Measurement of entanglement in quantum many body systems: A random matrix approach

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UNIVERSITY OF INNSBRUCK

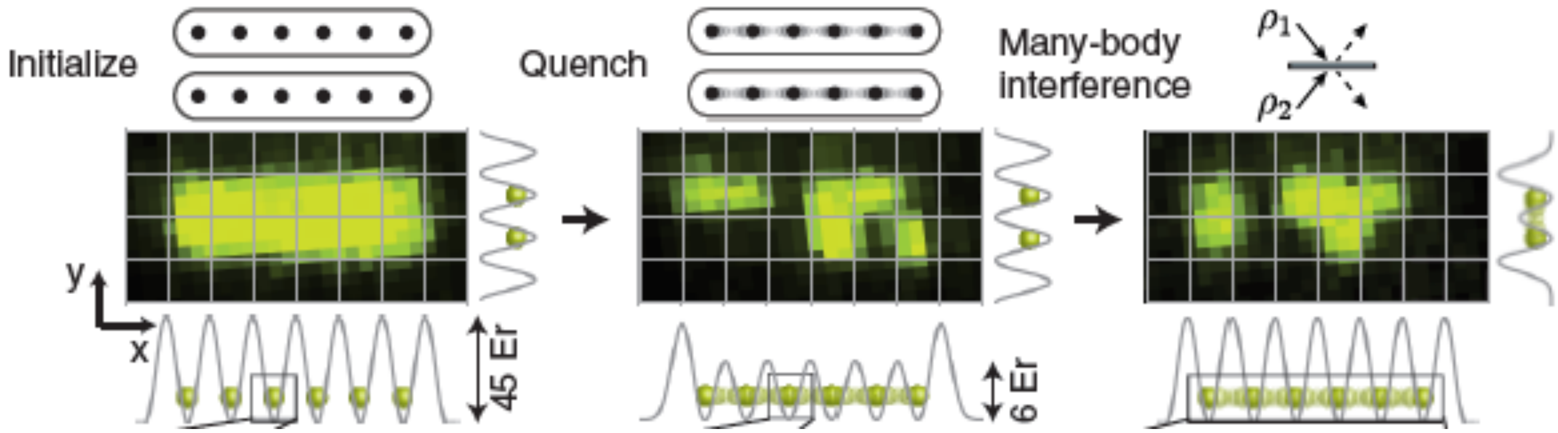
ÖAW

SFB FoQuS

UQUAM ERC Synergy Grant

Quantum many body systems

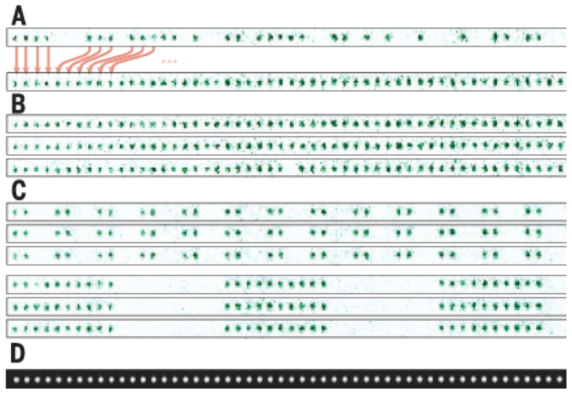
Hubbard models

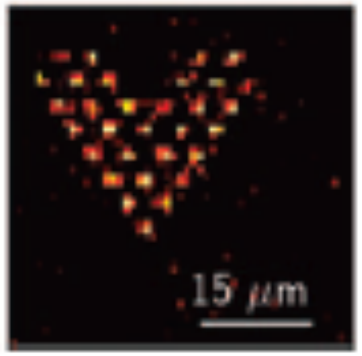


CUA,MPQ, JQI,
Innsbruck,…

M. Kaufman et al., Science (2016)

Rydberg Atoms





M. Endres et al., Science (2016)

D. Barredo et al., Science (2016)

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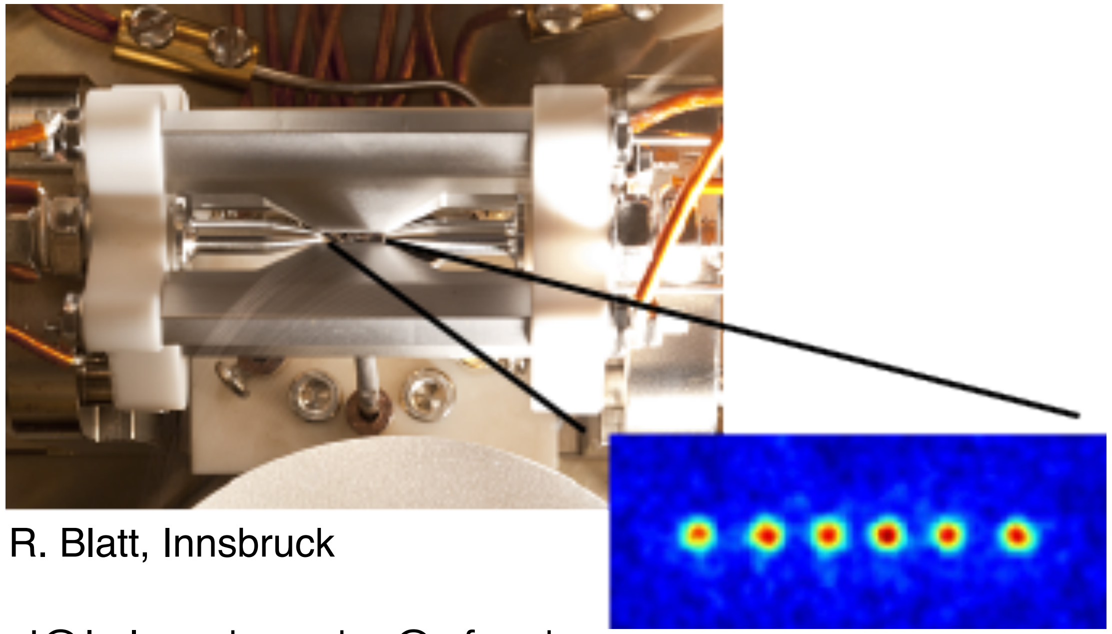
Ion Traps



R. Blatt, Innsbruck

JQI, Innsbruck, Oxford,…

Quantum many body systems

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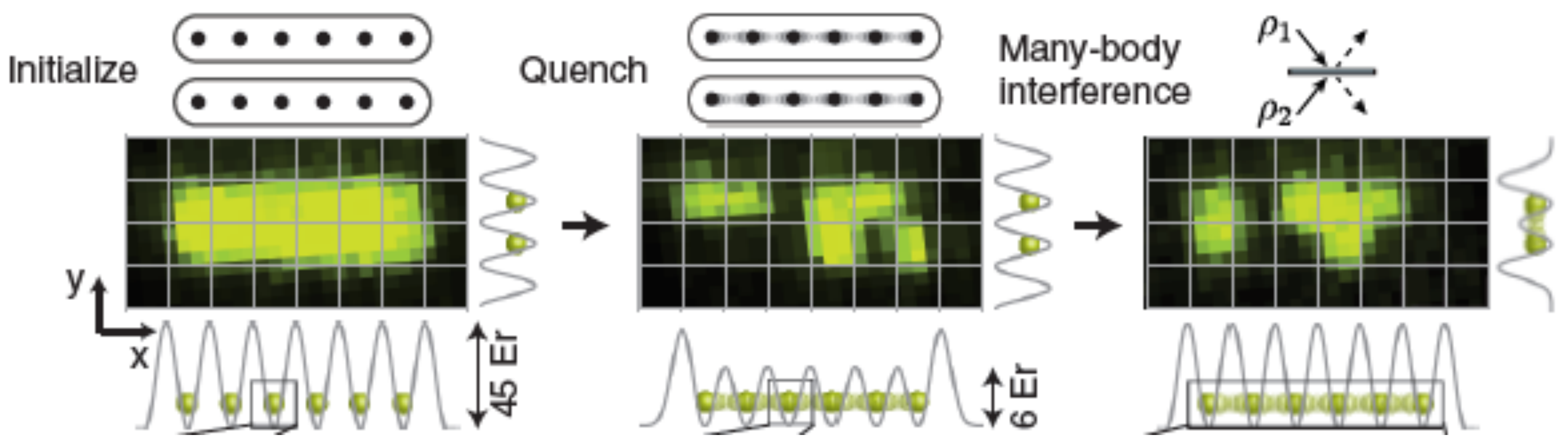

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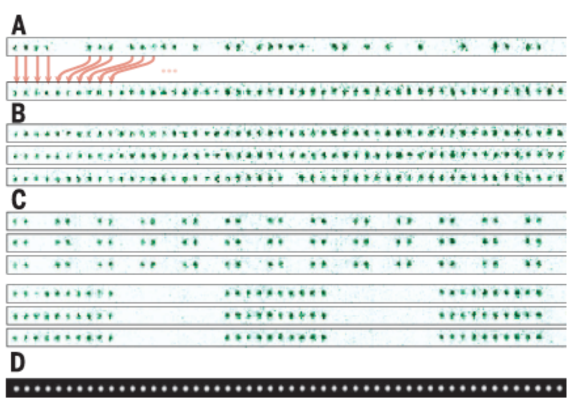

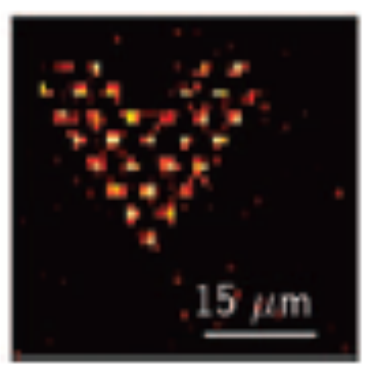

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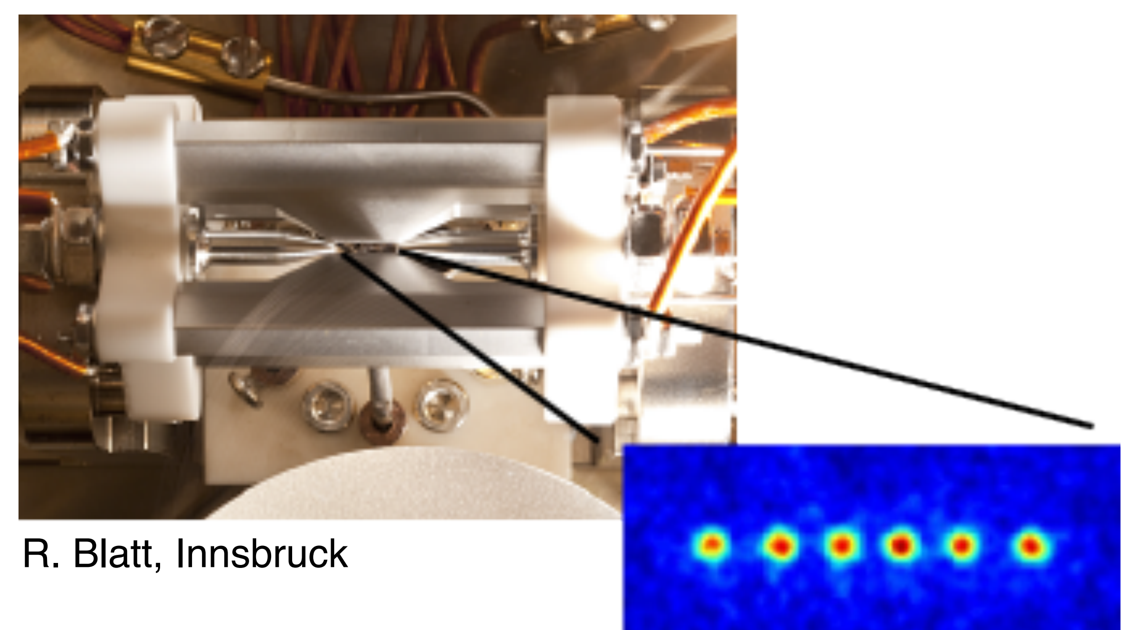

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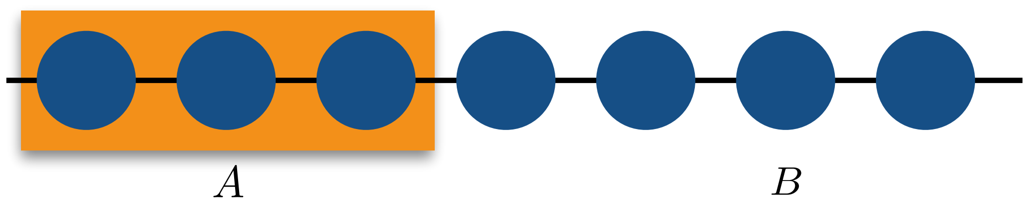
JQI, Innsbruck, Oxford,…

How to measure Renyi entropies demonstrating entanglement for interesting many body dynamics?

Renyi entropy as an entanglement measure

Two subsystems A and B are bipartite entangled iff

$$|\Psi_{AB}\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle$$



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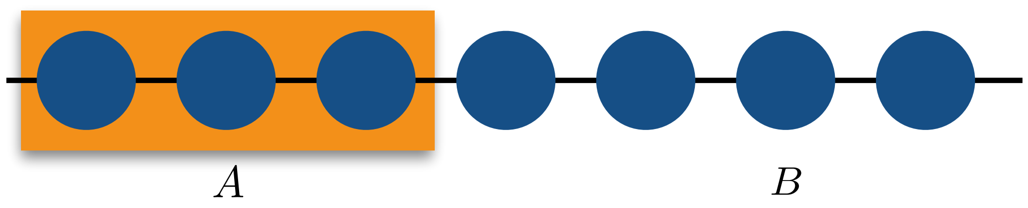

Sufficient condition for bipartite entanglement (also mixed states):

Purity of subsystems

$$\mathrm{Tr}\left[\rho_A^2\right] < \mathrm{Tr}\left[\rho_{AB}^2\right]$$

$$\mathrm{Tr}\left[\rho_B^2\right] < \mathrm{Tr}\left[\rho_{AB}^2\right]$$

Purity of full system

Second Renyi entropy: lower bound to von-Neumann entropy

$$S_2(A) = -\log \mathrm{Tr}\left[\rho_A^2\right] \leq -\mathrm{Tr}\left[\rho_A \log \rho_A\right] = S_{VN}(A)$$

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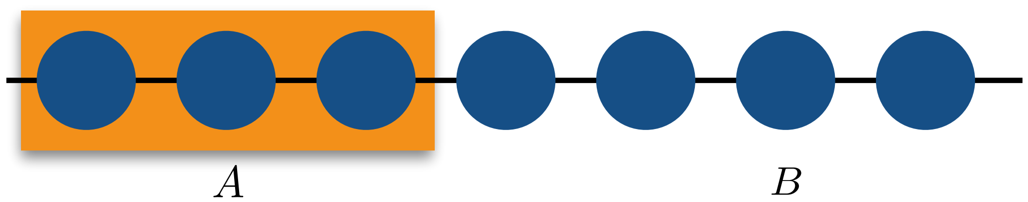

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Measure purity (and pot. higher order functionals) of a (reduced) density matrix **in a quantum many body system** (Spin chain, Bose/Fermi Hubbard, …)

Measurement of Renyi entropies

How to measure Renyi entropies for interesting many body dynamics?

Quench dynamics

State preparation

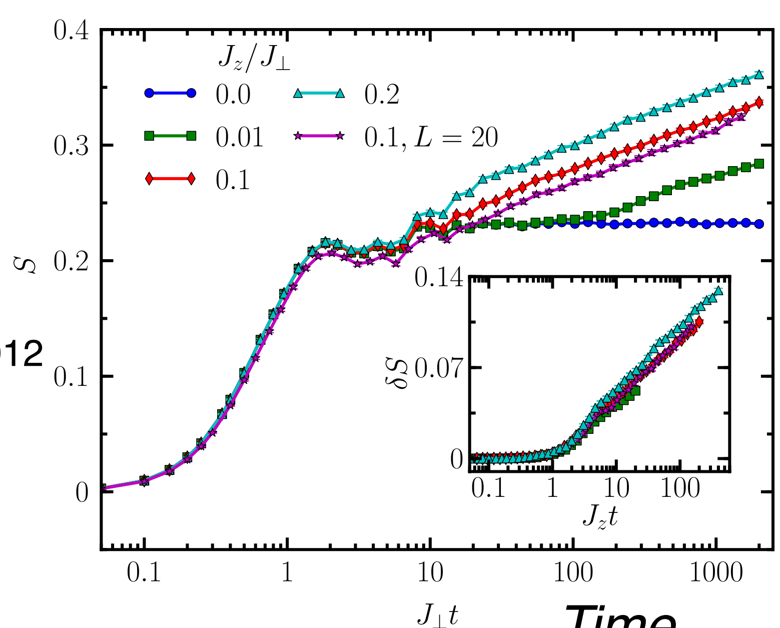
ρ_I

Many body localization,
thermalization,
…

ρ_t

Von Neumann Entropy

Bardarson, et al., PRL 2012



Time

Measurement of Renyi entropies

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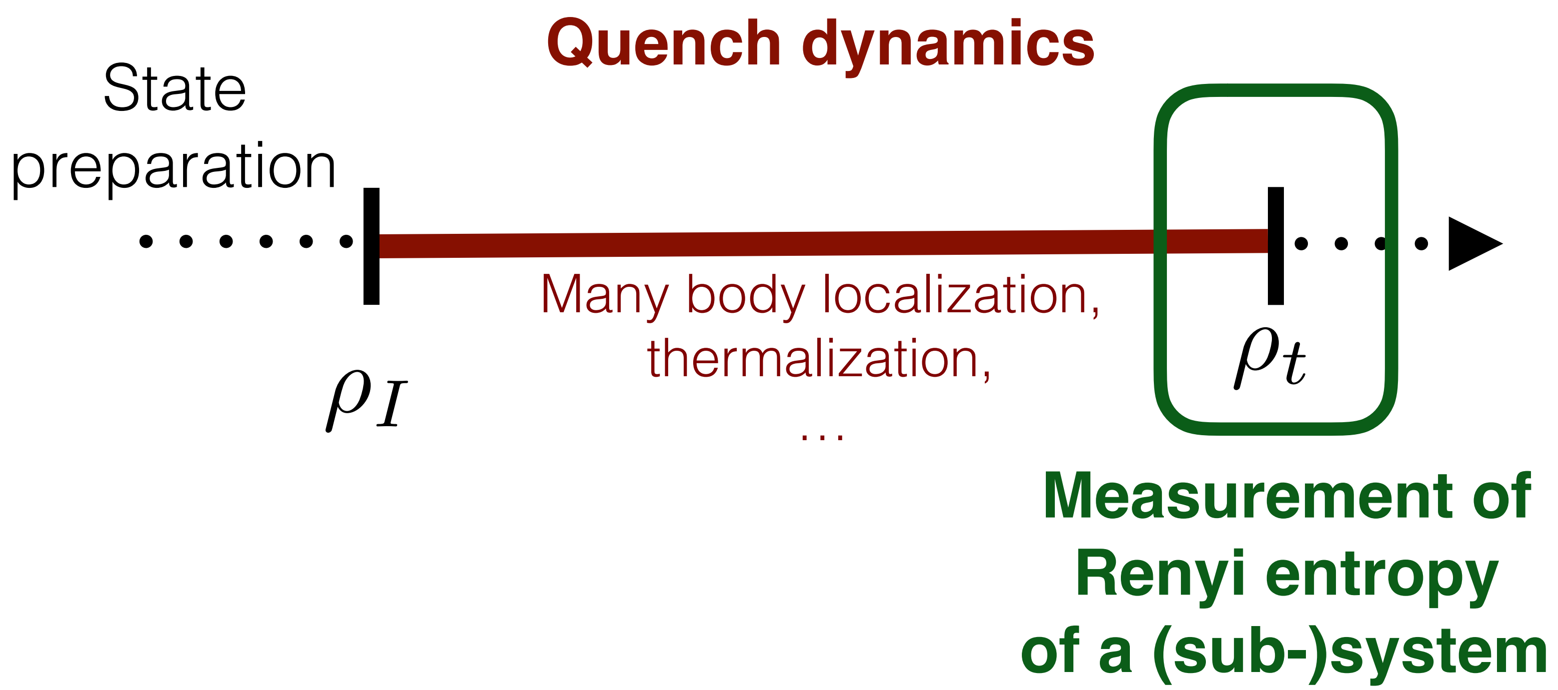

