Constructing the LIOMs

A.Scardicchio

Abdus Salam ICTP, Trieste

Explaining the phenomenology

- Energy and charge transport is suppressed
- Some memory of the initial state is conserved forever in **local** quantities
- Eigenstates close in energy have different "footprints" of local observables
- Eigenstates have area law entanglement (even at high *T*)
- Entanglement of an initial product state grows slowly, but to an extensive value

Local IOM

Local IOMs can explain all of this

We need to find them: numerically or analytically

Analytical:

derivation: V.Ros, M.Mueller, A.S. NPB 2015
review (including numerics): J.Imbrie, V.Ros, A.S. Annalen der Physik 2017

Local IOM

$$H = J\sum_i \vec{s}_i \cdot \vec{s}_{i+1} - \sum_i h_i s_i^z$$

in the MBL phase can be rewritten as

$$H = -\sum_i h' \tau_i^z - \sum_{ij} J_{ij}\tau_i^z \tau_j^z - \sum_{ijk} J_{ijk}\tau_i^z \tau_j^z \tau_k^z + ...$$

The operators τ_i^z are conserved quantities

$[H, \tau_i^z] = 0$ called *l-bits*

Local IOM

Which are local

$$\tau_1^z = \frac{1}{Z}(\sum_i A_i^{(1)} s_i^z + \sum_{i,j} A_{ij}^{(2)} s_i^+ s_j^- + \sum_{i,j,k} A_{ijk}^{(3)} s_i^+ s_j^z s_k^- + ...)$$

$$|A_i^{(1)}| < e^{-|i-1|/\xi_1} \qquad |A_{ij}^{(2)}| < e^{-(|i-1|+|j-1|)/\xi_2}$$

This LIOMs constrain the dynamics of the system in such a way that ergodicity cannot be achieved

Local IOM

We will construct explicitly the LIOM for

$$H = -t \sum_i c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + \sum_i \epsilon_i n_i + \lambda \sum_{i,j} v(|i-j|) n_i n_j$$

connecting the formulation in terms of LIOM to the perturbation theory of BAA

These LIOMs are number operators dressed with strings of excitations

$$I_1 \simeq n_1 + A_2 c_2^+ n_1 c_0 + A_3 c_3^+ n_2 n_1 c_0 + \ldots$$

Local IOM

First of all we diagonalize the quadratic part

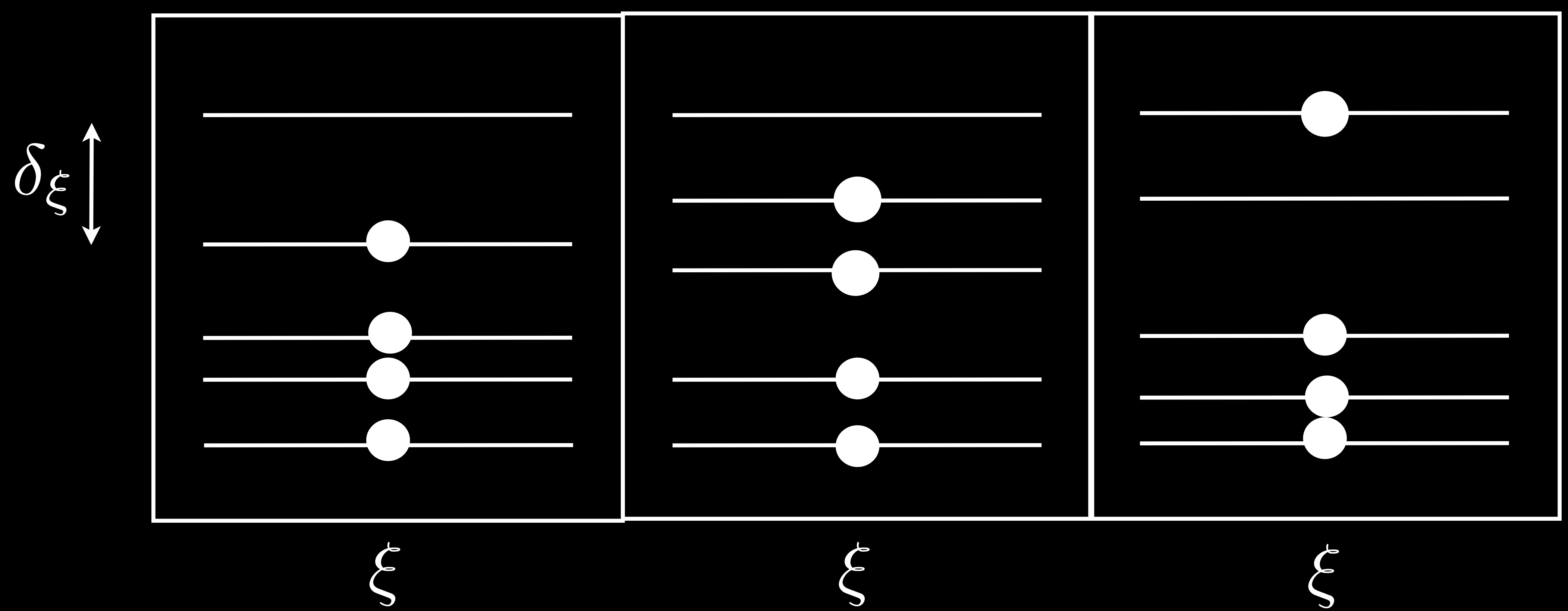
$$H = \sum_\alpha \epsilon_\alpha c_\alpha^\dagger c_\alpha + \lambda \sum_{\alpha,\beta,\gamma,\delta} u_{\alpha,\beta,\gamma,\delta} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta$$

$|\alpha\rangle$ single particle localized states

we coarse grain the system into "quantum dots" of size ξ and then we consider only matrix elements between the same or n.n. quantum dots

