One-body localization

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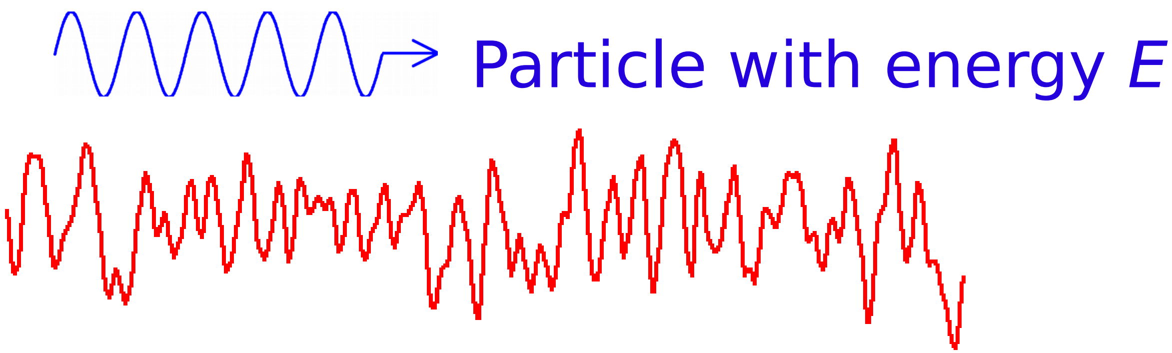
Outline

- Lecture 1
  - What is Anderson localization?
  - 1d systems
  - Scaling theory of Anderson localization
- Lecture 2
  - How to observe Anderson localization?
  - Cold atoms and disorder
  - Critical analysis of experimental results
- Lecture 3
  - Alternative characterizations of Anderson localization
  - Towards many-body localization

Classical dynamics in a disordered potential

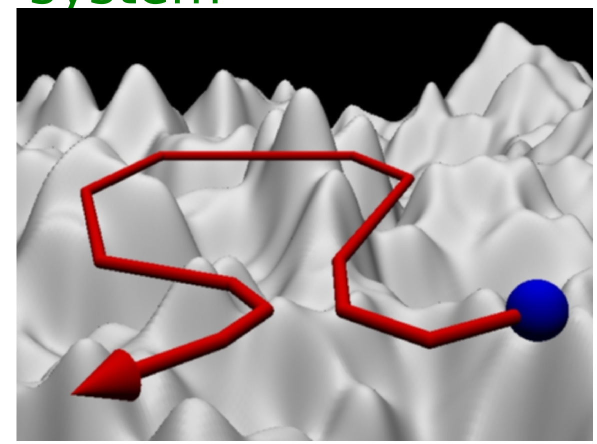
- Particle in a disordered (random) potential:

One-dimensional system



Disordered potential $V(z)$ (typical value V_0)

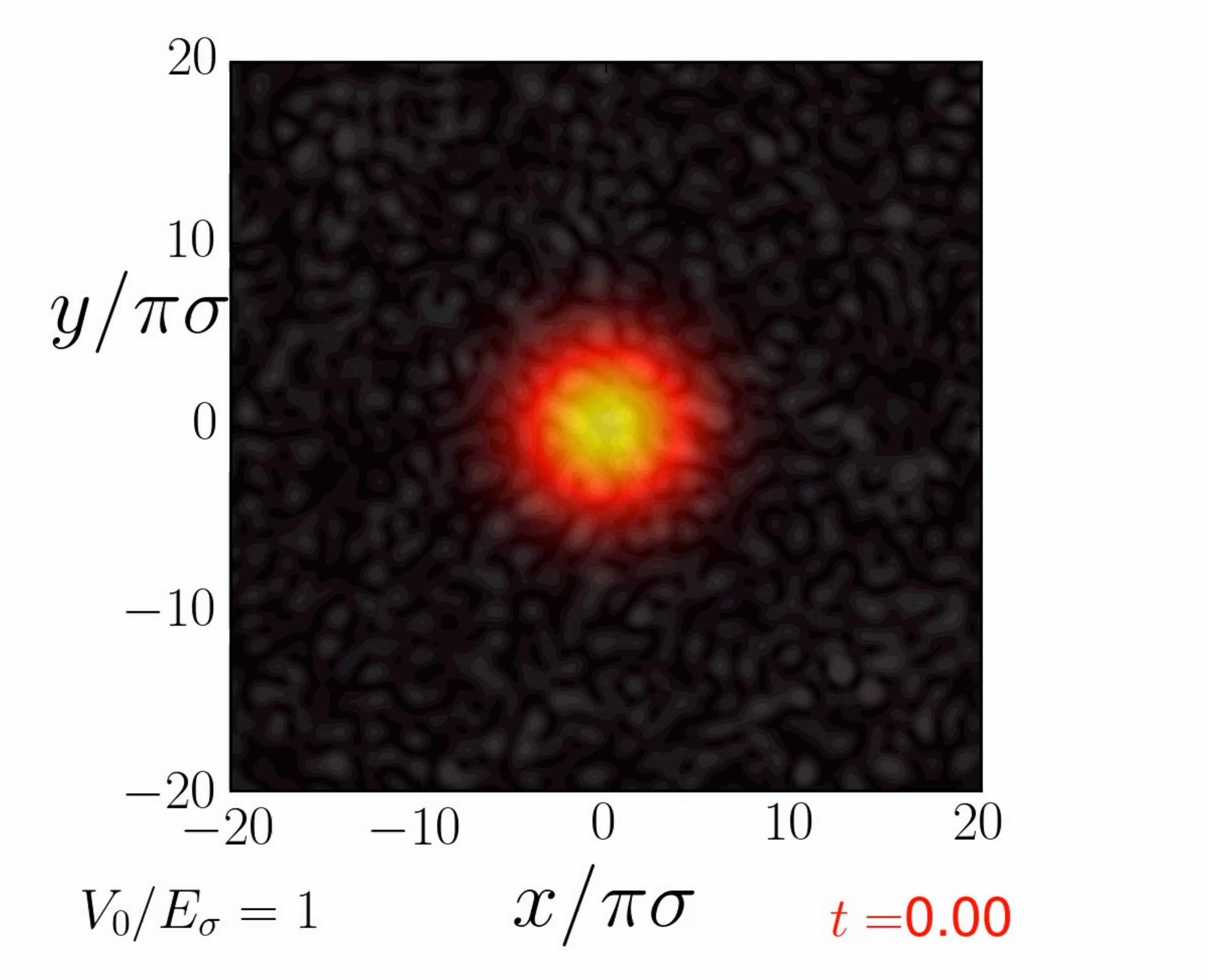
Two-dimensional system



- When $E \ll V_0$, the particle is classically trapped in the potential wells.
- When $E \gg V_0$, the classical motion is ballistic in 1d, typically diffusive in dimension 2 and higher.