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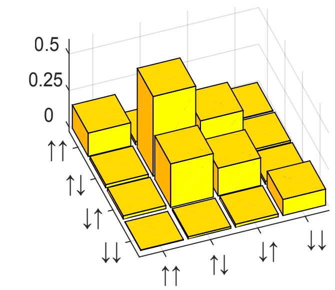
Quantum State Tomography



Gross et al Phys . Rev. Lett. 105, 150401
B. P. Lanyon et al., arXiv: 1612.08000

If available

Measurement of Renyi entropy of a (sub-)system

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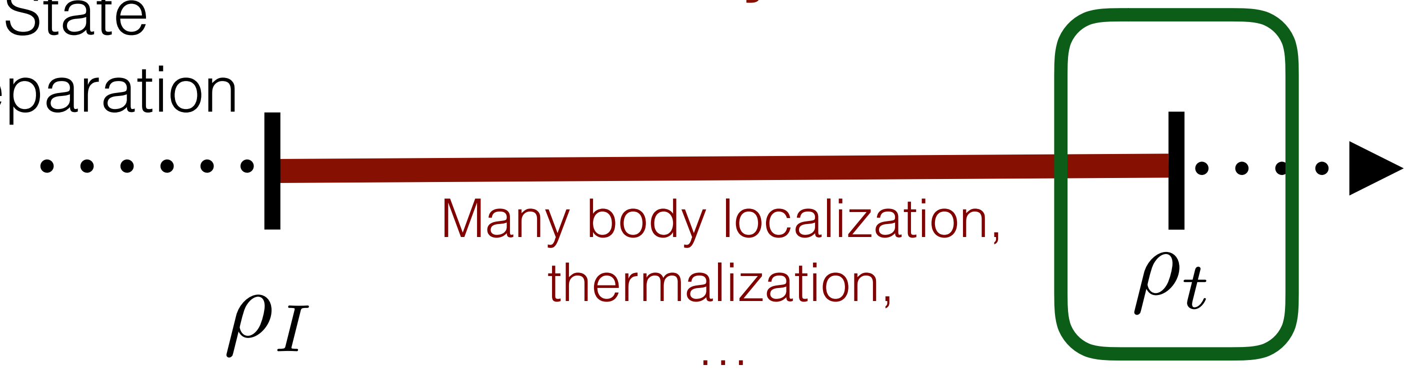

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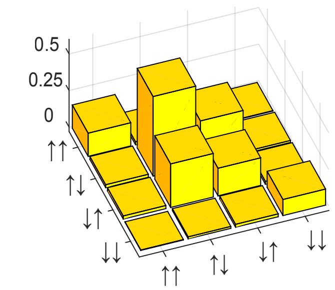

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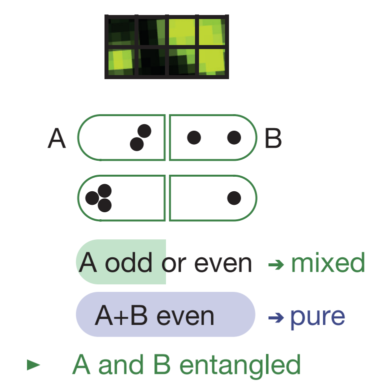

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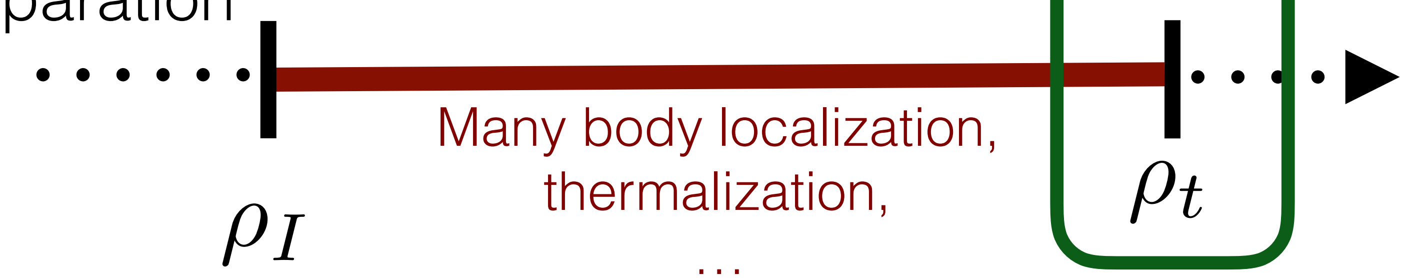
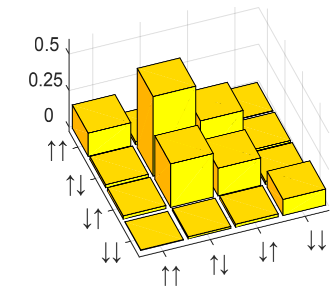


Many body localization, thermalization, …

Measurement of Renyi entropy of a (sub-)system

Quantum State Tomography



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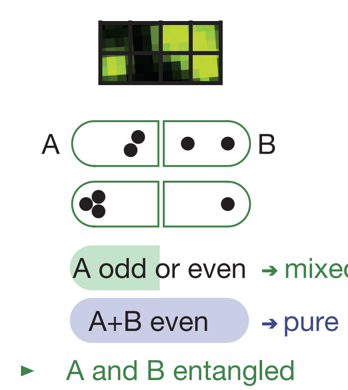

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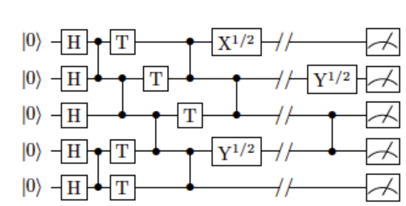
Random measurements on single copies



S. Boixo, et al., arXiv:1608.00263

van Enk, Beenaker, PRL 108, 110503 (2012)

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How to measure Renyi entropies for interesting many body dynamics?

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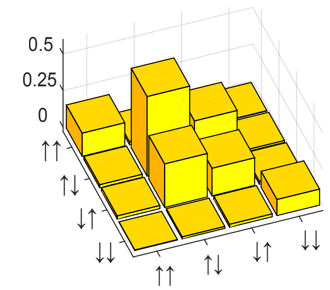

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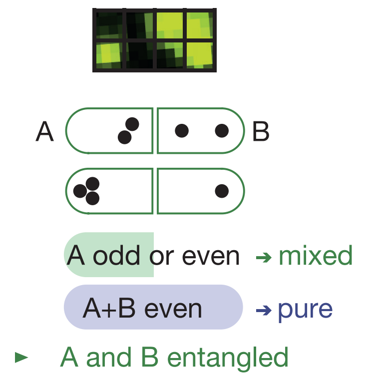

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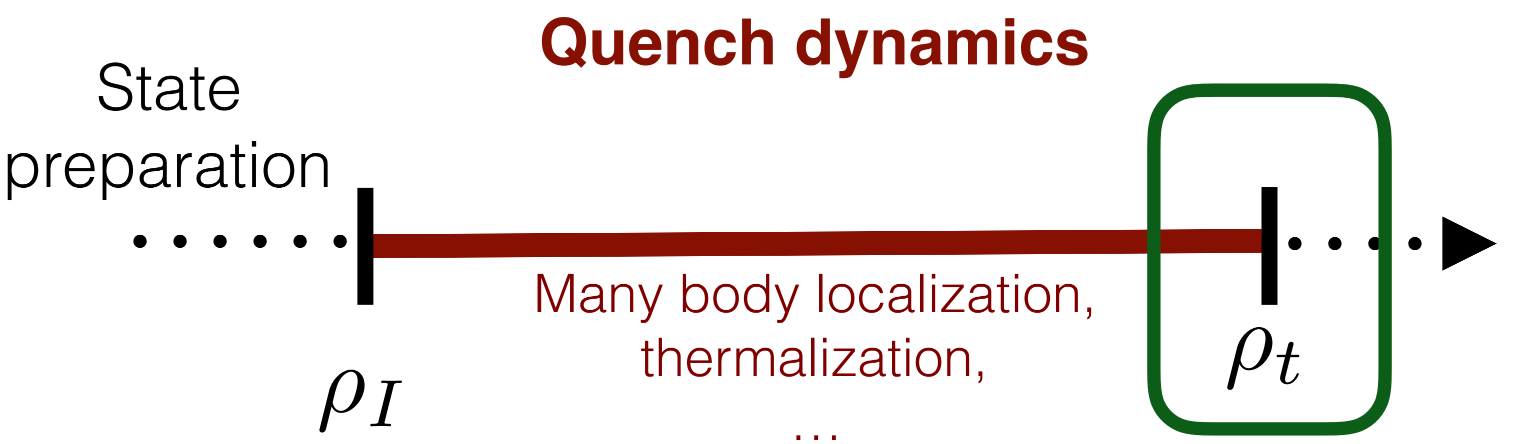
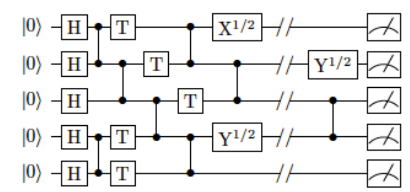


Measurement of Renyi entropy of a (sub-)system

How to realize in a physical system which exists today in the lab?
- Hubbard or Spin model

Random measurements on single copies



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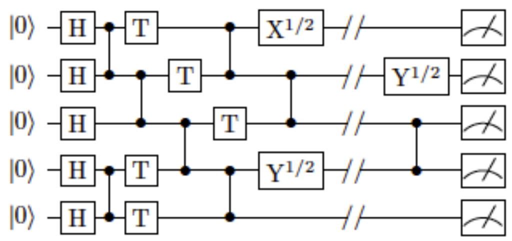
Outline

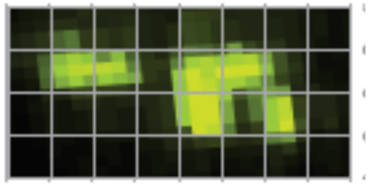
1. Review: Entanglement measurement with random measurements

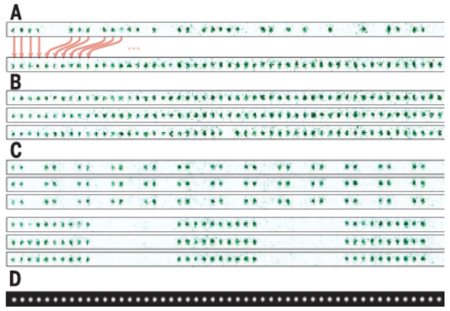
van Enk, Beenaker (PRL 2012)



2. Realization in a physical system (Hubbard or Spin model)

1. Measurement protocol
2. Generation of random unitaries
3. Scaling of statistical errors





3. An example: Entanglement growth in the many-body localized phase in the Bose Hubbard model

Measuring entanglement on single copies with random measurements

Idea: Extract purity (and higher functionals) from correlations between random measurements

van Enk, Beenaker (PRL 2012)

Measuring entanglement on single copies with random measurements

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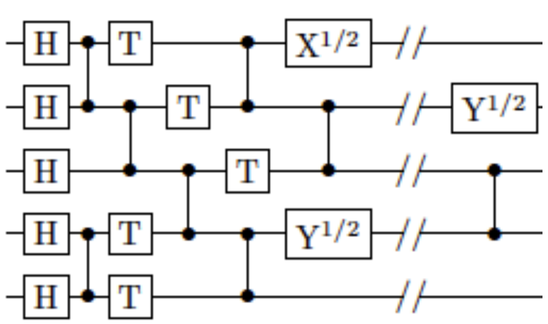
Random measurement:

Apply a random unitary to the state and measure in fixed basis

Random unitary

ρ_A +  + Measurement in fixed basis

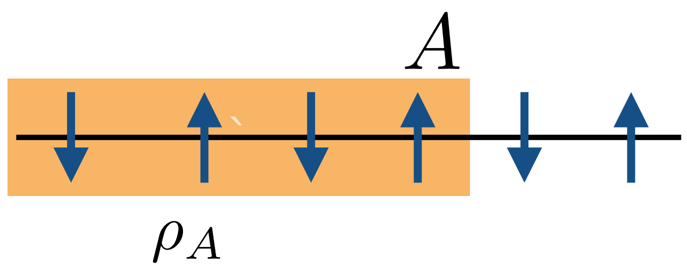
S. Boixo, et al., arXiv:1608.00263

Distributed according to Haar measure:
Circular unitary ensemble

Measuring Renyi entropies with random measurements

Protocol for chain of qubits:

ρ_A

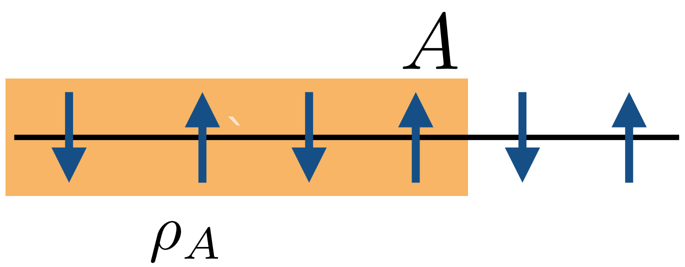


van Enk, Beenaker (PRL 2012)

Measuring Renyi entropies with random measurements

Protocol for chain of qubits:

Start with state ρ_A of a (sub-)system



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Measuring Renyi entropies with random measurements

Protocol for chain of qubits:

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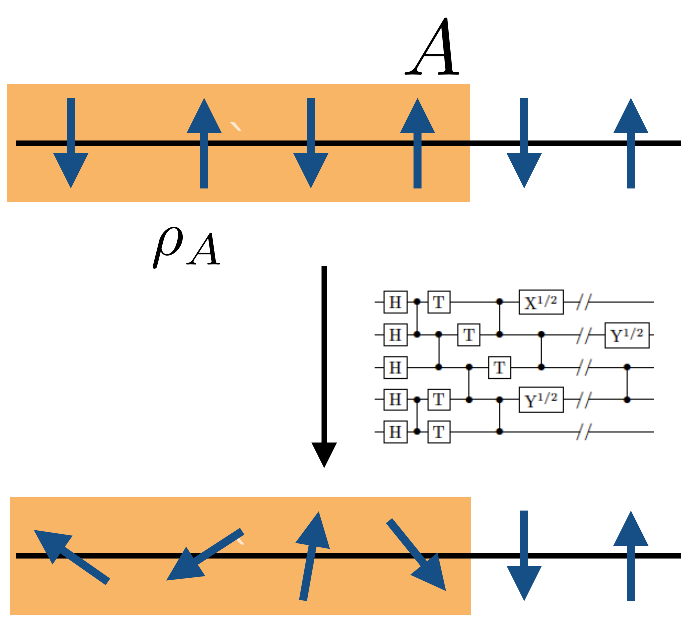
1. Apply random unitary U distributed according Haar measure generated by random gate sequence

$$\rho_A \longrightarrow \rho_U^f = U \rho_A U^\dagger$$



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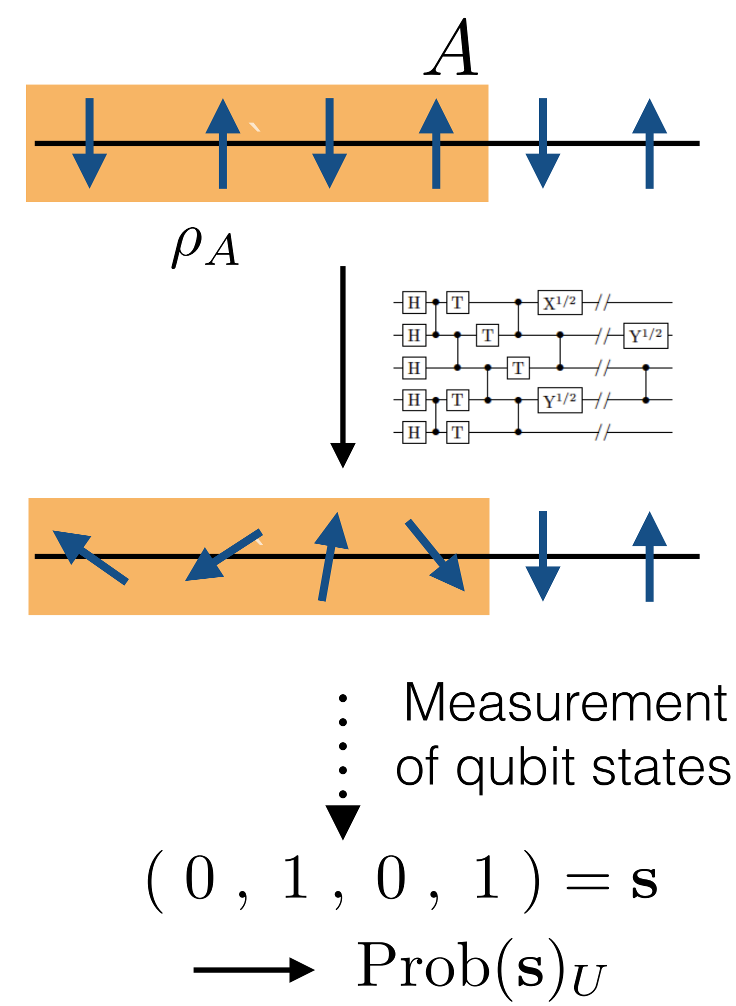

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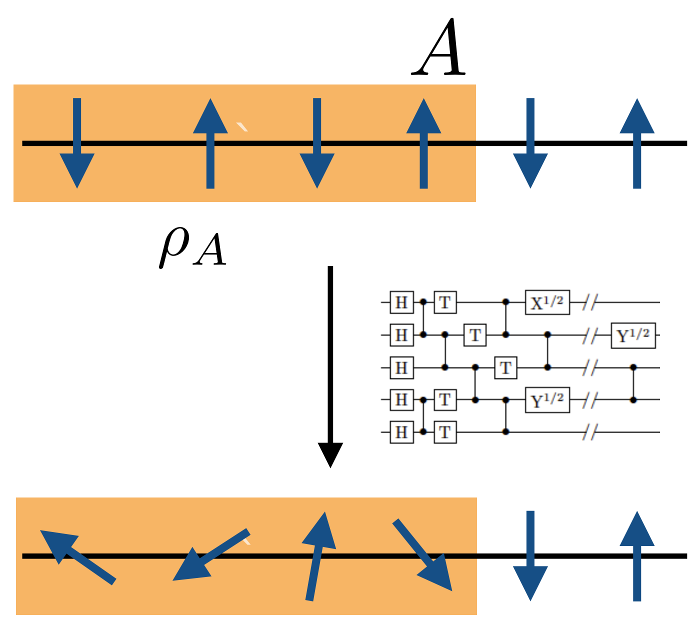
3. Repeat 1.+2. for many random unitaries and average to obtain

$$\langle \mathrm{Prob}(\mathbf{s})_U \rangle = \frac{1}{N_H} \qquad \langle \mathrm{Prob}(\mathbf{s})_U^2 \rangle = \frac{1 + \mathrm{Tr}\left[\rho_A^2\right]}{N_H(N_H+1)}$$



Measurement of qubit states

$$(\,0\,,\,1\,,\,0\,,\,1\,) = \mathbf{s}$$

$$\longrightarrow \mathrm{Prob}(\mathbf{s})_U$$

N_H: Hilbert space dimension of subsystem

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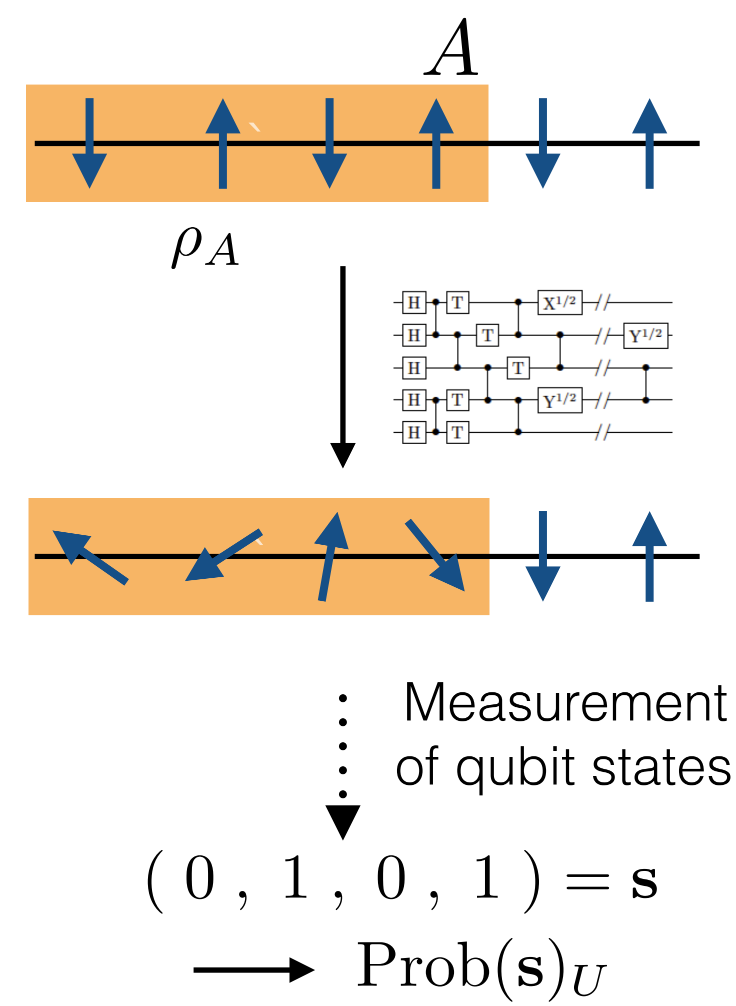

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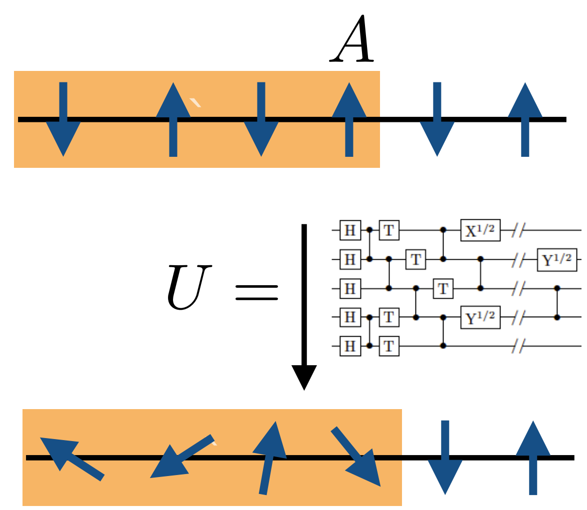
Measuring entanglement on single copies with random measurements

The protocol on a quantum computer:

Quantum state ρ_A of interest

1. Apply random unitary from Haar measure



2. Measurements of qubit states to obtain outcome probabilities

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$$\langle \mathrm{Prob}(\mathbf{s})_U \mathrm{Prob}(\mathbf{s})_U \rangle \sim \mathrm{Tr}\left[\rho_A^2\right]$$

How to realize in a physical system (Hubbard or Spin model)?

