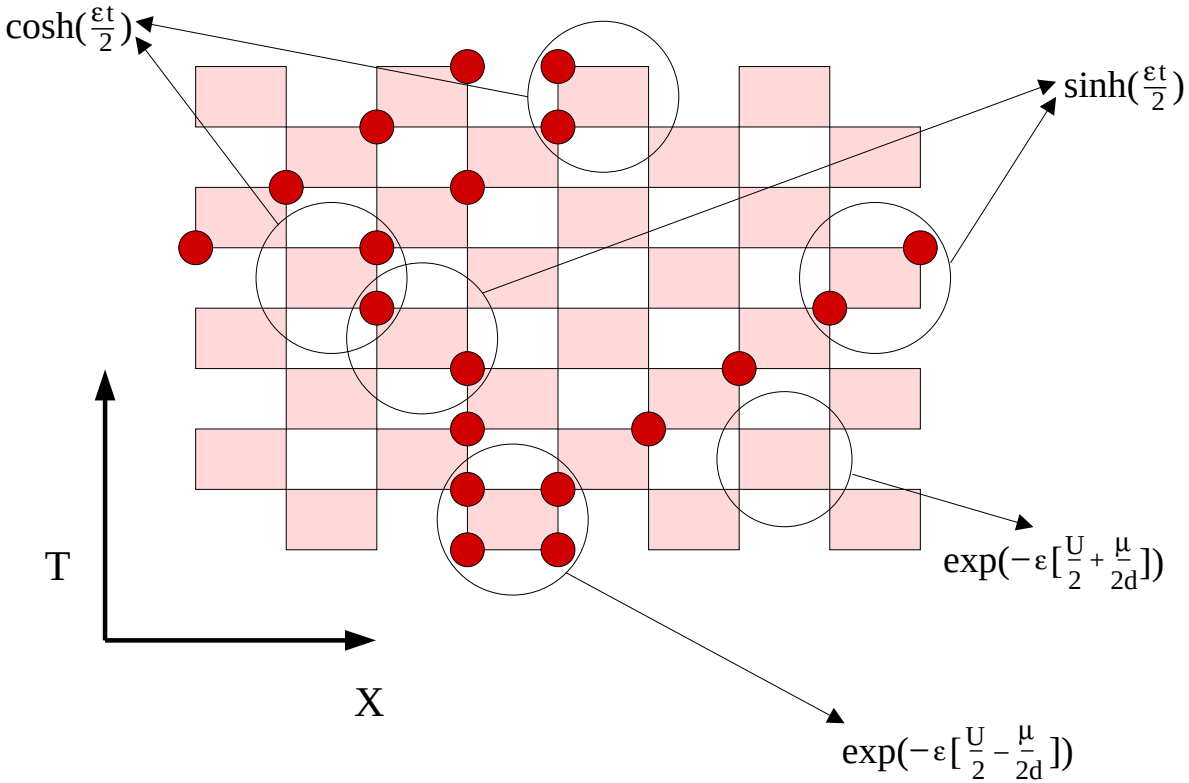
Path integral

$$
\begin{aligned}
Z_f &= \mathrm{Tr}[\exp(-\varepsilon H_1)\exp(-\varepsilon H_2)\ldots\exp(-\varepsilon H_M)]^N \\
&= \sum_{[n]} \mathrm{Sign}[n]\exp(-S[n])
\end{aligned}
$$



Sign problem of fermionic path integrals

$$Z_f = \mathrm{Tr}\exp(-\beta H) = \sum_{[n]} \mathrm{Sign}[n]\exp(-S[n]) \ , \quad \mathrm{Sign}[n] = \pm 1$$

Average sign is exponentially small

$$\langle \mathrm{Sign} \rangle = \frac{\sum_{[n]} \mathrm{Sign}[n]\exp(-S[n])}{\sum_{[n]} \exp(-S[n])} = \frac{Z_f}{Z_b} = \exp(-\beta V \Delta f)$$

The statistical error is exponentially large

$$\frac{\sigma_{\mathrm{Sign}}}{\langle \mathrm{Sign} \rangle} = \frac{\sqrt{\langle \mathrm{Sign}^2 \rangle - \langle \mathrm{Sign} \rangle^2}}{\sqrt{N}\langle \mathrm{Sign} \rangle} = \frac{\exp(\beta V \Delta f)}{\sqrt{N}} \ .$$

Some very hard sign problems are NP complete

M. Troyer, UJW, Phys. Rev. Lett. 94 (2005) 170201.

Homework 1:

Show that the Heisenberg Hamiltonian H commutes with the total spin $\vec{S}$

$$H = J \sum_{\langle xy \rangle} \vec{S}_x \cdot \vec{S}_y, \quad \vec{S} = \sum_x \vec{S}_x.$$

Show that ferromagnetic spin waves $|p_1 p_2\rangle$ are eigenstates of H and determine their energy-momentum dispersion relation,

$$|p_1 p_2\rangle = \sum_x \exp(i(p_1 x_1 + p_2 x_2)) S_x^- | \uparrow\uparrow \ldots \uparrow\rangle.$$

Homework 2:

Show that the anti-commutation relations $\{c_{x,s}^\dagger, c_{y,s'}\} = \delta_{xy}\delta_{ss'}$ of fermionic creation and annihilation operators imply angular momentum commutation relations

$$[S_x^a, S_y^b] = i\delta_{xy}\varepsilon_{abc} S_x^c, \quad \vec{S}_x = \sum_x c_x^\dagger \frac{\vec{\sigma}}{2} c_x, \quad c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix}.$$

Homework 3:

Show that the Hubbard Hamiltonian H commutes with the total spin $\vec{S}$ and with the particle number N

$$H = -t \sum_{\langle xy \rangle} (c_x^\dagger c_y + c_y^\dagger c_x) + U \sum_x (c_x^\dagger c_x - 1)^2, \quad c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix}$$

$$\vec{S} = \sum_x \vec{S}_x = \sum_x c_x^\dagger \frac{\vec{\sigma}}{2} c_x, \quad N = \sum_x n_x = \sum_x c_x^\dagger c_x.$$

Important lessons from part 1:

• Quantum computers or quantum simulators are potentially much more powerful than classical computers.

• The Heisenberg quantum spin model in thermal equilibrium can be simulated very efficiently using classical computers.

• The collective dynamics of discrete quantum spin degrees of freedom can give rise to an emergent quantum field theory for the low-energy spin wave Goldstone boson excitations.

• Classical simulations of strongly correlated electrons suffer from a very severe fermion sign problem.

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From Wilson's Lattice QCD to Quantum Link Models

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Continuum Limit of non-Abelian Quantum Link Models

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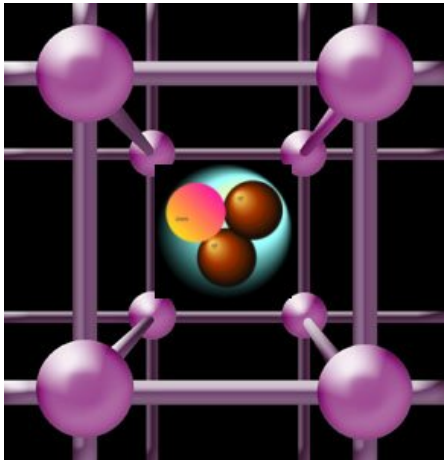
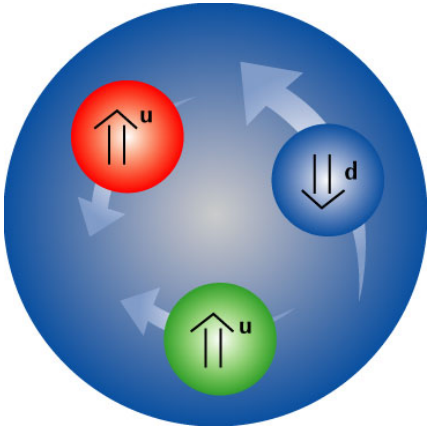
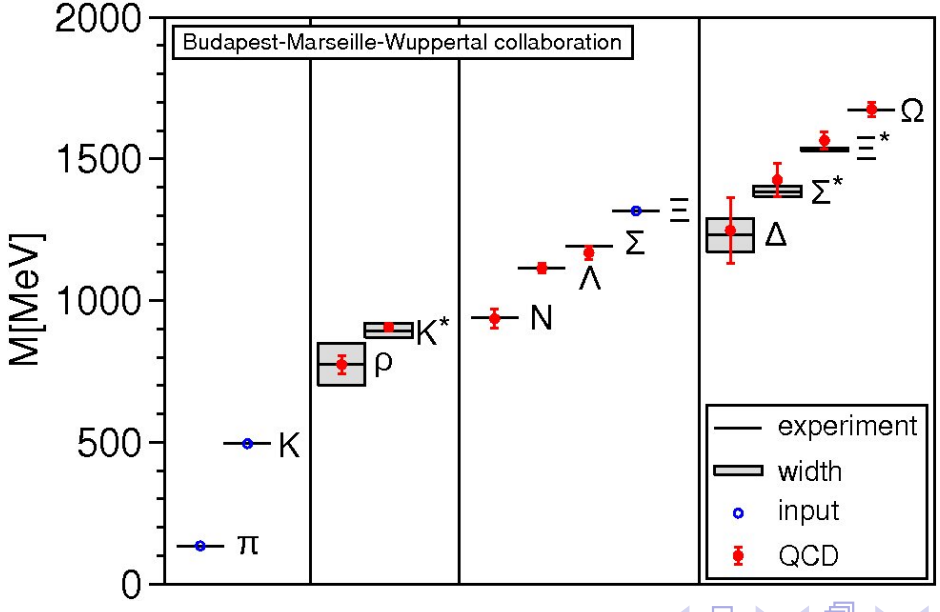
Kenneth Wilson's lattice QCD describes confinement of quarks and gluons inside protons und neutrons

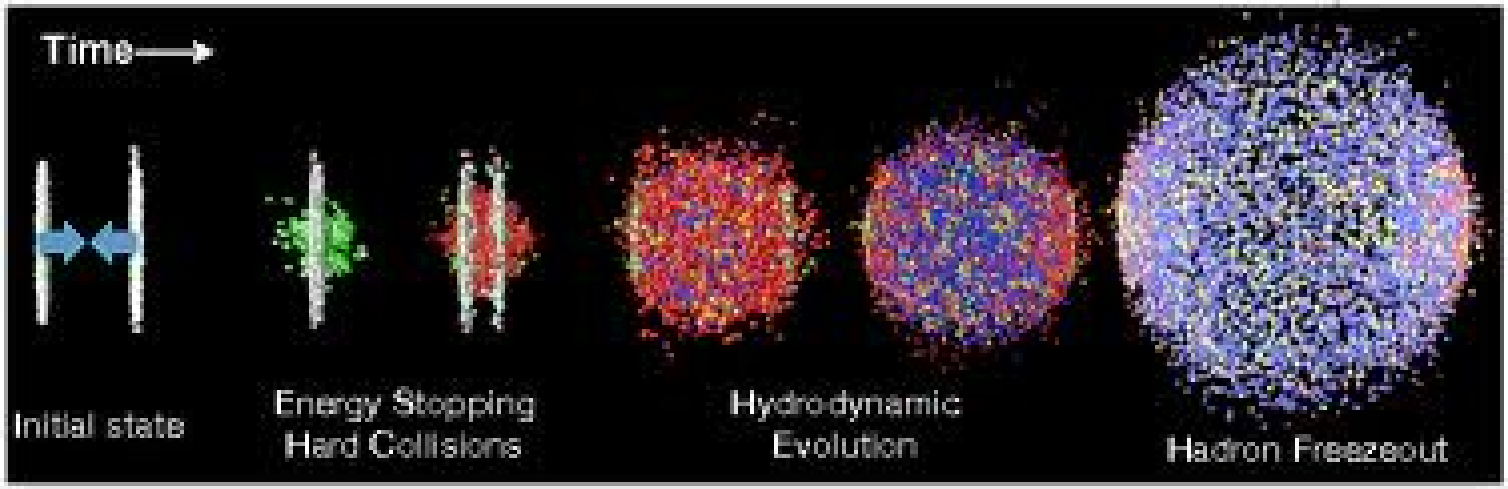


and confirms the experimentally measured mass spectrum





Can heavy-ion collision physics or nuclear astrophysics benefit from quantum simulations in the long run?