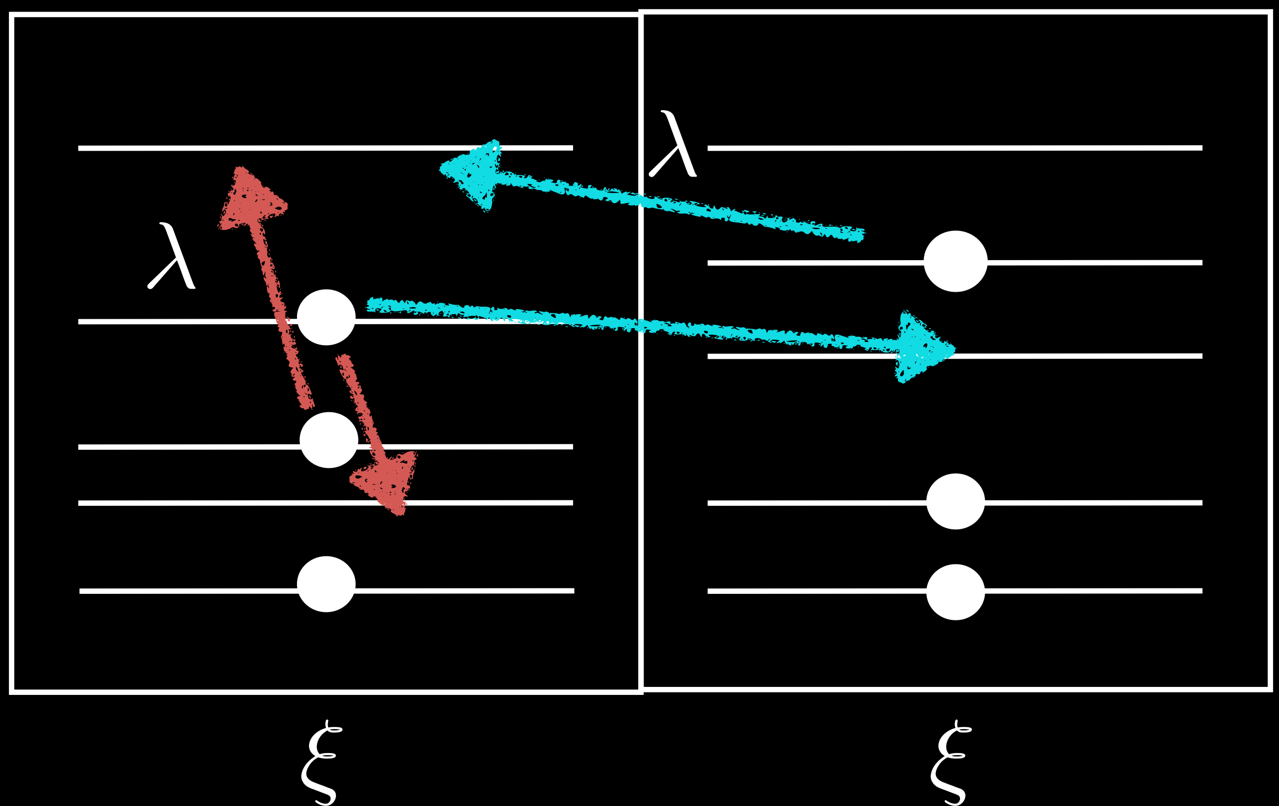
Local IOM

eigenstate at $\lambda = 0$



Local IOM

$$H = \sum_{\alpha} \epsilon_\alpha c^\dagger_\alpha c_\alpha + \lambda \sum_{\alpha,\beta,\gamma,\delta} u_{\alpha,\beta,\gamma,\delta} c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta$$



$$u \sim \delta_\xi$$

$$[H_0 + \lambda V, I_\alpha] = 0$$

Local IOM

$$I_1 = n_1 + A_{2,1} c_2^\dagger n_1 c_0 + A_{2,2} c_2^\dagger c_0 + ...$$
$$+ A_{3,1} c_3^\dagger c_2^\dagger c_1^\dagger c_2 c_1 c_0 + A_{3,2} c_3^\dagger c_2^\dagger c_2 c_0 + ...$$

