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At the limits of criticality-based quantum metrology: the minimal model exhibiting apparent super-Heisenberg scaling

M.M. Rams, P. Sierant, O. Dutta, P. Horodecki and J. Zakrzewski

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Quantum metrology

Estimation of small external parameter λ with the help of a quantum procedure

Generate family of quantum states which depend strongly on the parameter λ

Distance quantified by fidelity:

$$\mathcal{F}(\hat{\rho},\hat{\sigma}) = \mathrm{Tr}\left(\sqrt{\sqrt{\hat{\rho}}\hat{\sigma}\sqrt{\hat{\rho}}}\right)$$

Two scenarios discussed in the literature

unitary rotation generated by parameter dependent Hamiltonian

family of ground states of parameter dependent Hamiltonian

N ← number of elementary subsystems

$$\hat{H} = \hat{H}_1 + \lambda \sum_{n=1}^{N} \hat{h}_n = \hat{H}_1 + \lambda \hat{H}_0,$$

Bounds on precision

Measurement of some general observable $\hat{A}$

Precision it offers to identify the change δ_λ set by error propagation formula

$$\Delta^2_{\delta_\lambda}(\hat{A},\lambda) = \frac{\langle \hat{A}^2\rangle_{\rho(\lambda)} - \langle \hat{A}\rangle^2_{\rho(\lambda)}}{\left(\left. \frac{\partial \langle \hat{A}\rangle_{\rho(\lambda+\delta_\lambda)}}{\partial \delta_\lambda} \right|_{\delta_\lambda=0} \right)^2}$$

Quantum Cramer-Rao bound

$$\Delta^2_{\delta_\lambda}(\hat{A},\lambda) \geq \frac{1}{G(\lambda)}$$

Fisher information

In unitary rotation scenario

$$\hat{H} = \hat{H}_1 + \lambda \sum_{n=1}^{N} \hat{h}_n \longrightarrow \frac{1}{G(\lambda)} \sim N^{-2}$$

as Heisenberg limit

Fidelity susceptiblity

fidelity susceptibility

$$\mathcal{F}(\hat{\rho}(\lambda), \hat{\rho}(\lambda + \delta_\lambda)) = 1 - \frac{1}{2}\chi_F(\lambda)\delta_\lambda^2 + O(\delta_\lambda^3) + \dots$$

small change

Fidelity susceptibility and Fisher information are directly related

$$G(\lambda) = 4\chi_F(\lambda)$$

For family of ground states of parameter dependent Hamiltonian

most interesting is behavior in the vicinity of critical point

$$G(\lambda_c) \sim \chi_F(\lambda_c) \sim N^{2/d\nu}$$

$$N = L^d$$

$$G(\lambda) \sim \chi_F(\lambda) \sim N|\lambda - \lambda_c|^{d\nu - 2}$$

critical exponent related with the correlation length. Depends on universality class of the critical point

Model: XXZ Heisenberg spin chain in external field

Looking for a model where $d\nu$ is small → extreme susceptibility

$$\hat{H}(\lambda) = -\sum_{n=1}^{N-1} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + J_z \sigma_n^z \sigma_{n+1}^z\right) + \lambda \sum_{n=1}^{N} \sigma_n^x$$