Short-time dynamics of a Gaussian wavepacket

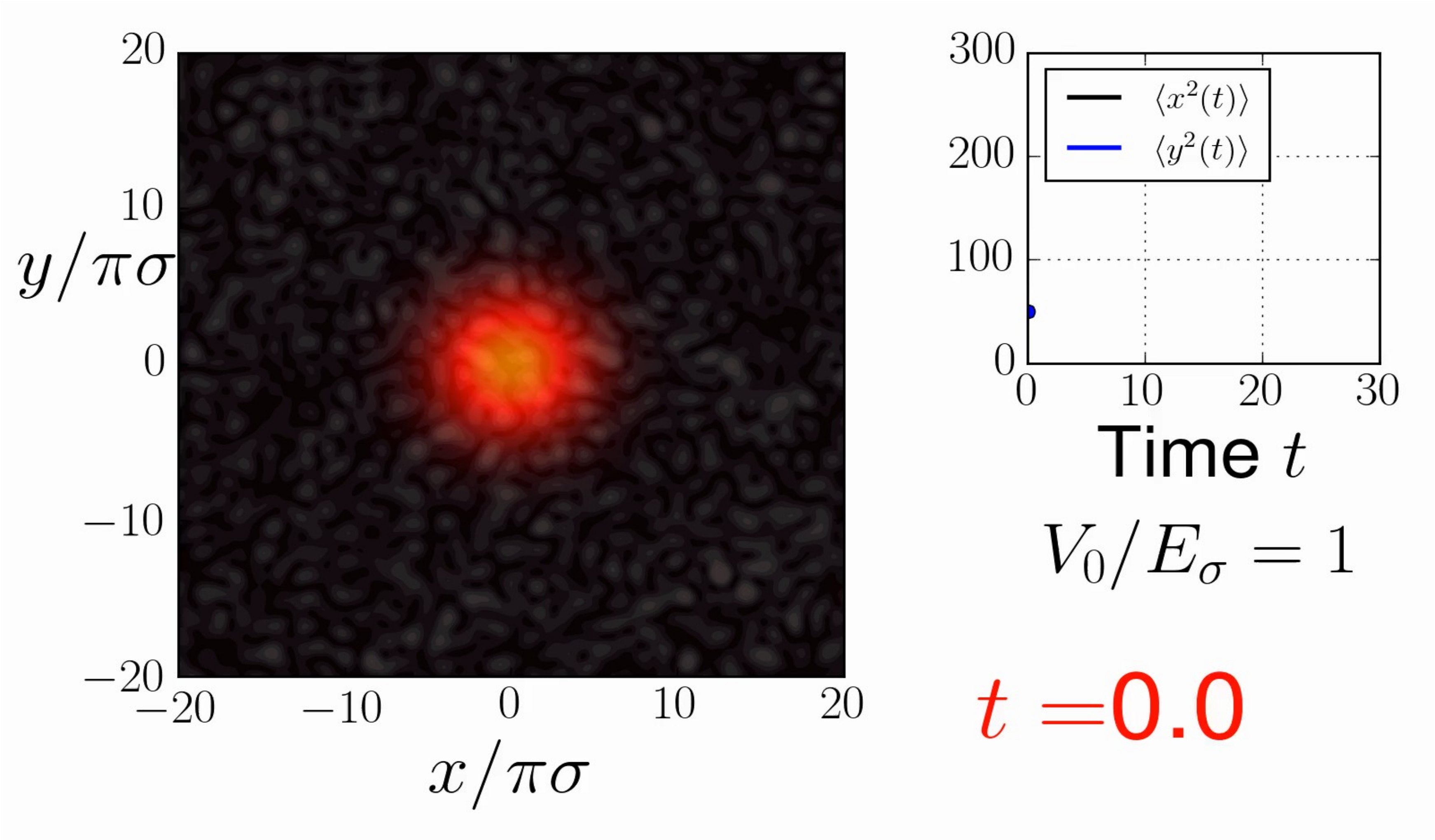
- In the presence of a moderate disorder



1. Very short time: The atoms fall into the potential minima and convert potential energy into kinetic energy

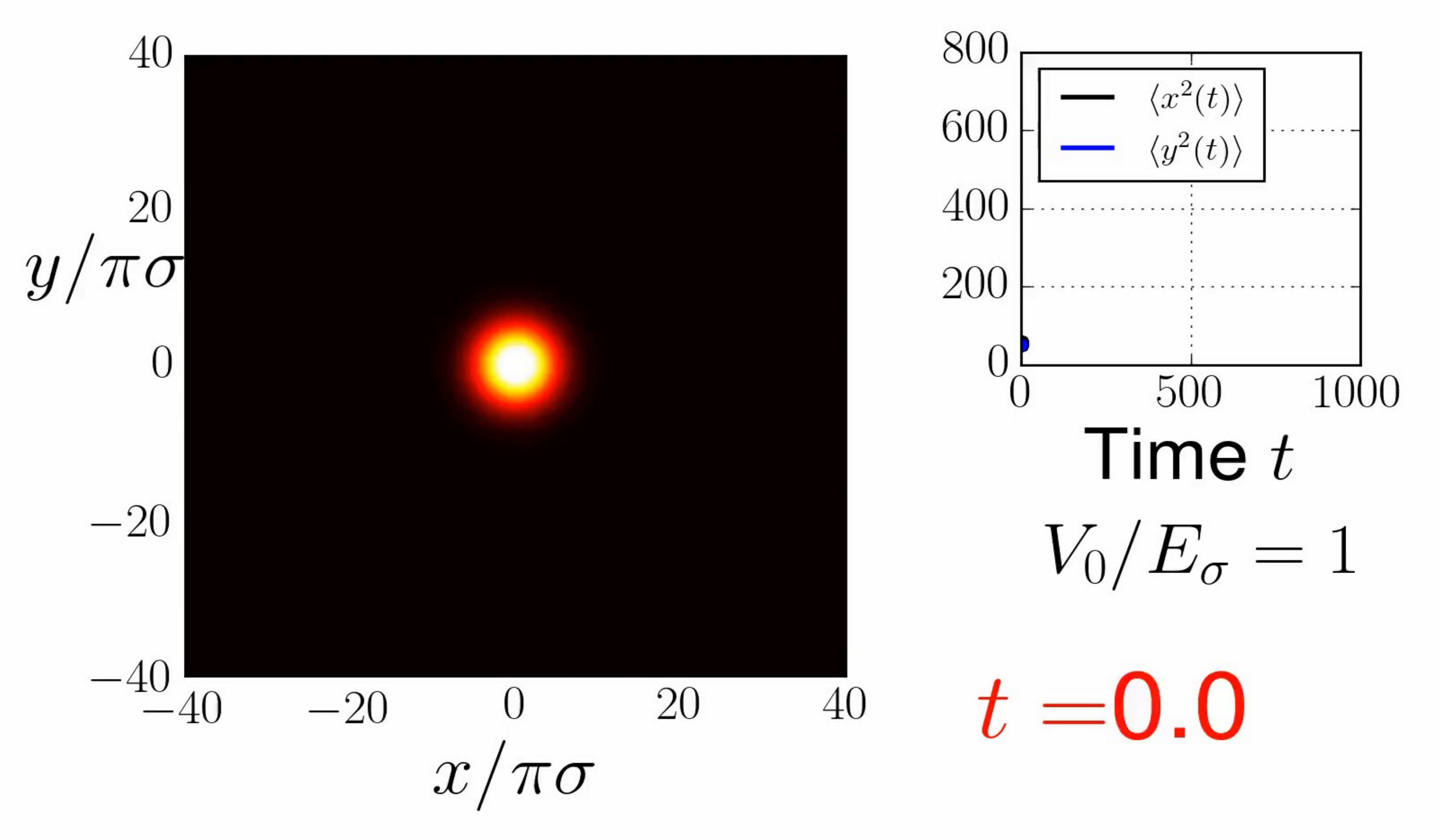
2. The atomic matter wave is later scattered by the potential hills