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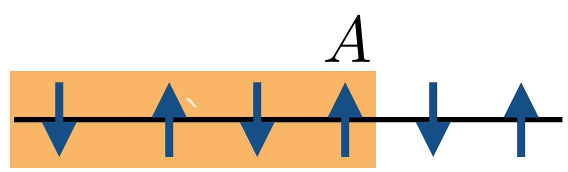

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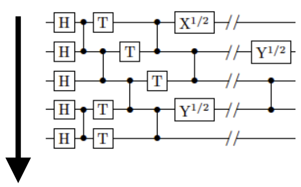
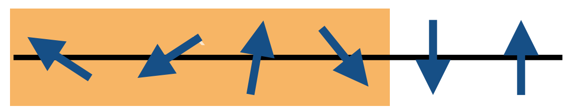
$U =$





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Key questions for physical realization:

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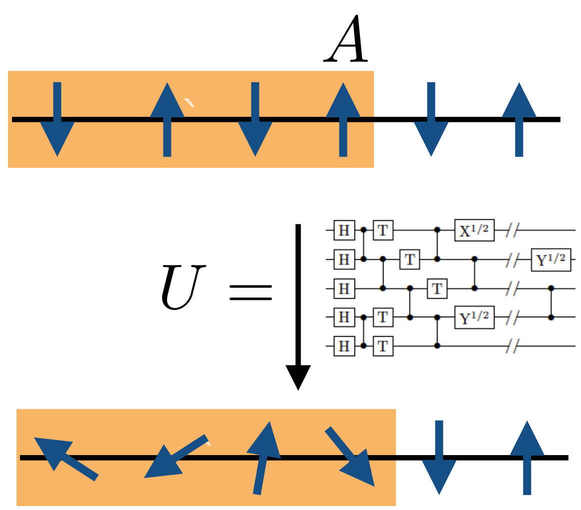

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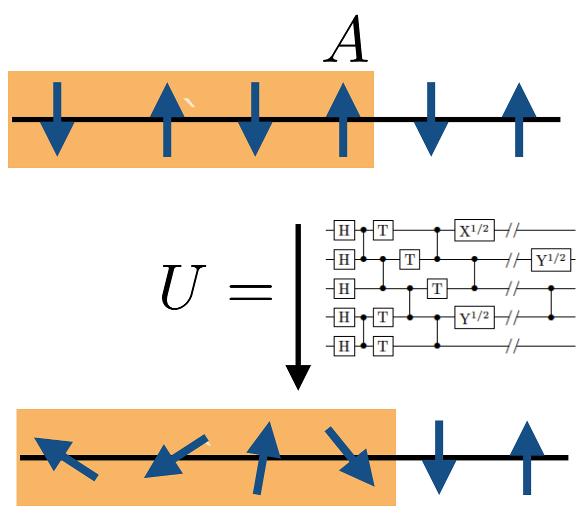

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Which **observables** to measure**?**

How many measurements per unitary and **how many unitaries?**

How to realize in a physical system (Hubbard or Spin model)?

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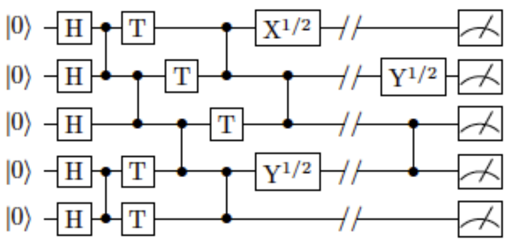

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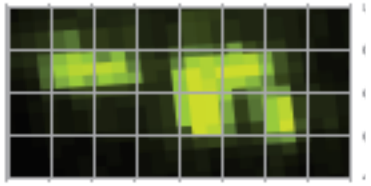
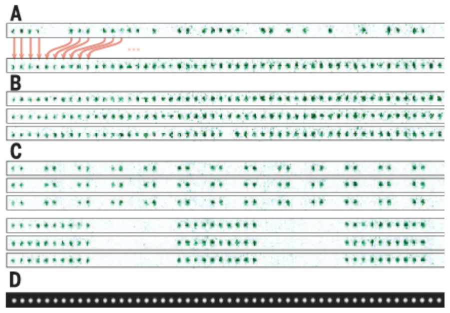
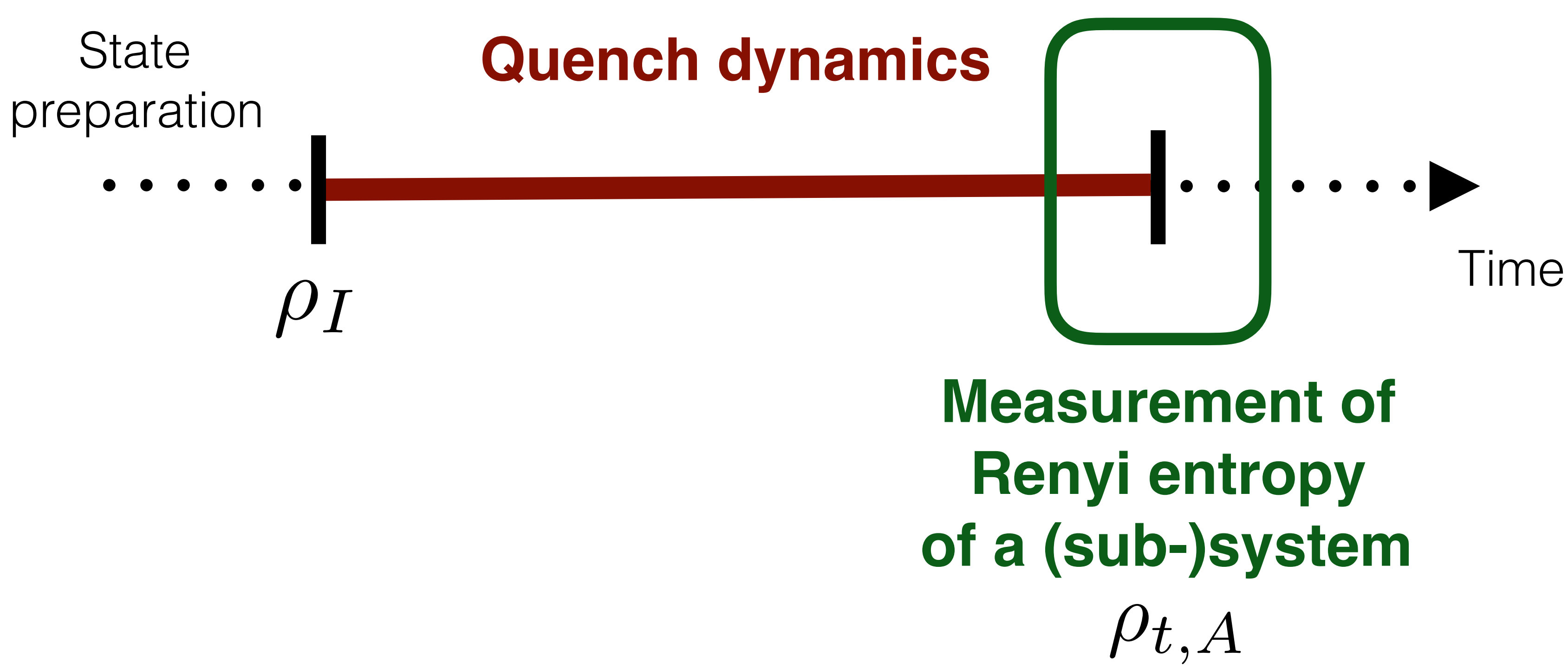




3. An example: Entanglement growth in the many-body localized phase in the BH model

Measurement protocol for Hubbard and Spin models



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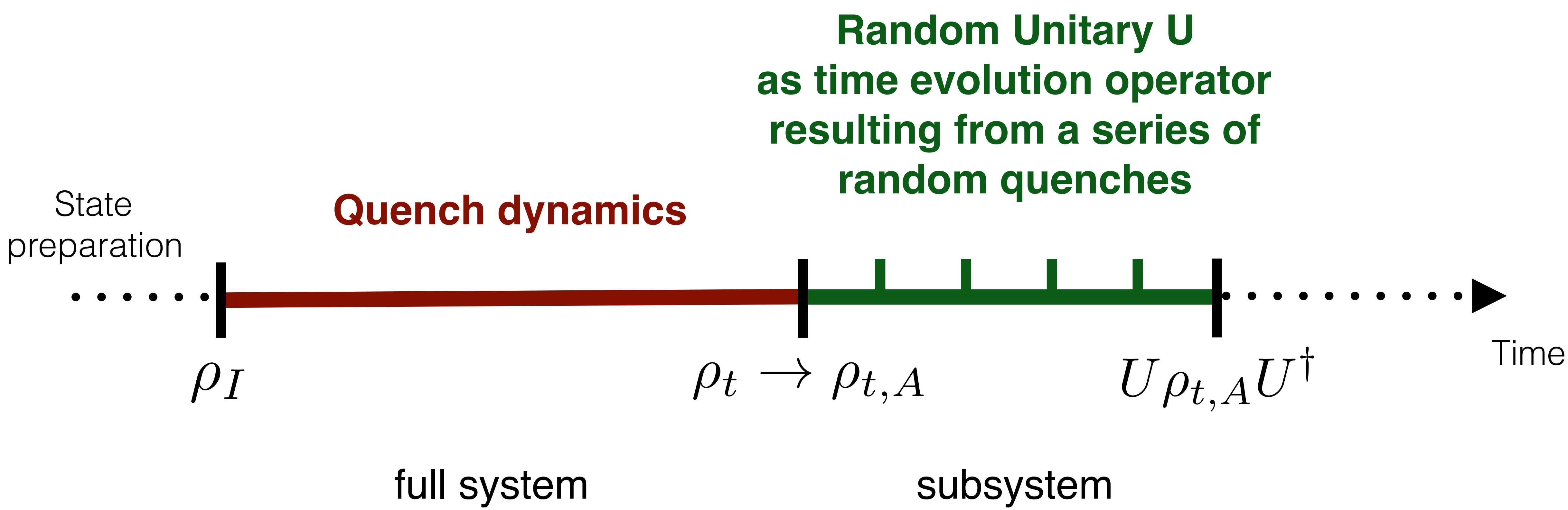

Generation of random unitaries

Idea: *Random unitary* as time evolution operator resulting from *a series of random quenches*

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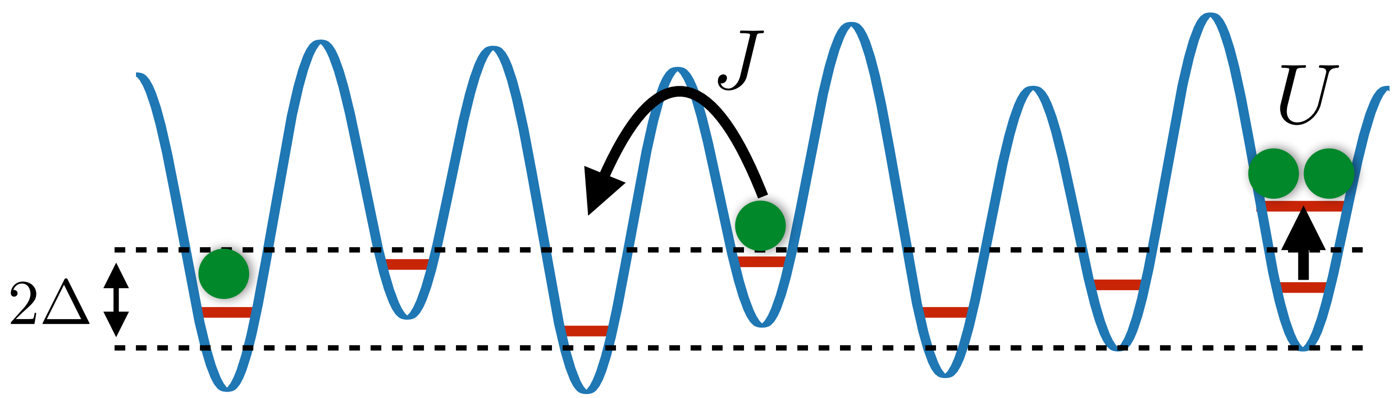
Bose Hubbard system with disorder potentials:



$$H_0 = -J \sum_i \left(a_i^\dagger a_i + \mathrm{h.c.} \right) + \frac{U}{2} \sum_i n_i (n_i - 1)$$

$$H_\alpha = H_0 - \sum_i \mu_i(\alpha)\, a_i^\dagger a_i$$ where $\mu_i(\alpha) \in [-\Delta, \Delta]$

Vary disorder in discrete steps in time $\longrightarrow$ $U = e^{-iH_1 T} \dots e^{-iH_\eta T}$

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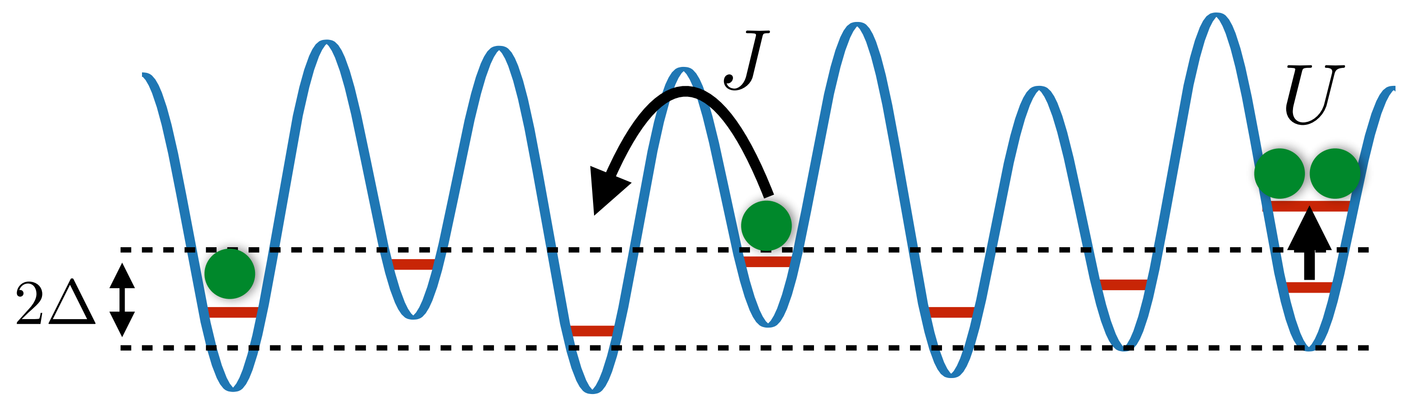

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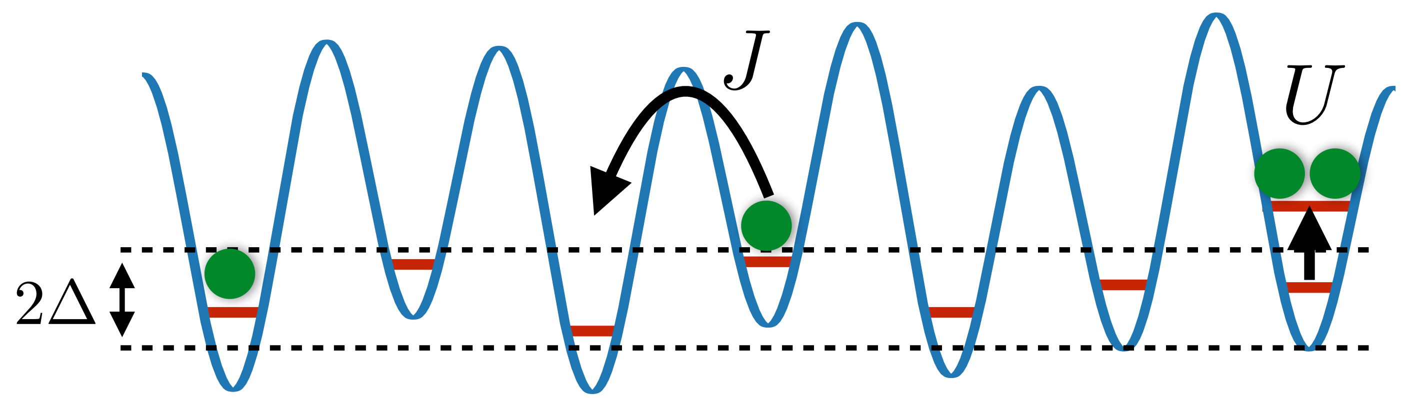

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Random unitary?

F. Haake, Quantum signatures of chaos $\longleftrightarrow$ Kicked top

Certification of random unitaries

Question: Is $U = e^{-iH_1T} \cdots e^{-iH_\eta T}$ a random unitary?

