Spectral Statistics for localized states with their nearest neighbours in Quantum Chaos

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Outline

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Introduction

- Quantum chaos: The study of quantum systems which exhibit chaos in the classical regime.
- The level spacing distribution is an indicator of quantum chaos. Random Matrix Theory (RMT) proves to be very useful in this regard.
- The presence of regular and chaotic regions in the classical phase space of a system affects the spectral statistics of its quantum version.
- Some of these systems exhibit semiclassical phenomena like localization of states.

Localization in quantum chaos

- Anomalous enhancement of eigenfunction intensity.
- Can we obtain any information about the localized states by investigating the corresponding eigenvalue statistics?
- We construct a random matrix model to simulate the coupling between a localized state and its nearest neighbours.
- In this work, localized states are identified by calculating the information entropy for each eigenstate, given by:

$$S_n^{\alpha} = \sum_{j=1}^{M} |a_{n,j}^{\alpha}|^2 \log |a_{n,j}^{\alpha}|^2$$

Outline

Random Matrix Theory- the essentials

- Hamiltonians having time-reversal symmetry are modeled using the Gaussian Orthogonal Ensemble (GOE) class of random matrices.
- The nearest neighbour spacing distribution (NNSD) for GOE systems is given by the Wigner distribution:

$$P_W(S) = \frac{\pi}{2} S e^{-\frac{\pi}{4} S^2}$$

- Here $S_i = E_{i+1} - E_i$. Instead, we may also look at the ratio of spacings $r_i = S_i / S_{i-1}$. For the GOE case, the NNSD would then be[1]:

$$P_W(r) = \frac{27}{8} \frac{(r + r^2)}{(1 + r + r^2)^{5/2}}$$

[1] *PRL*, *110*(8), 084101 (2013)

Matrix Model

- We consider a 3×3 real, symmetric matrix of the following type:

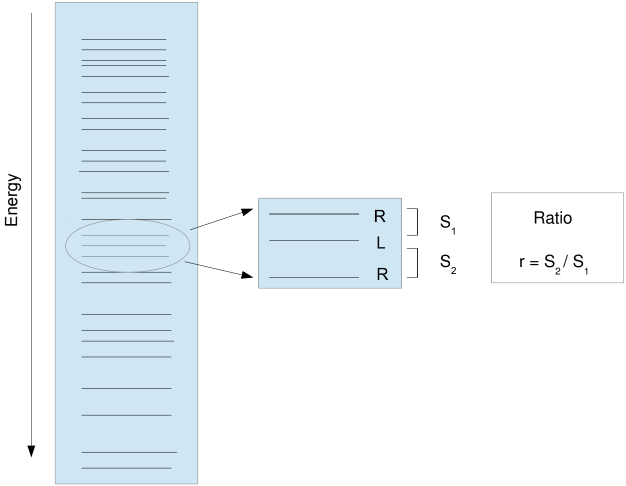
$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{bmatrix}$$

- Here, $\langle H_{ij} \rangle = 0$, $\langle H_{ii}^2 \rangle = 1$ for i, j=1,2,3,

  $\langle H_{12}^2 \rangle = \frac{1}{2}$, $\langle H_{13}^2 \rangle = \langle H_{23}^2 \rangle = \frac{k^2}{2}$ with $0 \le k \le 1$

- We consider H_{33} to be an eigenvalue corrresponding to a localized state, which is coupled to its nearest neighbours (the 2×2 block) via the coupling parameter of strength k.

Identification of localized states



Identification of localized states

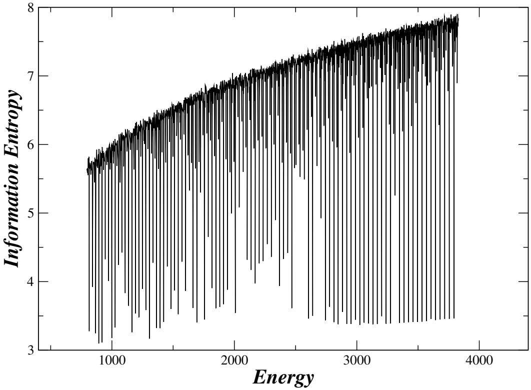


Figure : Plot of information entropy against energy of the quartic oscillator, for α=90.