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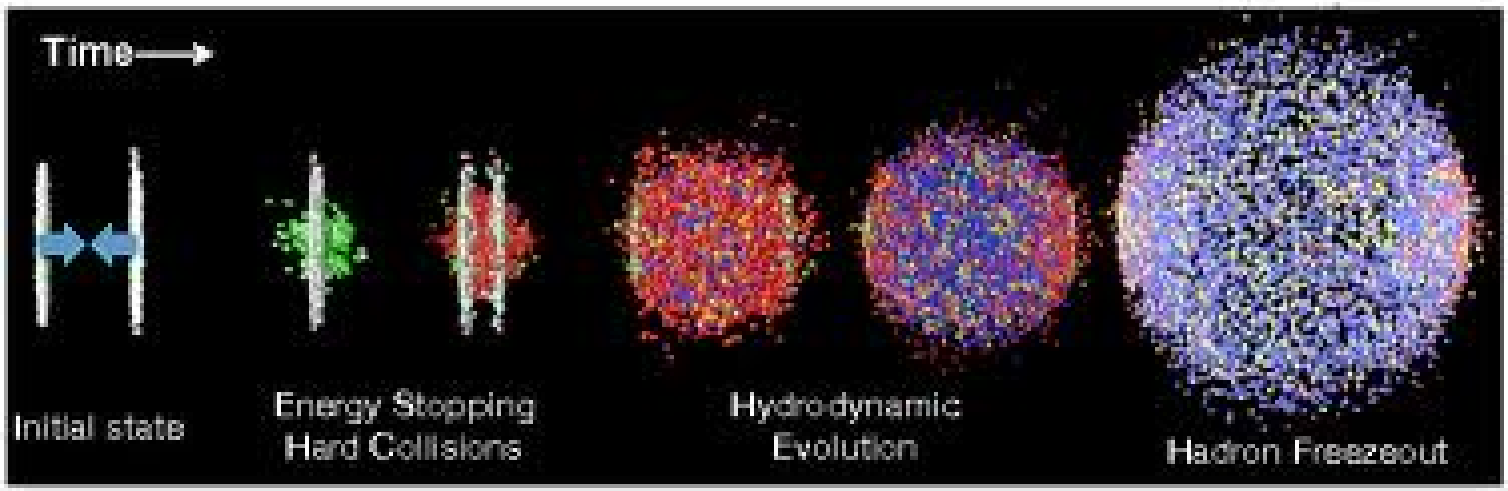

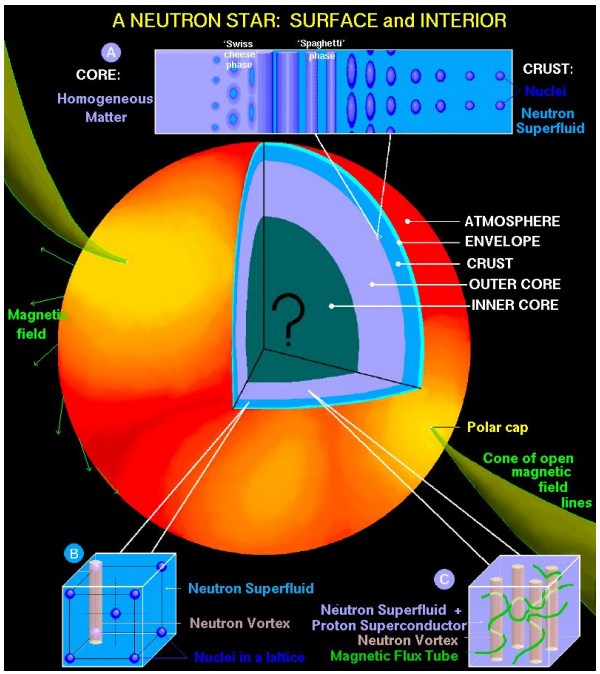

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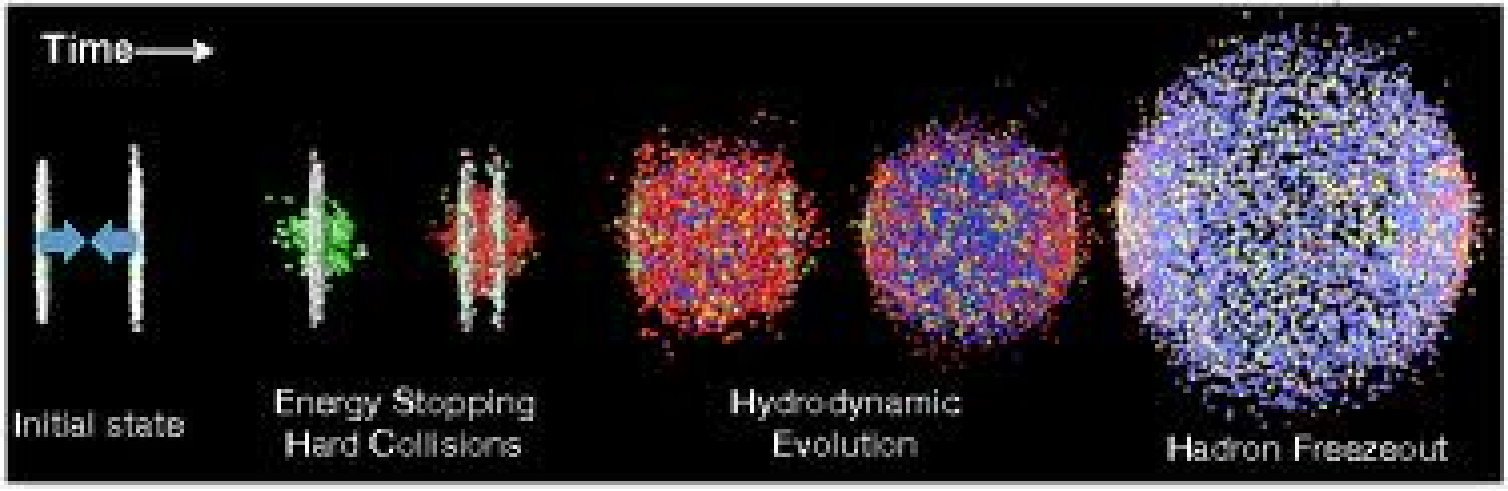

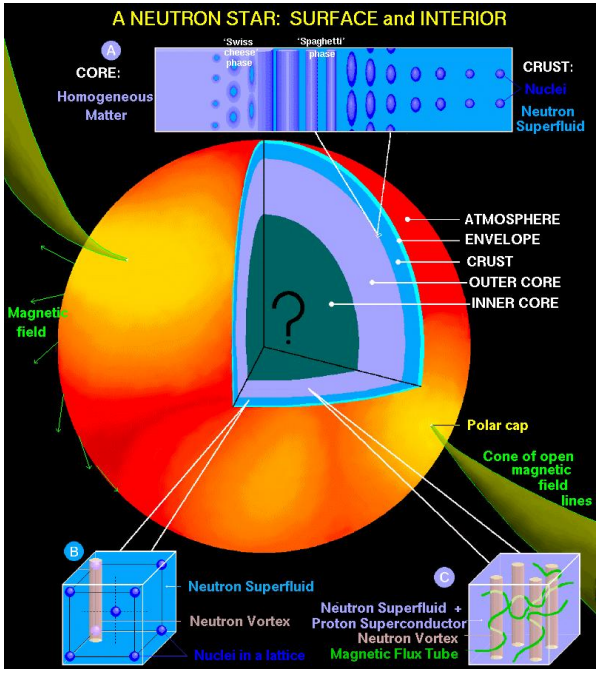

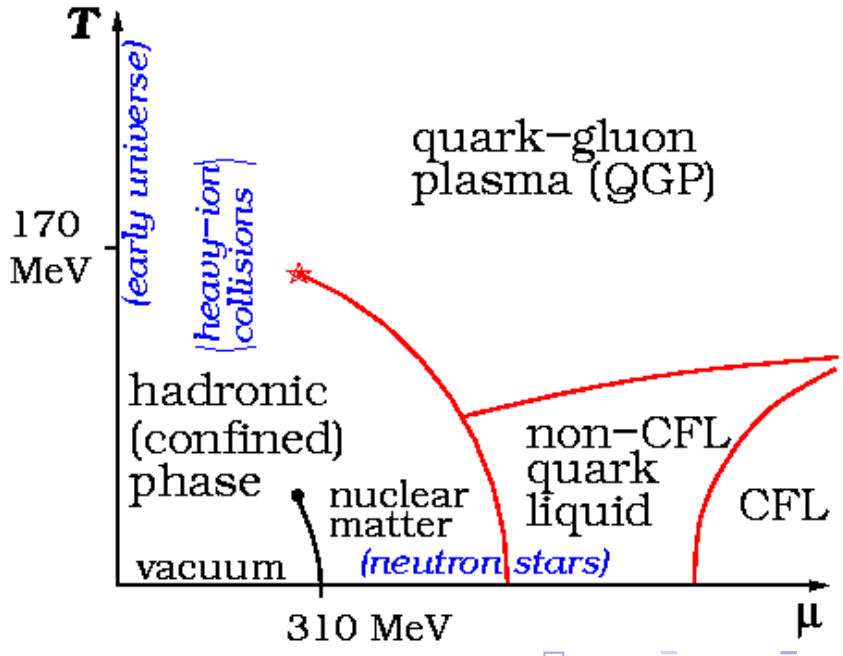

Different descriptions of dynamical Abelian gauge fields:

Maxwell's classical electromagnetic gauge fields

$$\vec{\nabla} \cdot \vec{E}(\vec{x}, t) = \rho(\vec{x}, t), \quad \vec{\nabla} \cdot \vec{B}(\vec{x}, t) = 0, \quad \vec{B}(\vec{x}, t) = \vec{\nabla} \times \vec{A}(\vec{x}, t)$$

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Quantum Electrodynamics (QED) for perturbative treatment

$$E_i = -i \frac{\partial}{\partial A_i}, \quad [E_i(\vec{x}), A_j(\vec{x}')] = i\delta_{ij}\delta(\vec{x}-\vec{x}'), \quad \left[\vec{\nabla} \cdot \vec{E} - \rho\right] |\Psi[A]\rangle = 0$$

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Wilson's $U(1)$ lattice gauge theory for classical simulation

$$U_{xy} = \exp\left(ie \int_x^y d\vec{l} \cdot \vec{A}\right) = \exp(i\varphi_{xy}) \in U(1), \quad E_{xy} = -i\frac{\partial}{\partial \varphi_{xy}},$$

$$[E_{xy}, U_{xy}] = U_{xy}, \quad \left[\sum_i (E_{x,x+\hat{i}} - E_{x-\hat{i},x}) - \rho\right] |\Psi[U]\rangle = 0$$

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$U(1)$ quantum link models for quantum simulation

$$U_{xy} = S^+_{xy}, \quad U^\dagger_{xy} = S^-_{xy}, \quad E_{xy} = S^3_{xy},$$

$$[E_{xy}, U_{xy}] = U_{xy}, \quad [E_{xy}, U^\dagger_{xy}] = -U^\dagger_{xy}, \quad [U_{xy}, U^\dagger_{xy}] = 2E^\dagger_{xy}$$

Hamiltonian formulation of Wilson's $U(1)$ lattice gauge theory

$$U = \exp(i\varphi),\ U^\dagger = \exp(-i\varphi) \in U(1)$$

Electric field operator E

$$E = -i\partial_\varphi,\ [E, U] = U,\ [E, U^\dagger] = -U^\dagger,\ [U, U^\dagger] = 0$$

Generator of $U(1)$ gauge transformations

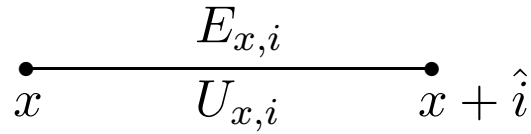
$$G_x = \sum_i (E_{x-\hat{i},i} - E_{x,i}), \quad [H, G_x] = 0$$

$U(1)$ gauge invariant Hamiltonian

$$H = \frac{g^2}{2} \sum_{x,i} E_{x,i}^2 - \frac{1}{2g^2} \sum_{x,i\neq j} (U_{x,i} U_{x+\hat{i},j} U^\dagger_{x+\hat{j},i} U^\dagger_{x,j} + \text{h.c.})$$

operates in an infinite-dimensional Hilbert space per link

$U(1)$ quantum links from spins $\frac{1}{2}$

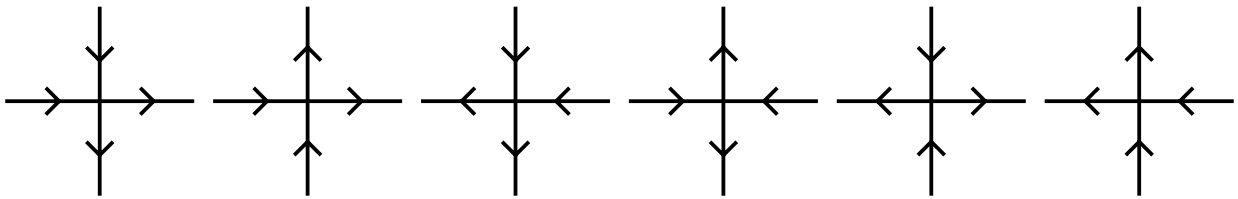


$$U = S_1 + iS_2 = S_+, \; U^\dagger = S_1 - iS_2 = S_-$$

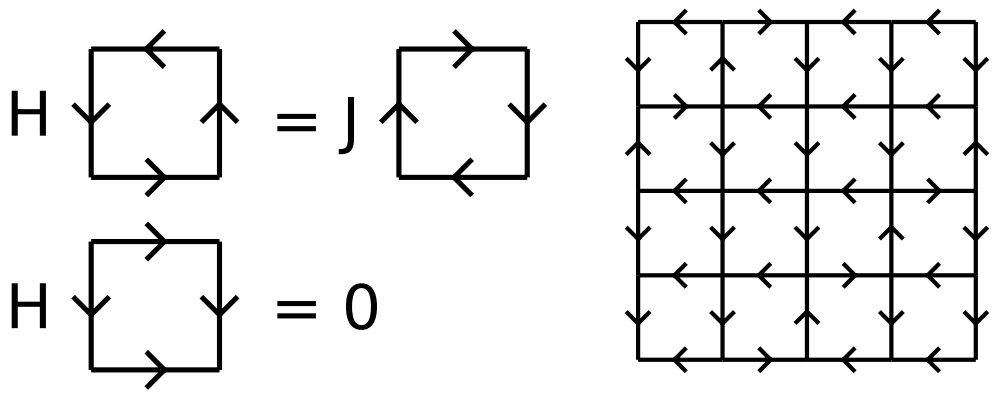
Electric flux operator E

$$E = S_3, \; [E, U] = U, \; [E, U^\dagger] = -U^\dagger, \; [U, U^\dagger] = 2E$$