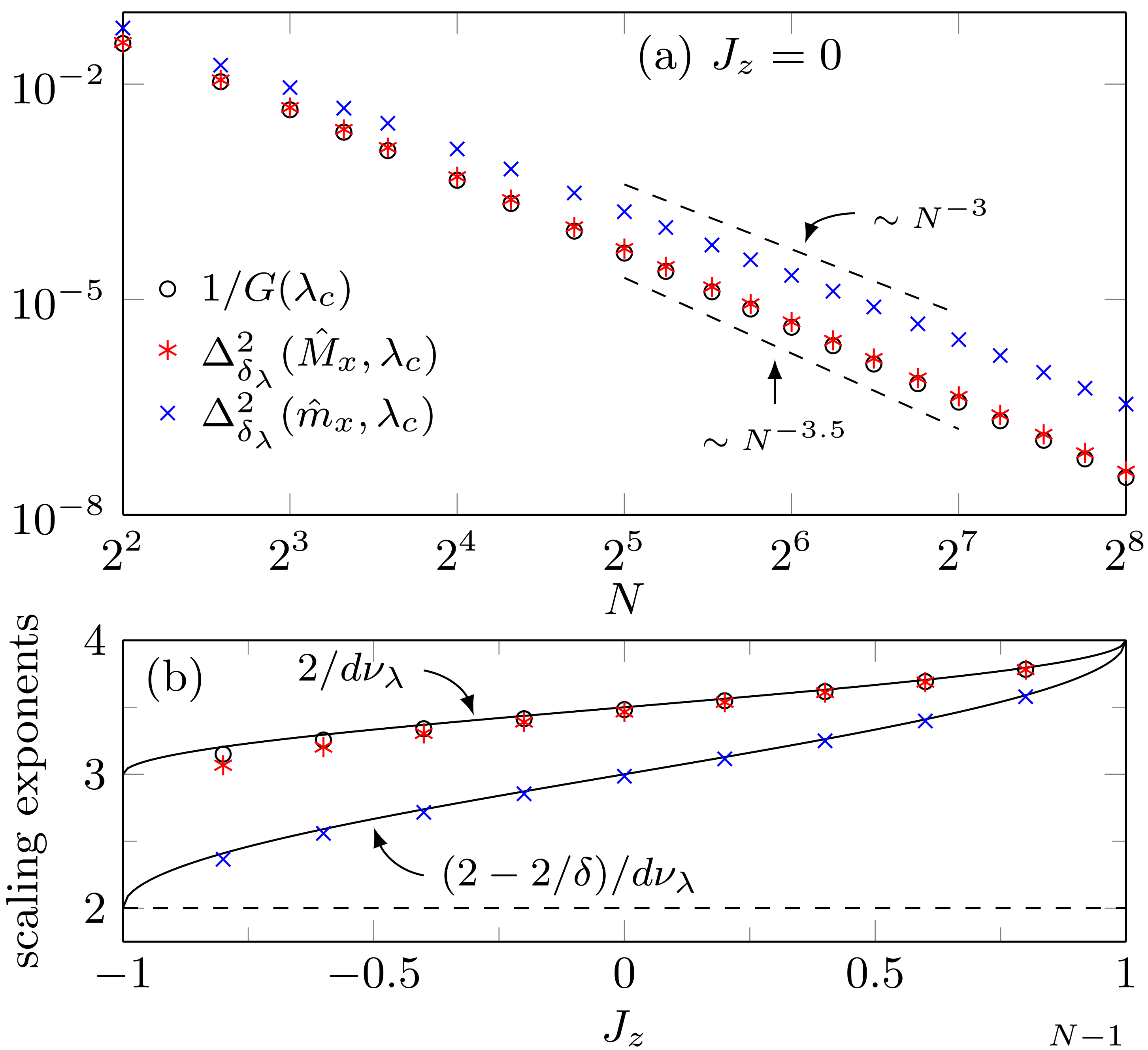
Critical exponent known $\quad \nu_\lambda = \frac{2}{4 - \arccos(J_z)/\pi} < 1$

Simple observables of interest:

$$\hat{M}_x = \frac{1}{N} \sum_{n=1}^{N} \hat{\sigma}_n^x \qquad\qquad \hat{m}_x = \hat{\sigma}_{N/2}^x$$

Possible realization in ultra-cold bosons in optical lattice potential

XXZ model: scaling at the critical point



$$G(\lambda_c) \sim \chi_F(\lambda_c) \sim N^{2/d\nu}$$

$$\hat{H}(\lambda) = -\sum_{n=1}^{N-1} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + J_z \sigma_n^z \sigma_{n+1}^z\right) + \lambda \sum_{n=1}^{N} \sigma_n^x$$