the number of terms at distance *r* 4^r

the amplitudes are random numbers

$$|A_r| = \max_i |A_{r,i}|$$

In the localized regime we expect

$$\exists \xi > 0 \quad \lim_{r\to\infty} P(|A_r| < e^{-r/\xi}) = 1$$

so that the operators are *(quasi-)*local

Local IOM

$$[H_0 + \lambda V, I_\alpha] = 0$$

In perturbation theory: $[H_0, I_\alpha^{(n)}] + [V, I_\alpha^{(n-1)}] = 0$

To make *analytic* progress, we focus on the tail of the operators and estimate

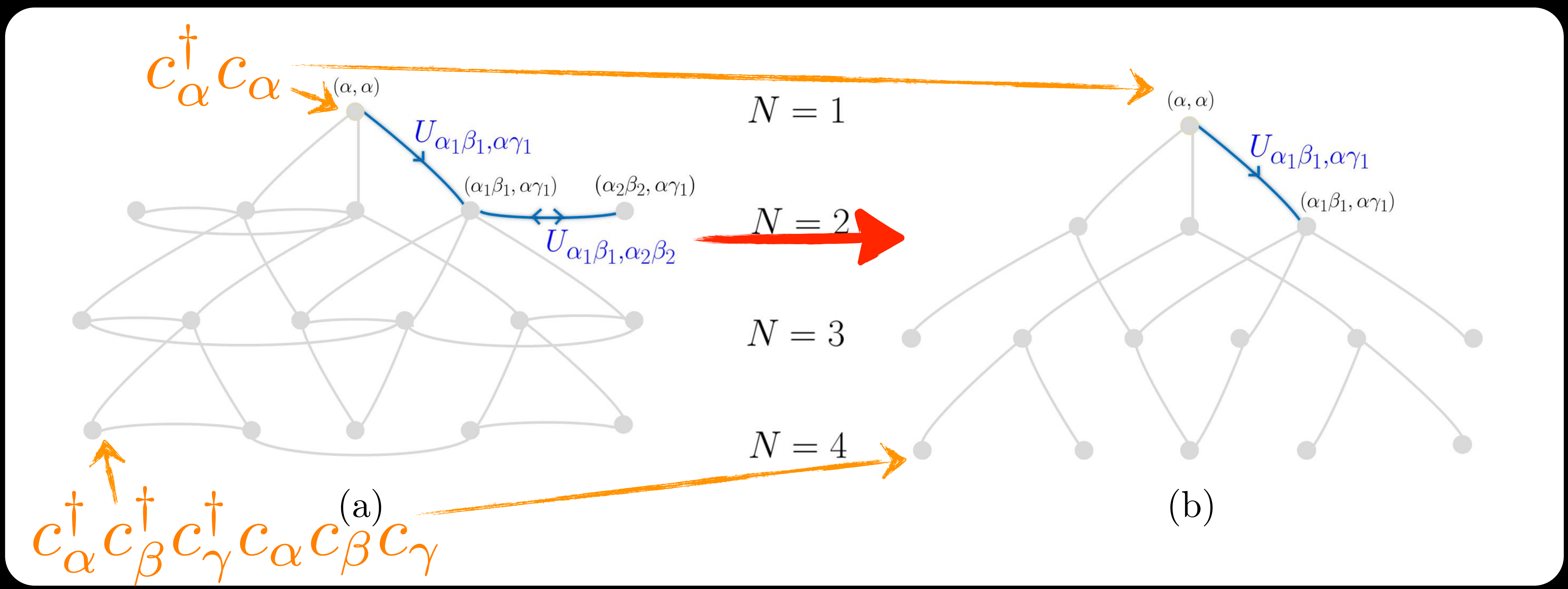
$$A_r = \lambda^r c_0(r) + \lambda^{r+1} c_1(r) + ...$$

to lowest order in perturbation theory

$$c_0(r) \simeq q^r$$

Perturbation theory

Hopping in operator space



Lowest order: shortest paths from a short to a long operator (forward approximation)

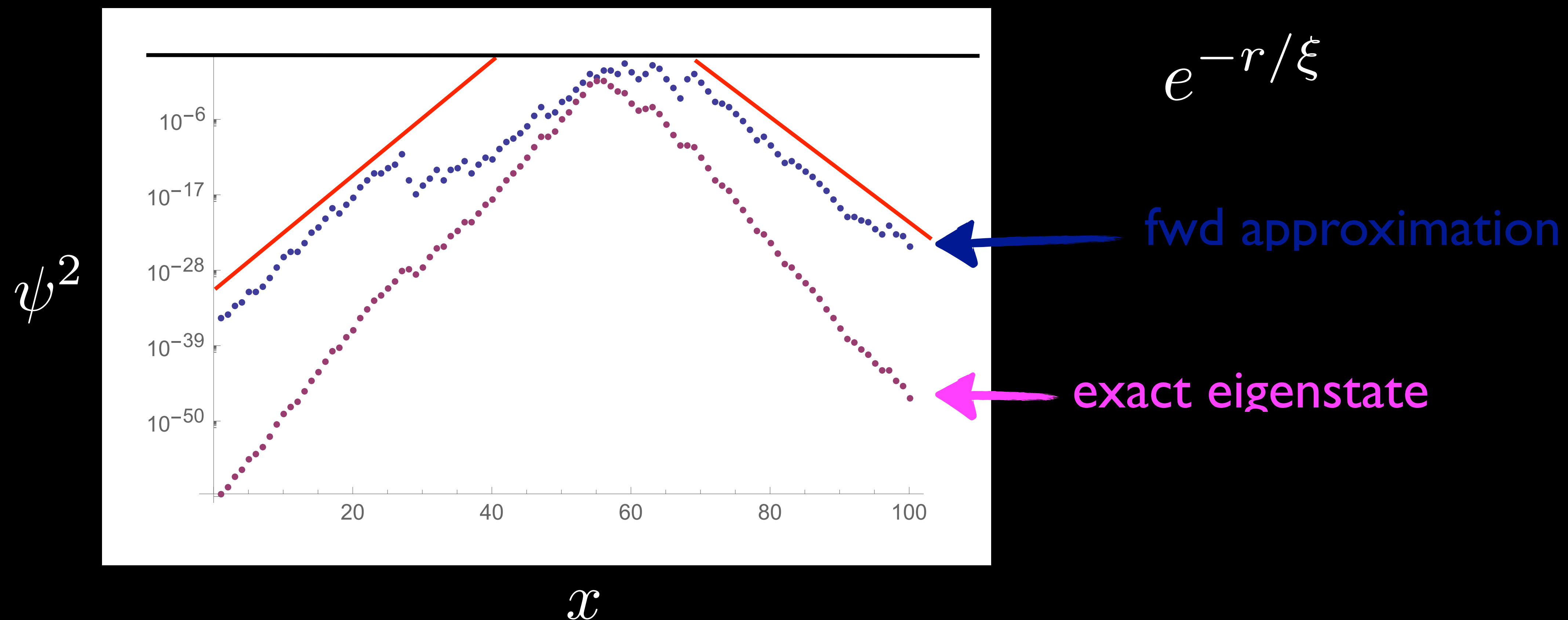
This should give a lower bound for the critical interaction

Perturbation theory

Forward approximation

$$\psi_\alpha(n) \simeq \sum_{p \in \mathrm{paths}(0,n)} \prod_{i \in p} \frac{t}{\epsilon_0 - \epsilon_i}$$