Gauss law



Ring-exchange plaquette Hamiltonian



D. Horn, Phys. Lett. B100 (1981) 149

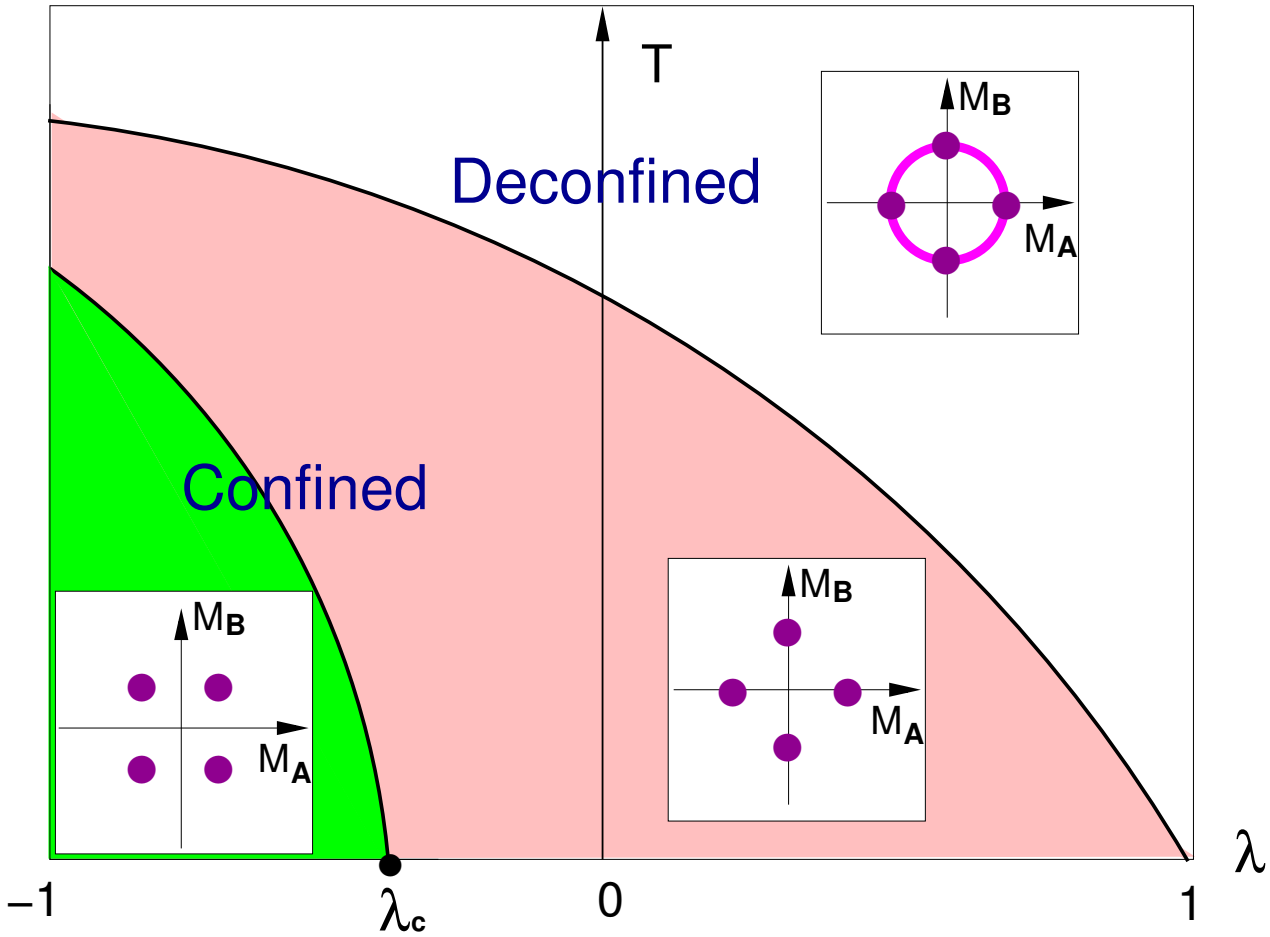
P. Orland, D. Rohrlich, Nucl. Phys. B338 (1990) 647

S. Chandrasekharan, UJW, Nucl. Phys. B492 (1997) 455

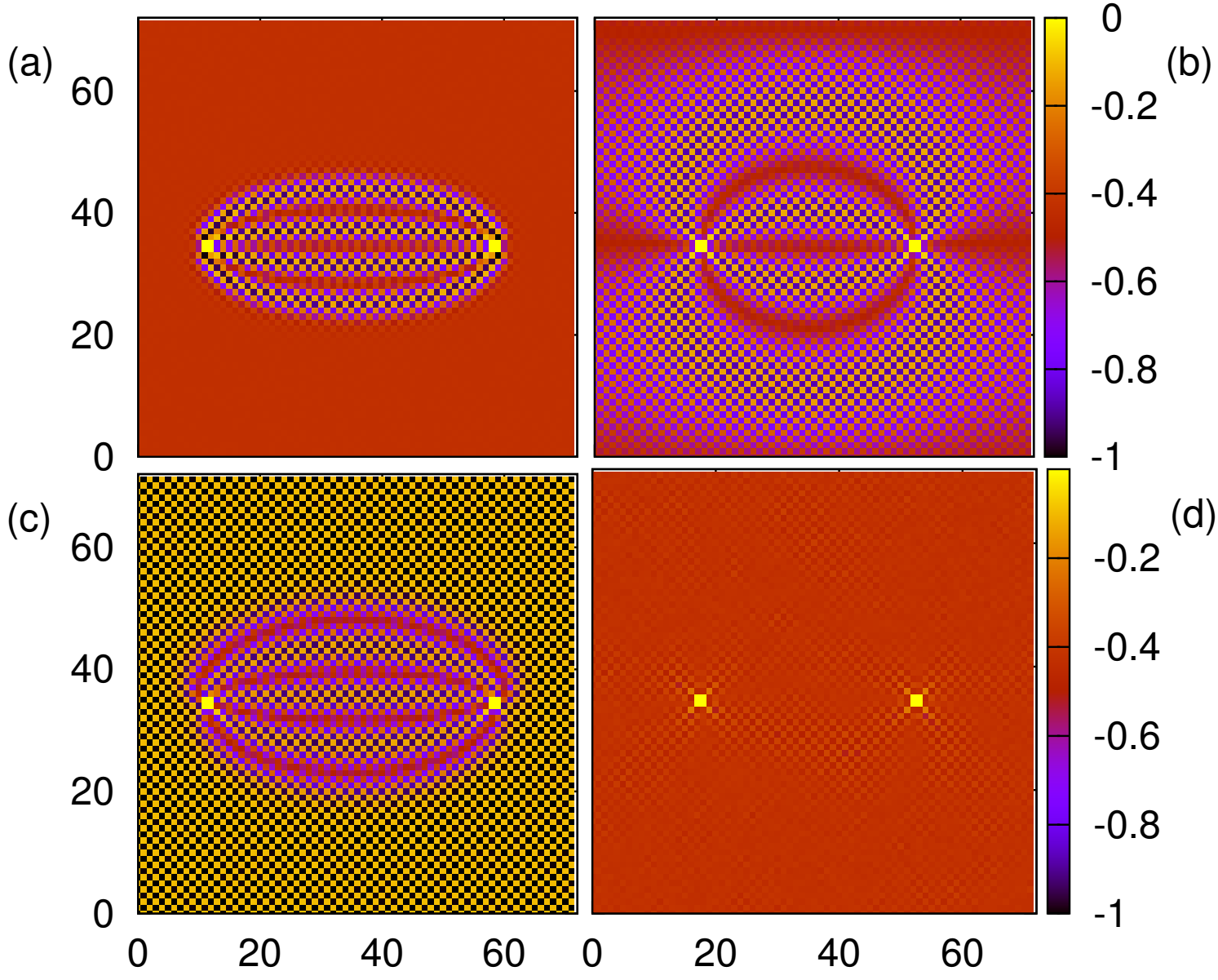
Hamiltonian with Rokhsar-Kivelson term

$$H = -J \left[\sum_{\square} (U_{\square} + U_{\square}^{\dagger}) - \lambda \sum_{\square} (U_{\square} + U_{\square}^{\dagger})^2 \right]$$

Phase diagram



Energy density of charge-anti-charge pair $Q = \pm 2$



D. Banerjee, F.-J. Jiang, P. Widmer, UJW, JSTAT (2013) P12010.

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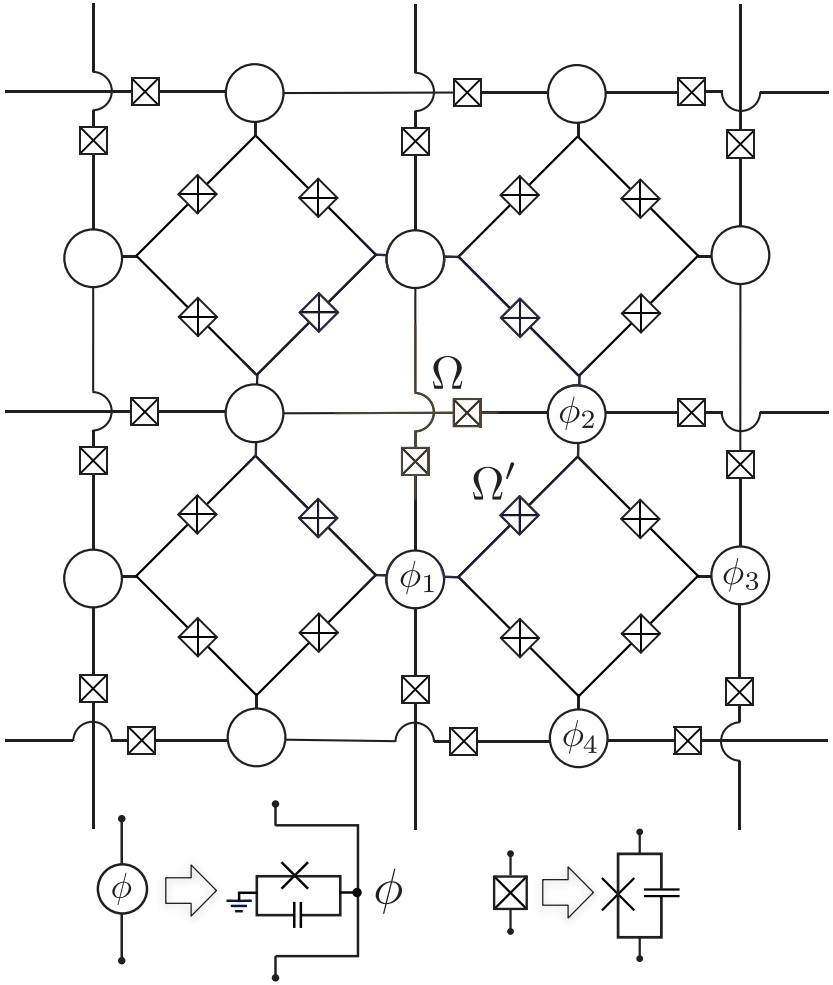
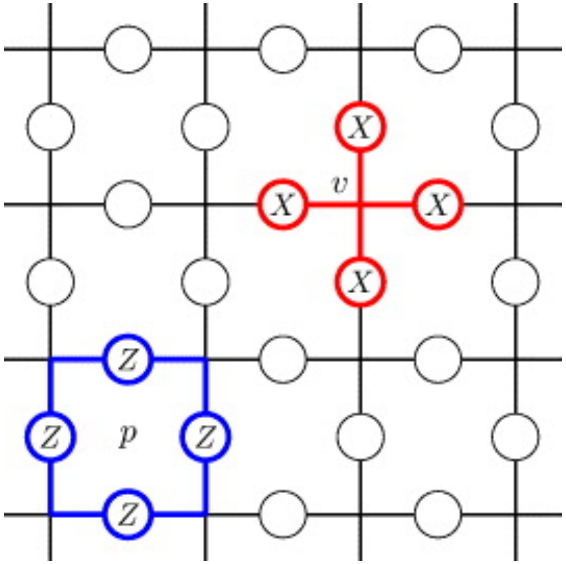
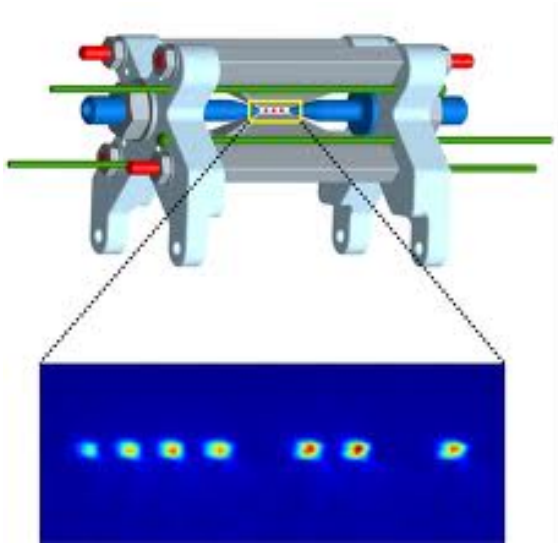
"String theory on a chip" with superconducting circuits



D. Marcos, P. Rabl, E. Rico, P. Zoller,
Phys. Rev. Lett. 111 (2013) 110504.
D. Marcos, P. Widmer, E. Rico, M. Hafezi, P. Rabl, UJW, P. Zoller,
Annals of Physics 351 (2014) 634.