Beyond the single scattering time



Diffusive motion: $\langle r^2(t)\rangle \propto Dt$

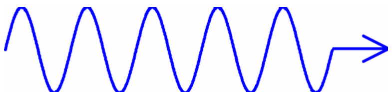
Long time dynamics: towards 2d Anderson localization



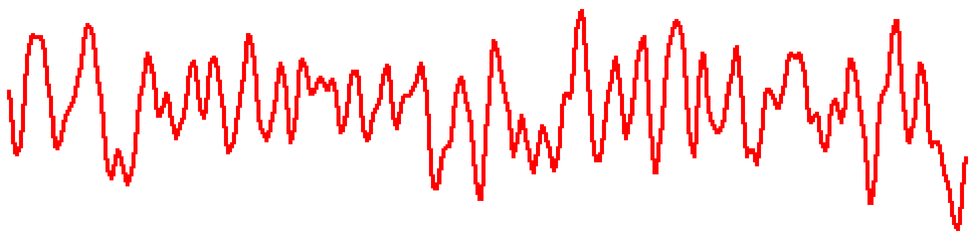
Anderson (a.k.a. Strong) localization

- Particle in a disordered (random) potential:

One-dimensional system

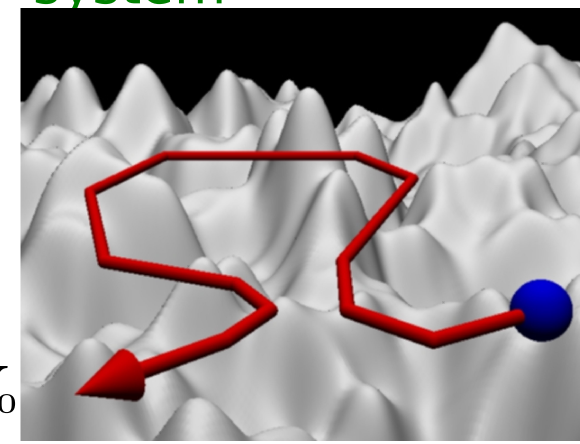


Particle with energy *E*



Disordered potential $V(z)$ (typical value V_0

Two-dimensional system



- When $E \ll V_0$, the particle is classically trapped in the potential wells.
- When $E \gg V_0$, the classical motion is ballistic in 1d, typically diffusive in dimension 2 and higher.
- Quantum interference may inhibit diffusion at long times =>

Anderson localization

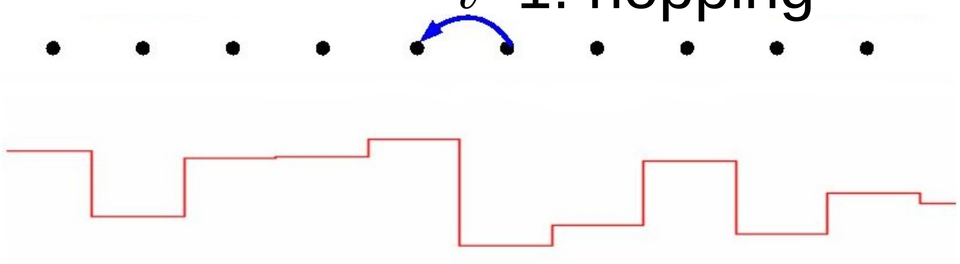
Anderson localization in 1d

- Anderson localization is the generic behavior, even for small disorder!
- Simple Anderson model: discretized Schroedinger equation on a 1d lattice.

$$H = \sum_n \epsilon_n |n\rangle\langle n| + t|n\rangle\langle n+1| + t|n+1\rangle\langle n|$$

$$H = \sum_n \epsilon_n c_n^\dagger c_n + t c_{n+1}^\dagger c_n + t c_n^\dagger c_{n+1}$$

t=1: hopping



Sites

ϵ_n : diagonal disorder

- Schroedinger equation at energy E:

$$\psi_{n+1} + (\epsilon_n - E)\psi_n + \psi_{n-1} = 0$$

- Can be rewritten with the transfer matrix:

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = T_n \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} \qquad T_n = \begin{pmatrix} E - \epsilon_n & -1 \\ 1 & 0 \end{pmatrix}$$

Anderson localization in 1d

- Propagation of the transfer matrix:

$$\begin{pmatrix}\psi_{n+1}\\ \psi_n\end{pmatrix} = \mathcal{T}\begin{pmatrix}\psi_1\\ \psi_0\end{pmatrix}$$

$$\mathcal{T} = T_n\, T_{n-1}\cdots T_1 = \begin{pmatrix}E-\epsilon_n & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}E-\epsilon_{n-1} & -1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}E-\epsilon_1 & -1\\ 1 & 0\end{pmatrix}$$

- $\mathcal{T}$ is the product of independent random matrices with unit determinant => the eigenvalues of $\mathcal{T}$ typically behave like $\exp(\pm\lambda n)$ at large n.
- Boundary conditions forbid exponential growth at large distance.