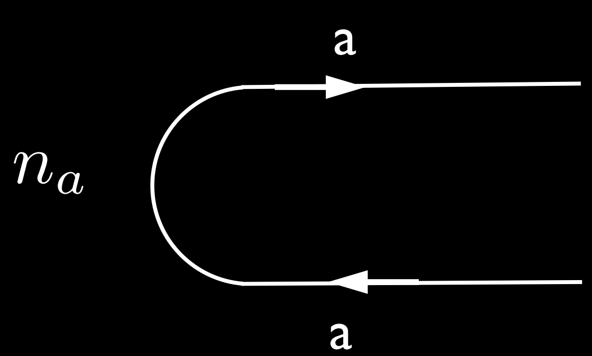
Resonances are less important in the exact solution than in the fwd approx



This is equivalent to the ImSCBA in BAA's perturbation theory (see also Abou-Chacra, Anderson, Thouless 1973)

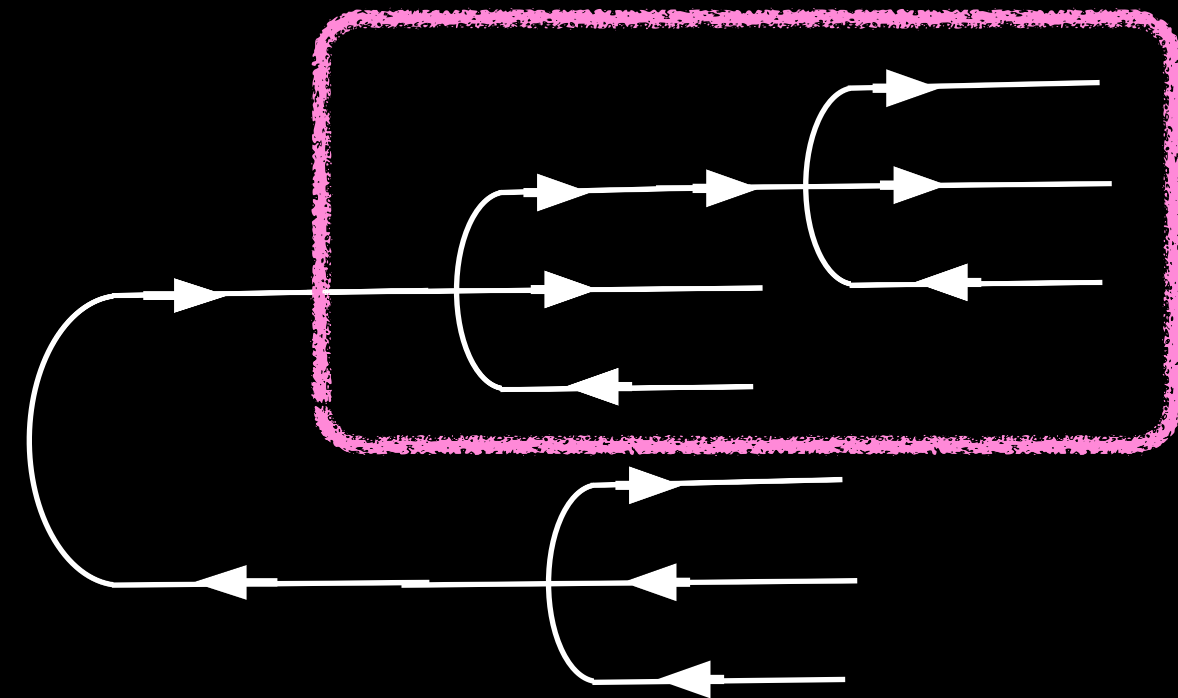
Perturbation theory

We only include terms in the *n*-th order operator which look like this:



λ^0

or

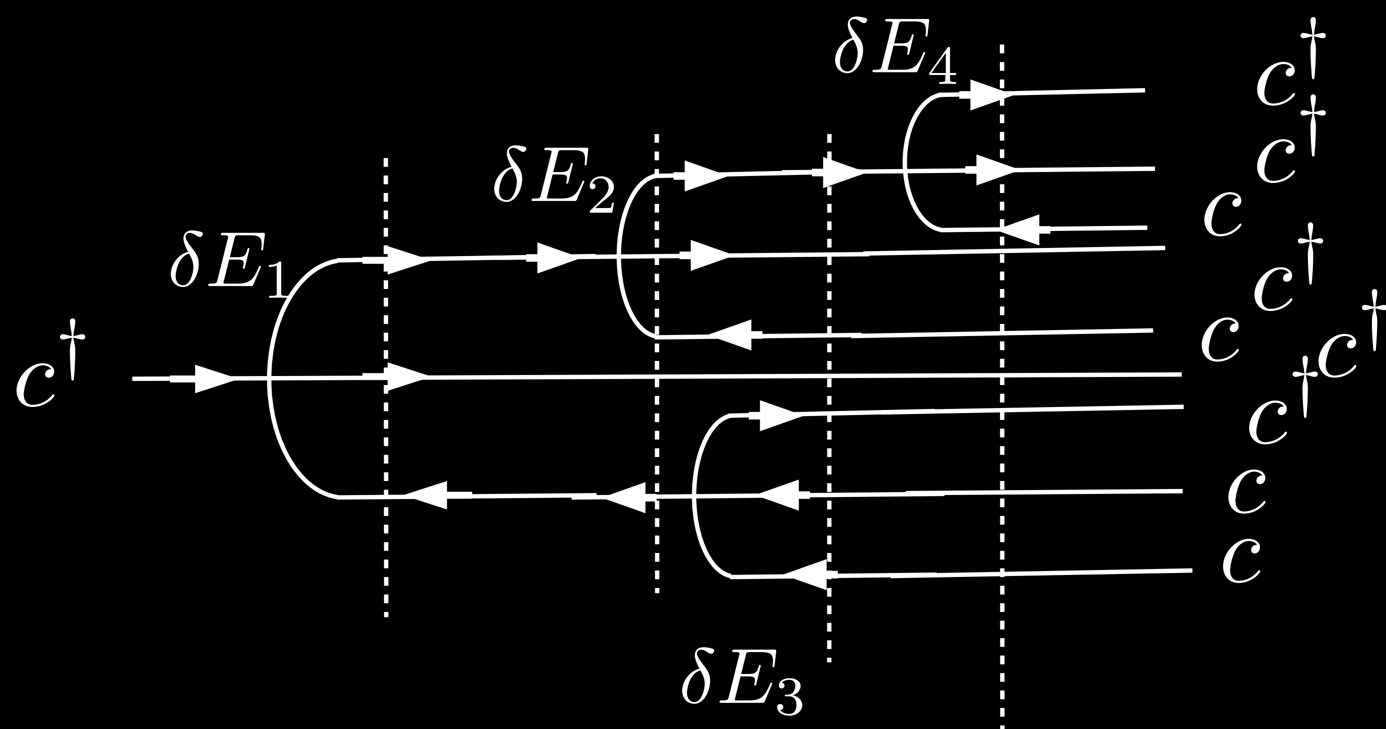


λ^3 $4\ c^\dagger,\ 4\ c$

Consider the two branching trees

Perturbation theory

One sub-tree generates an amplitude:



$$\delta E_1 = (\epsilon_a - \epsilon_b + \epsilon_c - \epsilon_d)^{(1)}$$

$$A = \frac{\lambda\delta_\xi}{E+\delta E_1}\frac{\lambda\delta_\xi}{E+\delta E_1+\delta E_2}\frac{\lambda\delta_\xi}{E+\delta E_1+\delta E_2+\delta E_3}\frac{\lambda\delta_\xi}{E+\delta E_1+\delta E_2+\delta E_3+\delta E_4}$$

Correlated denominators

Problem: it looks like there are *n!* terms at order *n* but they are actually correlated

$$A = \frac{1}{E_1(E_1+E_2)(E_1+E_2+E_3)} + \frac{1}{E_1(E_1+E_3)(E_1+E_3+E_2)} + \frac{1}{E_3(E_3+E_1)(E_1+E_2+E_3)}$$
$$= \frac{1}{E_3}\frac{1}{E_1(E_1+E_2)}$$

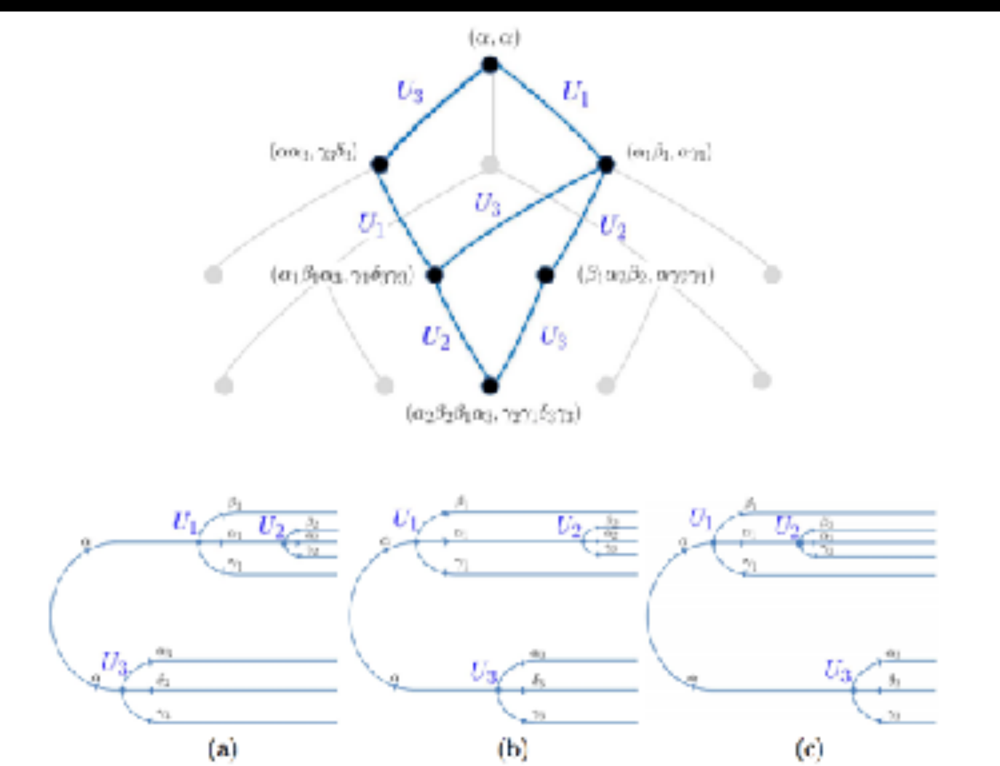


Figure 3: Loops in the many-body lattice corresponding to different processes with the same final state, and the corresponding ordered graphs. The graphs differ only in the order in which the interactions U_1, U_2, U_3 act. The weights of such paths are strongly correlated: they are all proportional to the same product of matrix elements, $U_1U_2U_3$, and have highly correlated denominators. The sum over all these ordered graphs constitutes a diagram.