Scaling of error propagation formula at the critical point

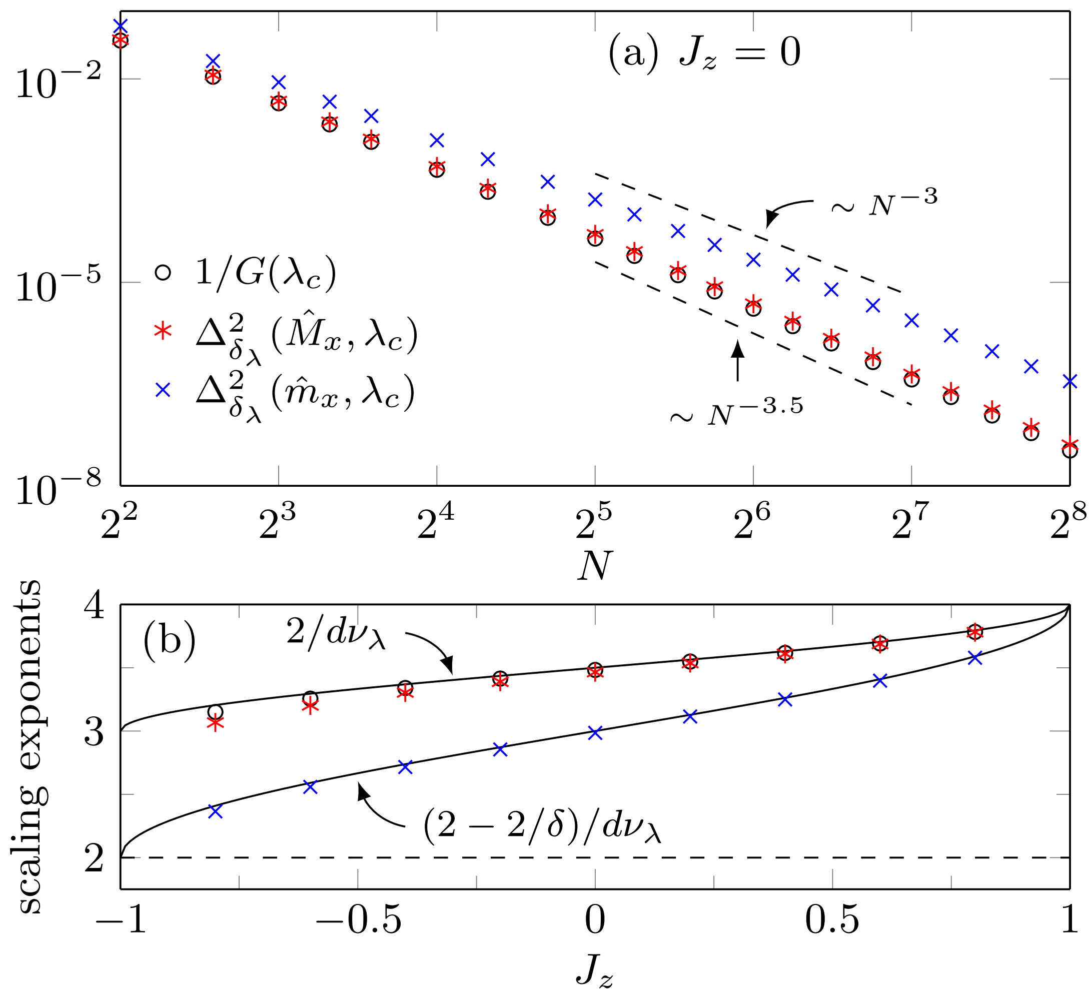
$$\Delta^2_{\delta_\lambda}(\hat{A},\lambda) = \frac{\langle \hat{A}^2\rangle_{\rho(\lambda)} - \langle \hat{A}\rangle^2_{\rho(\lambda)}}{\left(\frac{\partial \langle \hat{A}\rangle_{\rho(\lambda+\delta_\lambda)}}{\partial \delta_\lambda}\Big|_{\delta_\lambda=0}\right)^2}$$