Digital quantum simulation of Kitaev's toric code (a $\mathbb{Z}(2)$ quantum link model) with trapped ions

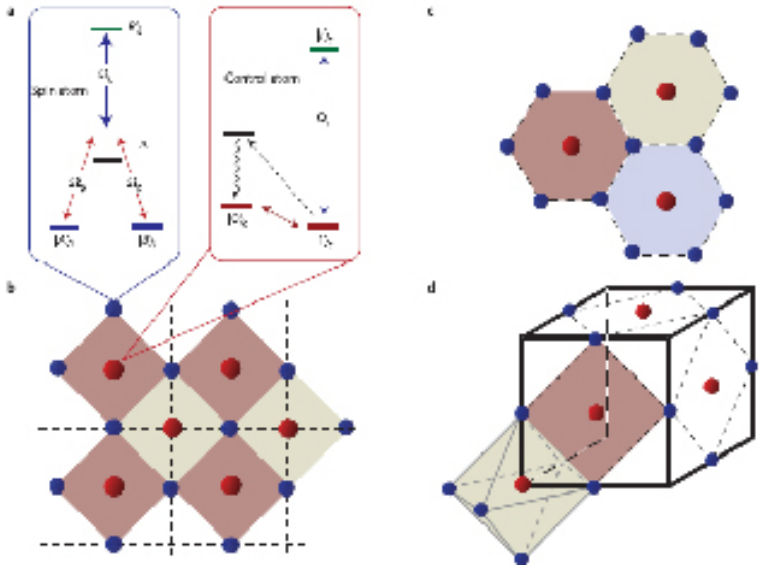


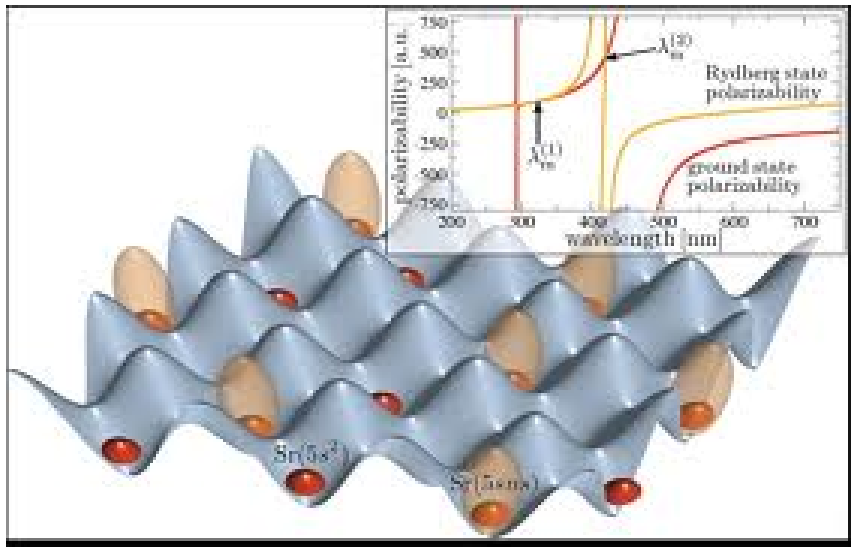


• Precisely controllable many-body quantum device, which can execute a prescribed sequence of quantum gate operations.
• State of simulated system encoded as quantum information.
• Dynamics is represented by a sequence of quantum gates, following a stroboscopic Trotter decomposition.

A. Y. Kitaev, Ann. Phys. 303 (2003) 2.
B. P. Lanyon, C. Hempel, D. Nigg, M. Müller, R. Gerritsma, F. Zähringer, P. Schindler, J. T. Barreiro, M. Rambach, G. Kirchmair, M. Hennrich, P. Zoller, R. Blatt, C. F. Roos, Science 334 (2011) 6052.

$U(1)$ quantum link models can also be simulated with Rydberg atoms in an optical lattice

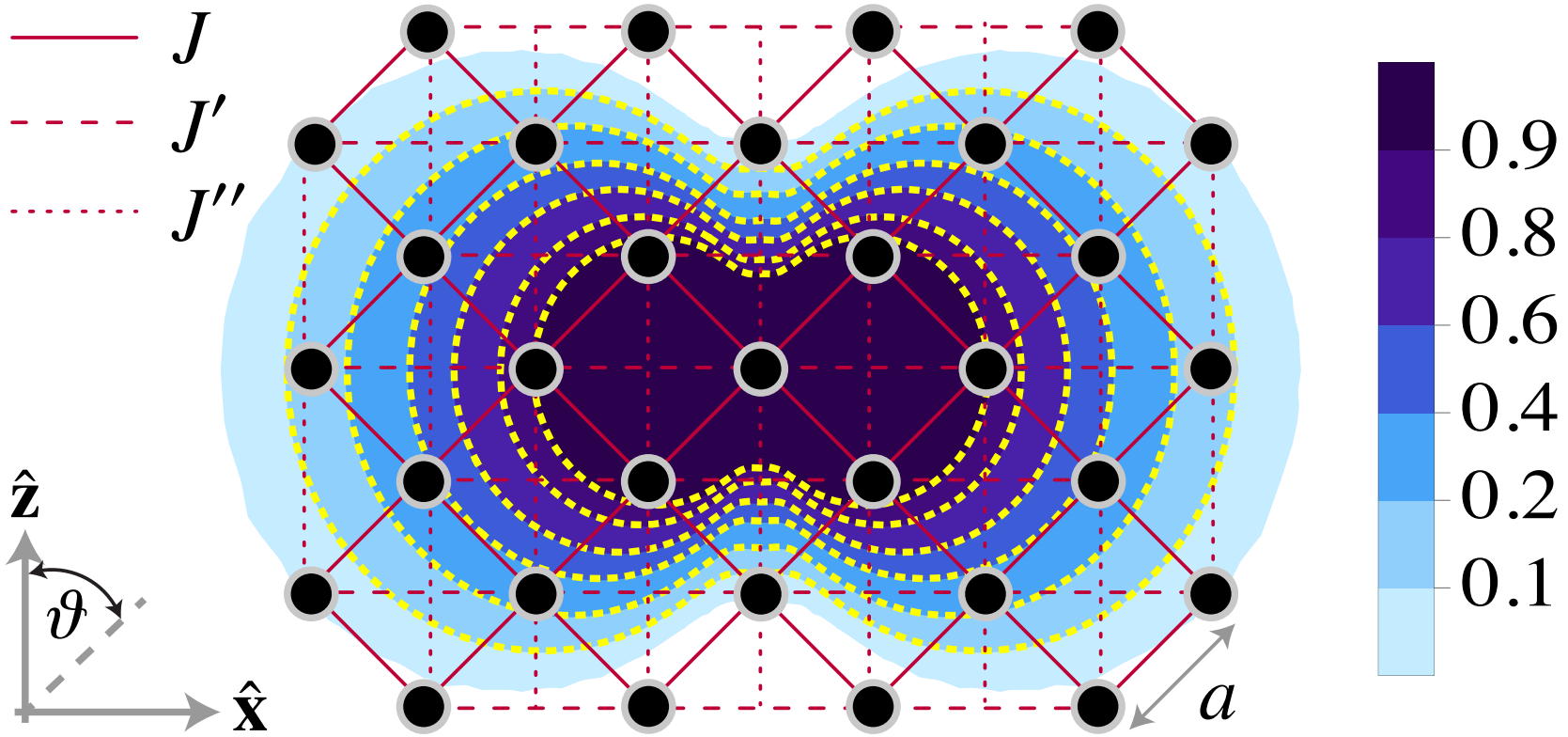




- Lasers can excite atoms to high-lying Rydberg states.
- Rydberg atoms are large and have collective interactions.
- Ensemble Rydberg atoms represent qubits at link centers.
- Control atoms at lattice sites ensure the Gauss' law.

M. Müller, I. Lesanovsky, H. Weimer, H. P. Büchler, P. Zoller, Phys. Rev. Lett. 102 (2009) 170502.
H. Weimer, M. Müller, I. Lesanovsky, P. Zoller, H. P. Büchler, Nat. Phys. 6 (2010) 382.
L. Tagliacozzo, A. Celi, P. Orland, M. Lewenstein, Nature Communications 4 (2013) 2615.
L. Tagliacozzo, A. Celi, A. Zamora, M. Lewenstein, Ann. Phys. 330 (2013) 160.

P-state excited Rydberg atoms in an optical lattice



A. G. Glaetzle, M. Dalmonte, R. Nath, I. Rousochatzakis, R. Moessner, P. Zoller, Phys. Rev. X4 (2014) 041037.

Hamiltonian for staggered fermions and $U(1)$ quantum links

$$H = -t \sum_x \left[\psi_x^\dagger U_{x,x+1} \psi_{x+1} + \text{h.c.} \right] + m \sum_x (-1)^x \psi_x^\dagger \psi_x + \frac{g^2}{2} \sum_x E_{x,x+1}^2$$

Bosonic rishon representation of the quantum links

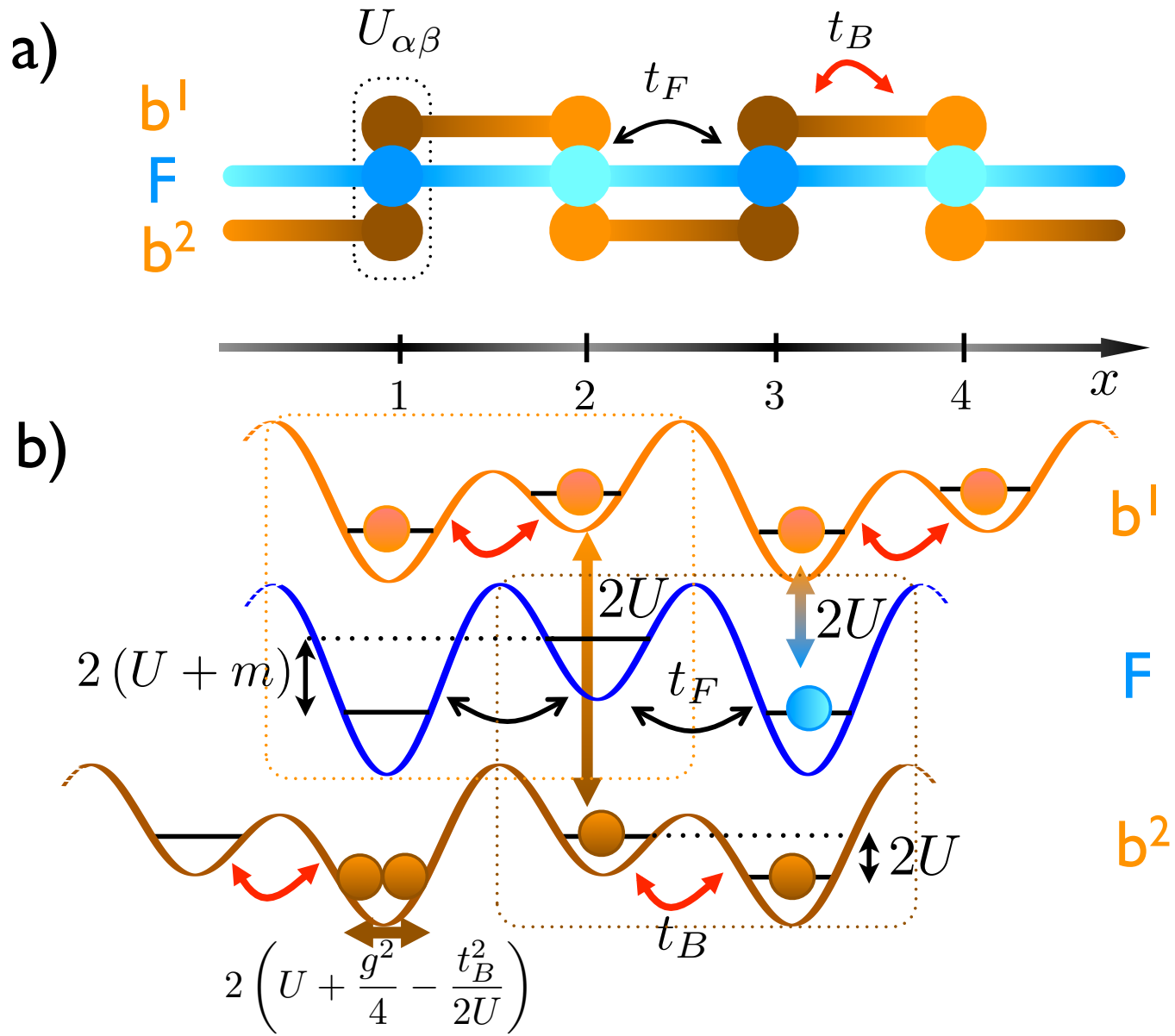
$$U_{x,x+1} = b_x b_{x+1}^\dagger, \;\; E_{x,x+1} = \frac{1}{2} \left(b_{x+1}^\dagger b_{x+1} - b_x^\dagger b_x \right)$$