Independent amplitudes correspond to independent physical processes

Correlated denominators

The many-body amplitude

$$A = \prod_{a=1,\dots,n} \frac{\lambda\delta_\xi}{E + \sum_{i=1}^{a} \delta E_i}$$

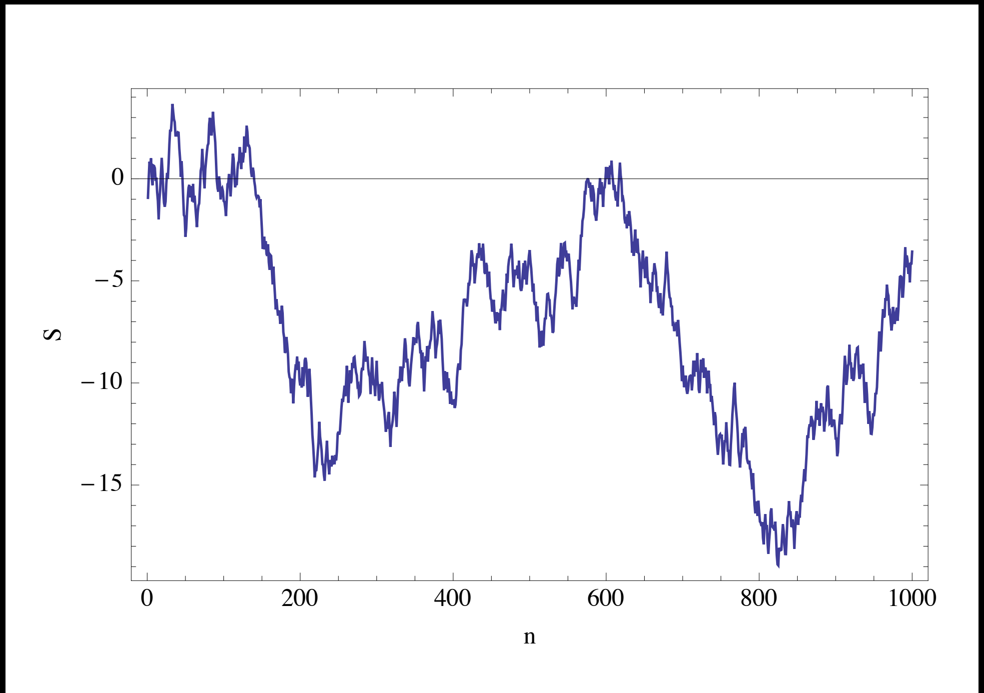
is different from the single-particle one

$$A = \prod_{a=1,\dots,n} \frac{\lambda\delta_\xi}{E + \delta E_a}$$

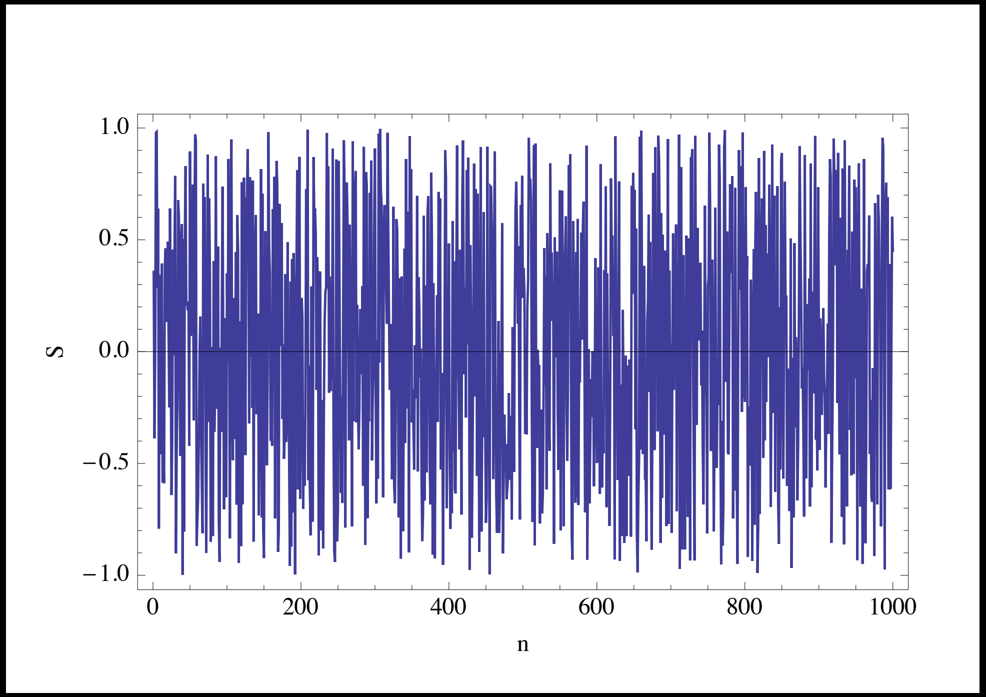
We need to find $P(A)$ and we cannot use the techniques used for single particle AL

Correlated denominators

Very different probability distributions...



many body



single particle

Consider $Y = -\ln|A|$

Correlated denominators

We can compute the Laplace transform

$$G_N(k) = \mathbb{E}[e^{-kY}]$$

and eventually invert it to get

$$P(Y) = \int_B \frac{dk}{2\pi i} e^{kY} G_N(k)$$

we anticipate that we are going to do a saddle point calculation with

$$Y \sim N$$

Correlated denominators

$$G_N(k) = \int d^N x\, (2\pi)^{-N/2}\, e^{-\sum_i \frac{x_i^2}{2}}\, e^{k \sum_i \log|s_i|}$$

$$G_N(k) = \int d^N x \prod_i f(s_i - s_{i-1}) e^{k \sum_i \log|s_i|}$$

$$G_N(k) = \int ds_N O^{N-1}[f](s_N)|s_N|^k$$

$$O[f](s) = \int dt f(s-t)|t|^k dt$$

Correlated denominators

We cast the Laplace transform in a transfer matrix calculation

$$G_N(k) = \left(\frac{\delta_\xi^{2k}}{2\pi}\right)^{N/2} \langle\psi'|\mathcal{H}^N|\psi\rangle.$$

$$\mathcal{H}_{n,m} = \frac{\Gamma(\frac{1+k}{2}+m+n)}{\sqrt{\Gamma(1+2m)}\sqrt{\Gamma(1+2n)}}$$