$\longrightarrow$ **Apply the protocol to a known input state and compare estimated to true purity to test the ensemble**

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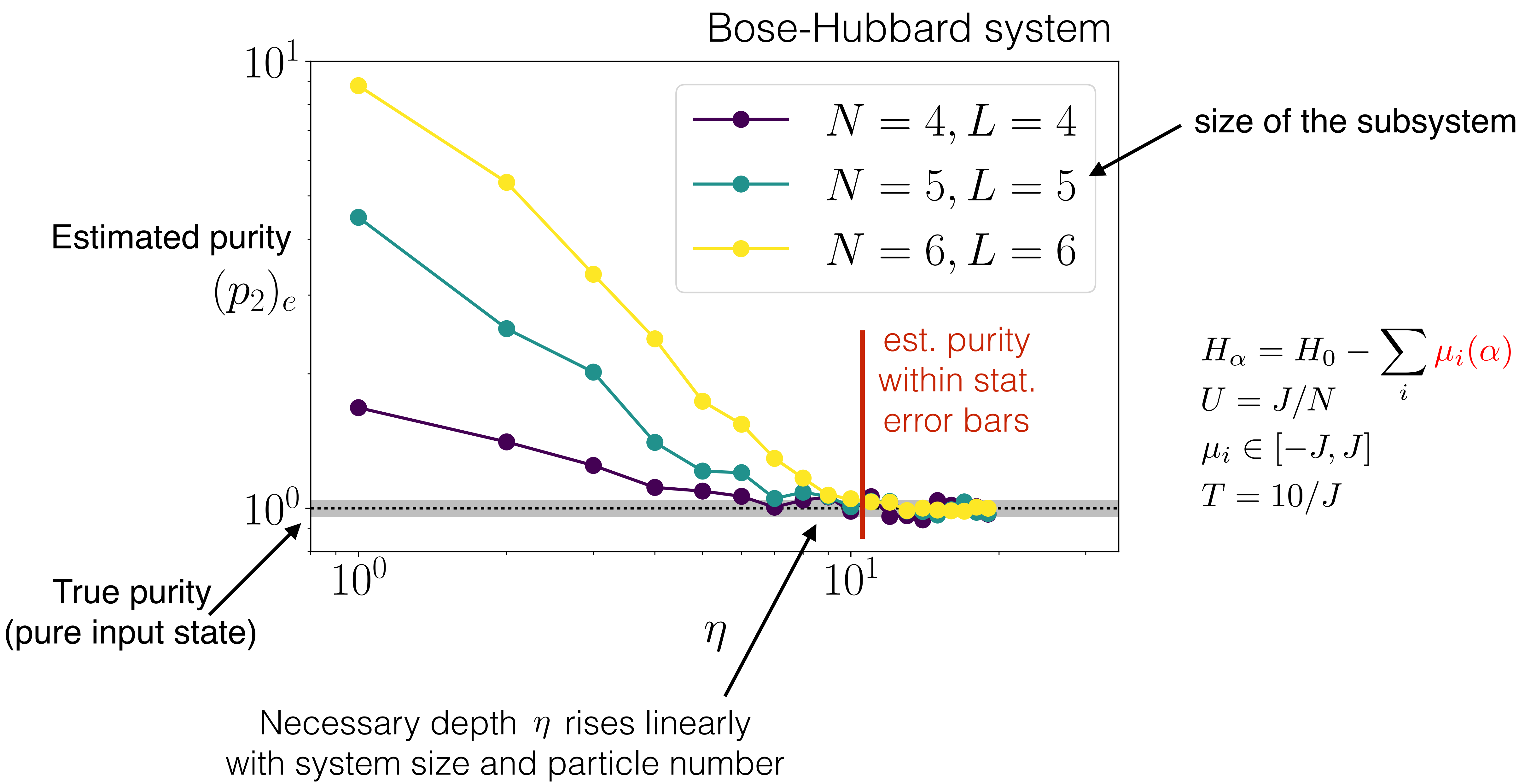

$$H_\alpha = H_0 - \sum_i \mu_i(\alpha)\, a_i^\dagger a_i$$
$$U = J/N$$
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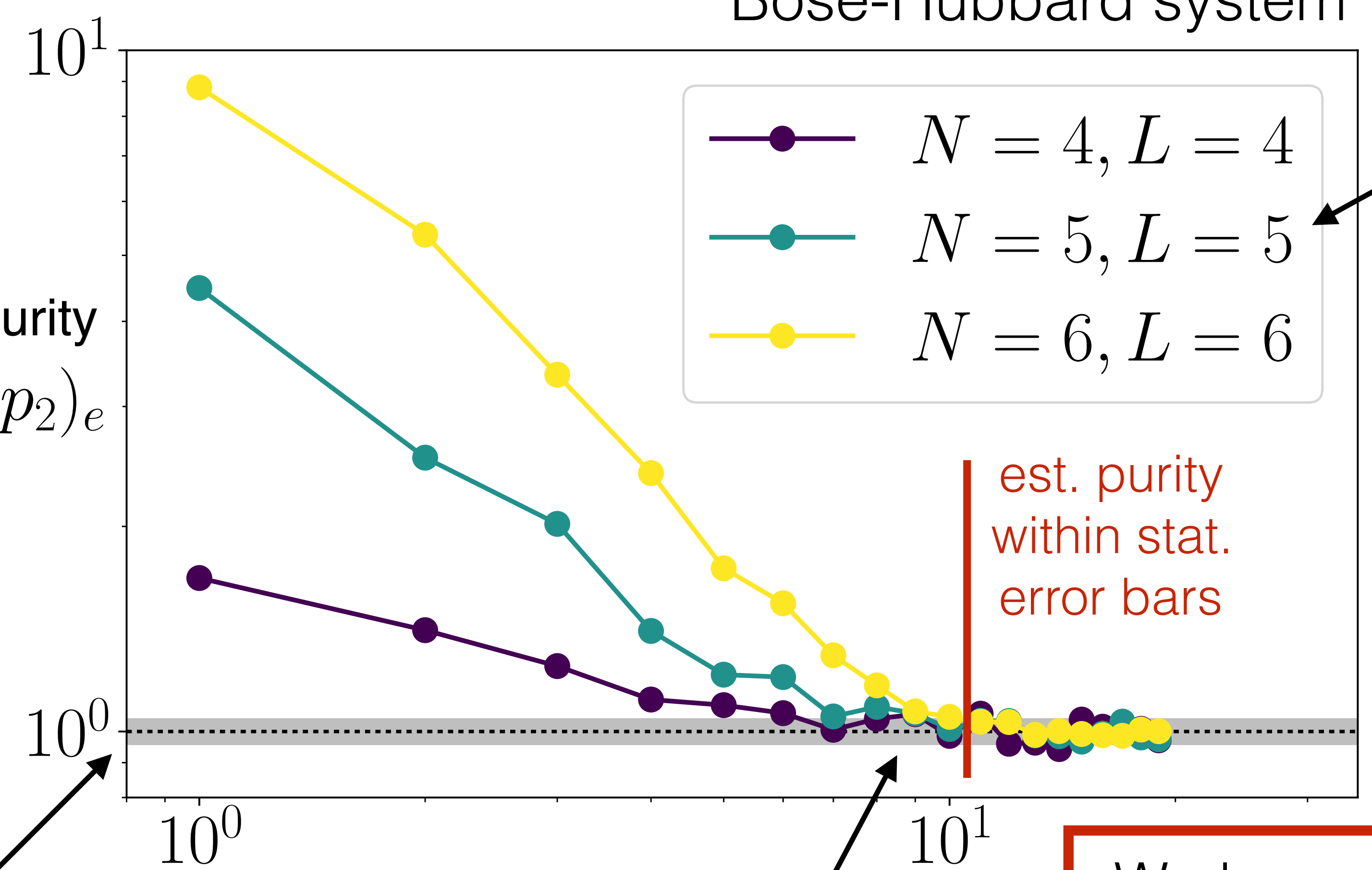
Bose-Hubbard system



Estimated purity $(p_2)_e$

size of the subsystem

True purity (pure input state)

η

Necessary depth η rises linearly with system size and particle number

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We have just looked at second order correlation

$$\langle \mathrm{Prob}(\mathbf{s})_U \mathrm{Prob}(\mathbf{s})_U \rangle \sim (p_2)_e$$

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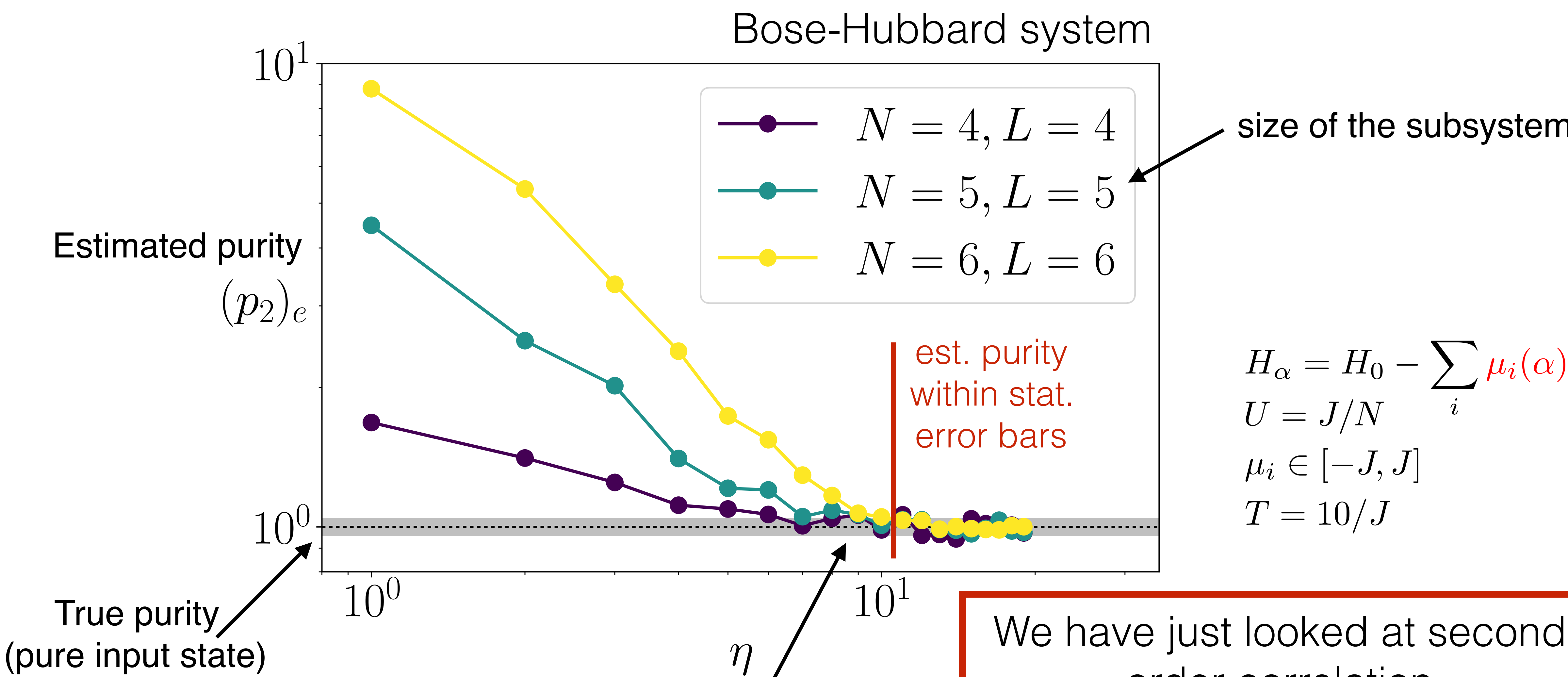

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$$\langle \mathrm{Prob}(\mathbf{s})_U \mathrm{Prob}(\mathbf{s})_U \rangle \sim (p_2)_e$$

$\longrightarrow$ Test on unitary 2-design

Measurement protocol for Hubbard and Spin models

Random Unitary U
as time evolution operator
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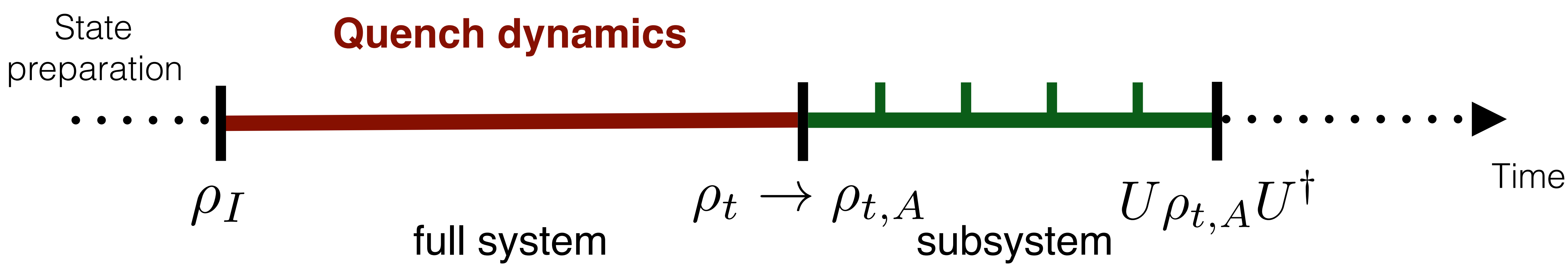
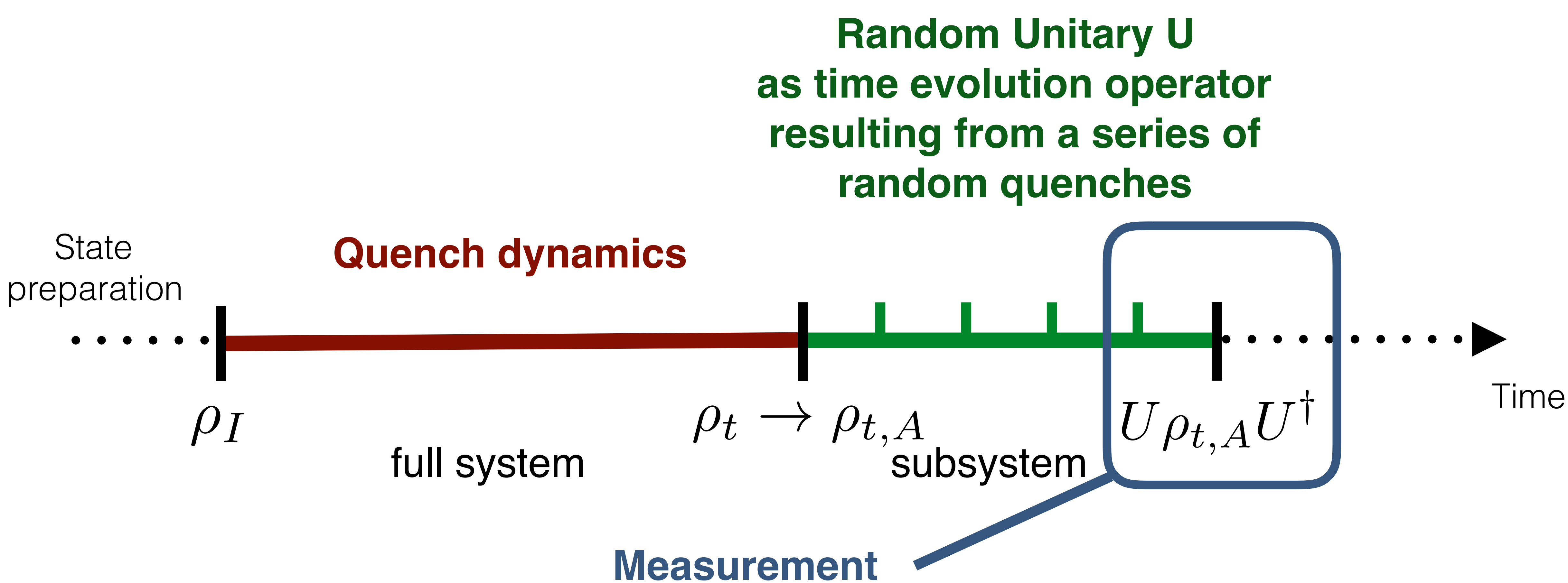


- **Bosonic and Fermionic Hubbard models**
- **Spin chains (Rydberg atoms)**
- **...**

Measurement protocol for Hubbard and Spin models



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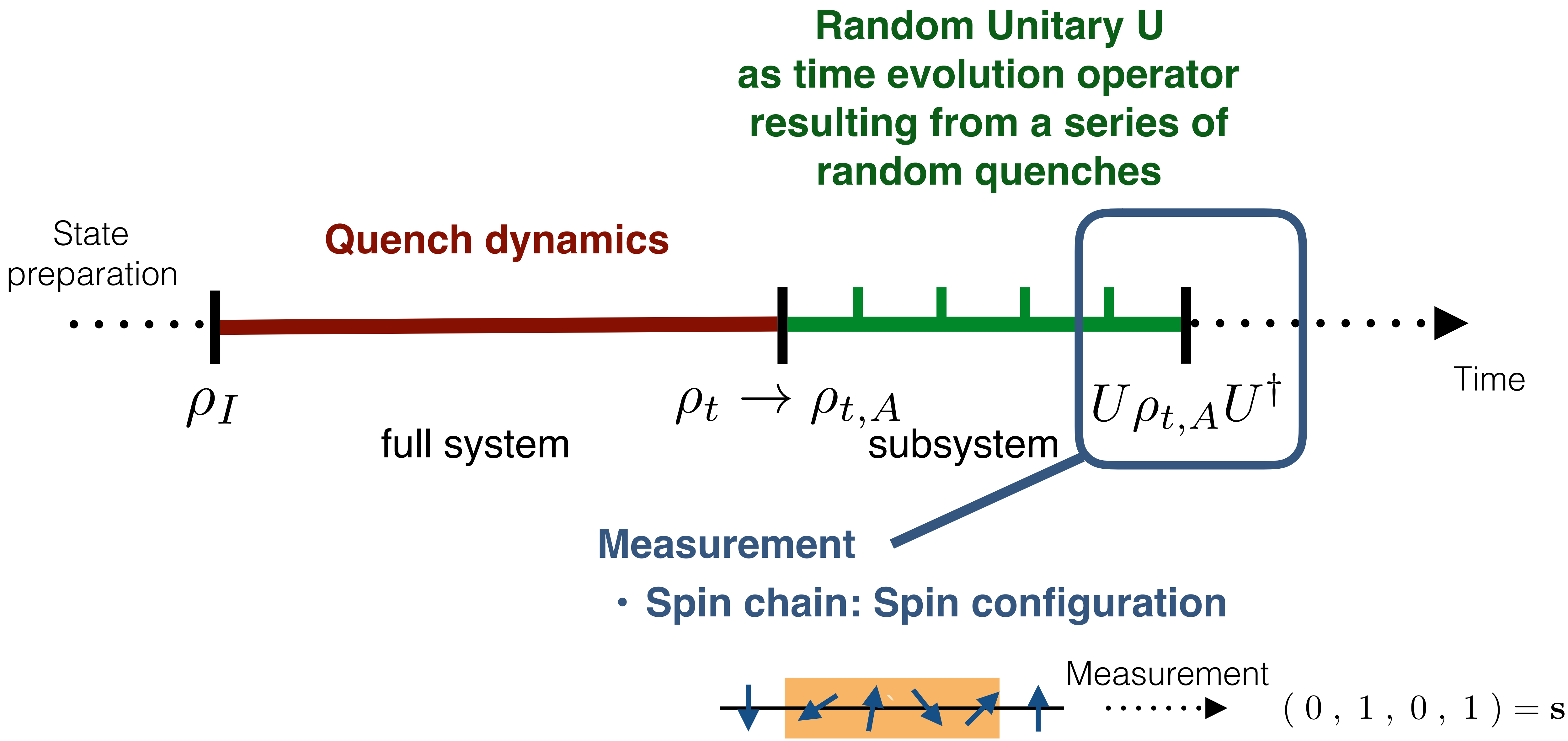