Quantum Billiards

- Obtaining the eigenvalues for a closed quantum billiard system involves solving the Helmholtz equation with Dirichlet boundary conditions. That is,

$$\left[\vec{\nabla}^2 + k^2\right] \vec{E} = 0$$

- The chaos parameter in this problem is the shape of the billiard, varying which, the transition from integrability to chaos can be observed.
- Localized states called scars first studied in this context[3].
- Depending on shape, different kinds of localized states observed, including whispering gallery modes, bouncing ball modes, bowtie modes etc.

[3] *PRL*, *53*(16), 1515 (1984)

Stadium Billiard

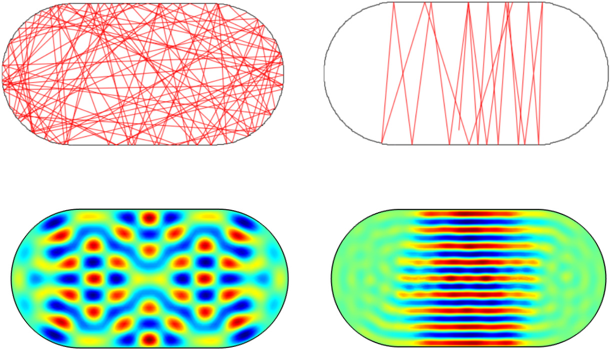


Figure : Trajectories in configuration space and corresponding eigenstates for a chaotic and a localized state for the billiard.

Results

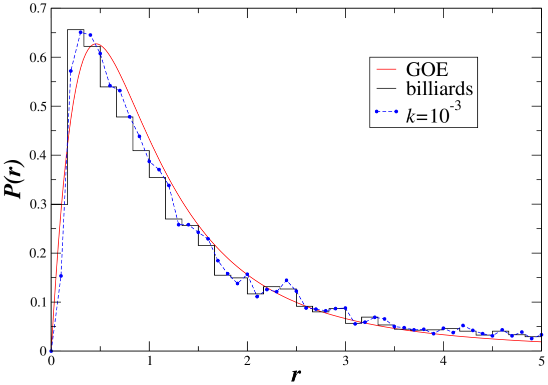


Figure : Distribution of the ratio of spacings between a localized state and its nearest neighbours for the stadium billiard plotted along with the numerical results obtained from the 3 $\times$ 3 model with the appropriate value of parameter k, and the theoretical curve for GOE.

Coupled Quartic Oscillator

- Here, we consider a coupled quartic oscillator system which is classically chaotic for different values of the coupling parameter α.
- The Hamiltonian for this system is given by

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + x^4 + y^4 + \alpha x^2 y^2$$

- The classical phase space has both regular and chaotic regions. Regular periodic orbits exist even as $\alpha \to \infty$.
- Quantum system exhibits localization along channel periodic orbits.

Coupled Quartic Oscillator

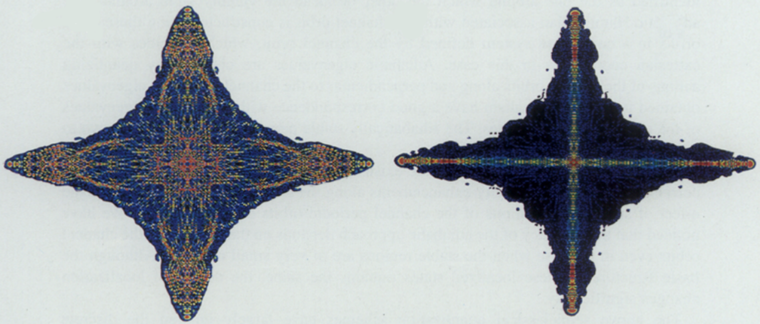


Figure : Configuration space intensities for a chaotic state and a localized state for $\alpha = 90$. Colour code —red is the maximum, blue is low and black is zero.

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[2] *Pramana*, *48*(2), 439, 1997

Results

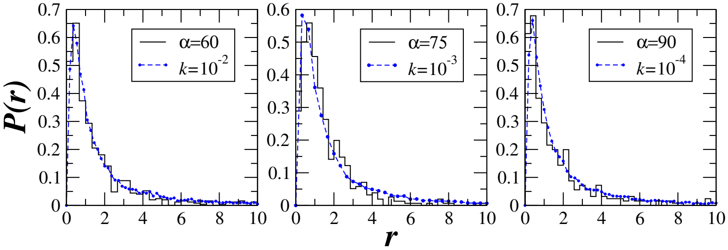


Figure : Distribution of the ratio of spacings between a localized state and its nearest neighbours for three values of α for the quartic oscillator shown along with the numerical results obtained from the 3×3 model with the appropriate value of parameter k.