So we need to find the largest eigenvalue of *H*

Correlated denominators

This can be done (large *Y* means $k \to -1$) and we find that

$$P(Y) = \int \frac{dk}{2\pi i} e^{N(y+\log(\mu))} \simeq \left(\frac{Y}{y_0 N}\right)^N e^{-Y/y_0(1-\gamma/(2(Y/N)^2)}$$

This is the probability distribution of a single amplitude

We now need to find *how many terms are there in the sum*

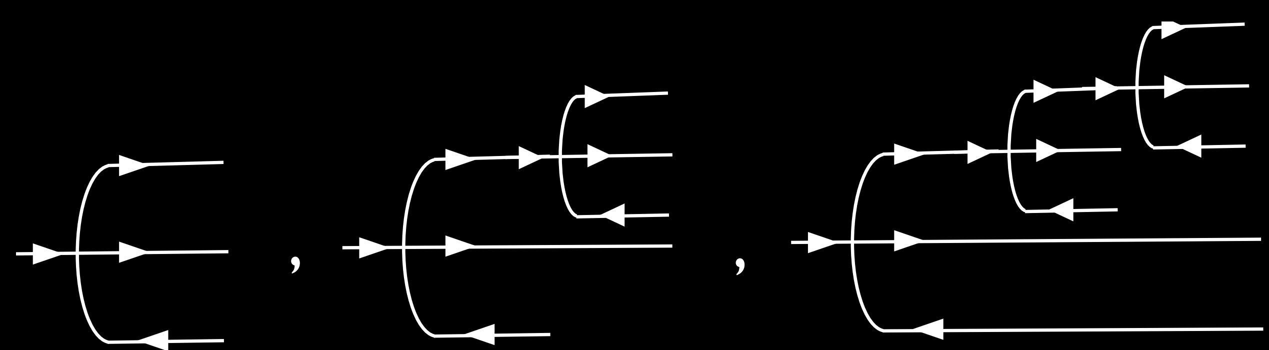
$$I_\alpha^{(n)} = \sum_{|\mathcal{I}|=|\mathcal{J}|=n} A_{\mathcal{I},\mathcal{J}} O_{\mathcal{I},\mathcal{J}}$$

of the order *n* correction

Counting diagrams

Topology + assignment of indices $\alpha, \beta, \gamma, \delta$

Topology:



Classic combinatorics problem (generalized Catalan numbers)

$$T_n = \sum_{m_1, m_2, m_3, \sum_i m_i = n} T_{m_1} T_{m_2} T_{m_3}$$

Counting diagrams

$$T(x) = \sum_n T_n x^n \qquad T_n = \sum_{m_1, m_2, m_3, \sum_i m_i = n} T_{m_1} T_{m_2} T_{m_3}$$

$$T(x) = 1 + xT(x)^3 \qquad x = \frac{T-1}{T^3} \equiv f(T)$$

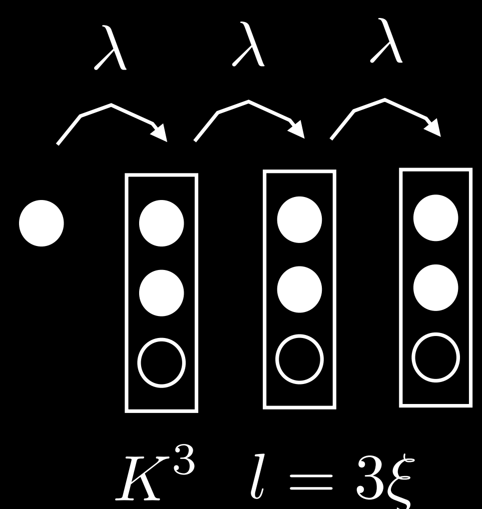
Lagrange inversion theorem:

$$T_n = \lim_{T \to 1} \frac{1}{n!} \frac{d^{n-1}}{dT^{n-1}} \left(\frac{T-1}{f(T)-0} \right)^n$$

$$T_n = \frac{1}{2n+1} \binom{3n}{n} \sim \left(\frac{27}{4} \right)^n$$

Counting diagrams

Assignment of indices defines the spatial structure of the excitations



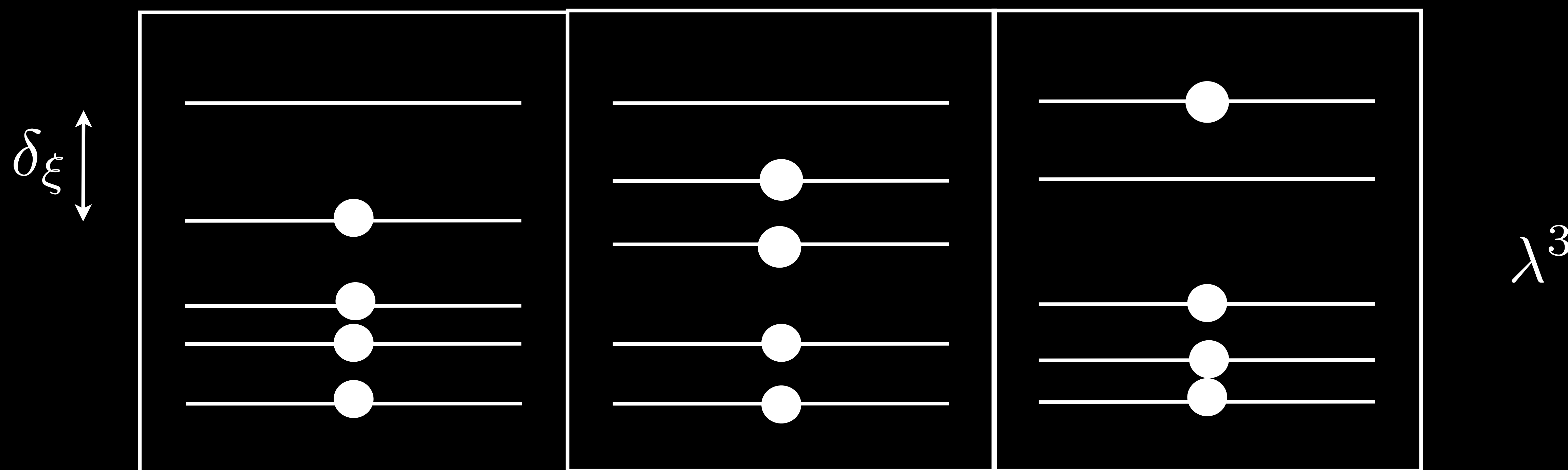
e.g. ballistic excitation

$$K = \xi/a$$

It is convenient to consider the picture of a particle which hops from volume to volume leaving a trail of excitations