$$\psi_n \sim \exp\left(-\frac{|n-n_0|}{2\xi_{\rm loc}}\right)$$

$\xi_{\rm loc}$: localization length

Exponential localization of the wavefunction

$$\xi_{\rm loc} \approx \frac{4-E^2}{\langle\epsilon_n^2\rangle}$$

$\xi_{\rm loc} = 2\ell \quad \ell:$ mean free path

Anderson localization in 1d

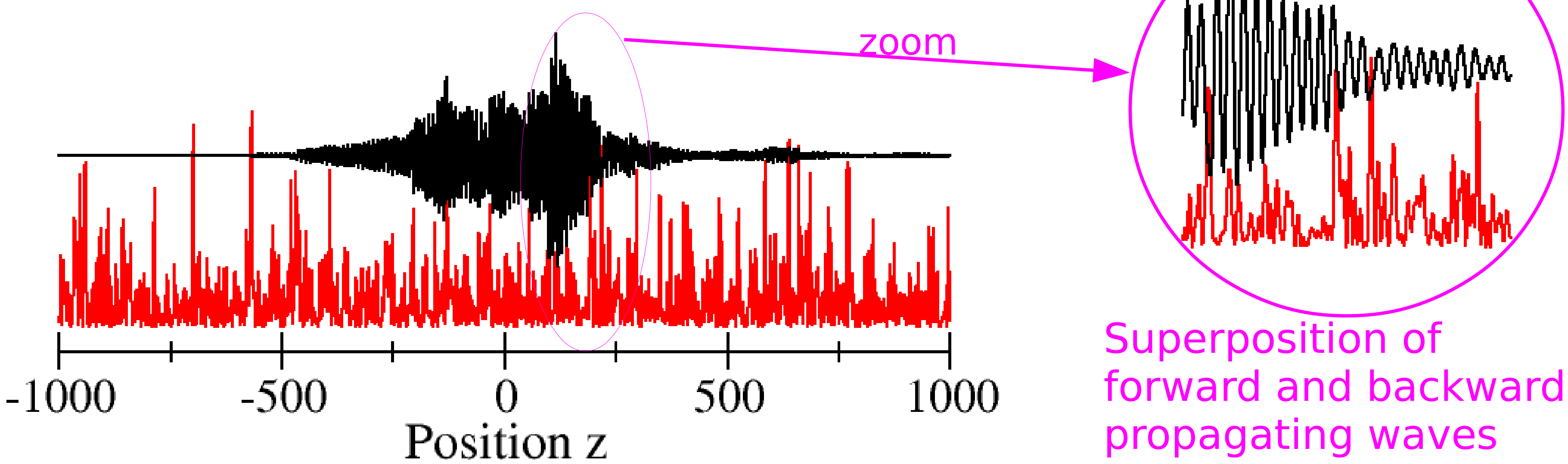
- Continuous Schroedinger equation:

$$-\frac{\hbar^2}{2m}\Delta\psi(z) + V(z)\psi(z) = E\psi(z)$$

Spatial discretization: $\frac{\psi((n+1)\delta) + \psi((n-1)\delta) - 2\psi(n\delta)}{\delta^2}$

=> back to Anderson model with correlated disorder

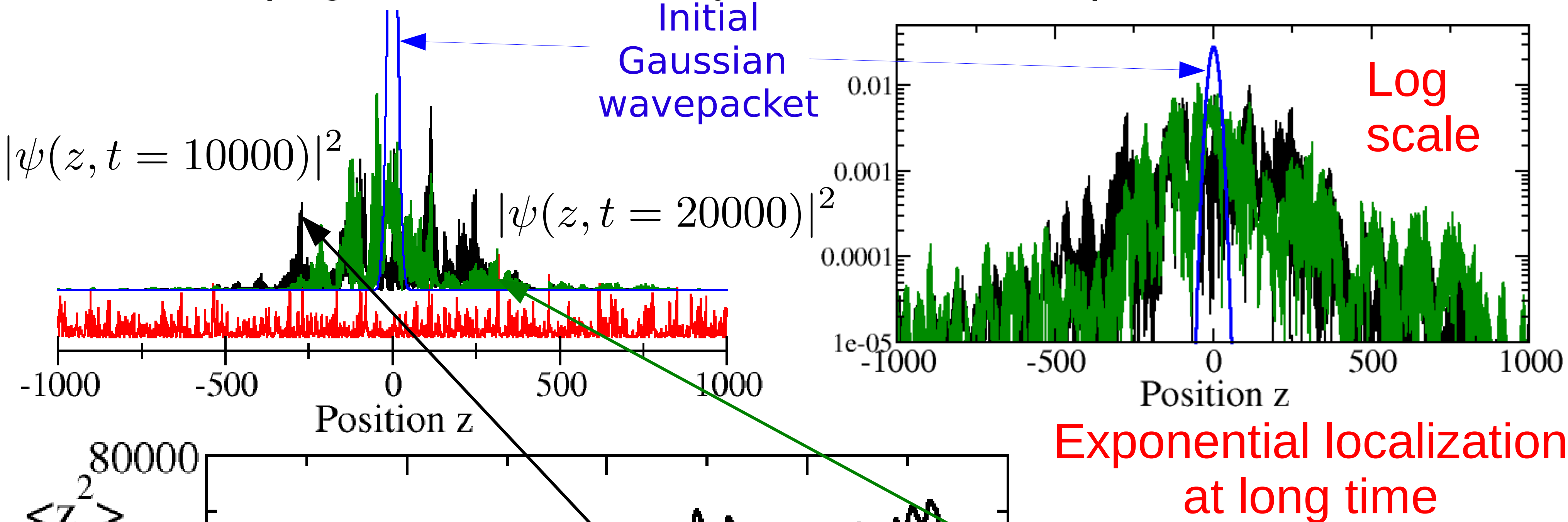
- Typical exponential localization of an eigenstate:

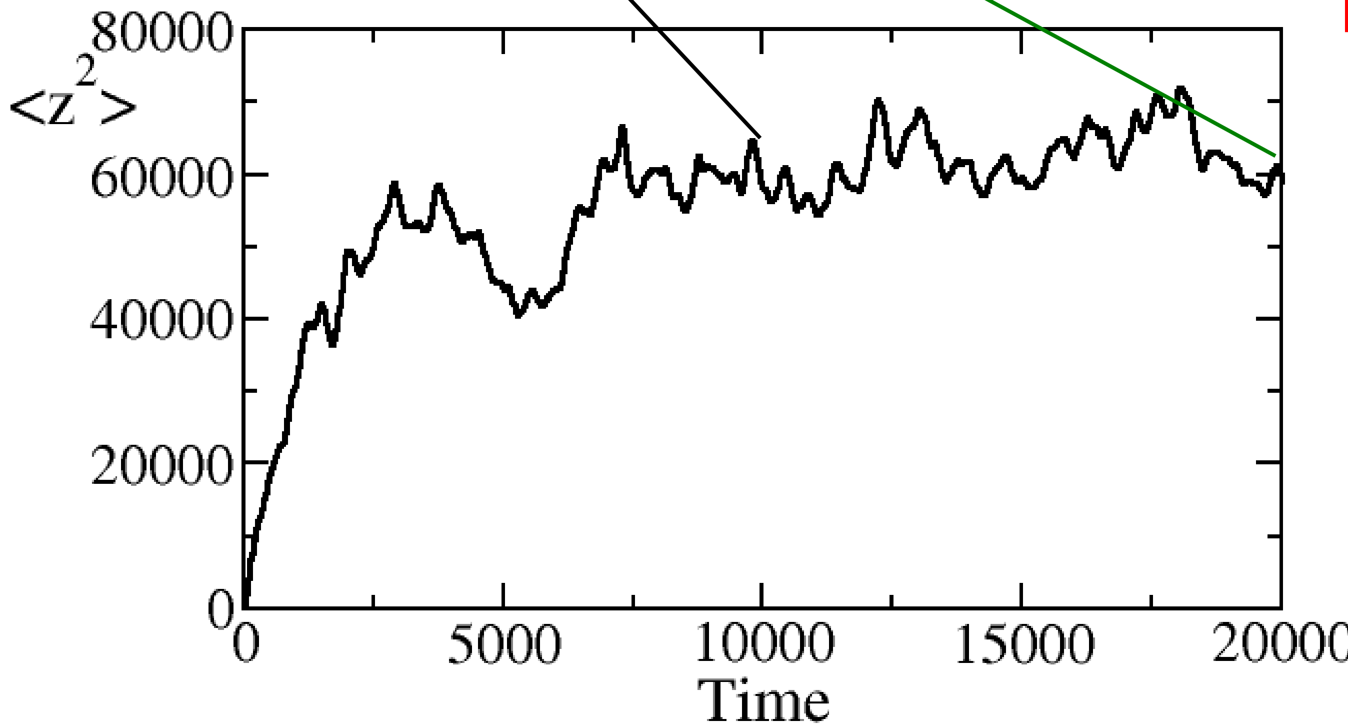


- Strong fluctuations on top of average exponential decay.