Outline

Analytical form for $P(r)$

- Consider real symmetric (for $\beta = 1$) or Hermitian (for $\beta = 2$) matrices H from the probability measure

$$\mathcal{P}(H)d[H] \propto \exp\left(-\frac{\beta}{2}\, tr\ \Sigma^{-2}H^2\right) d[H].$$

- For β=1, the joint probability density of eigenvalue follows as

$$P(k;\lambda_1,\lambda_2,\lambda_3) \propto |\Delta(\{\lambda\})| \int_{\mathcal{O}_3} d\mu(O) \exp\left(-\frac{1}{2}\Sigma^{-2}O\Lambda^2 O^T\right)$$

where the integral is over the group of 3×3 orthogonal matrices with $d\mu(O)$ representing the corresponding Haar measure.

Analytic form for $P(r)$

- For $\beta{=}2$, the joint probability density of (unordered) eigenvalue in this case follows as

$$P(k;\lambda_1,\lambda_2,\lambda_3) \propto \Delta^2(\{\lambda\}) \int_{\mathcal{U}_3} d\mu(U) \exp\left(-\Sigma^{-2} U \Lambda^2 U^\dagger\right),$$

where The unitary group integral can be performed using the Harish-Chandra-Itzykson-Zuber integral:

$$\int_{\mathcal{U}_N} dU \exp\left(-s\ tr\ XUYU^\dagger\right) \propto \frac{\det\left[\exp(-s\, x_j y_k)\right]_{j,k=1,\ldots,N}}{\Delta(\{x\})\Delta(\{y\})}.$$

Outline

Gaussian Unitary Ensemble

- The Gaussian Unitary Ensemble of random matrices correspond to systems that do not possess time reversal symmetry.
- The distribution of the ratio of spacings in this case, is given by[1]:

$$\frac{81\sqrt{3}}{4\pi}\frac{(r+r^2)^2}{(1+r+r^2)^4}$$

- Examples: driven Rydberg atoms(*Eur. Phys. J. D*, (2000)), chaotic graphene billiards(*Chaos*,2011), Rydberg excitons(*Nature Materials*,2016) etc.

[1] *PRL*, *110*(8), 084101 (2013)

Gaussian Unitary Ensemble: Random Matrix Model

- We consider a 3×3 Hermitian matrix of the following type:

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{bmatrix}$$

- The matrix elements are zero-mean Gaussians (real and complex for diagonal and off-diagonal) with variances: $\langle H_{11}^2 \rangle = \langle H_{22}^2 \rangle = 1$, $\langle Re(H_{12})^2 \rangle = \langle Im(H_{12})^2 \rangle = \frac{1}{4}$, $\langle Re(H_{13}^2) \rangle = \langle Im(H_{13}^2) \rangle = \langle Re(H_{23}^2) \rangle = \langle Im(H_{23}^2) \rangle = \frac{k^2}{4}$, $\langle H_{33}^2 \rangle = \frac{1}{2} \frac{k^2}{2-k^2}$

GUE: Analytical result[2]

- For β=2, the joint probability density of (unordered) eigenvalue in this case follows as

$$P(k;\lambda_1,\lambda_2,\lambda_3) \propto \Delta^2(\{\lambda\}) \int_{\mathcal{U}_3} d\mu(U) \exp\left(-\Sigma^{-2} U \Lambda^2 U^\dagger\right),$$

where The unitary group integral can be performed using the Harish-Chandra-Itzykson-Zuber integral:

$$\int_{\mathcal{U}_N} dU \exp\left(-s \ tr \ XUYU^\dagger\right) \propto \frac{\det\left[\exp(-s\, x_j y_k)\right]_{j,k=1,\ldots,N}}{\Delta(\{x\})\Delta(\{y\})}.$$

[2]The form of p(k;r) has been obtained in collaboration with Santosh Kumar (Shiv Nadar University,Noida)

Analytic form for $P(r)$

- The probability density function of the ratio of consecutive spacings $r = (\lambda_2 - \lambda_3)/(\lambda_3 - \lambda_1)$ can then be found as

$$p(k;r) = \int_{-\infty}^{\infty} d\lambda_3 \int_{-\infty}^{\lambda_3} d\lambda_1 \int_{\lambda_3}^{\infty} d\lambda_2 \, \delta\left(r - \frac{\lambda_2 - \lambda_3}{\lambda_3 - \lambda_1}\right) \times \widetilde{P}(k; \lambda_1, \lambda_2, \lambda_3)$$

- Here, $\lambda_1 < \lambda_3 < \lambda_2$, where λ_3 is the eigenvalue corresponding to a localized state.