Local IOM

The important processes are those in which an excitation can travel staying (almost) in resonance



$$\Delta E = (E_1 - E_2) + (E_3 - E_4) + (E_5 - E_6) \lesssim \delta_\xi$$

$T \ll W \;:\; K \sim T/\delta_\xi$ $\qquad$ $T \gg W \;:\; K \sim W/\delta_\xi$

We need to estimate the probability that a resonance occurs far away

Counting diagrams

Assigning indices: describing the trail of excitations

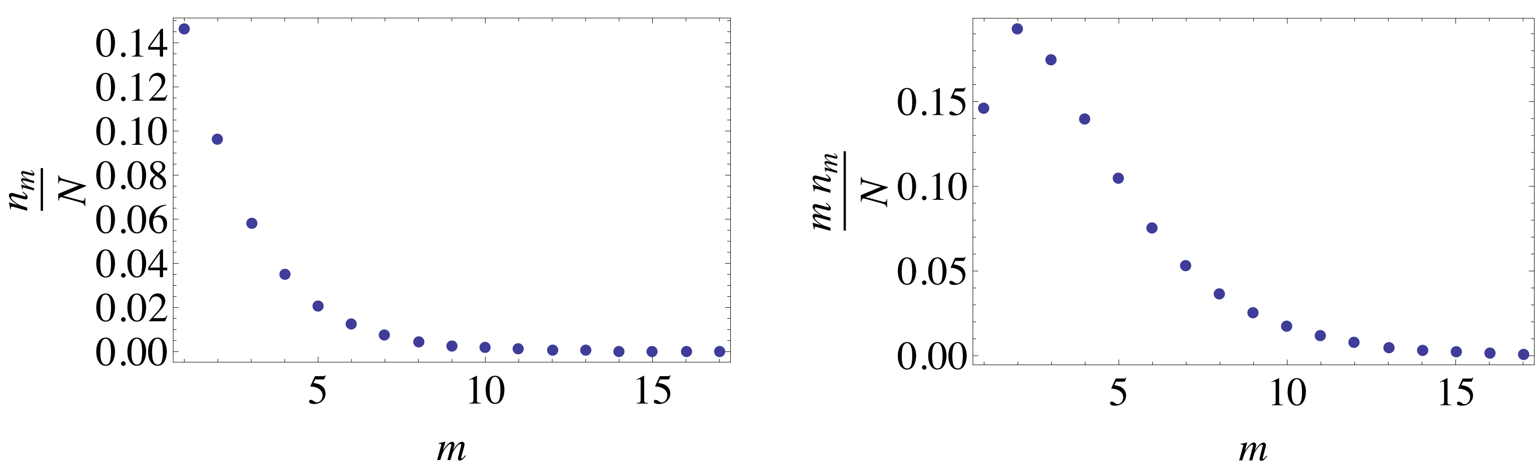
In every box *i* there are *m excitations*

$$\mathcal{N}_N \approx \overline{\mathcal{P}(d)} \sum_{\{m_i\} | \sum_i m_i = N} \frac{1}{2} \prod_{i=1}^{n} \left[2\mathcal{K}^{m_i} m_i \mathcal{T}_{m_i} \right],$$

We need to maximize this number over the *m's*

$$\mathcal{N}_N \approx (\mathcal{K} e^{\mu})^N \approx (10.6\,\mathcal{K})^N$$

Counting diagrams



(a) (b)

Figure 6: The plot (a) shows the distribution of the number n_m/N of groups of m particle-hole pairs in necklace diagrams dominating $\mathcal{N}_N$. The plot (b) shows the probability mn_m/N that a given pair belongs to a group containing m pairs.

Recapitulate

- 1) Find distribution of amplitudes of a single diagram giving a long operator inside I_α
- 2) Count the number of diagrams
- 3) Count the number of spatial processes pertaining to a given assignment of indices

Result

for

$$\lambda < \lambda_c = \frac{\sqrt{2\pi}}{\nu(1-\nu)2eC}\frac{1}{K\ln K}$$

$18.97 < C < 36.25$

$K \sim T/\delta_\xi$

$K \sim W/\delta_\xi$

we can find operators I_α (one per site) $I_\alpha = 0, 1$

$$[H, I_\alpha] = 0$$

Then the eigenstates can be written as bit strings

$$|E_m\rangle = |0, 1, 0, 0, 0..., 1\rangle$$

each bit is the eigenvalue of a local operator

$$\mathrm{Tr}(I_\alpha c_r^\dagger c_r) \sim e^{-d(\alpha,r)/\ell}$$

Convergence of p.t. for LIOMs

We found that the IOM are local for

$$\lambda < \lambda_c = \frac{\sqrt{2\pi}}{\nu(1-\nu)2eC}\frac{1}{K\ln K}$$

For $\lambda > \lambda_c$ there are several different scenarios

a) All LIOMs die, becoming non-local
b) Some LIOMs "die," some don't (*a la* KAM)

This problem is open

Conclusions

- MBL phenomenology can be recovered by conjecturing the existence of local IOMs
- We can show, under the same approximations of BAA, that LIOMs exist for weak interactions/strong disorder
- We can find the radius of convergence of the perturbation theory