Anderson localization in 1d

- Propagation of an initially localized wave-packet:

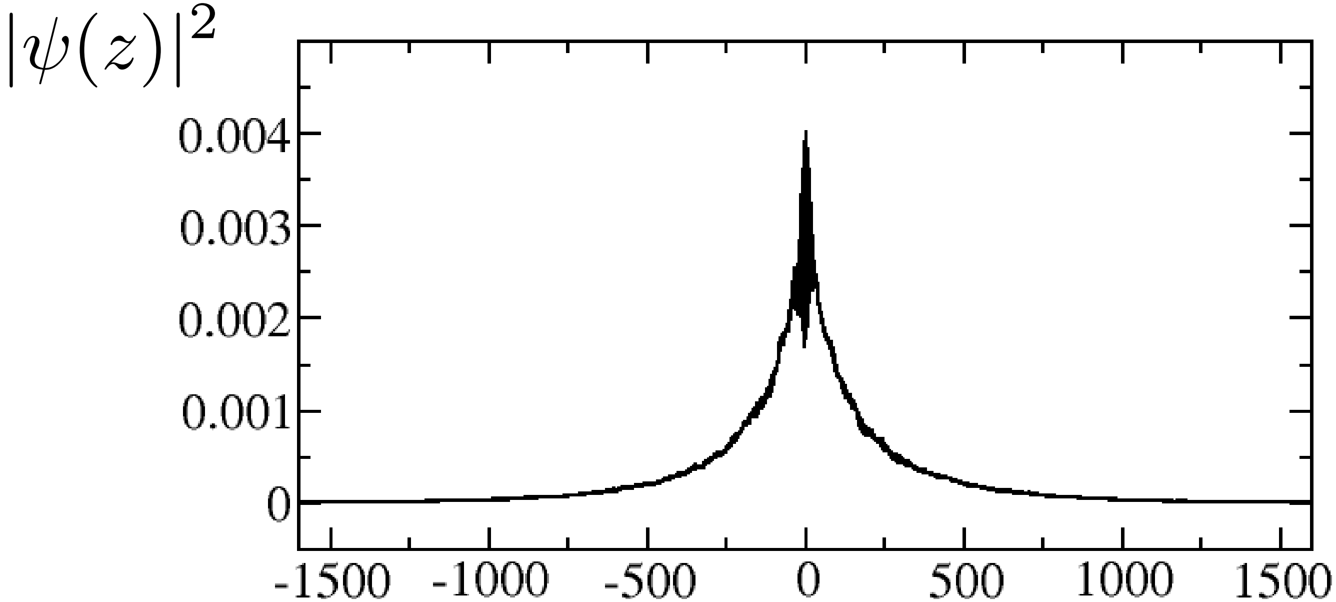




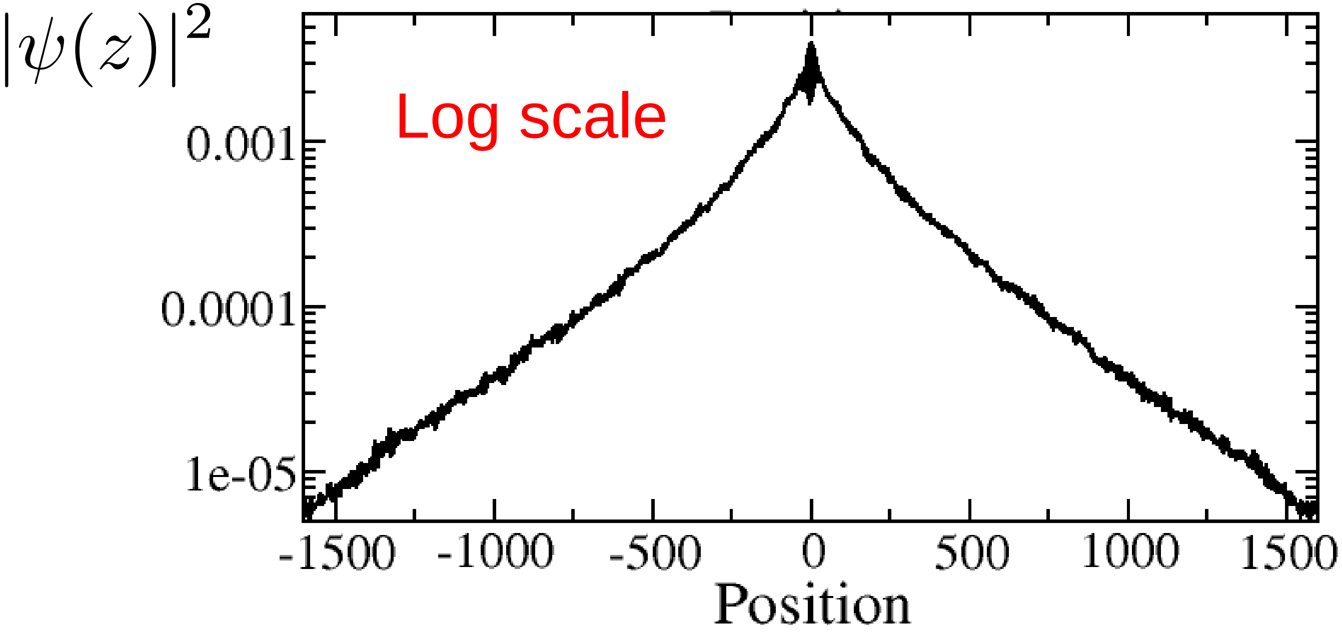
- Both $\langle z^2 \rangle(t)$ and $|\psi(z,t)|^2$ display signatures of Anderson localization

Anderson localization in 1d

- When averaged over time and/or different realizations of the disorder, the fluctuations are smoothed out:



Averaged over 800 realizations of the disorder



Approximate exponential localization

1d Anderson localization

- "Dilute" system:

$$k = \frac{\sqrt{2mE}}{\hbar}$$

wavevector of the particle

$$k\ell \gg 1$$

weak disorder approximation

ℓ : scattering mean free path

- Everything can be computed analytically using e.g. the DMPK (Dorokov, Mello, Pereyra, Kumar) equations.
  - "Optical thickness" of the sample: $t = \frac{L}{\xi}$ (L: system size; ξ: localization length=2ℓ)
  - Typical transmission *T*: $\langle \ln(T) \rangle = -t$
  - $\mathrm{Var}(\ln(T)) \propto t$ => $\ln(T)$ is self-averaging at large *t*.
- But:
  - $\langle T \rangle \approx \exp(-t/4)$
  - $\sqrt{\mathrm{Var}(T)}/\langle T \rangle \approx \exp(t/8) \gg 1$
  - Approximately log-normal distribution:

T is not self-averaging

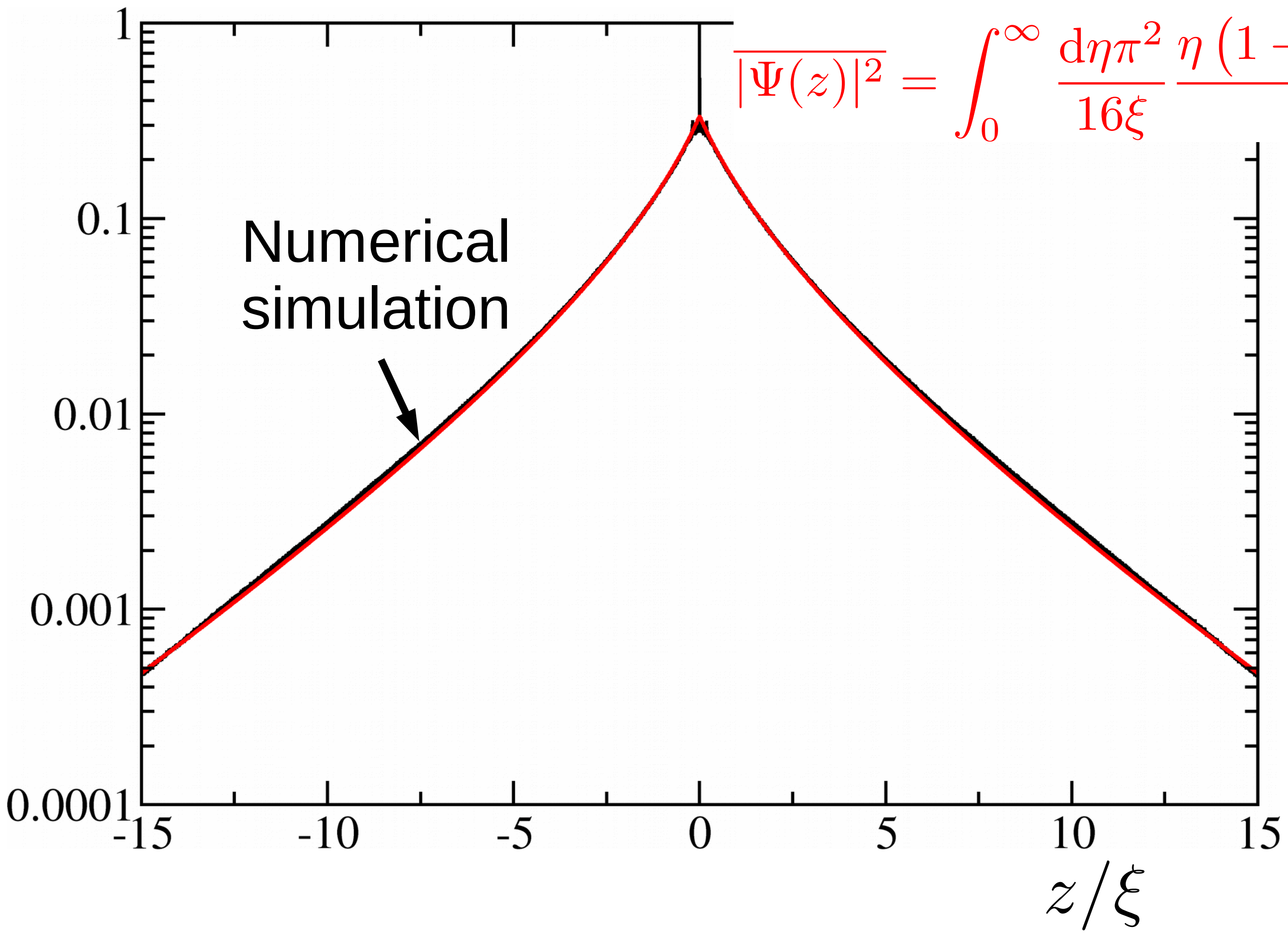
$$P(\ln T) \approx \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{(\ln T + t)^2}{4t}\right)$$

1d Anderson localization

- The full (localized at infinite time) density profile is perfectly well predicted by the DMPK equations.
- N.B.: Temporal dynamics can also be computed.

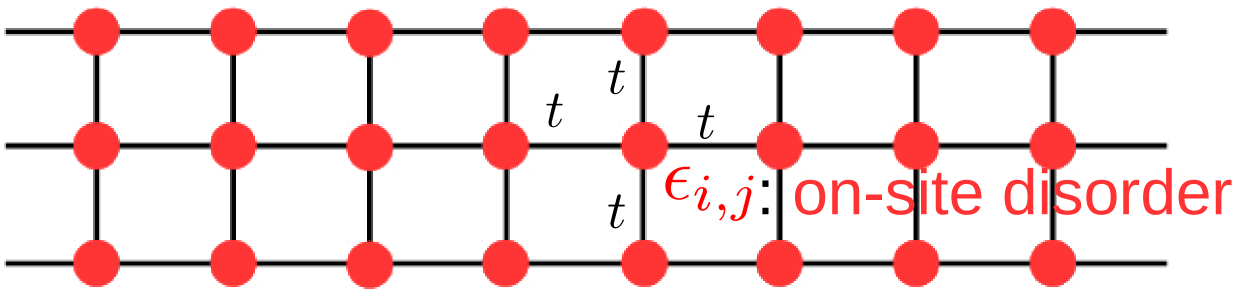
DMPK prediction

$$\overline{|\Psi(z)|^2} = \int_0^\infty \frac{d\eta \pi^2}{16\xi} \frac{\eta \left(1+\eta^2\right)^2 \sinh(\pi\eta) e^{-\left(1+\eta^2\right)|z|/4\xi}}{[1+\cosh(\pi\eta)]^2}$$



Quasi-1d Anderson localization

- N transverse channels:



- Transfer matrix is now a $2N \times 2N$ matrix.
- Exact solution known in the weak scattering regime
- Localization length:

$$\xi = 2N\ell$$

N : number of channels
ℓ : mean free path

- Classical dynamics (and quantum dynamics before localization sets in) is diffusive with diffusion coefficient:

$$D = v\ell = \frac{\hbar}{m} k\ell$$

particle velocity

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Length scales – time scales

- Microscopic quantities
  - ℓ : mean free path
  - τ : mean scattering time
  - $\lambda = 2\pi/k$: de Broglie wavelength

$$\ell = v\tau = \frac{\hbar k \tau}{m}$$

- Macroscopic quantities:
  - Diffusion coefficient
  - Localization length ξ
  - System size L

$$D = \frac{v\ell}{d} = \frac{\hbar}{md} k\ell$$

- Two very important time scales for a finite system
  - Thouless time: time for a diffusive particle to cross the system

$$T_{\mathrm{Th}} = \frac{L^2}{D}$$

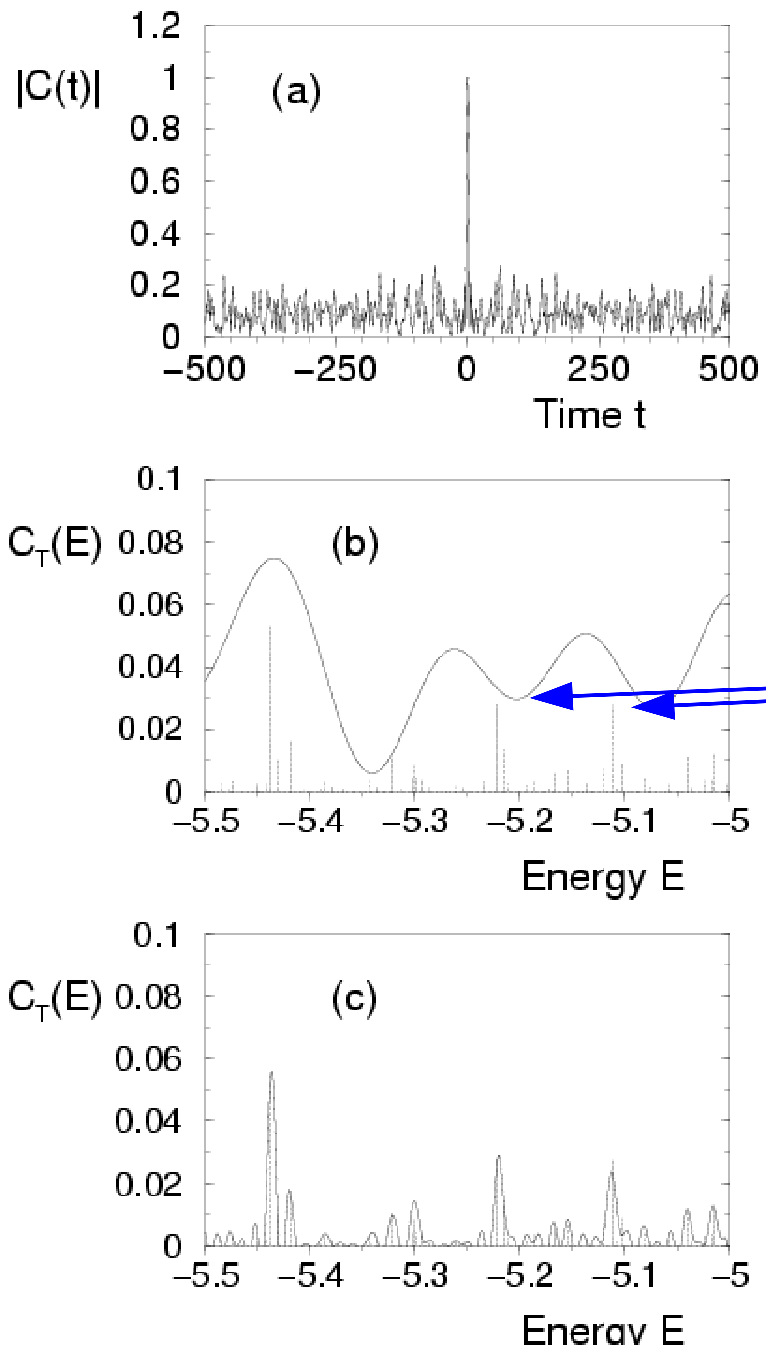
  - Heisenberg time: time to resolve the discrete energy spectrum

$$T_{\mathrm{H}} = \frac{h}{\Delta E} = h\rho(E)L^d$$

mean level spacing

$\rho(E)$: density of states per unit volume

Time scales: the Heisenberg time



Quantum autocorrelation function for a chaotic or disordered system

Fourier transform at short time

The peaks associated with individual energy levels are not resolved!

Fourier transform at long time (longer than the Heisenberg time)

The peaks are resolved!

Heisenberg vs. Thouless

- If $T_{\rm H} \ll T_{\rm Th}$, the system "feels" the discreteness of the energy spectrum before touching the edges => expect localization with $\xi < L$
- If $T_{\rm H} \gg T_{\rm Th}$, do no expect localization.

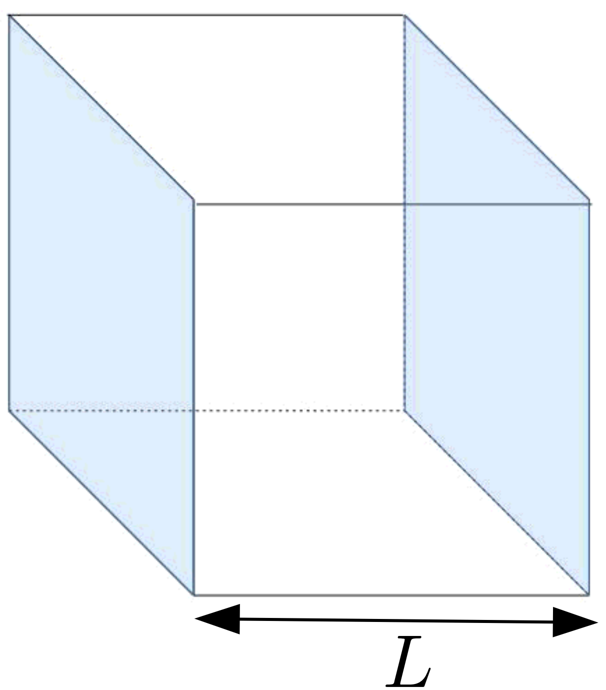
Einstein conductivity

Conductance

$$\frac{T_{\rm H}}{T_{\rm Th}} = h\rho(E)DL^{d-2} = \frac{\boxed{e^2\rho(E)D}\; L^{d-2}}{e^2/h} = \frac{G}{e^2/h} = g$$