Analytic form for $P(r)$

The final result is of the form:

$$p(k;r) = \frac{\sqrt{2-k^2}}{4\pi k(1-k^2)^2} r(r+1) \sum_{j=1}^{3} \left[\frac{b_j(5a_j^2+2b_j^2)}{a_j^4(a_j^2+b_j^2)^2} + \frac{3}{(a_j^2+b_j^2)^{5/2}} \sinh^{-1}\left(\frac{b_j}{a_j}\right) - \frac{c_j(5a_j^2+2c_j^2)}{a_j^4(a_j^2+c_j^2)^2} - \frac{3}{(a_j^2+c_j^2)^{5/2}} \sinh^{-1}\left(\frac{c_j}{a_j}\right) \right].$$

Analytic form for $P(r)$

Here, the coefficients are defined as:

$$a_1 = \frac{\sqrt{2[1+r(r+1)(2-k^2)]}}{\sqrt{2+k^2}}, \quad a_2 = \frac{\sqrt{2[1+r(r+k^2)]}}{\sqrt{2+k^2}}, \quad a_3 = \frac{\sqrt{2[2+r(r+2)-k^2(r+1)]}}{\sqrt{2+k^2}},$$

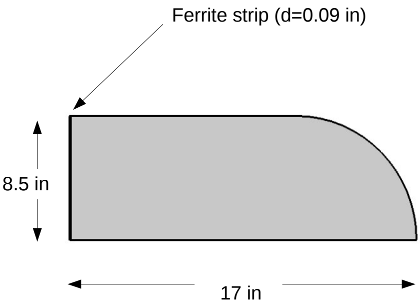
$$b_1 = \frac{k^2(3r+1)-2(r+1)}{2k\sqrt{2+k^2}}, \quad b_2 = \frac{2+k^2(2r-1)}{2k\sqrt{2+k^2}}, \quad b_3 = \frac{2-k^2(2r+3)}{2k\sqrt{2+k^2}},$$

$$c_1 = \frac{k^2(3r+2)-2r}{2k\sqrt{2+k^2}}, \quad c_2 = \frac{k^2(r-2)-2r}{2k\sqrt{2+k^2}}, \quad c_3 = \frac{2(r+1)-k^2(r+3)}{2k\sqrt{2+k^2}}.$$

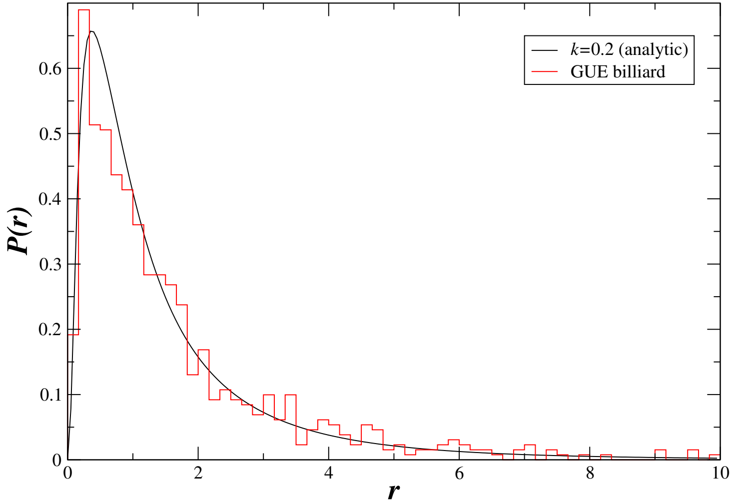
Experimental Observation in 2D Microwave Cavity

- So, Paul, et al. "Wave chaos experiments with and without time reversal symmetry: GUE and GOE statistics." *Physical Review Letters* **74.14** (1995): 2662.
- Electromagnetic wave equation in a thin microwave cavity with a magnetized ferrite stip is in the same universality class (GUE) as the Schrodinger equation without time reversal symmetry.
- Simulations done with COMSOL Multiphysics, which uses Finite Element Method.

Geometry



Results



Conclusion

- A random matrix model for the coupling between a localized state and its nearest neighbours was put forth for GOE and GUE systems.
- The distribution of the ratios of level spacings (corresponding to localized states) was evaluated for some well-known quantum systems.
- This was compared with data obtained numerically(for GOE) and analytically(for GUE) for different values of the parameter, to find an appropriate fit.
- This parameter k can be considered a numerical measure of the strength of the coupling.

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