Microscopic Hubbard model Hamiltonian

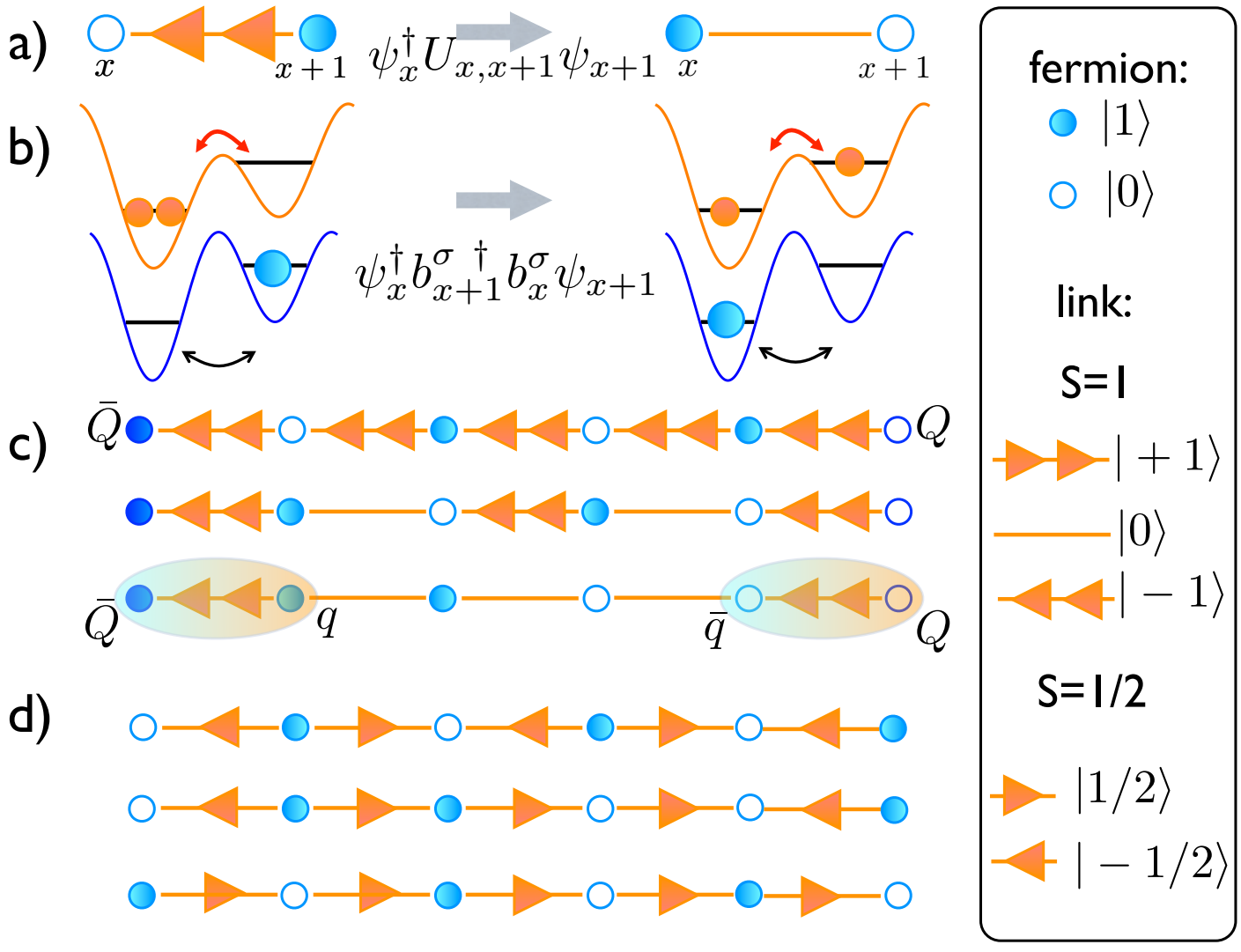
$$\begin{aligned} \widetilde{H} &= \sum_x h_{x,x+1}^B + \sum_x h_{x,x+1}^F + m \sum_x (-1)^x n_x^F + U \sum_x \widetilde{G}_x^2 \\ &= -t_B \sum_{x \text{ odd}} b_x^{1\dagger} b_{x+1}^1 - t_B \sum_{x \text{ even}} b_x^{2\dagger} b_{x+1}^2 - t_F \sum_x \psi_x^\dagger \psi_{x+1} + \text{h.c.} \\ &+ \sum_{x,\alpha,\beta} n_x^\alpha U_{\alpha\beta} n_x^\beta + \sum_{x,\alpha} (-1)^x U_\alpha n_x^\alpha \end{aligned}$$

D. Banerjee, M. Dalmonte, M. Müller, E. Rico, P. Stebler, UJW, P. Zoller, Phys. Rev. Lett. 109 (2012) 175302.

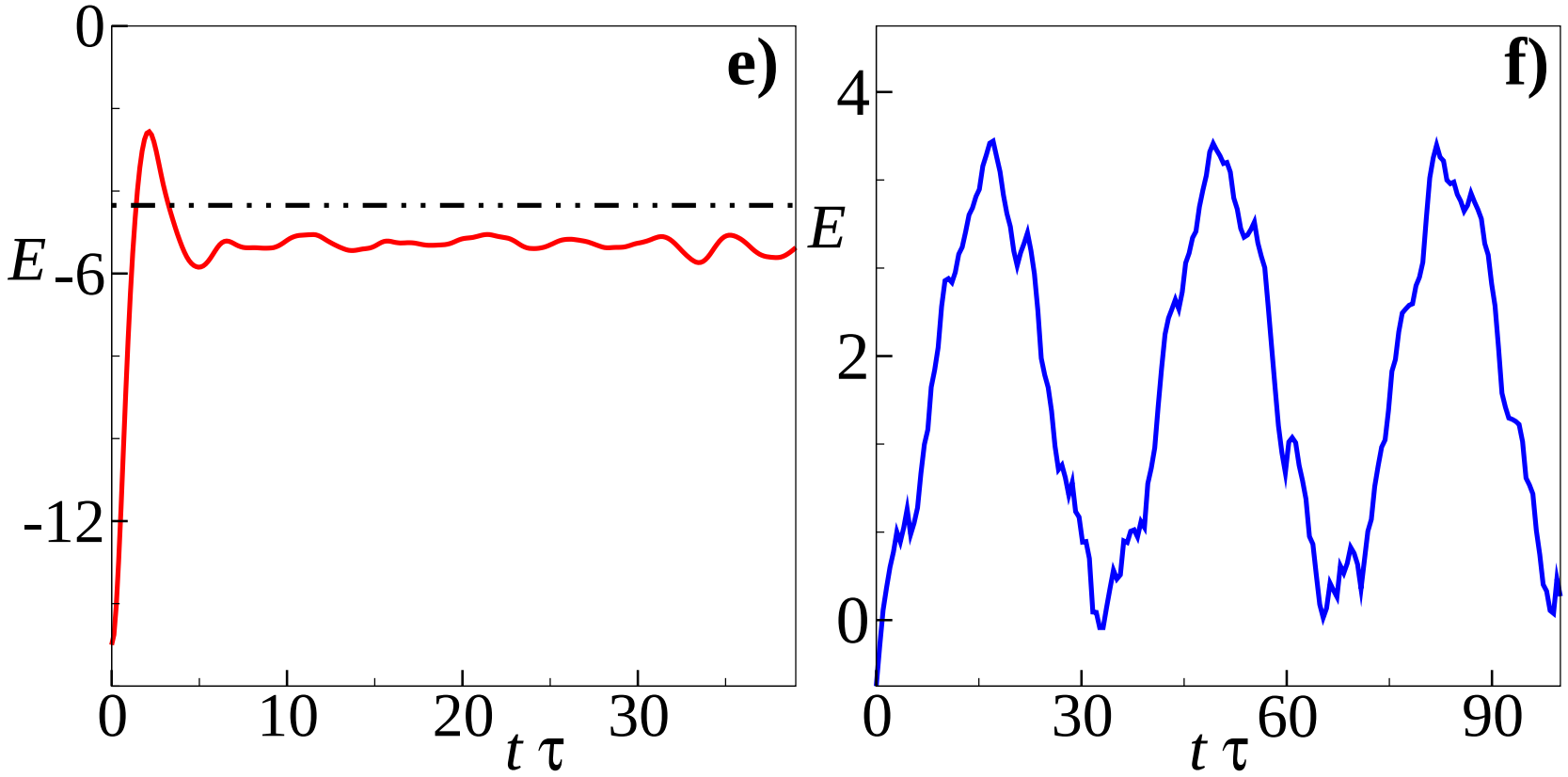
Optical lattice with Bose-Fermi mixture of ultra-cold atoms



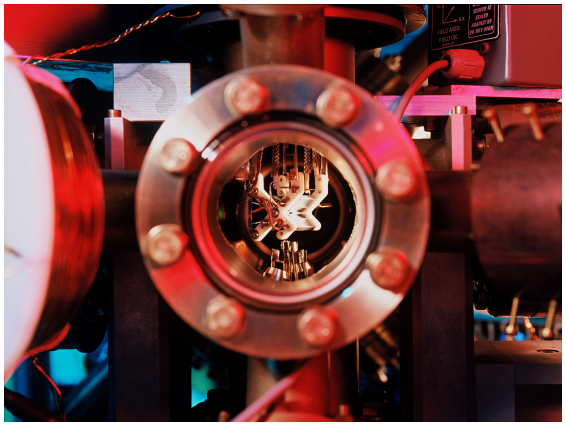
From string breaking to false vacuum decay

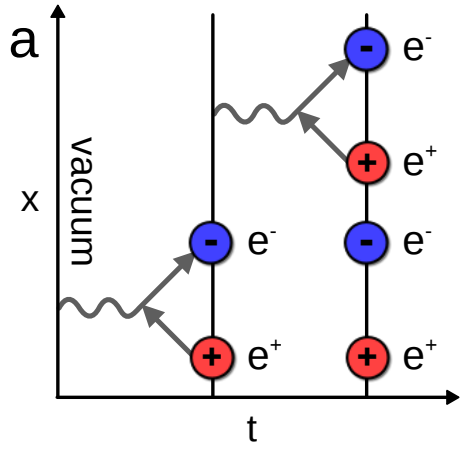


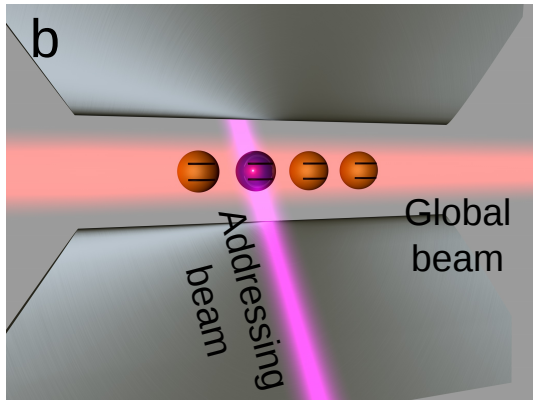
Quantum simulation of the real-time evolution of string breaking and of coherent vacuum oscillations



Digital 4-qubit ion-trap quantum computation of pair creation







• Quantum computer consisting of four trapped Ca ions that act as four qubits, which are manipulated by external laser beams.

• Precisely controllable many-body quantum device, executing a prescribed sequence of quantum gate operations.

• State of simulated system is encoded as quantum information.

• Dynamics is represented by a sequence of quantum gates, following a stroboscopic Trotter decomposition.

E. A. Martinez, C. A. Muschik, P. Schindler, D. Nigg, A. Erhard, M. Heyl, P. Hauke, M. Dalmonte, T. Monz, P. Zoller, R. Blatt, Nature 534 (2016) 516.