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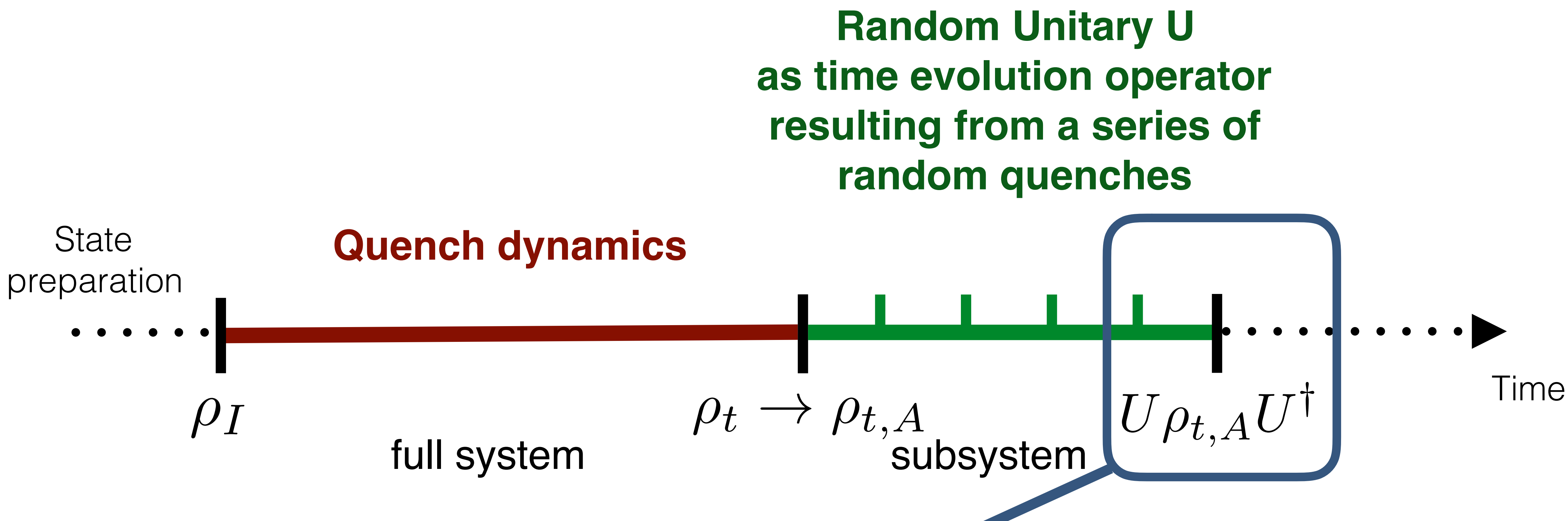
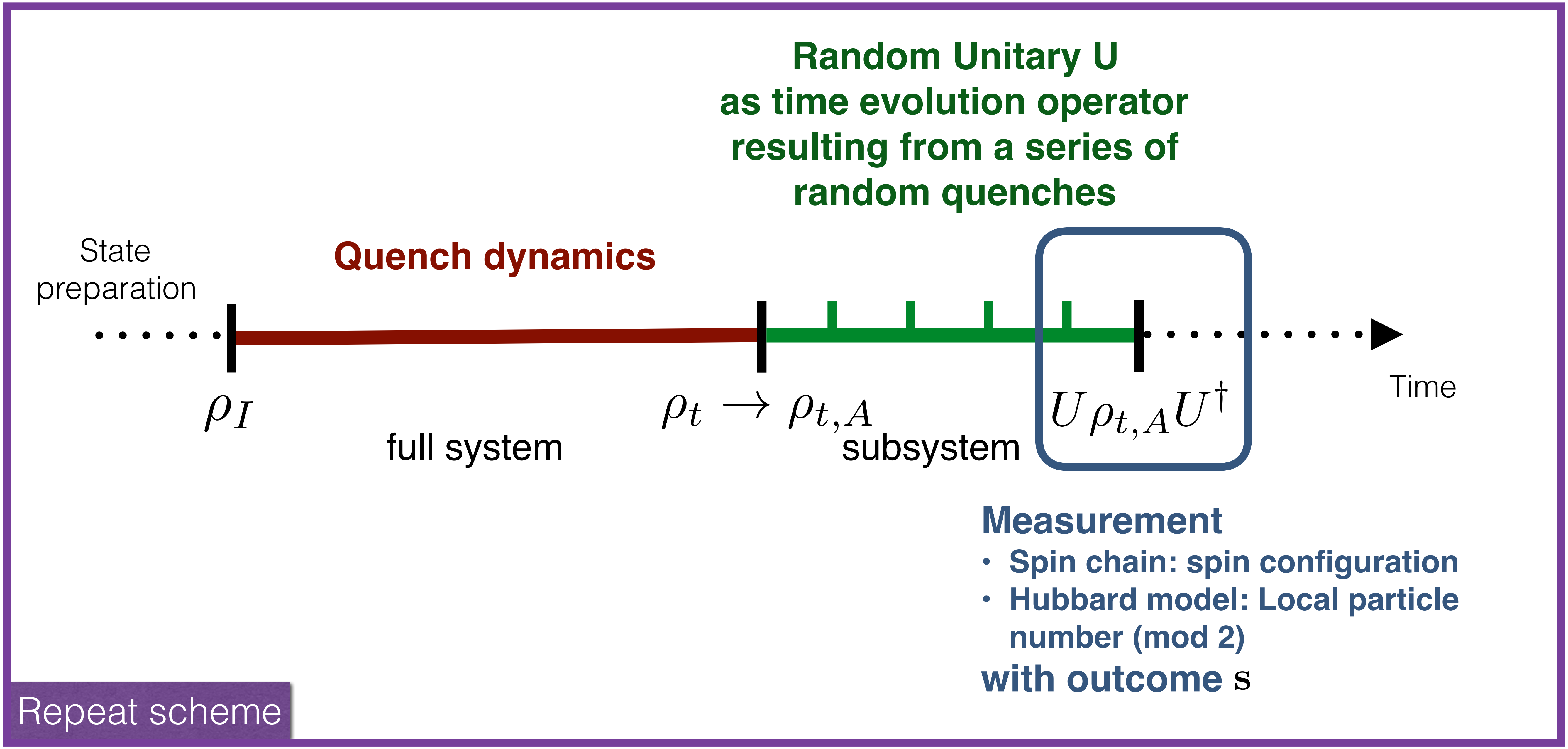


Measurement

- **Spin chain: Spin configuration**

- **Hubbard model: Local particle number (modulo 2)**

Measurement protocol for Hubbard and Spin models



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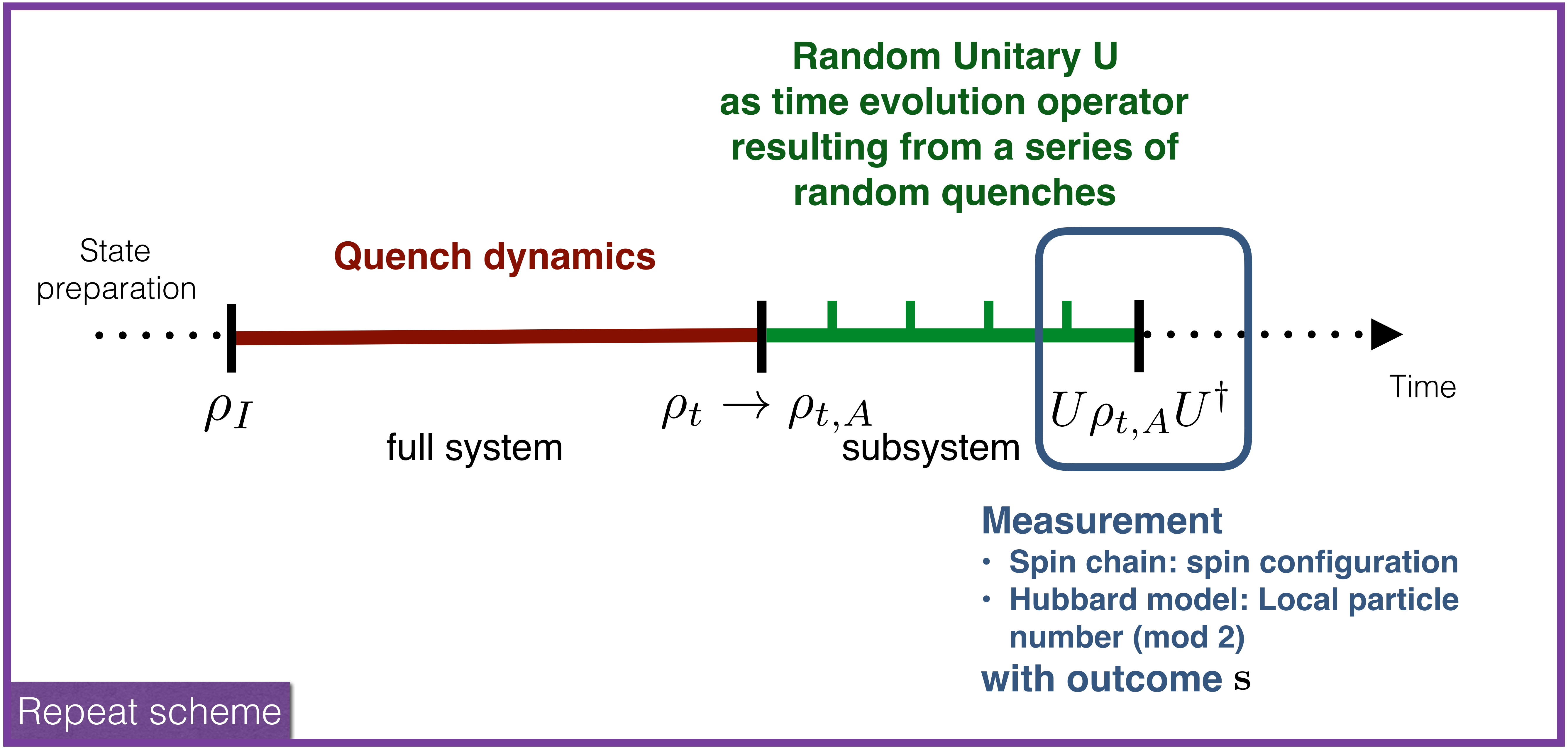

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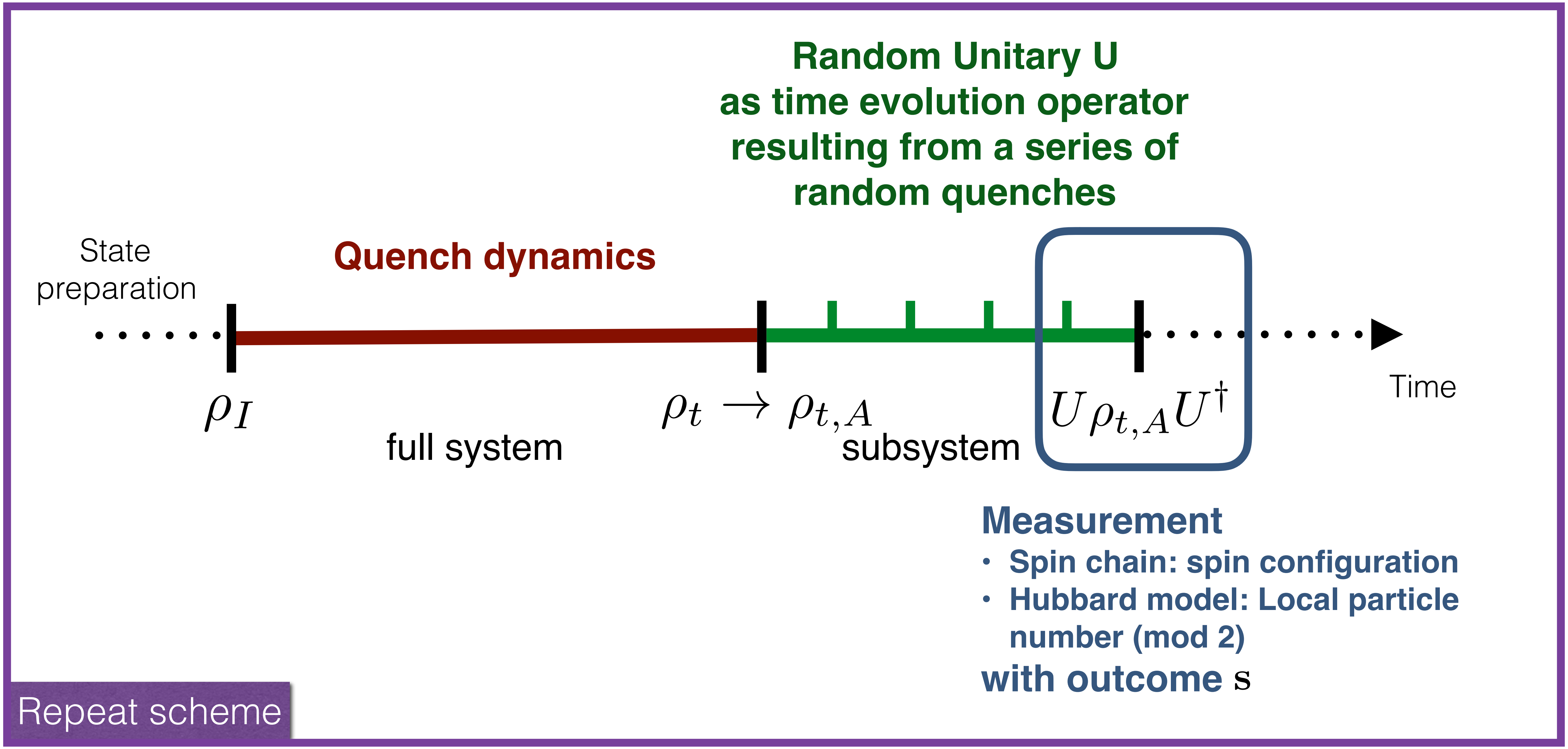

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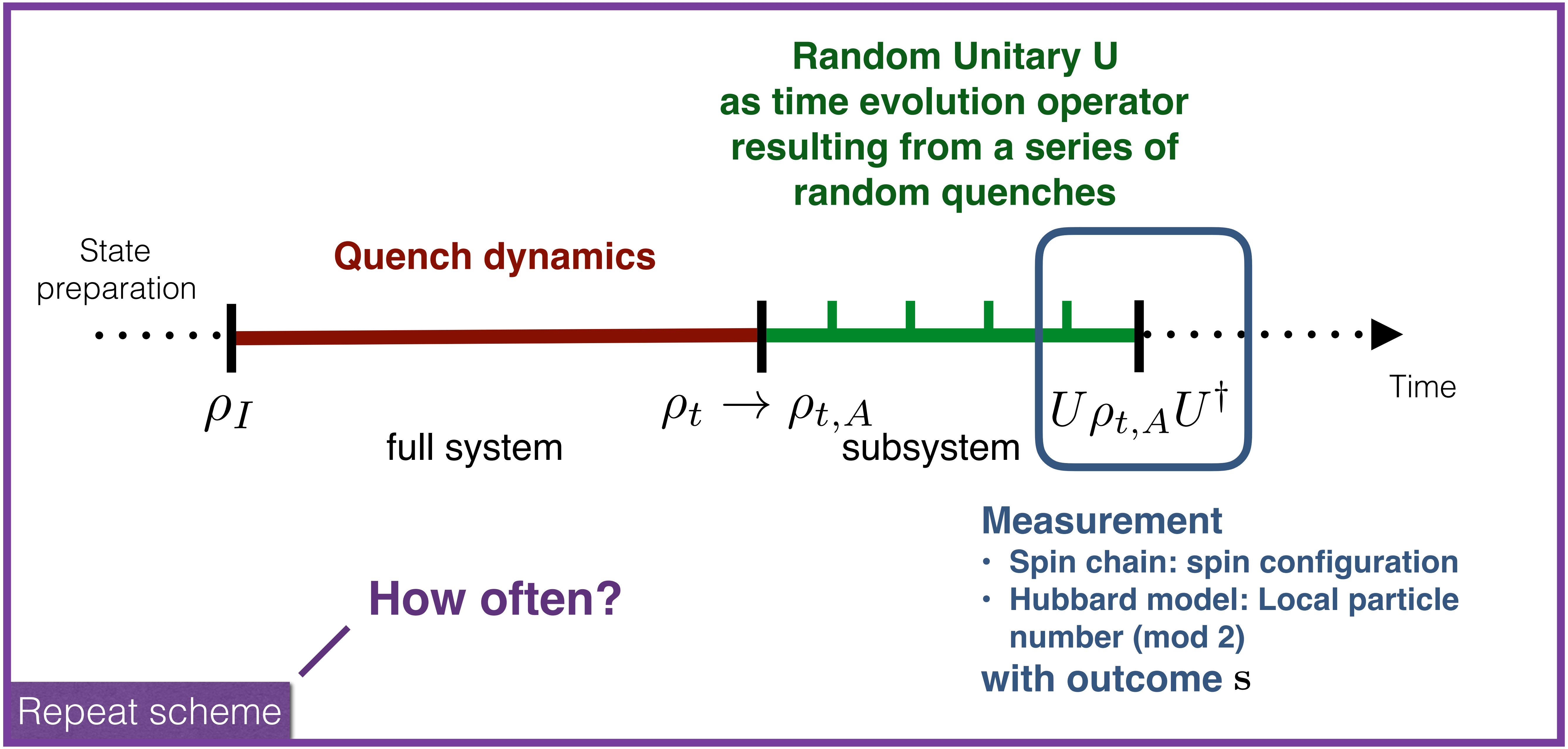

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Scaling of statistical errors

Error for estimated purity

$|(p_2)_e - p_2|$ for a finite number

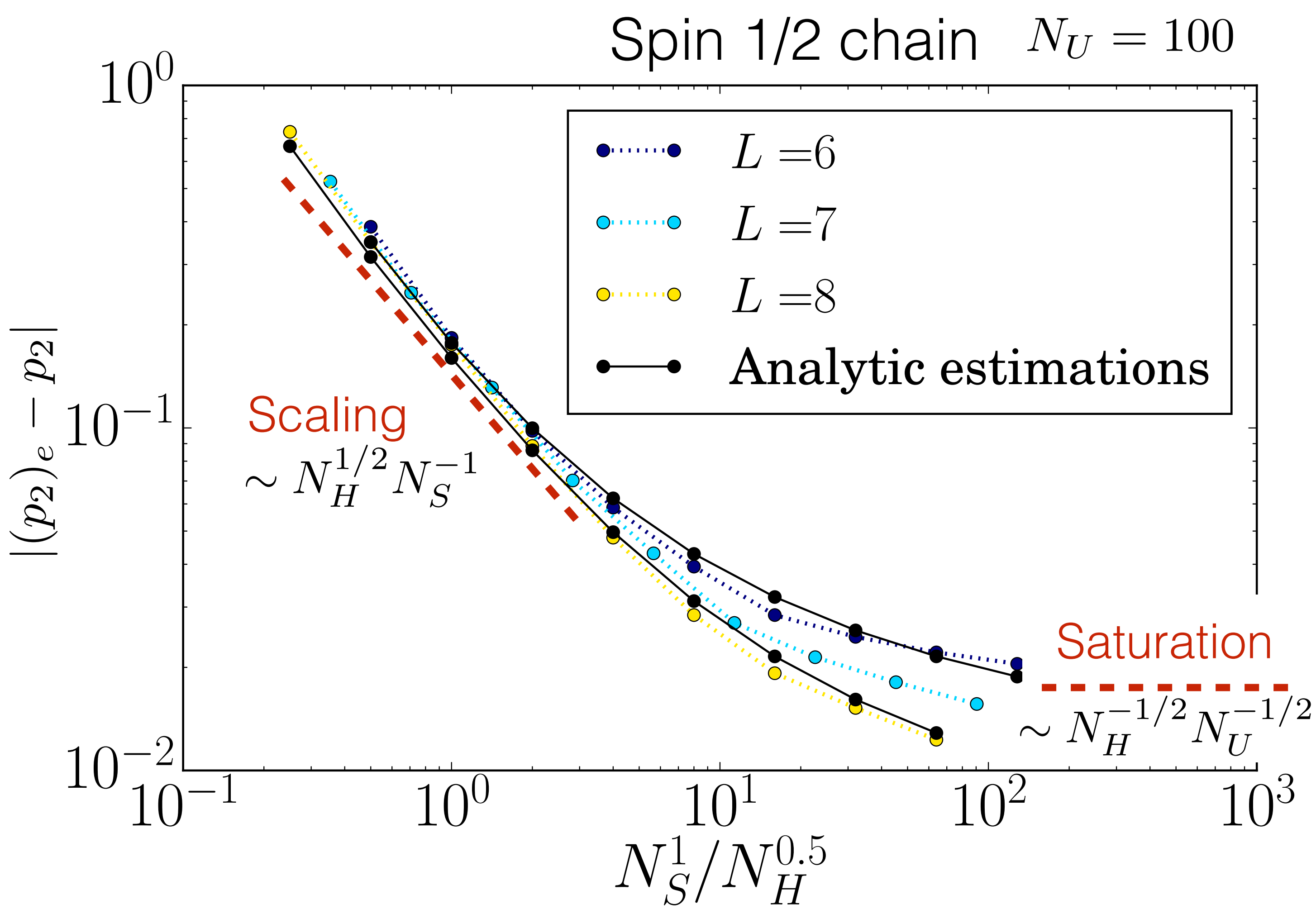
N_S of measurements per unitary
N_U of unitaries

Scaling of statistical errors

Error for estimated purity

$$|(p_2)_e - p_2| \sim \frac{1}{\sqrt{N_H N_U}}\left(1 + \frac{N_H}{N_S}\right)$$

N_S: number of measurements per unitary
N_U: number of unitaries
N_H: Hilbert space dimension of subsystem