the same scaling as Fisher information

$$\Delta^2_{\delta_\lambda}(\hat{M}_x,\lambda_c) \sim N^{-2/d\nu_\lambda}$$

$$\Delta^2_{\delta_\lambda}(\hat{m}_x,\lambda_c) \sim N^{-(2-2/\delta)/d\nu_\lambda}$$

$$\hat{M}_x = \frac{1}{N}\sum_{n=1}^{N}\hat{\sigma}^x_n \qquad \hat{m}_x = \hat{\sigma}^x_{N/2}$$



Derivation assuming:

— finite size scaling

— hyperscaling relations (small d)

— correlation function vanishing slowly enough

Robustness

Temperature T=0 but the external field not tuned exactly to the critical point

large detuning:

$$\Delta^2_{\delta_\lambda}(\hat{M}_x, \lambda) \sim N^{-1}|\lambda - \lambda_c|^{2-d\nu_\lambda}$$

$$\Delta^2_{\delta_\lambda}(\hat{m}_x, \lambda) \sim N^{0}|\lambda - \lambda_c|^{2-2/\delta}$$

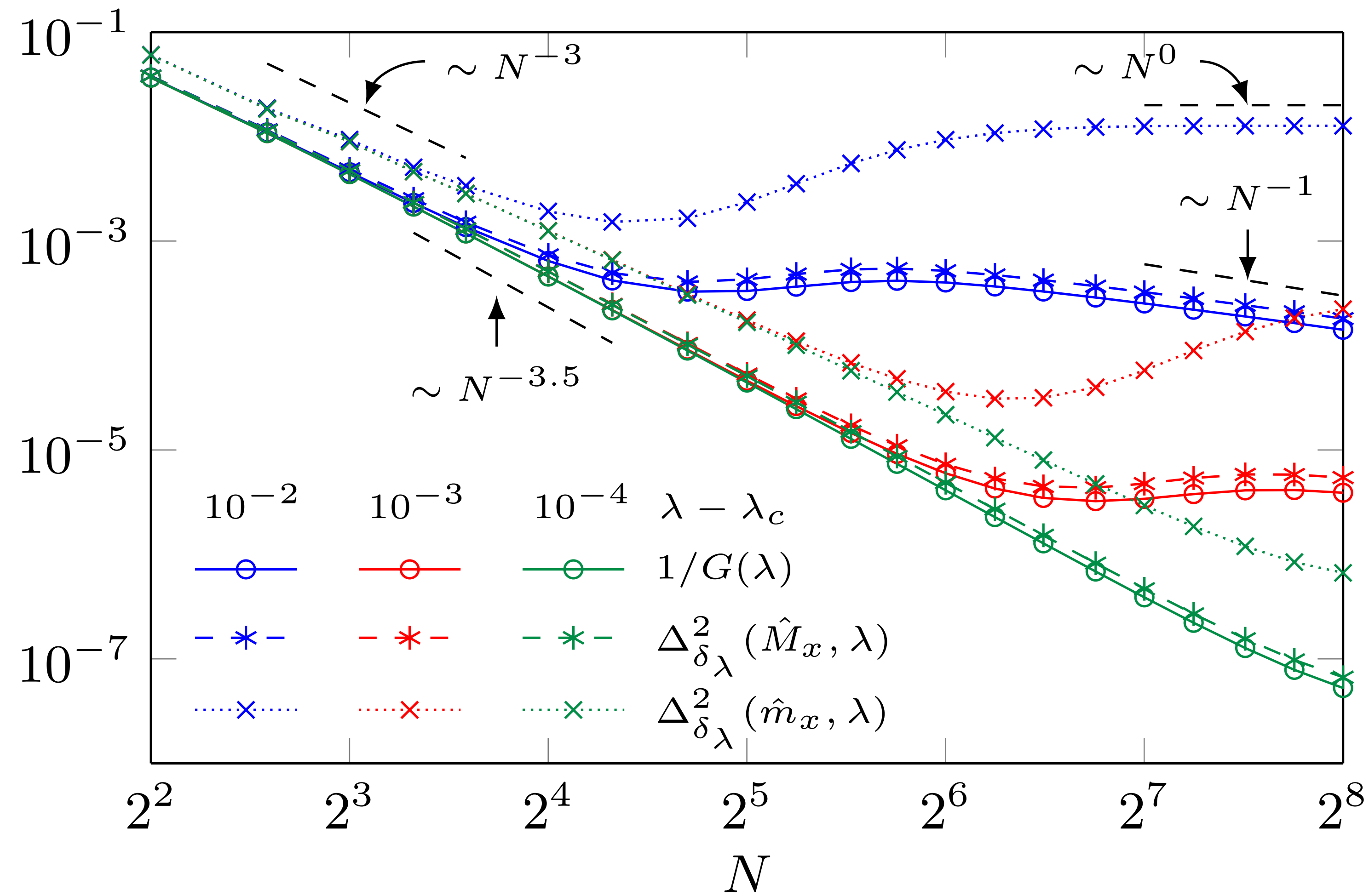
Crossover:

$$N/\xi^d_\lambda \sim N|\lambda - \lambda_c|^{d\nu_\lambda} \sim 1$$

small detuning:

$$\Delta^2_{\delta_\lambda}(\hat{M}_x, \lambda_c) \sim N^{-2/d\nu_\lambda}$$

$$\Delta^2_{\delta_\lambda}(\hat{m}_x, \lambda_c) \sim N^{-(2-2/\delta)/d\nu_\lambda}$$



Robustness

External field tuned to the critical point but the temperature T is nonzero

large T:

$$\Delta^2_{\delta_\lambda}(\hat{M}_x, \lambda_c, T) \sim N^{-1} T^{(2-d\nu_\lambda)/z\nu_\lambda}$$

$$\Delta^2_{\delta_\lambda}(\hat{m}_x, \lambda_c, T) \sim N^{0} T^{(2-2/\delta)/z\nu_\lambda}$$

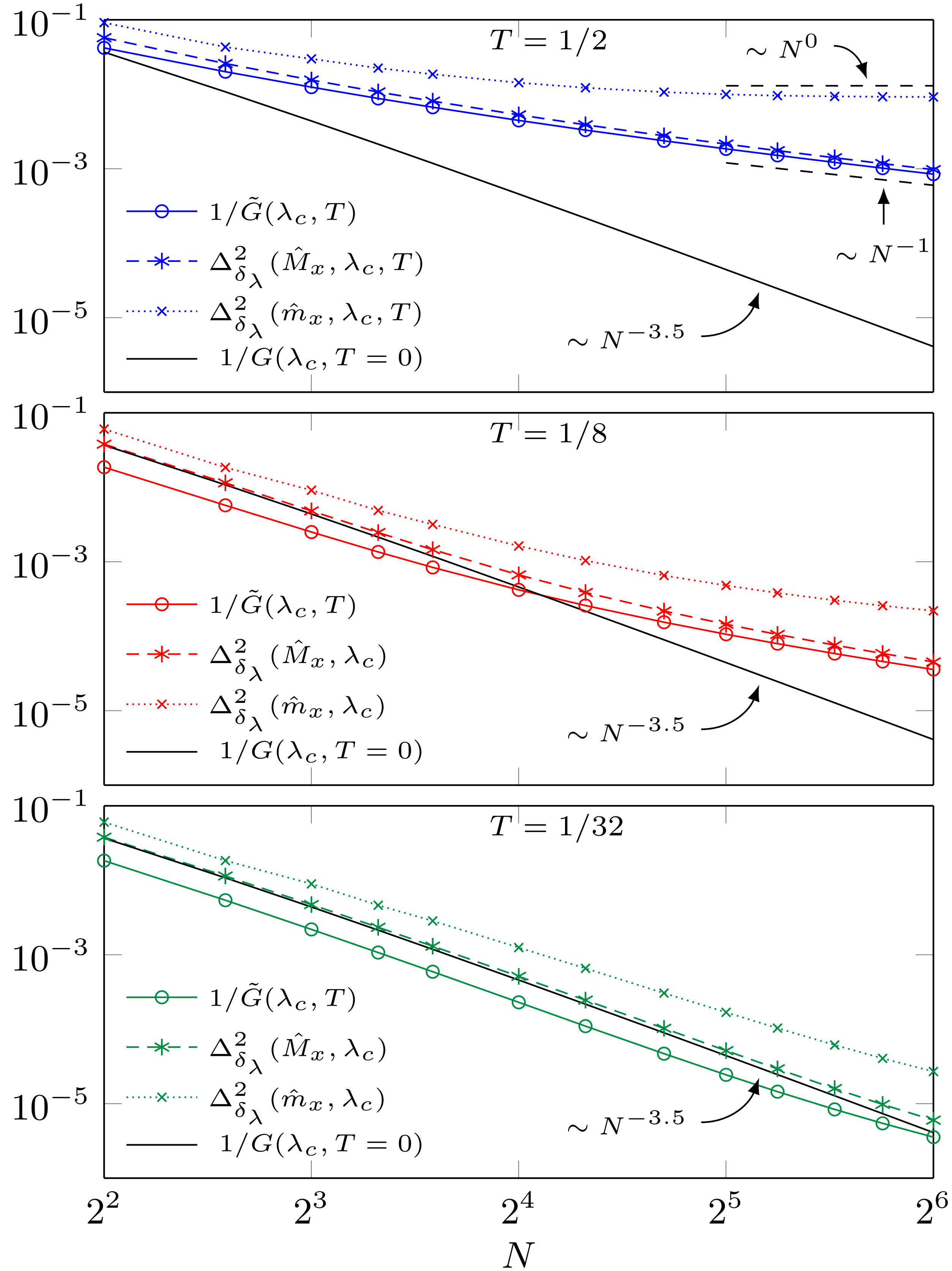
Crossover:

$$N T^{d/z} \sim 1$$

small T:

$$\Delta^2_{\delta_\lambda}(\hat{M}_x, \lambda_c) \sim N^{-2/d\nu_\lambda}$$

$$\Delta^2_{\delta_\lambda}(\hat{m}_x, \lambda_c) \sim N^{-(2-2/\delta)/d\nu_\lambda}$$



Preparation time in the vicinity of the critical point

In unitary rotation scenario

$$\hat{H} = \hat{H}_1 + \lambda \sum_{n=1}^{N} \hat{h}_n = \hat{H}_1 + \lambda \hat{H}_0, \quad \longrightarrow \quad G^{-1} \geq \frac{1}{t^2 ||\hat{H}_0||^2} \sim \frac{1}{t^2 N^2}$$

(Boixo et al. 2007)

When we consider the family of the ground states we need to (adiabatically) evolve between them.

Apply Kibble-Zurek (adiabatic) argument to estimate how slowly λ can change

$$t \sim \tau_Q \delta_\lambda \sim N^{z/d}$$

$$N^{-2/d\nu_\lambda} \sim 1/G(\lambda_c) \geq 1/t^2 ||\hat{H}_0||^2 \sim N^{-2(z+d)/d}$$

Everything is consistent

Conclusion

→ We discussed „minimal" Hamiltonian leading to apparent super-Heisenberg scaling at criticality

→ Analytic finite-size estimates in a full agreement with quasi-exact MPS calculations both for $T=0$ and $T>0$.

→ When we consider the family of the ground states we need to (adiabatically) evolve between them.

→ Recovery of the Heisenberg limit

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Thank you!, Gracias!, Merci , Danke, Dziękuję