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$U(N)$ quantum link operators

$$U^{ij} = S_1^{ij} + iS_2^{ij},\ U^{ij\dagger} = S_1^{ij} - iS_2^{ij},\ i,j \in \{1,2,\ldots,N\},\ [U^{ij},(U^\dagger)^{kl}] \neq 0$$

$SU(N)_L \times SU(N)_R$ gauge transformations of a quantum link

$$[L^a, L^b] = if_{abc}L^c,\ [R^a, R^b] = if_{abc}R^c,\ a,b,c \in \{1,2,\ldots,N^2-1\}$$

$$[L^a, R^b] = [L^a, E] = [R^a, E] = 0$$

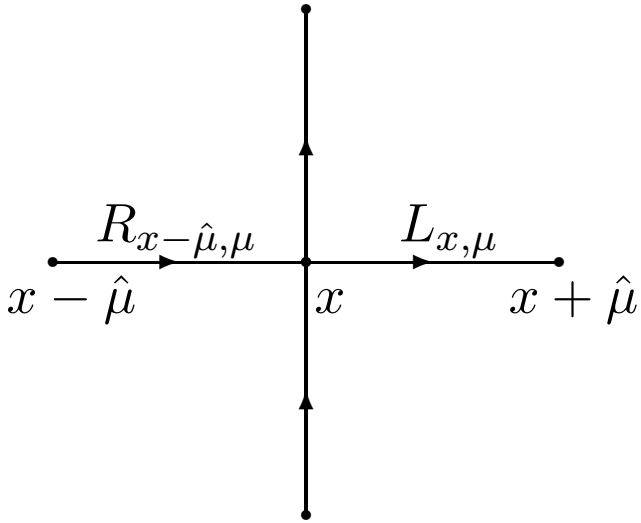
Infinitesimal gauge transformations of a quantum link

$$[L^a, U] = -\lambda^a U,\ [R^a, U] = U\lambda^a,\ [E, U] = U$$

Algebraic structures of $U(N)$ quantum link models

$U^{ij},\ L^a,\ R^a, E,\ 2N^2 + 2(N^2-1) + 1 = 4N^2 - 1$ $SU(2N)$ generators

R. Brower, S. Chandrasekharan, UJW, Phys. Rev. D60 (1999) 094502



Generator of $SU(N)$ gauge transformations

$$G_x^a = \sum_\mu (R^a_{x-\hat{\mu},\mu} + L^a_{x,\mu})$$

$U(N)$-invariant Hamiltonian "action" operator

$$H = -J \sum_{x,\mu<\nu} \mathrm{Tr}(U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\nu} + \mathrm{h.c.}),\ [H, G_x^a] = 0$$

Functional integral of a quantum link model

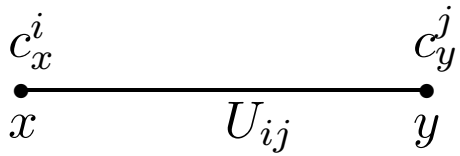
$$Z = \mathrm{Tr}\exp(-\beta H)$$

defines a quantum field theory using discrete variables

Fermionic rishons at the two ends of a link

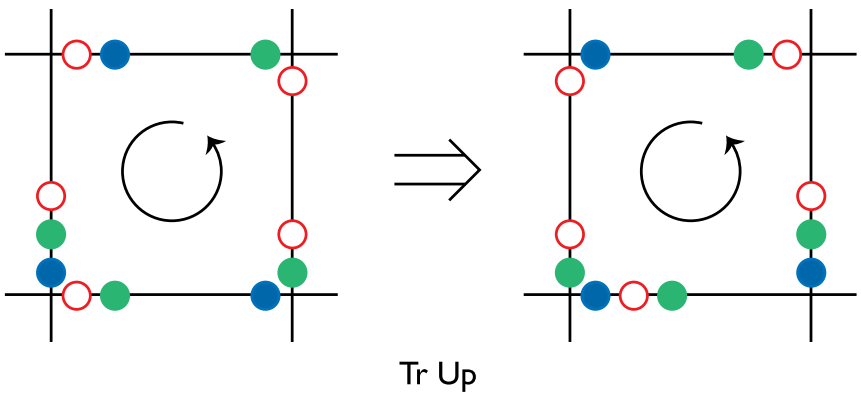
$$\{c_x^i, c_y^{j\dagger}\} = \delta_{xy}\delta_{ij},\ \{c_x^i, c_y^j\} = \{c_x^{i\dagger}, c_y^{j\dagger}\} = 0$$

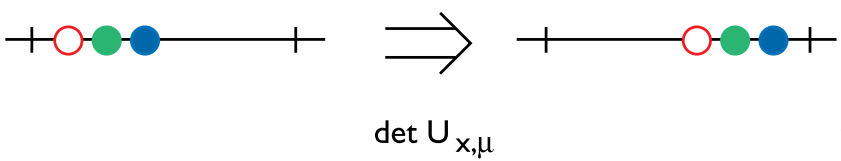
Rishon representation of link algebra



$$U_{xy}^{ij} = c_x^i c_y^{j\dagger},\ L_{xy}^a = c_x^{i\dagger}\lambda_{ij}^a c_x^j,\ R_{xy}^a = c_y^{i\dagger}\lambda_{ij}^a c_y^j,\ E_{xy} = \frac{1}{2}(c_y^{i\dagger}c_y^i - c_x^{i\dagger}c_x^i)$$

Can a "rishon abacus" implemented with ultra-cold atoms be used as a quantum simulator?





d-dimensional $SU(N)$ gauge theory with staggered fermions

$$
\begin{aligned}
H &= -t \sum_{\langle xy \rangle} \left(s_{xy} \psi_x^{i\dagger} U_{xy}^{ij} \psi_y^j + \text{h.c.} \right) + m \sum_x (-1)^x \psi_x^{i\dagger} \psi_x^i \\
&+ \frac{g^2}{2} \sum_{\langle xy \rangle} \left(L_{xy}^a L_{xy}^a + R_{xy}^a R_{xy}^a \right) + \frac{g'^2}{2} \sum_{\langle xy \rangle} E_{xy}^2 \\
&- \frac{1}{4g^2} \sum_{\langle wxyz \rangle} \left(U_{wx} U_{xy} U_{yz} U_{zw} + \text{h.c.} \right) - \gamma \sum_{\langle xy \rangle} \left(\det U_{xy} + \text{h.c.} \right)
\end{aligned}
$$

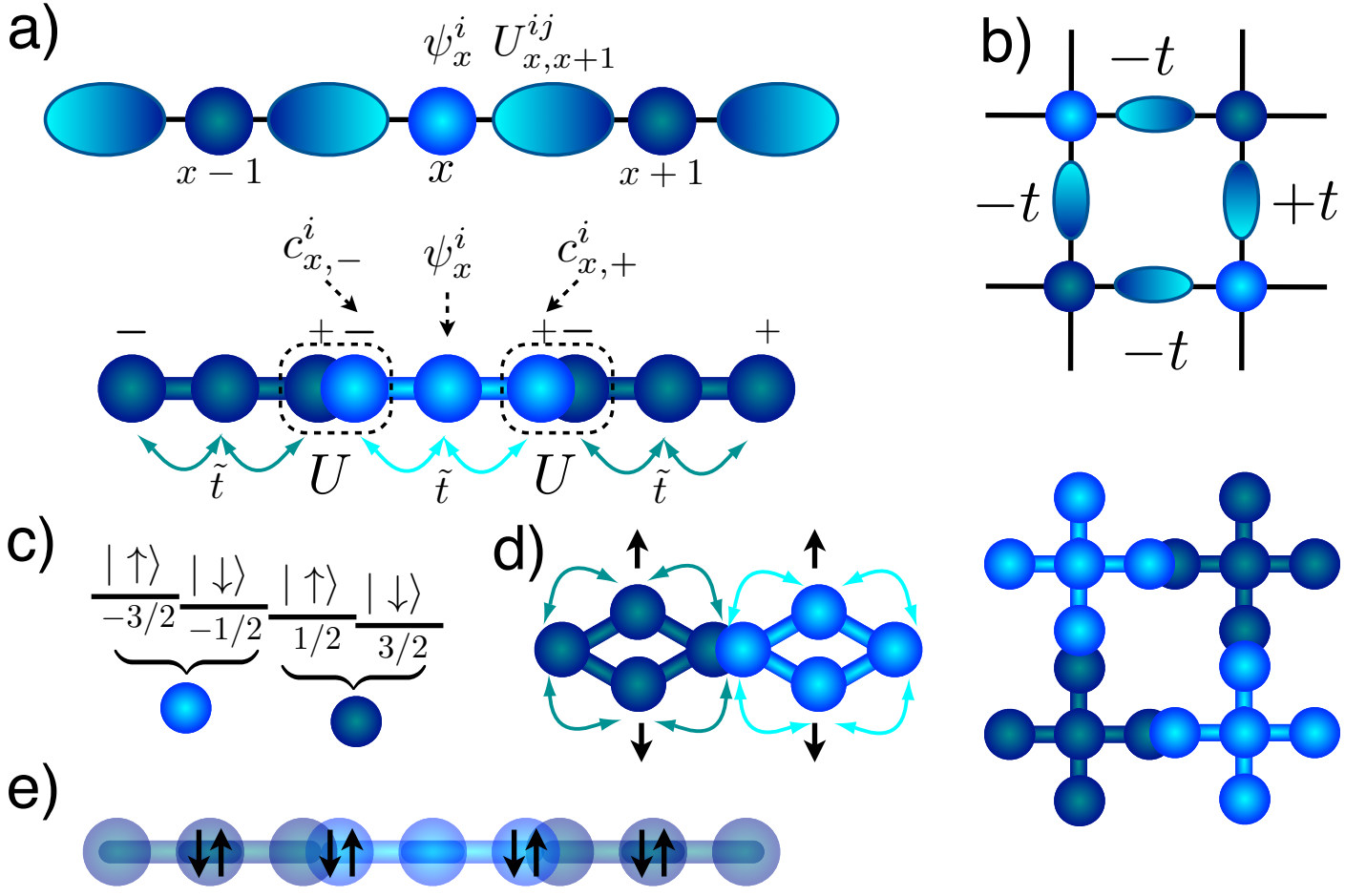
Meson, constituent quark, and glueball operators

$$
M_x = \psi_x^{i\dagger} \psi_x^i, \quad Q_{x,\pm k} = c_{x,\pm k}^{i\dagger} \psi_x^i, \quad \Phi_{x,\pm k,\pm l} = c_{x,\pm k}^{i\dagger} c_{x,\pm l}^i
$$

form a local $U(2d+1)$ algebra at each site x, thus providing a formulation in terms of manifestly $SU(N)$ gauge invariant objects. However, the conserved rishon number gives rise to a $U(1)$ gauge symmetry on the links

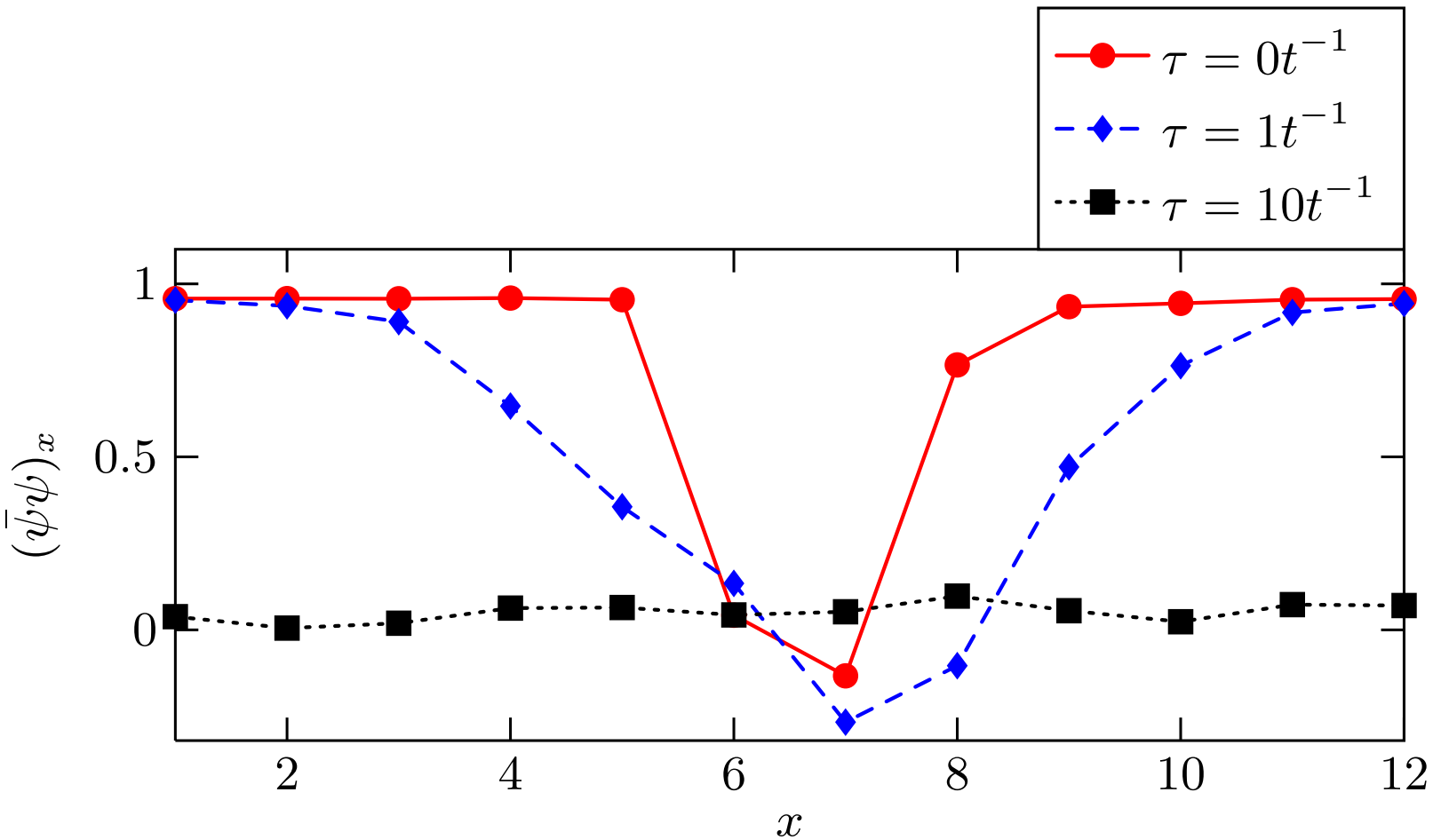
$$
N_{xy} = c_x^{i\dagger} c_x^i + c_y^{i\dagger} c_y^i, \quad [N_{xy}, H] = 0
$$