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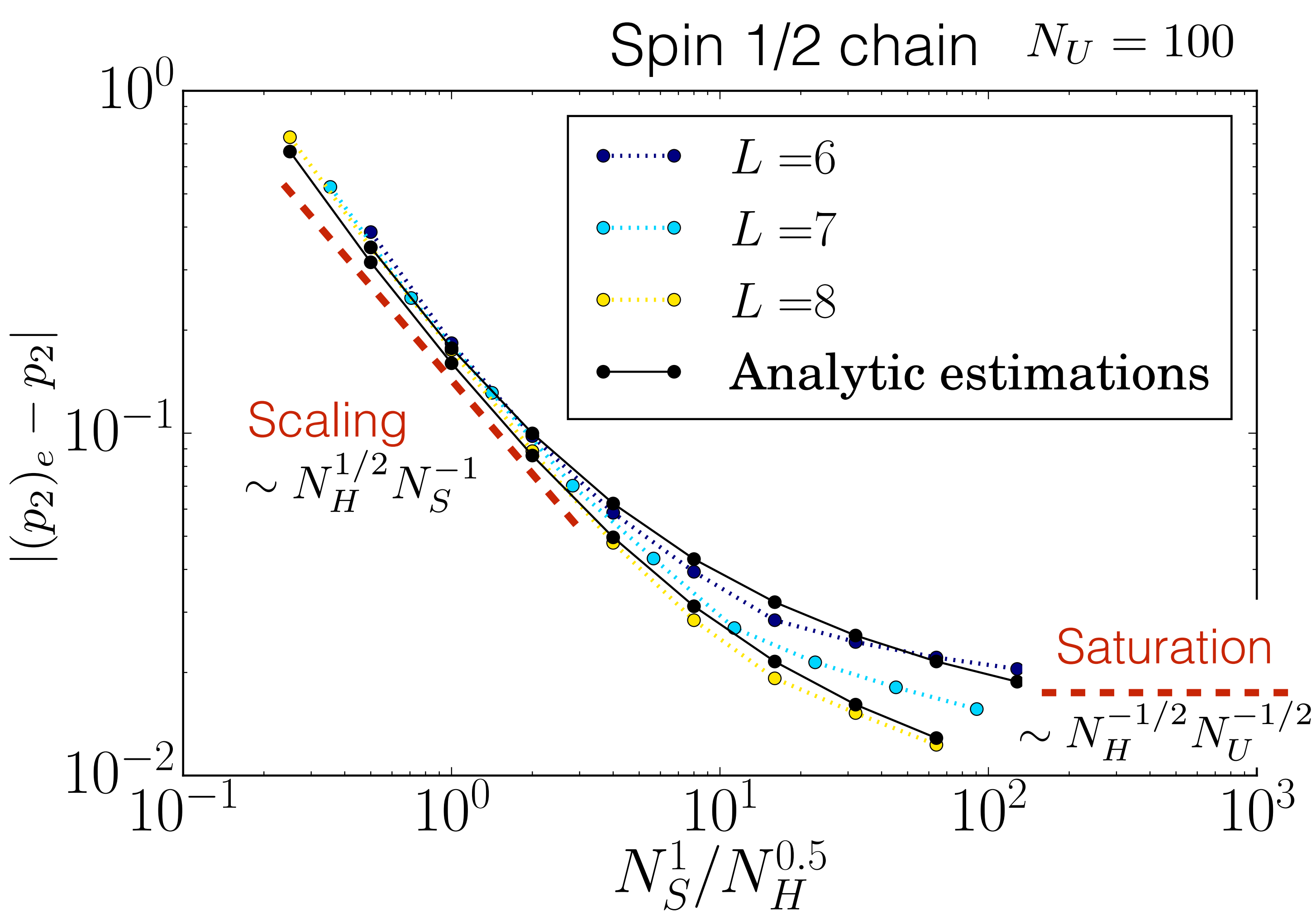
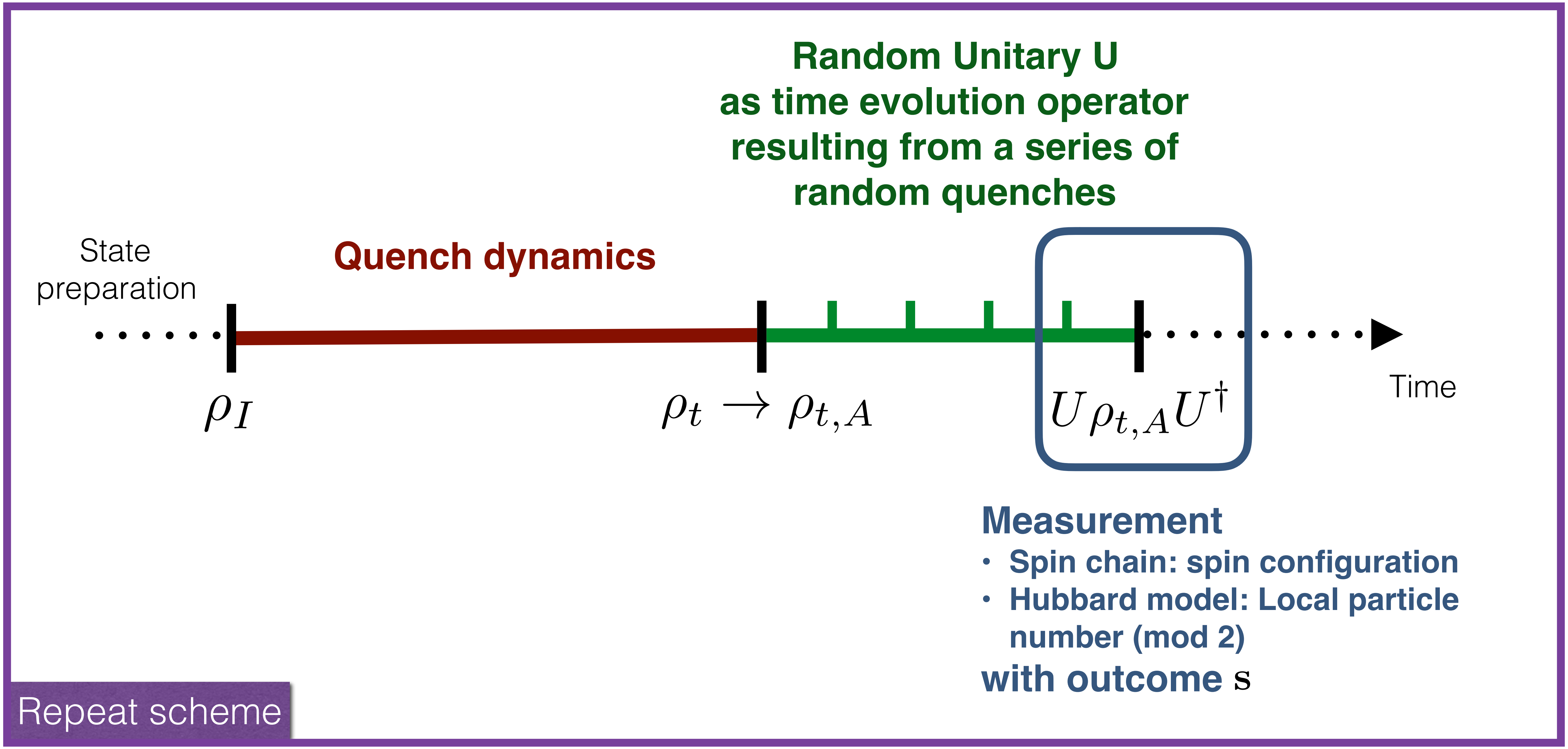


Number of measurements per unitary to determine p_2 up to error $\sim 1/\sqrt{N_U}$

$$N_S \sim N_H^{1/2}$$

Measurement protocol for Hubbard and Spin models



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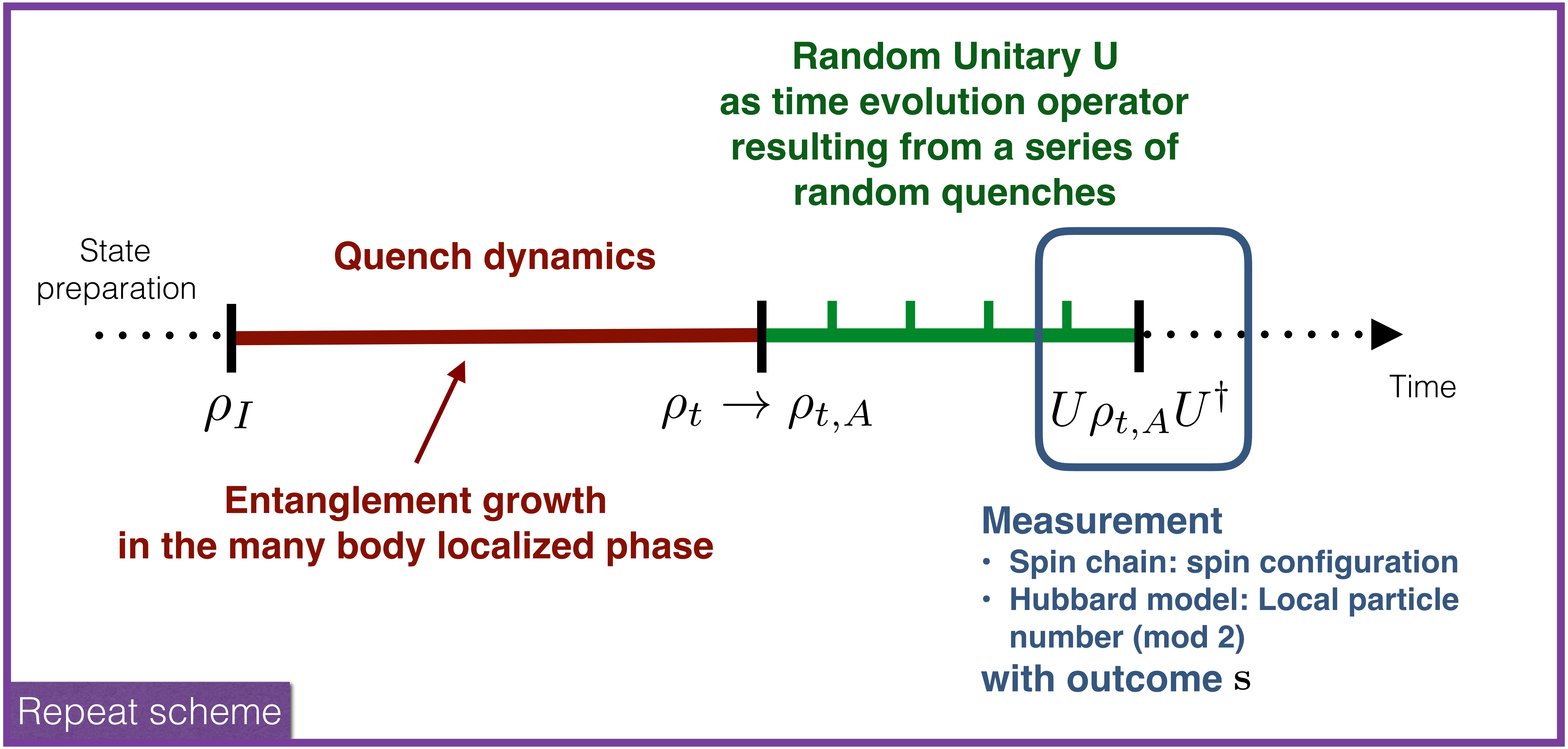

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Many Body Localization in the BH model

Many Body Localization

A quantum phase characterized by the interplay of disorder (localization) and interactions

Slow (logarithmic, but nonzero) growth of entanglement

D.M. Basko, I.L. Aleiner, B.L. Altshuler Annals of Physics 321, 1126 (2006)

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Bose-Hubbard model with *strong, static disorder* potentials

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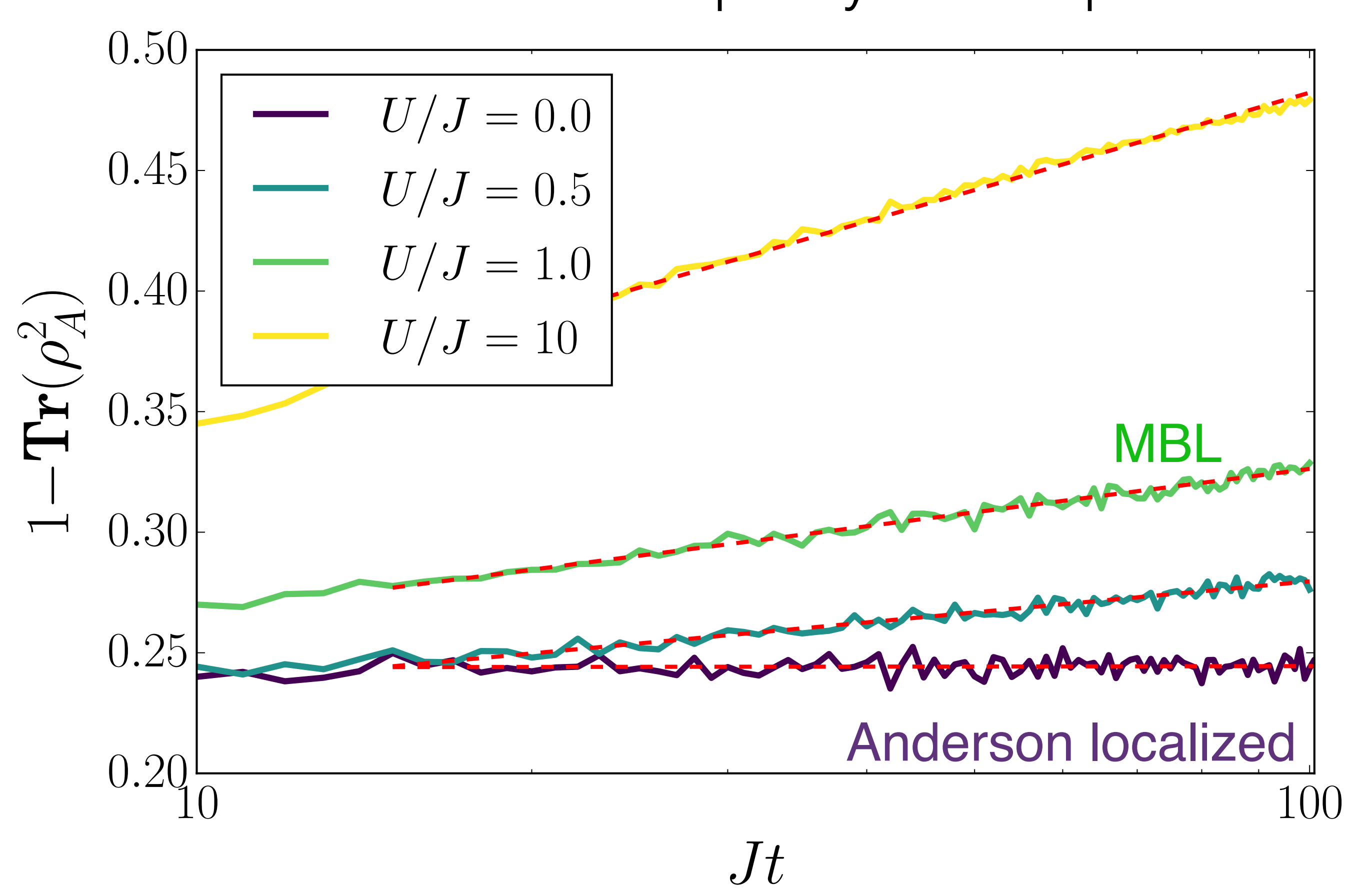
Time evolution of purity at half partition



tMPS calculation, Average over 1000 disorder realizations, L=16, N=8

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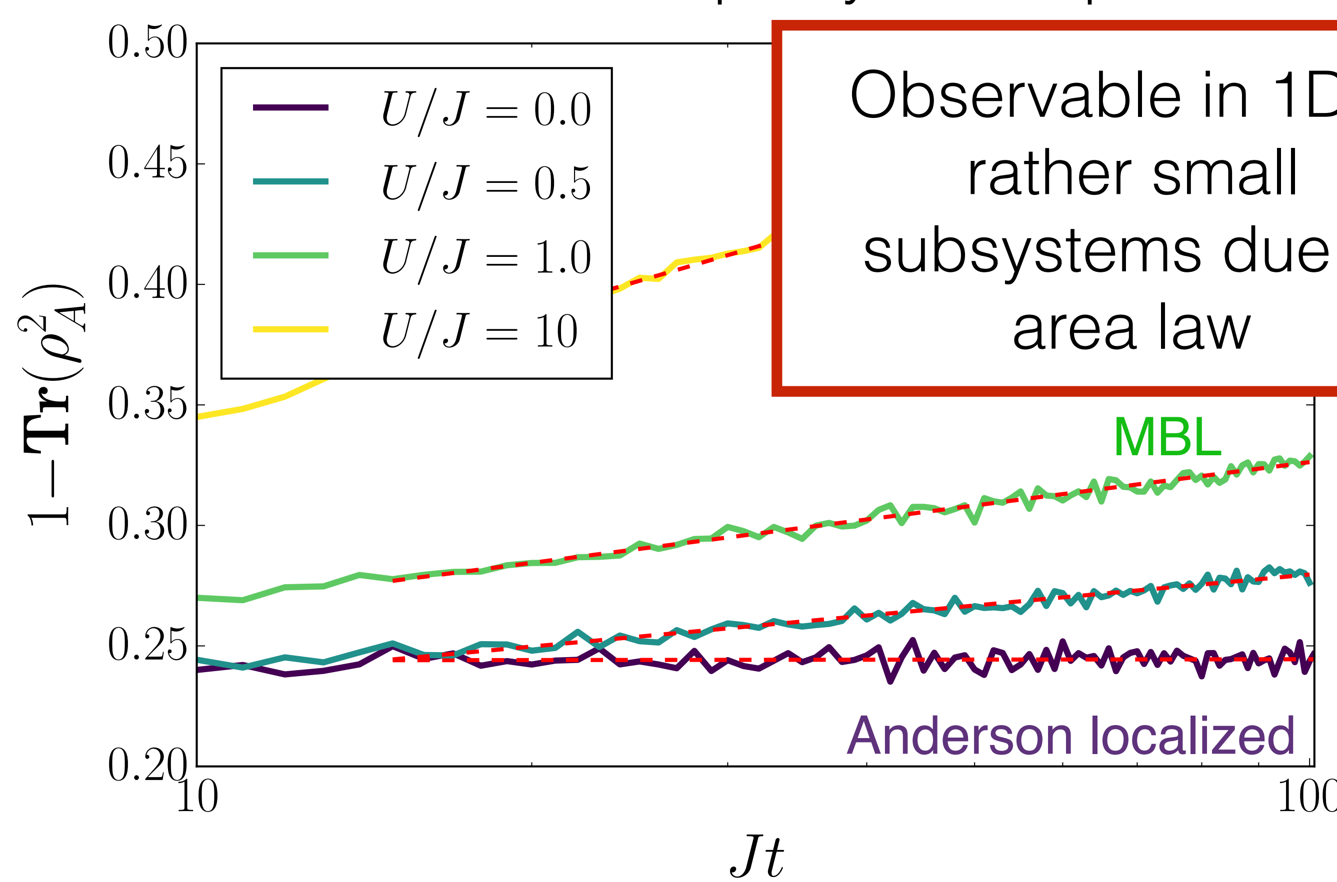


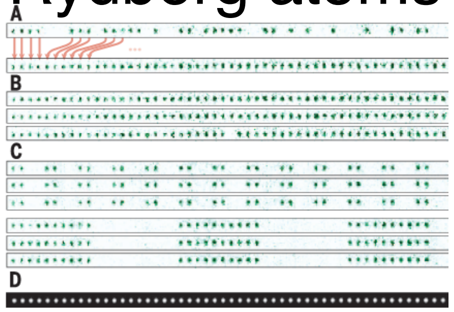
Observable in 1D in rather small subsystems due to area law

tMPS calculation, Average over 1000 disorder realizations, L=16, N=8

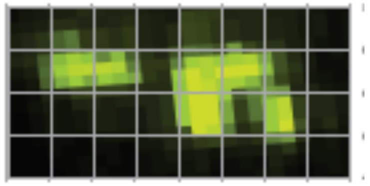
Conclusion

Entanglement measurement on single copies with random measurements

Rydberg atoms



Hubbard models



- Implementation in variety of **physical** systems is possible, e.g. **Spin Chains** (Rydberg atoms) or **Bose/Fermi Hubbard type** systems in arbitrary dimension
- Key ingredient: Random unitary operators
- Key constraint: Statistical errors limit to **moderate Hilbertspace dimensions**
- For known input states, the protocol provides possibility of **certification of random unitary ensembles**

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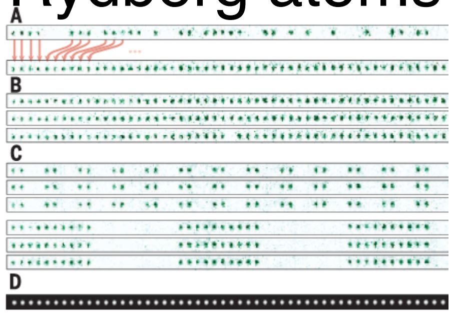

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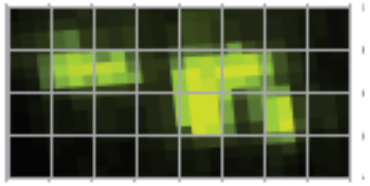

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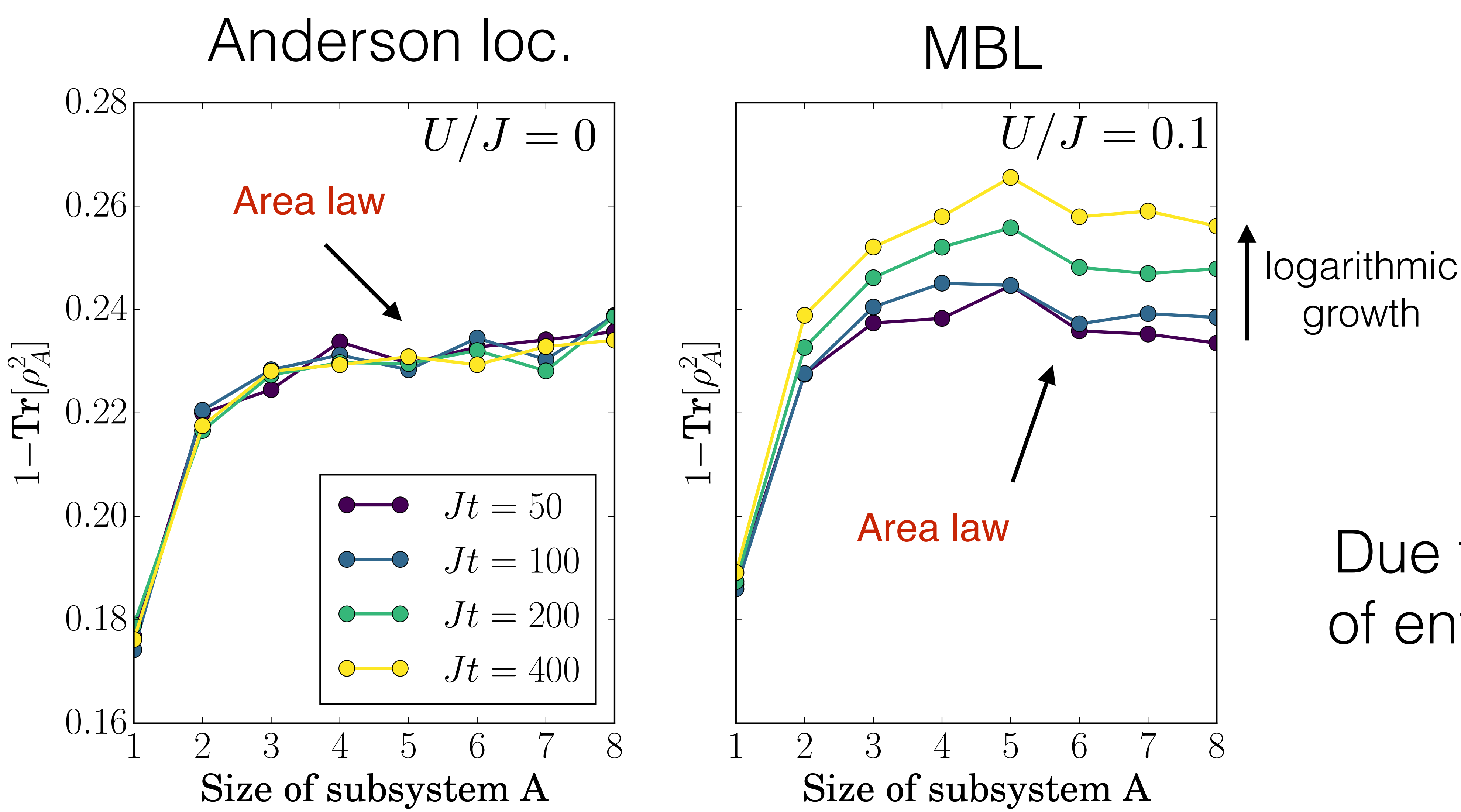
Thank you very much for your attention!

Area law in MBL

Question: Could one observe entanglement growth in **small subsystems**?



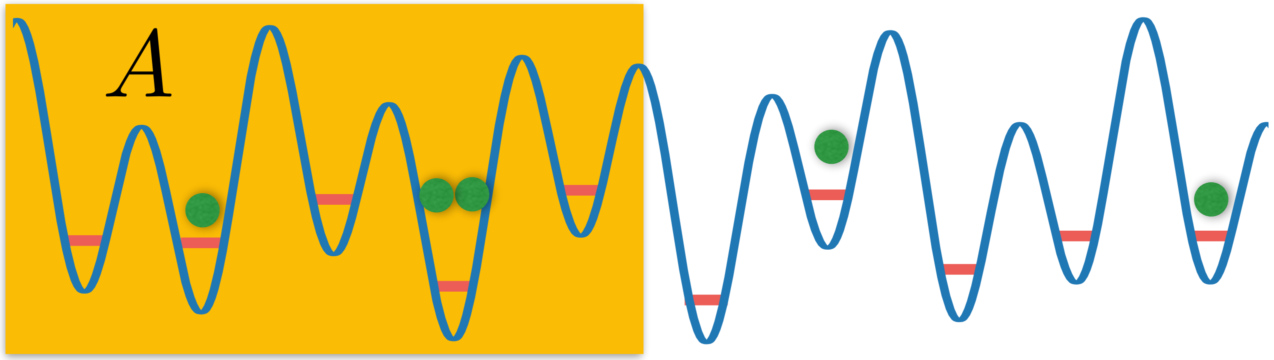
Yes!

Due to area law of entanglement in 1D

tMPS calculation, Average over 1000 disorder realisations, BH model L=16, N=8

Application of the measurement protocol

Single disorder realization in the MBL regime

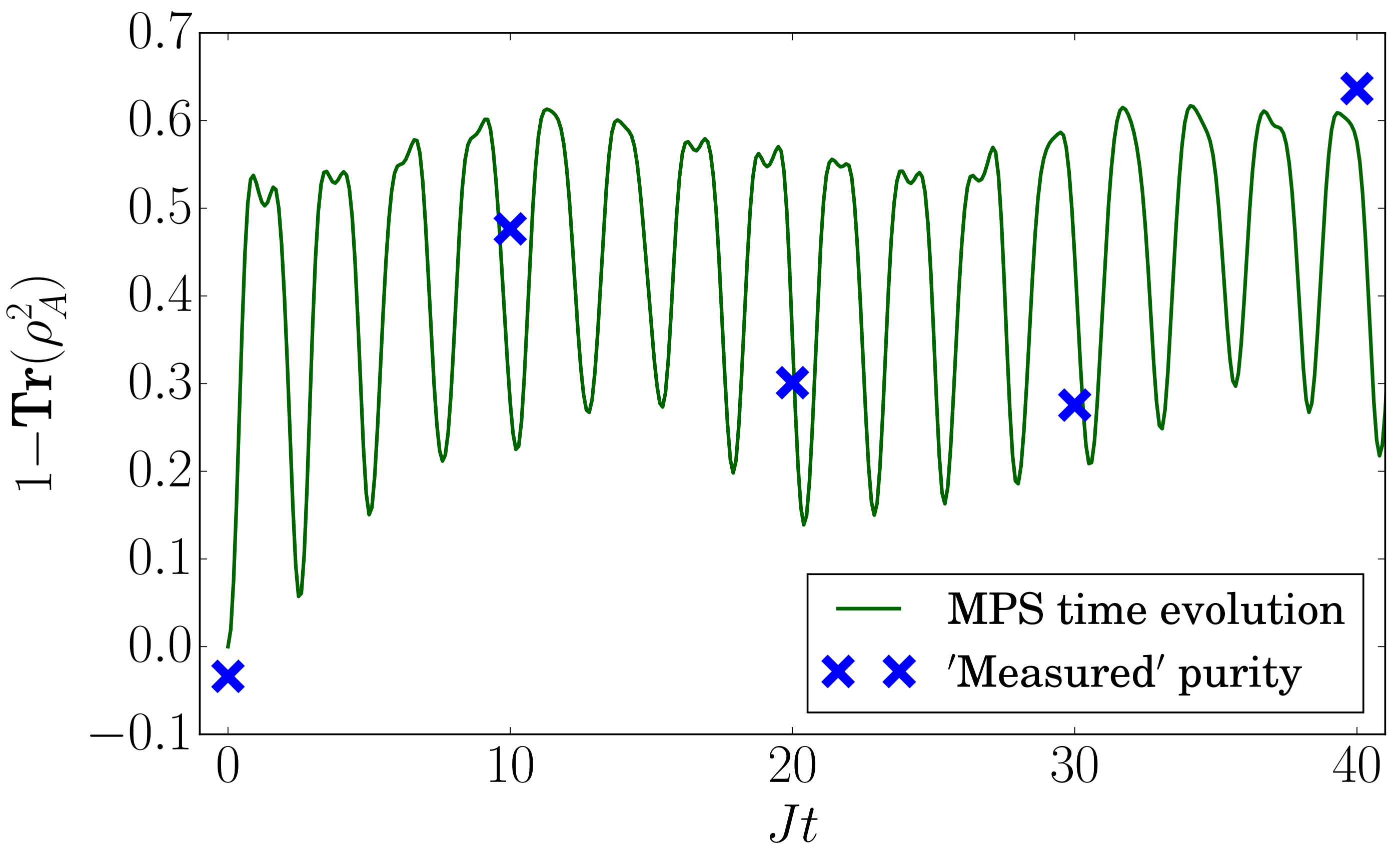


L=10, N=5,

$U/J = 0.1$

$\mu_i \in [-10J, 10J]$

Purity at half partition



Enhanced statistical errors due to mixture of different particle numbers

→ Extraction of purity for each particle sector is possible

→ Improvement of statistics via number resolving measurements!

100 generated Unitaries
100 Measurements per unitary

Random unitaries and t-designs

Unitary t-design:

Subset of random matrices $S \subseteq \mathrm{CUE}(N_H)$

with $\underbrace{\langle U_{ij}U^*_{kl}\ldots U_{mn}U^*_{op}}_{2t\ \text{matrix elements}}\rangle_S = \langle U_{ij}U^*_{kl}\ldots U_{mn}U^*_{op}\rangle_{\mathrm{CUE}(N_H)}$

Testing on a 2-design:

Revert the protocol to test the created unitary ensemble on being a t-design using known input states

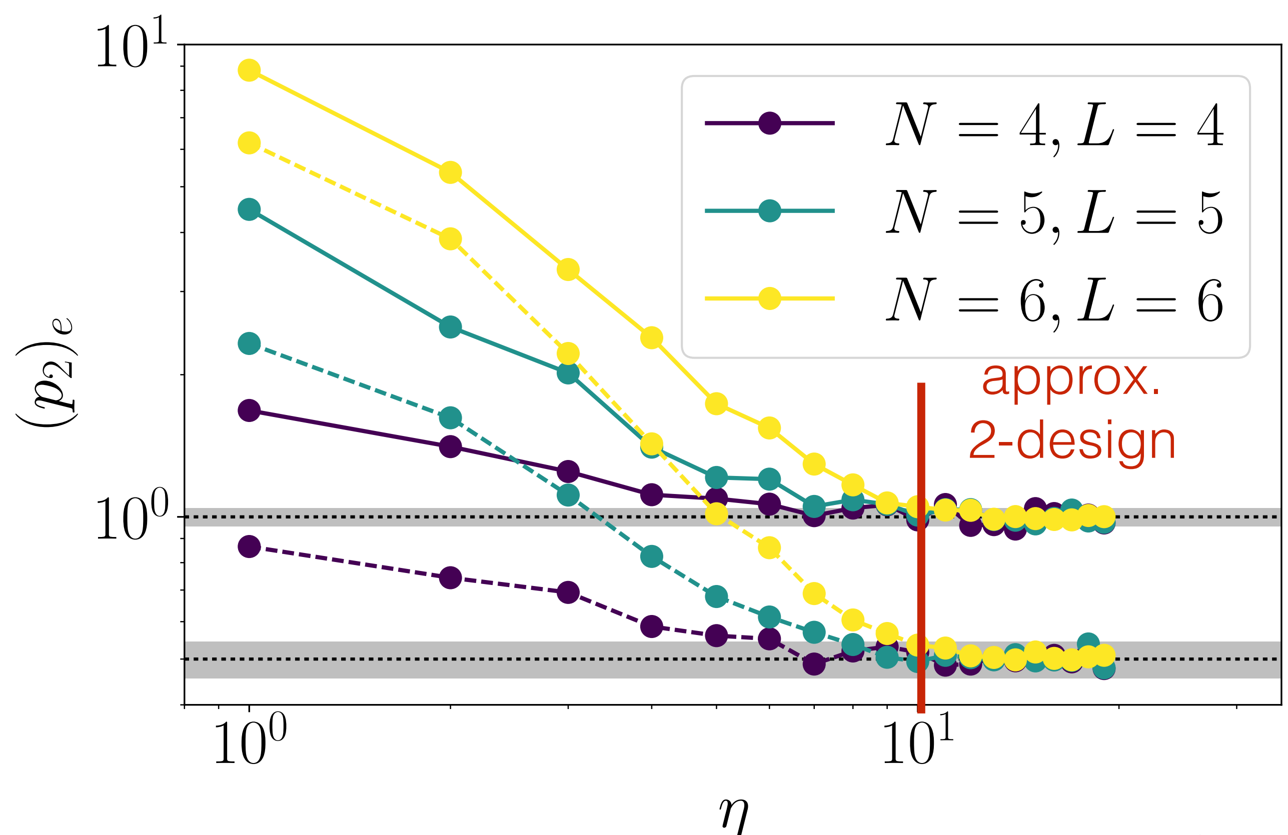
$$\langle \mathrm{Prob}(\mathbf{s})\rangle_S \overset{!}{=} \frac{1}{N_H}$$

holds if unitaries form 1-design

$$\langle \mathrm{Prob}(\mathbf{s})^2\rangle_S \overset{!}{=} \frac{1+\mathrm{Tr}\left[\rho^2\right]}{N_H(N_H+1)}$$

holds if unitaries form 2-design

Bose-Hubbard model



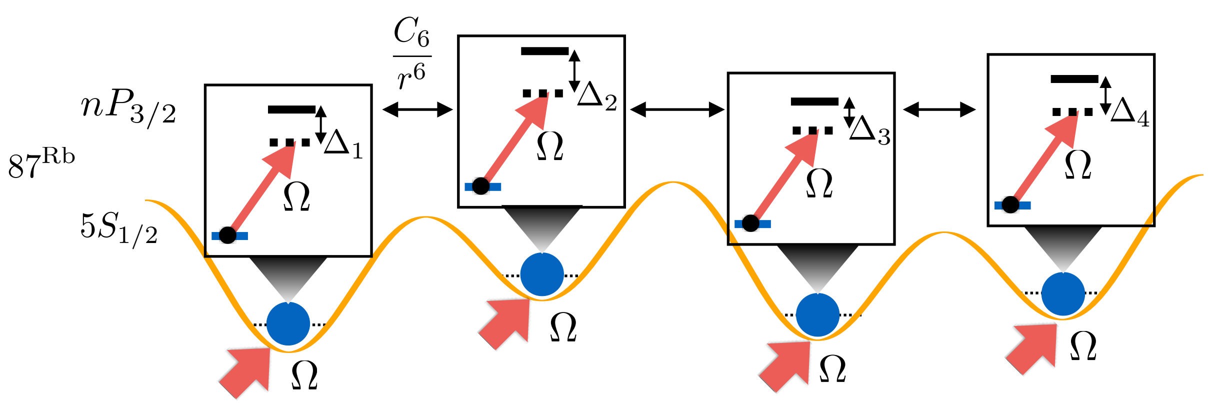
$U = J/N$

$\mu_i \in [-J, J]$

$T = 10/J$

Random unitaries in a Rydberg chain

1D Rydberg chain with random quenches

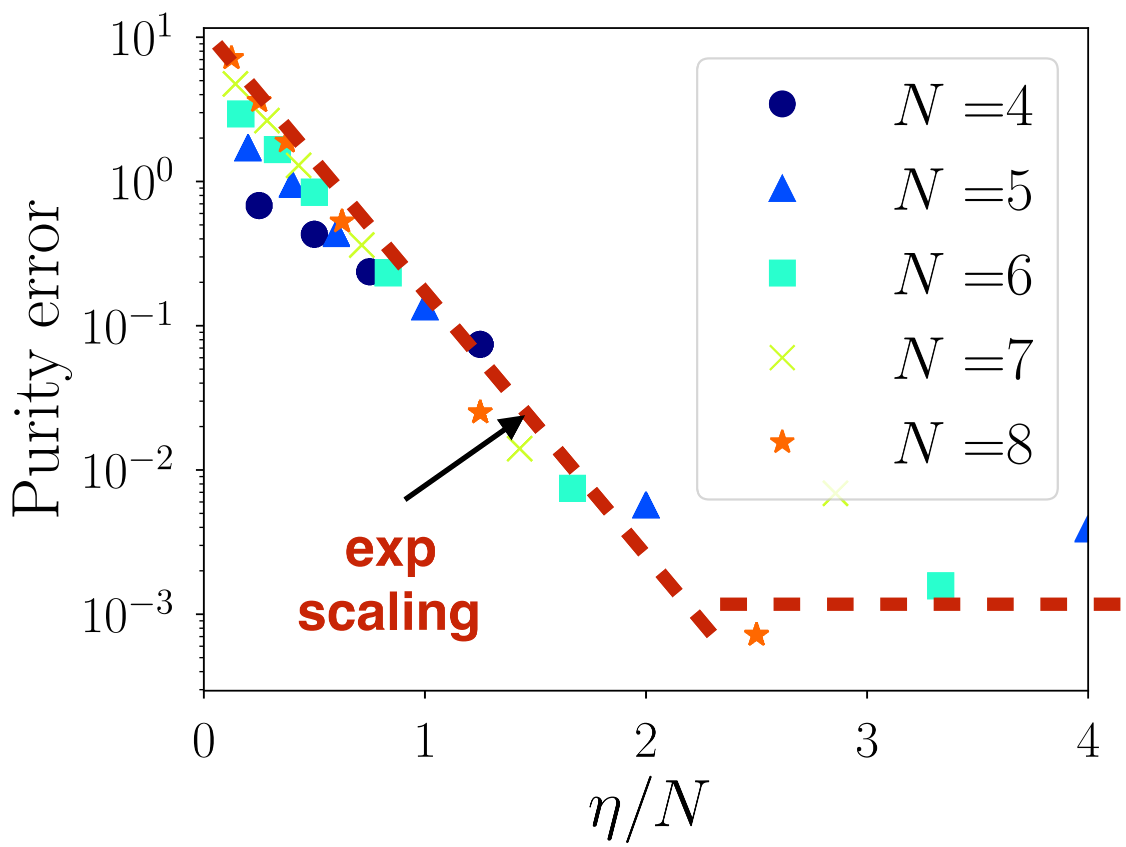


$$H_\alpha = \Omega \sum_i \sigma_i^x + \sum_i \Delta_i(\alpha)\sigma_i^z + \sum_{i<j} \frac{C_6}{|i-j|^6}\sigma_i^z \sigma_j^z$$

$\Delta_i(\alpha)$ form normal distribution with standard deviation Δ

Vary disorder in discrete steps in time $\longrightarrow$ $U = e^{-iH_1 T} \dots e^{-iH_\eta T}$

Apply protocol to known input state:



Saturation within the statistical uncertainty

N number of atoms

$T = 1/\Omega$

$\Omega = C_6$
$\Delta = C_6$

Number of random quenches needed
$\eta = 2N$

Random unitary matrices

Random unitaries from the Circular Unitary Ensemble (CUE)

Unitaries distributed according the Haar measure on the unitary group

$$U \in \mathrm{CUE}(N_H) \longleftrightarrow \Re(U_{ij}), \Im(U_{ij}) \sim \mathcal{N}\left(0, \frac{1}{N_H}\right)$$

N_H: Hilbert space dimension of subsystem

up to unitary constraints

Moments of random matrices

$$\langle U_{ij} U^*_{kl} \rangle = \frac{\delta_{kl}\delta_{ln}}{N_H}$$

$$\langle U_{ij} U^*_{kl} U_{mn} U^*_{op} \rangle \approx \langle U_{ij} U^*_{kl} \rangle \langle U_{mn} U^*_{op} \rangle + \langle U_{ij} U^*_{op} \rangle \langle U_{mn} U^*_{kl} \rangle$$

Wicks theorem

$$= \frac{\delta_{ik}\delta_{jl}\delta_{mo}\delta_{np} + \delta_{io}\delta_{jp}\delta_{mk}\delta_{nl}}{N_H^2}$$

van Enk, Beenaker (PRL 2012)