Optical lattice with ultra-cold alkaline-earth atoms (^{87}Sr or ^{173}Yb) with color encoded in nuclear spin



D. Banerjee, M. Bögli, M. Dalmonte, E. Rico, P. Stebler, UJW, P. Zoller, Phys. Rev. Lett. 110 (2013) 125303

Expansion of a "fireball" mimicking a hot quark-gluon plasma



Homework 4:

Show that the Hamiltonian of the 2-d $U(1)$ quantum link model commutes with the local generators of gauge transformations $G_x = \sum_i (E_{x-\hat{i},i} - E_{x,i})$

$$H = -\frac{1}{2g^2} \sum_{x,i\neq j} (U_{x,i} U_{x+\hat{i},j} U^{\dagger}_{x+\hat{j},i} U^{\dagger}_{x,j} + \text{h.c.}).$$

Homework 5:

Show that the $SU(2)$ embedding algebra of the $U(1)$ quantum link model can be realized with bosonic rishons

$$U_{xy} = b_x b_y^{\dagger}, \; E_{xy} = \frac{1}{2}(b_y^{\dagger} b_y - b_x^{\dagger} b_x)$$

Homework 6:

Show that the $SU(2N)$ embedding algebra of the $U(N)$ quantum link model can be realized with fermionic rishons

$$U^{ij}_{xy} = c^i_x c^{j\dagger}_y, \; L^a_{xy} = c^{i\dagger}_x \lambda^a_{ij} c^j_x, \; R^a_{xy} = c^{i\dagger}_y \lambda^a_{ij} c^j_y, \; E_{xy} = \frac{1}{2}(c^{i\dagger}_y c^i_y - c^{i\dagger}_x c^i_x)$$

Important lessons from part 2:

• Wilson's lattice QCD allows the precise determination of static hadron properties using Monte Carlo simulations. It also allows to simulate QCD at finite temperature. Simulations of dynamical processes or of the physics at non-zero baryon density suffer from very severe sign problems.

• Gauge theories with exact continuous gauge symmetry can be formulated in terms of discrete quantum link degrees of freedom.

• Thanks to their finite-dimensional Hilbert space per link, quantum links can be embodied by ultra-cold fermionic or bosonic atoms in an optical lattice or by superconducting flux cubits.

• Non-Abelian quantum link models are formulated in terms of discrete quantum links, but have an exact $SU(N)$ gauge symmetry.

• Quantum links have fermionic "rishon" constituents that can be embodied by alkaline-earth atoms.

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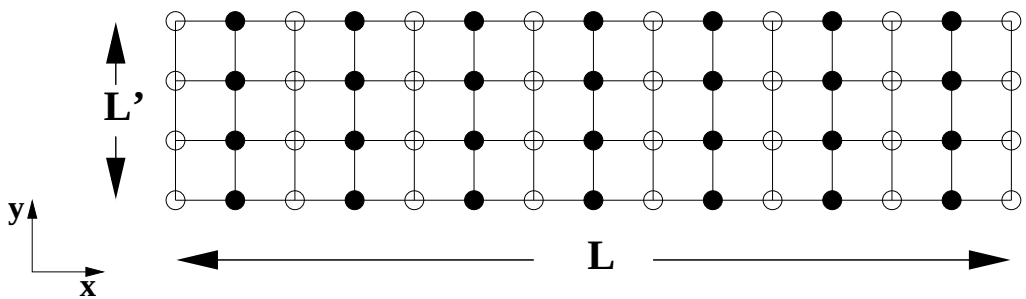
$\mathbb{C}P(N-1)$ Models from $SU(N)$ quantum spins

$$T^a_x, \quad a \in \{1, 2, ..., N^2-1\}, \quad [T^a_x, T^b_y] = i\delta_{xy} f_{abc} T^c_x$$

Spin ladder Hamiltonian

$$H = -J \sum_{x \in A} [T^a_x T^{a*}_{x+\hat{1}} + T^a_x T^a_{x+\hat{2}}] - J \sum_{x \in B} [T^{a*}_x T^a_{x+\hat{1}} + T^{a*}_x T^{a*}_{x+\hat{2}}]$$



Conserved $SU(N)$ spin

$$T^a = \sum_{x \in A} T^a_x - \sum_{x \in B} T^{a*}_x, \quad [T^a, H] = 0$$

C. Laflamme, W. Evans, M. Dalmonte, U. Gerber, H. Meja-Daz, W. Bietenholz, and P. Zoller, arXiv:1507.06788.

Goldstone boson fields in $\mathbb{C}P(N-1) = SU(N)/U(N-1)$

$$P(x)^\dagger = P(x), \quad \mathrm{Tr}P(x) = 1, \quad P(x)^2 = P(x)$$

Low-energy effective action

$$\begin{aligned} S[P] &= \int_0^\beta dt \int_0^L dx \int_0^{L'} dy \ \mathrm{Tr}\left\{\rho_s' \partial_y P \partial_y P \right. \\ &+ \left. \rho_s \left[\partial_x P \partial_x P + \frac{1}{c^2}\partial_t P \partial_t P\right] - \frac{1}{a} P \partial_x P \partial_t P \right\} \end{aligned}$$

Topological charge

$$Q[P] = \frac{1}{\pi i}\int_0^\beta dt \int_0^L dx \, \mathrm{Tr}[P\partial_x P \partial_t P] \in \Pi_2[SU(N)/U(N-1)] = Z$$

B. B. Beard, M. Pepe, S. Riederer, UJW, Phys. Rev. Lett. 94 (2005) 010603.

Very large correlation length

$$\xi \propto \exp(4\pi L' \rho_s / cN) \gg L'$$

Dimensional reduction to the $(1+1)$-d $\mathbb{C}P(N-1)$ model

$$S[P] = \int_0^\beta dt \int_0^L dx \, \mathrm{Tr} \left\{ \frac{1}{g^2} \left[\partial_x P \partial_x P + \frac{1}{c^2} \partial_t P \partial_t P \right] - n P \partial_x P \partial_t P \right\}$$

Emergent θ-vacuum angle

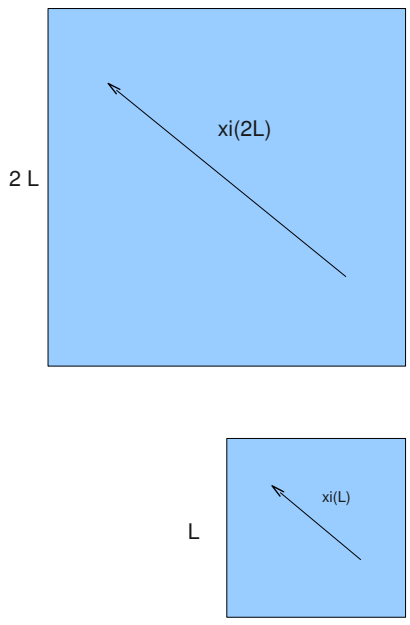
$$\theta = n\pi$$

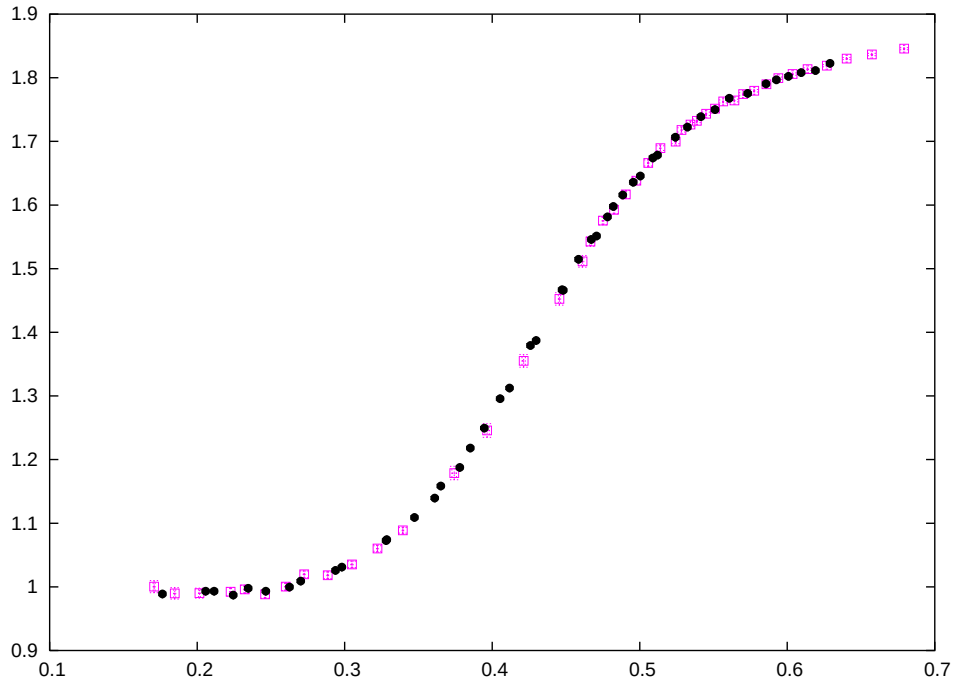
Correlation lengths in the $\mathbb{C}P(2)$ model

$$\xi(2) = 4.0(1), \quad \xi(4) = 17.6(2), \quad \xi(6) = 61(2)$$

Universal Step-Scaling Function in $\mathbb{C}P(2)$ Model

M. Lüscher, P. Weisz, and U. Wolff, Nucl. Phys. B359 (1991) 221

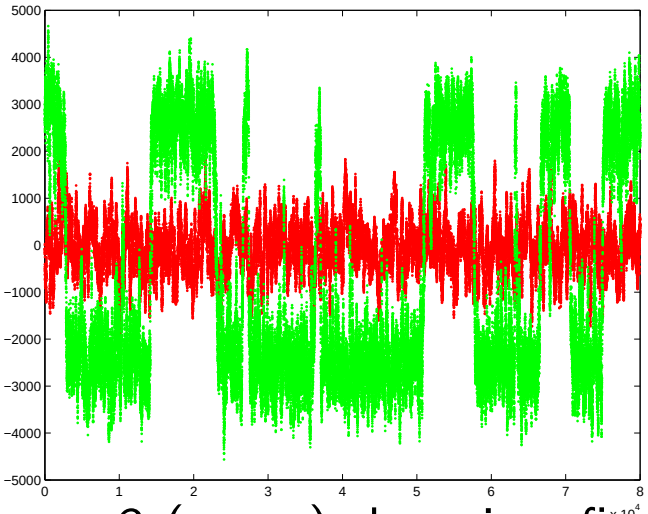




B.

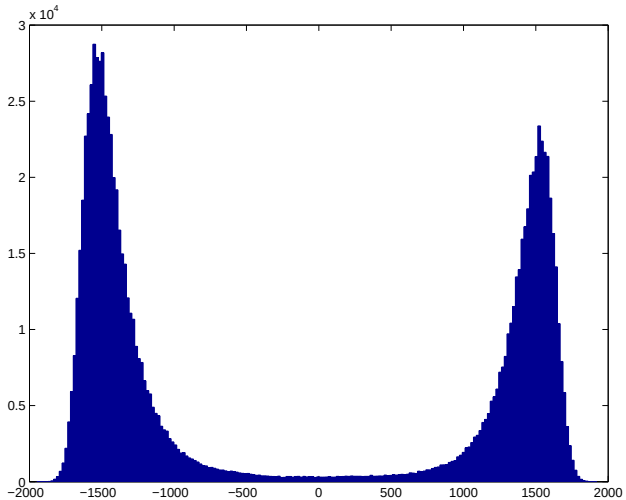
B. Beard, M. Pepe, S. Riederer, UJW, Phys. Rev. Lett. 94 (2005) 010603.