Justification of the protocol

Application to the protocol

Projector describing measurement

$$\rho_A \longrightarrow \rho_A^f = U\rho_A U^\dagger \xrightarrow[\text{with outcome } \mathbf{s}]{\text{Measurement}} \mathrm{Prob}\,(\mathbf{s})_U = \mathrm{Tr}\left[U\rho_A U^\dagger P_{\mathbf{s}}\right]$$

$$\mathrm{Prob}\,(\mathbf{s})_U^2 = \mathrm{Tr}\left[\rho_A^f P_{\mathbf{s}}\right]\mathrm{Tr}\left[\rho_A^f P_{\mathbf{s}}\right] = \mathrm{Tr}_{1\otimes 2}\left[\rho_A^f P_{\mathbf{s}} \otimes \rho_A^f P_{\mathbf{s}}\right] = \mathrm{Tr}_{1\otimes 2}\left[\, U\rho_A U^\dagger P_{\mathbf{s}} \;\otimes\; U\rho_A U^\dagger P_{\mathbf{s}} \,\right]$$

2 virtual copies

2 virtual copies

Average over the unitaries

$$\langle \mathrm{Prob}\,(\mathbf{s})_U^2 \rangle = \langle \mathrm{Tr}_{1\otimes 2}\left[\, U\rho_A U^\dagger P_{\mathbf{s}} \;\otimes\; U\rho_A U^\dagger P_{\mathbf{s}} \,\right]\rangle \approx \frac{1}{N_H^2}\left(\mathrm{Tr}\left[P_{\mathbf{s}}\right]^2 + \mathrm{Tr}\left[\rho_A^2\right]\mathrm{Tr}\left[P_{\mathbf{s}}\right]\right)$$

2 virtual copies

2nd moment of CUE

$$\langle U_{ij}U_{kl}^* U_{mn} U_{op}^* \rangle \approx \frac{\delta_{ik}\delta_{jl}\delta_{mo}\delta_{np} + \delta_{io}\delta_{jp}\delta_{mk}\delta_{nl}}{N_H^2}$$

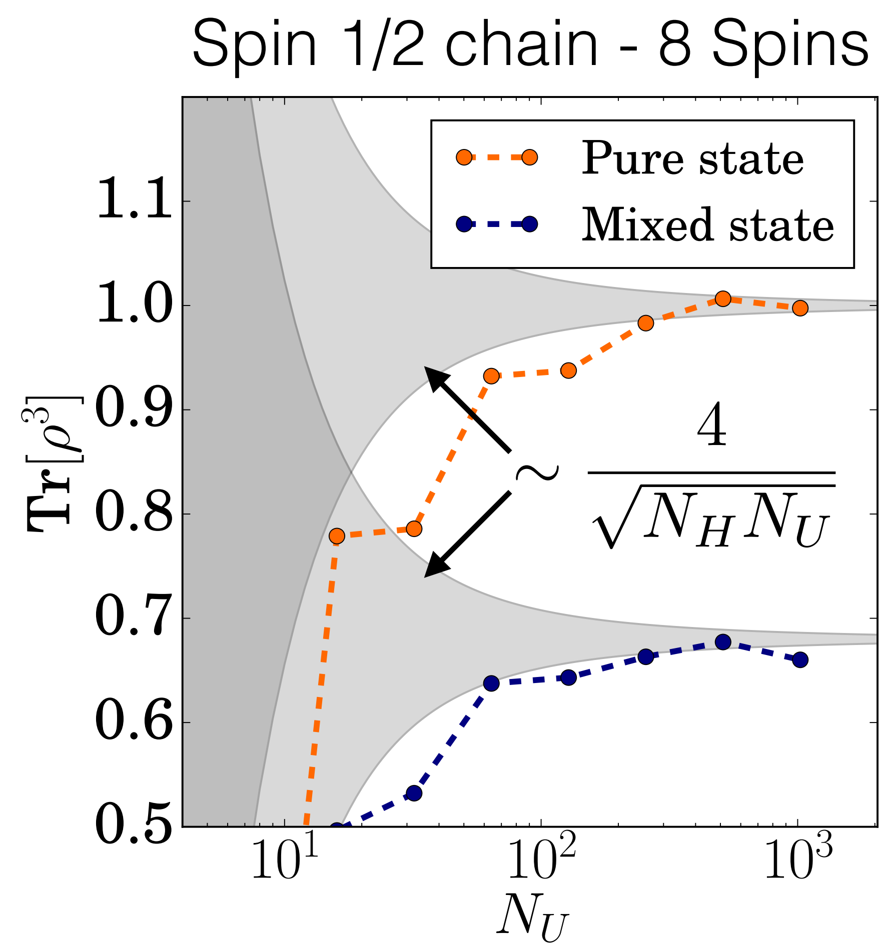
Higher order functionals

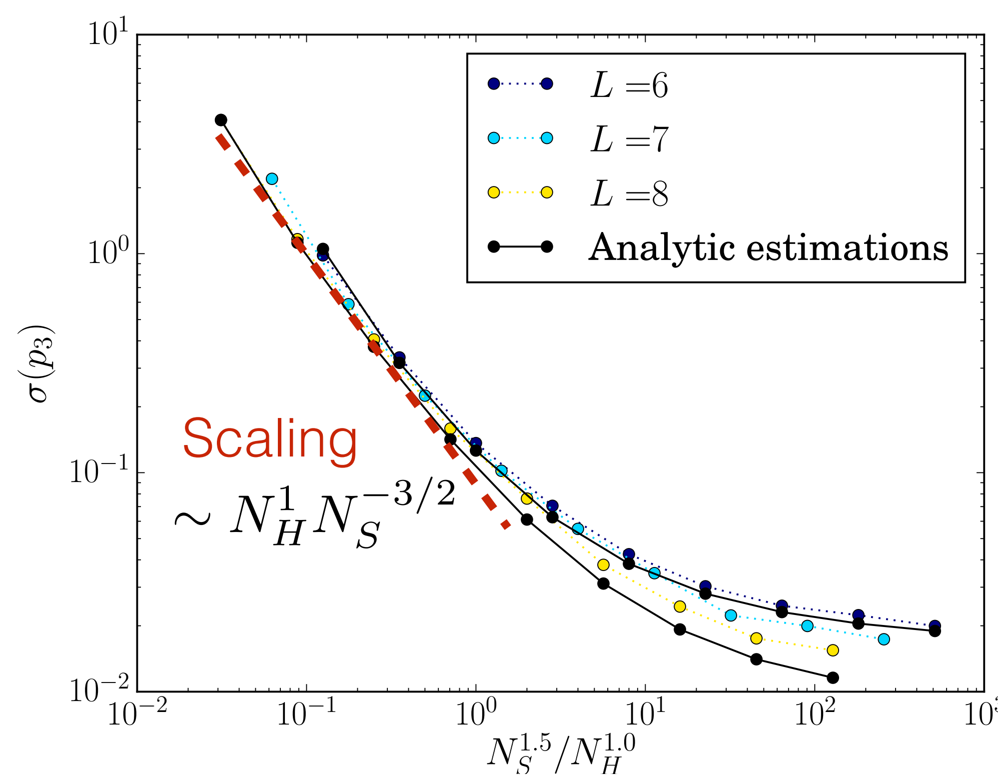
Error for estimated higher polynomials

$$p_\beta \equiv \mathrm{Tr}\left[\rho^\beta\right] \sim \frac{1}{N_H^\beta}\langle \mathrm{Prob}(\mathbf{s})_U^\beta \rangle$$

$$|(p_\beta)_e - p_\beta| \sim \frac{1}{\sqrt{N_H N_U}}\left(\mathcal{O}\left(\beta e^\beta\right) + \left(\frac{N_H}{N_S}\right)^{\frac{\beta}{2}}\right)$$

- Similar polynomial scaling for fixed β
- Exponential rising number of unitaries with increasing β needed
- Third, fourth polynomial might be feasible!





What observables can be measured?

Projector describing measurement

Purity from measurements of an arbitrary observable

$$\rho_A \longrightarrow \rho_A^f = U\rho_A U^\dagger \quad \xrightarrow[\text{with outcome } \mathbf{s}]{\text{Measurement}} \quad \mathrm{Prob}\,(\mathbf{s})_U = \mathrm{Tr}\left[U\rho_A U^\dagger P_{\mathbf{s}}\right]$$

Number of states projected to outcome

Average over unitaries

$$\langle \mathrm{Prob}\,(\mathbf{s})_U^2 \rangle = \langle \mathrm{Tr}\left[U\rho_A U^\dagger P_{\mathbf{s}}\right] \mathrm{Tr}\left[U\rho_A U^\dagger P_{\mathbf{s}}\right] \rangle \approx \frac{1}{N_H^2}\left(\mathrm{Tr}\left[P_{\mathbf{s}}\right]^2 + \mathrm{Tr}\left[\rho_A^2\right]\mathrm{Tr}\left[P_{\mathbf{s}}\right]\right)$$

Error for estimated purity (averaged over all outcomes)

$$|(p_2)_e - p_2| \sim \frac{1}{\sqrt{N_O N_U}}\left(1 + \frac{N_H}{N_S}\right)$$

N_S : number of measurements per unitary
N_U : number of unitaries
N_H : Hilbert space dimension of subsystem
N_O : Number of possible measurement outcomes

Number of measurements per unitary to determine p_2 up to error $\sim 1/\sqrt{N_U}$

$$\boxed{N_S \sim N_H/\sqrt{N_O}}$$

Scaling of statistical errors

Why should we care?

	Pure state $\rho_U^f = U\rho U^\dagger$	Fully mixed state $\rho^f = \mathbb{1}/N_H$
$\langle \mathrm{Prob}(\mathbf{s})_U \rangle$	$\frac{1}{N_H}$	$\frac{1}{N_H}$
$\langle \mathrm{Prob}(\mathbf{s})_U^2 \rangle$	$\frac{1}{N_H^2}(1 + \mathrm{Tr}\left[\rho^2\right]) = \frac{2}{N_H^2}$	$\frac{1}{N_H^2}$

$\longrightarrow$ Probabilities are very small due to randomization!

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Two sources of statistical errors:

1. For a given random unitary U, obtain probabilities $\mathrm{Prob}(\mathbf{s})_U$

   $\longrightarrow$ Sampling from the random quantum state $\rho_U^f = U\rho U^\dagger$

   $\longrightarrow$ At the heart of current approaches to quantum advantage

   $\longrightarrow$ How many measurements are needed?

S. Boixo, et al., arXiv:1608.00263
A. P. Lund, et al. , arXiv:1702.03061

2. Average over many random unitaries

   $\longrightarrow$ How many unitaries are needed?

Scaling of statistical errors

Error due to a finite number N_S of measurements

For a fixed random unitary U, one performs a (finite) number N_S of measurements to estimate the outcome probabilities with an error

$$|\,(\mathrm{Prob}(\mathbf{s}))_e - \mathrm{Prob}(\mathbf{s})| \sim \frac{\sigma'}{\sqrt{N_S}} \approx \frac{1}{\sqrt{N_S N_H}}$$

Central limit theorem

N_H: Hilbert space dimension of subsystem

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Central limit theorem

N_H: Hilbert space dimension of subsystem

Error due to a finite number N_U of unitaries

Average over N_U unitaries, to determine an estimate of the second moment with error

$$|\langle \mathrm{Prob}(\mathbf{s})^2 \rangle_e - \langle \mathrm{Prob}(\mathbf{s})^2 \rangle| \sim \frac{\sigma'}{\sqrt{N_U}} \approx \frac{1}{\sqrt{N_U}} \left(C(N_H, \rho) + \frac{1}{N_H N_S} \right)$$

Central limit theorem

where $C(N_H, \rho) \sim \dfrac{1}{N_H^2}$ is **maximal** for pure states

Scaling of statistical errors

Error for estimated purity

$$|(p_2)_e - p_2| \sim \frac{1}{\sqrt{N_H N_U}} \left(1 + \frac{N_H}{N_S}\right)$$

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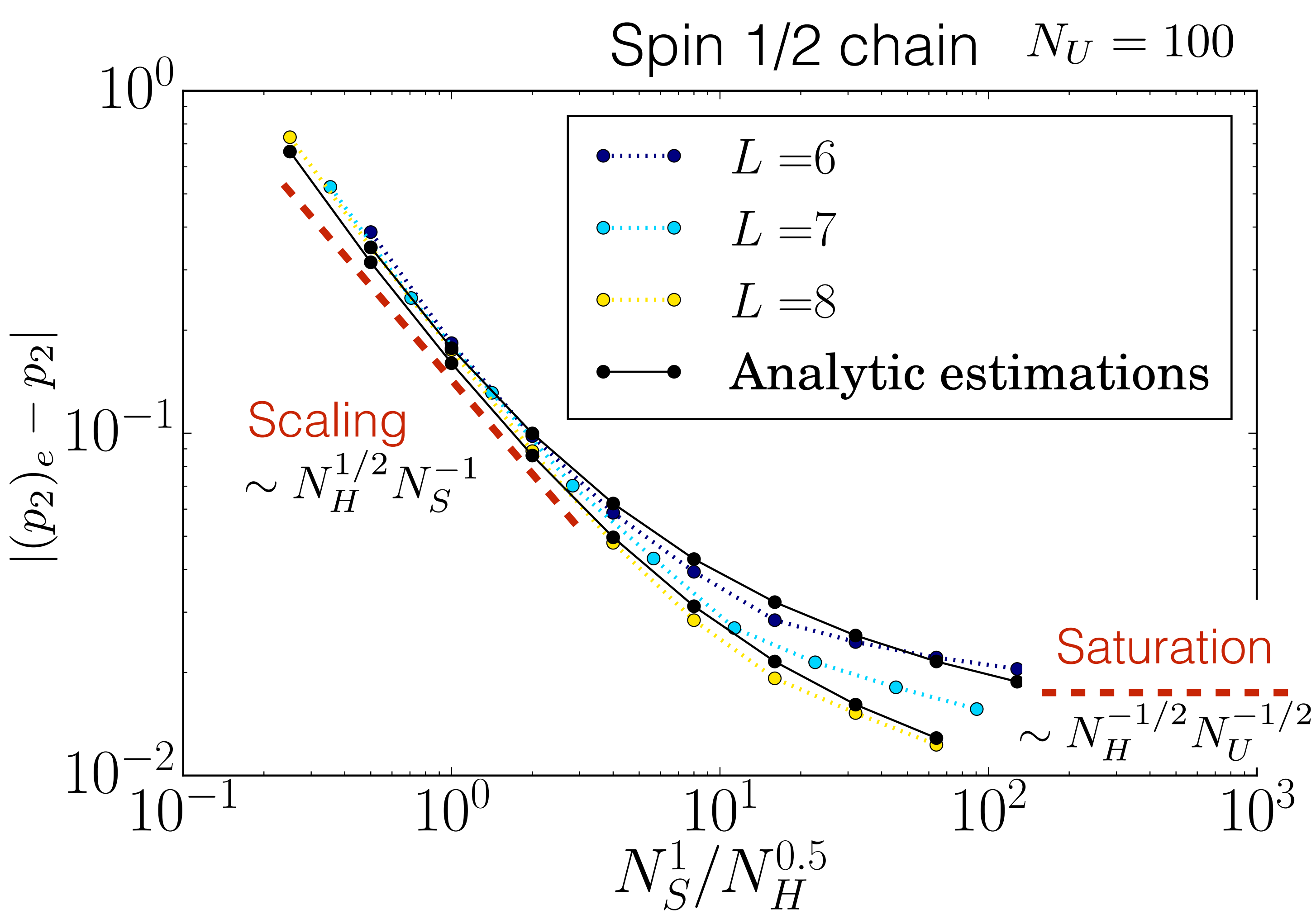


Number of measurements per unitary to determine p_2 up to error $\sim 1/\sqrt{N_U}$

$$N_S \sim N_H^{1/2}$$

Decoherence

The Liouvillian describing the evolution is made of two terms:

$$\mathcal{L} = \mathcal{L}_U + \mathcal{L}_D$$

Unitary evolution **Dissipation**

In first order in $\mathcal{L}_D/\mathcal{L}_U$ the final density matrix is given by

$$\boxed{\rho_A^f = e^{\mathcal{L}_D t/2}\,\mathcal{U}\,e^{\mathcal{L}_D t/2}\rho_A}$$

$$\mathcal{U} = e_+^{\int \mathcal{L}_U(t')dt'}$$

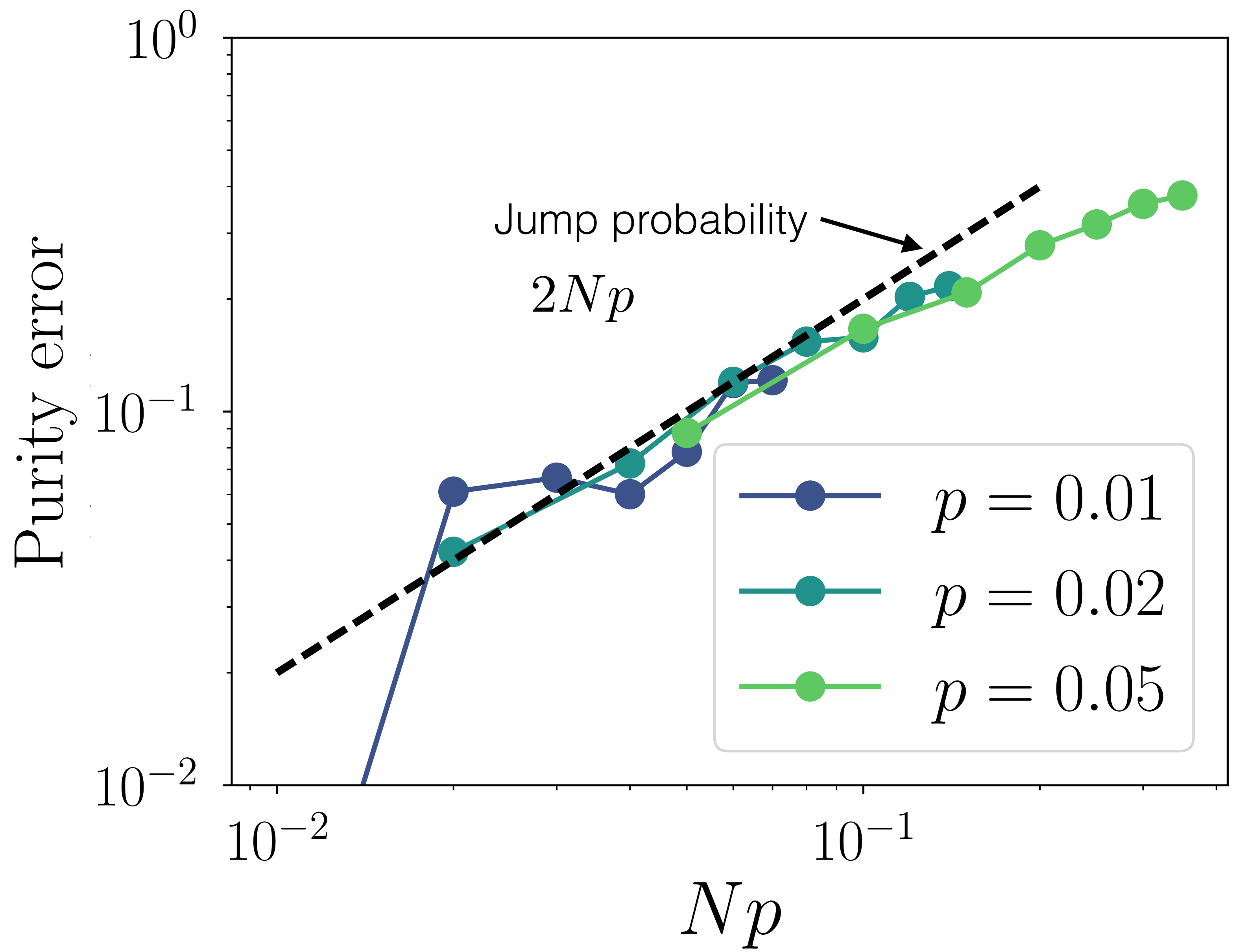
with

Unitary evolution	**Dissipation (Spins)**	
$\mathcal{U}\rho = U\rho U^\dagger$	$e^{\mathcal{L}_D t/2} = \otimes_i e^{\mathcal{L}_{D,I} t/2}$	
	$e^{\mathcal{L}_{D,I} t/2}\rho = (1-p)\rho + p\sigma_i^z\rho\sigma_i^z$	**Dephasing channel**
	$e^{\mathcal{L}_{D,I} t/2}\rho = (1-3p)\rho + p\sum_{\eta=x,y,z}\sigma_i^\eta\rho\sigma_i^\eta$	**Depolarizing channel**

Decoherence

Dephasing channel $$e^{\mathcal{L}_{D,I}t/2}\rho = (1-p)\rho + p\sigma_i^z\rho\sigma_i^z$$

$$\rho_A = \frac{1}{2}(|\uparrow..\uparrow\rangle + |\downarrow..\downarrow\rangle)(\langle\uparrow..\uparrow| + \langle\downarrow..\downarrow|$$



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