Monte Carlo history of the order parameter

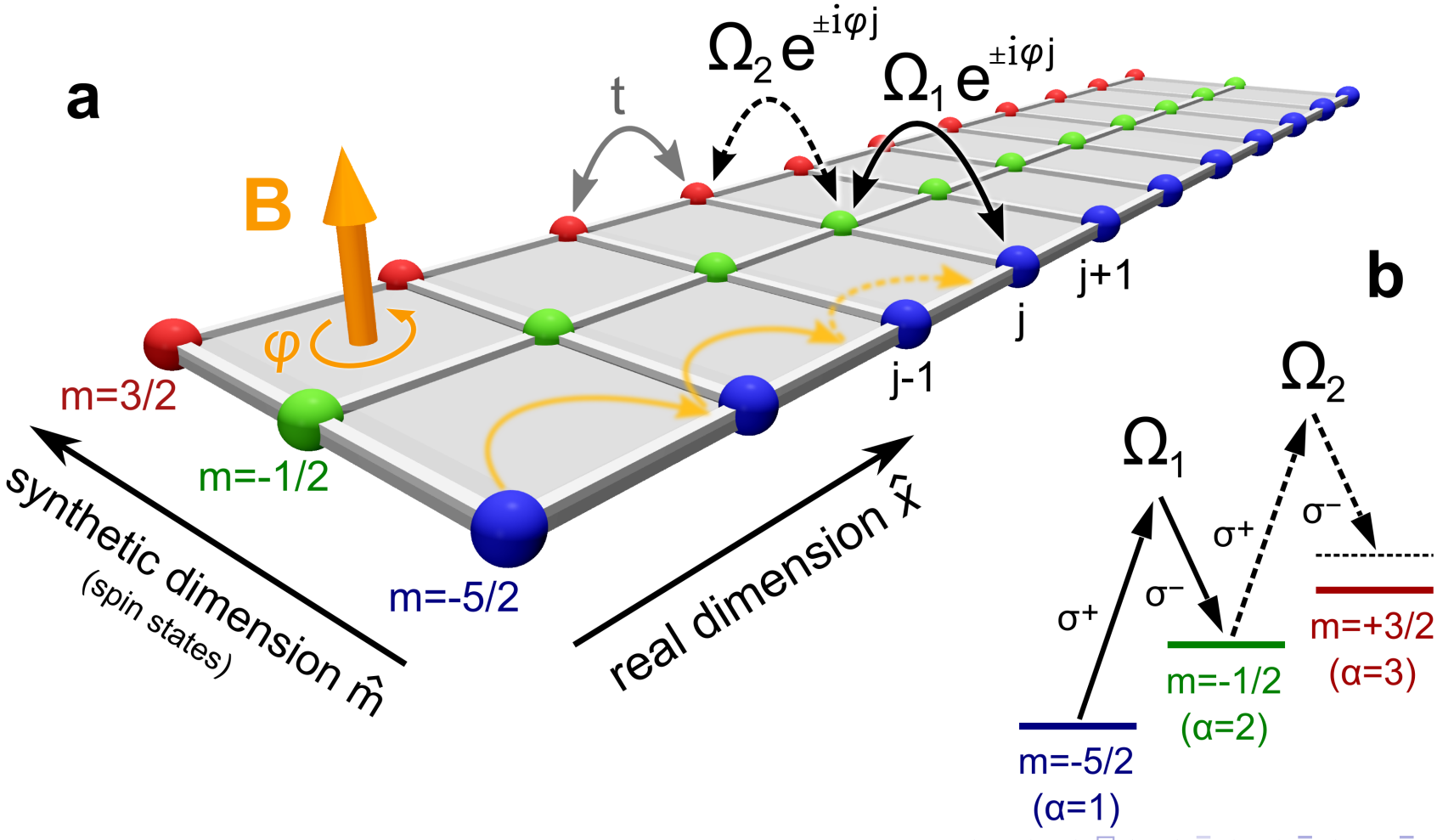


For $n = 3$ (green) there is a first order phase transition with spontaneous C and P breaking in the $(1 + 1)$-d $\mathbb{C}P(3)$ model at $\theta = \pi$.

For $n = 4$ (red) $\theta = 0$ and there is no phase transition.



Quantum Simulation with Alkaline-Earth Atoms



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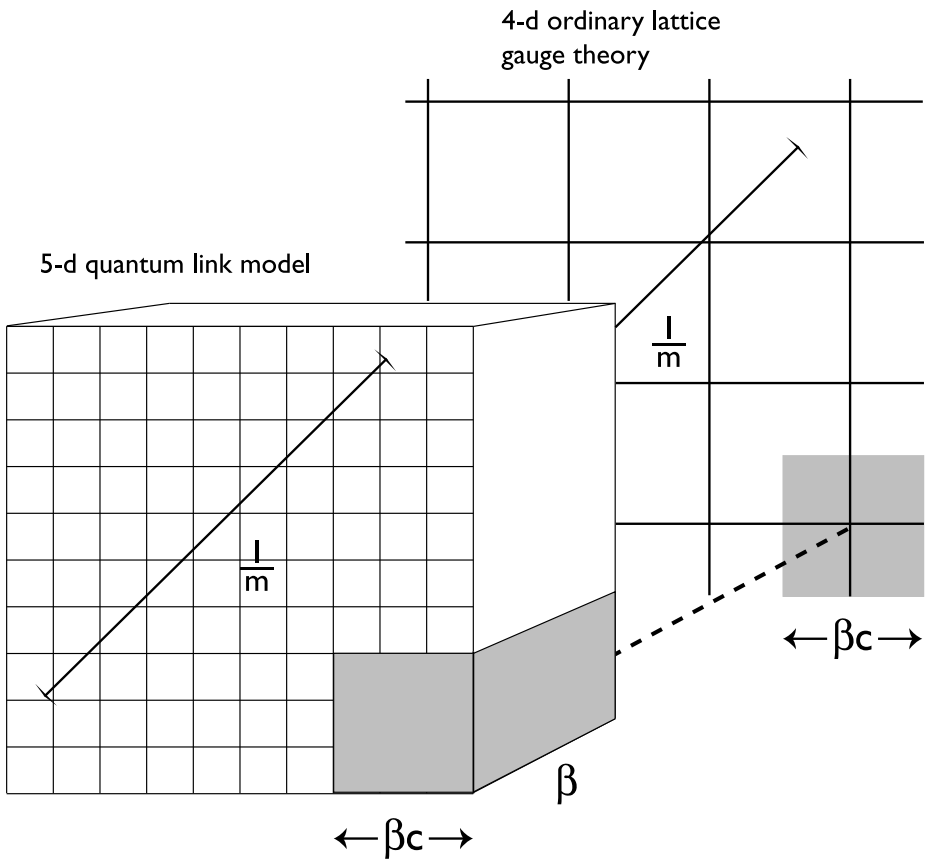
Low-energy effective action of a quantum link model

$$S[G_\mu] = \int_0^\beta dx_5 \int d^4x \, \frac{1}{2e^2} \left(\mathrm{Tr}\ G_{\mu\nu} G_{\mu\nu} + \frac{1}{c^2} \mathrm{Tr}\ \partial_5 G_\mu \partial_5 G_\mu \right), \ G_5 = 0$$

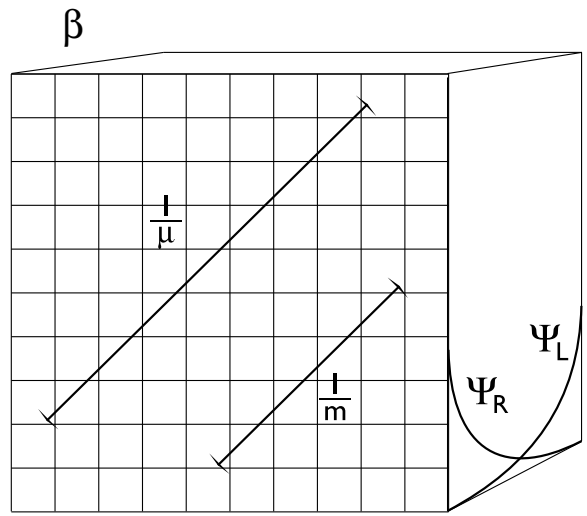
undergoes dimensional reduction from $4+1$ to 4 dimensions

$$S[G_\mu] \to \int d^4x \, \frac{1}{2g^2} \mathrm{Tr}\ G_{\mu\nu} G_{\mu\nu}, \ \frac{1}{g^2} = \frac{\beta}{e^2}, \ \frac{1}{m} \sim \exp\left(\frac{24\pi^2 \beta}{11 N e^2} \right)$$



Quarks as Domain Wall Fermions

$$
\begin{aligned}
H &= J \sum_{x,\mu\neq\nu} \mathrm{Tr}[U_{x,\mu} U_{x+\hat{\mu},\nu} U^{\dagger}_{x+\hat{\nu},\mu} U^{\dagger}_{x,\nu}] + J' \sum_{x,\mu} [\det U_{x,\mu} + \det U^{\dagger}_{x,\mu}] \\
&+ \frac{1}{2} \sum_{x,\mu} [\Psi^{\dagger}_{x} \gamma_0 \gamma_\mu U_{x,\mu} \Psi_{x+\hat{\mu}} - \Psi^{\dagger}_{x+\hat{\mu}} \gamma_0 \gamma_\mu U^{\dagger}_{x,\mu} \Psi_x] + M \sum_{x} \Psi^{\dagger}_{x} \gamma_0 \Psi_x \\
&+ \frac{r}{2} \sum_{x,\mu} [2\Psi^{\dagger}_{x} \gamma_0 \Psi_x - \Psi^{\dagger}_{x} \gamma_0 U_{x,\mu} \Psi_{x+\hat{\mu}} - \Psi^{\dagger}_{x+\hat{\mu}} \gamma_0 U^{\dagger}_{x,\mu} \Psi_x].
\end{aligned}
$$



4–d lattice

$$
\mu = 2M \exp(-M\beta), \; \frac{1}{m} \propto \exp\left(\frac{24\pi^2\beta}{(11N - 2N_f)e^2}\right), \; M > \frac{24\pi^2}{(11N - 2N_f)e^2}
$$

Homework 7:

Show that the $-T^{a*}$ (with $a \in \{1, 2, ..., N^2 - 1\}$), form a representation of $SU(N)$, assuming that the T^a do, i.e. $[T^a, T^b] = if_{abc}T^c$. Show that the conjugate anti-fundamental representation $-T^{a*}$ is unitarily equivalent to the fundamental representation T^a only for $SU(2)$.

Homework 8:

Show that the $SU(N)$ spin ladder Hamiltonian commutes with the total $SU(N)$ spin $T^a = \sum_{x \in A} T^a_x - \sum_{x \in B} T^{a*}_x$.

Important lessons from part 3:

• $\mathbb{C}P(N-1)$ models emerge by dimensional reduction from $SU(N)$ quantum spin ladders which can be quantum simulated with alkaline-earth atoms.

• Continuous gluon fields emerge as low-energy collective excitations of the discrete quantum link variables, just as magnetic spin wave fields emerge from discrete quantum spins.

• 4-d QCD emerges by dimensional reduction from a $(4+1)$-d $SU(3)$ quantum link model. Quarks arise as domain wall fermions.

Analog quantum simulator proposals

H. P. Büchler, M. Hermele, S. D. Huber, M. P. A. Fisher, P. Zoller, Phys. Rev. Lett. 95 (2005) 040402.

E. Zohar, B. Reznik, Phys. Rev. Lett. 107 (2011) 275301.

E. Zohar, J. I. Cirac, B. Reznik, Phys. Rev. Lett. 109 (2012) 125302; Phys. Rev. Lett. 110 (2013) 055302; Phys. Rev. Lett. 110 (2013) 125304.

D. Banerjee, M. Dalmonte, M. Müller, E. Rico, P. Stebler, UJW, P. Zoller, Phys. Rev. Lett. 109 (2012) 175302.

D. Banerjee, M. Bögli, M. Dalmonte, E. Rico, P. Stebler, UJW, P. Zoller, Phys. Rev. Lett. 110 (2013) 125303

Digital quantum simulator proposals

M. Müller, I. Lesanovsky, H. Weimer, H. P. Büchler, P. Zoller, Phys. Rev. Lett. 102 (2009) 170502; Nat. Phys. 6 (2010) 382.

L. Tagliacozzo, A. Celi, P. Orland, M. Lewenstein, Nature Communications 4 (2013) 2615.

L. Tagliacozzo, A. Celi, A. Zamora, M. Lewenstein, Ann. Phys. 330 (2013) 160.

Review on quantum simulators for lattice gauge theories

UJW, Annalen der Physik 525 (2013) 777, arXiv:1305.1602.

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• The path towards quantum simulation of QCD will be a long one. However, with a lot of interesting physics along the way.