Dimensionless conductance

Area L^{d-1}



$g \ll 1$: localized

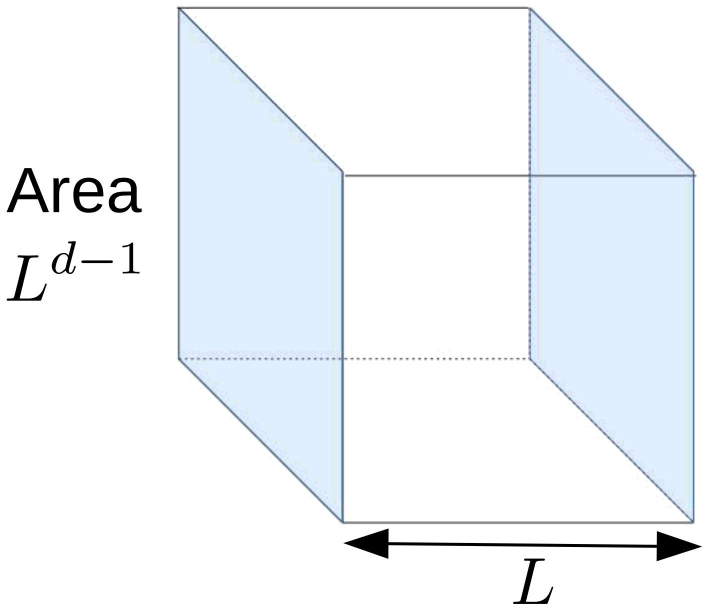
Anderson insulator

$g \gg 1$: delocalized

Ohmic metal

What did we learn so far?

- Anderson localization is a generic localization behavior for quantum disordered one-body systems.
- It is due to quantum interference between multiply scattered paths.
- It exists in dimension 1, 2, 3...
- It can be studied and understood in details in 1d.
- Scaling properties of the quantum dynamics show that the dimensionless conductance g is **THE** important parameter



$$g = \frac{T_{\text{Heisenberg}}}{T_{\text{Thouless}}}$$

$g \ll 1$: localized

Anderson insulator

$g \gg 1$: delocalized

Ohmic metal

Scaling theory of localization

- Proposed by the “gang of four” (Abrahams, Anderson, Licciardello, Ramakrishnan, 1979).
- Approximate theory neglecting (large) fluctuations.
- Assume that localization/delocalization properties are determined by the dimensionless conductance g and its dependence on the system size L.
- Define:

$$\beta = \frac{\mathrm{d}\ln g}{\mathrm{d}\ln L}$$

- Hypothesis: one parameter scaling law.

$$\boxed{\beta = \beta(g)} \text{ only}$$

Scaling theory in 1d

- Landauer formula for the dimensionless conductance:

$$g = \frac{T}{1-T} \qquad T: \ \text{transmission}$$

- Neglect all fluctuations and use:

$$T_{\text{typ}} = \exp(\langle \ln(T) \rangle) = \exp(-L/2\ell)$$

- Then:

$$g(L) = \frac{1}{\exp(L/2\ell) - 1} \qquad \begin{matrix} 2\ell/L & L \ll \ell \\ \exp(-L/2\ell) & L \gg \ell \end{matrix}$$

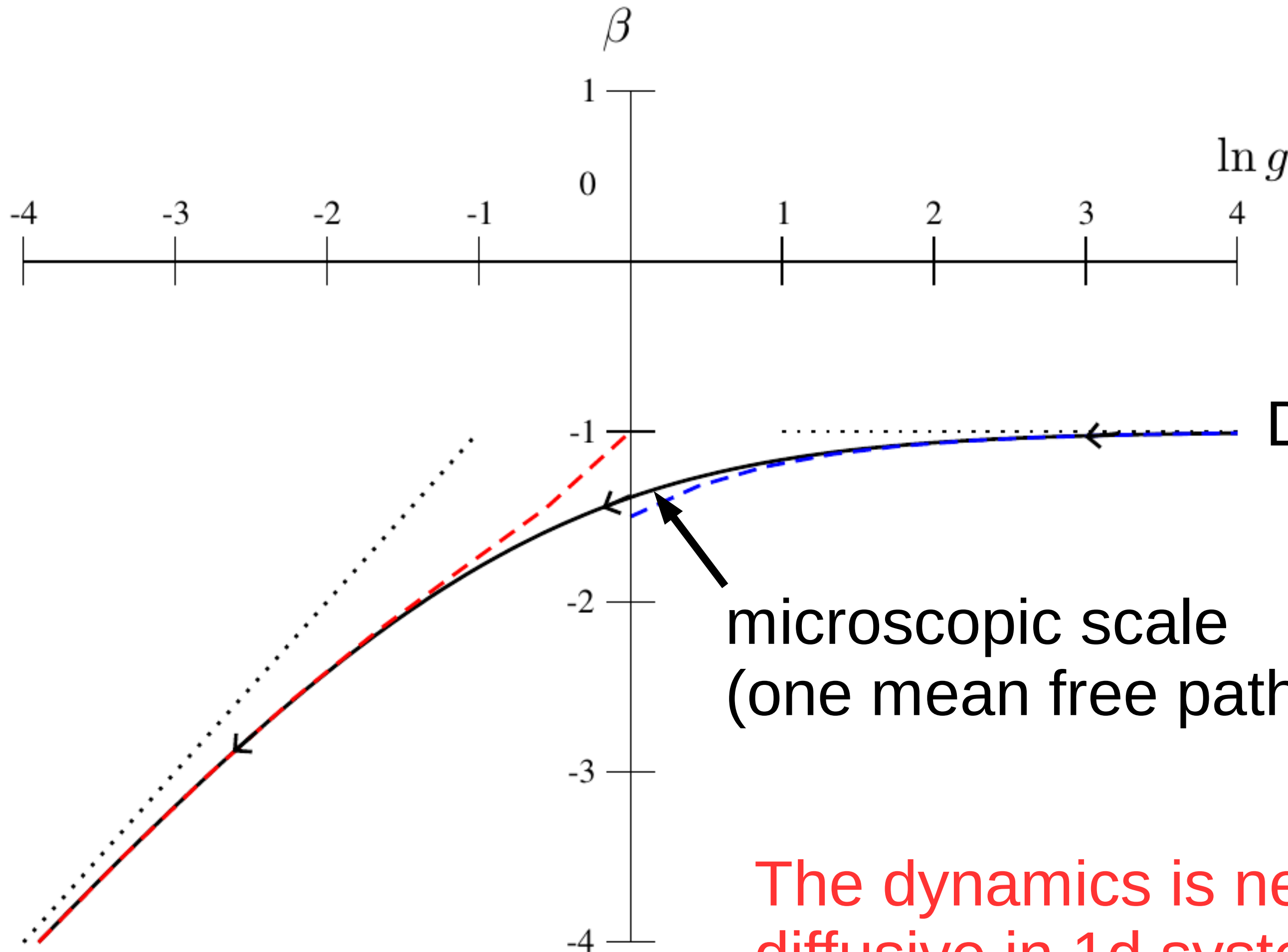
$$\beta(L) = -\frac{L}{2\ell}\frac{1}{1-\exp(-L/2\ell)}$$

$$\beta(g) = -(1+g)\ln\left[1+g^{-1}\right]$$

$$g \gg 1 \qquad \beta \approx -1$$

$$g \ll 1 \qquad \beta \approx \ln g$$

Scaling function in 1d



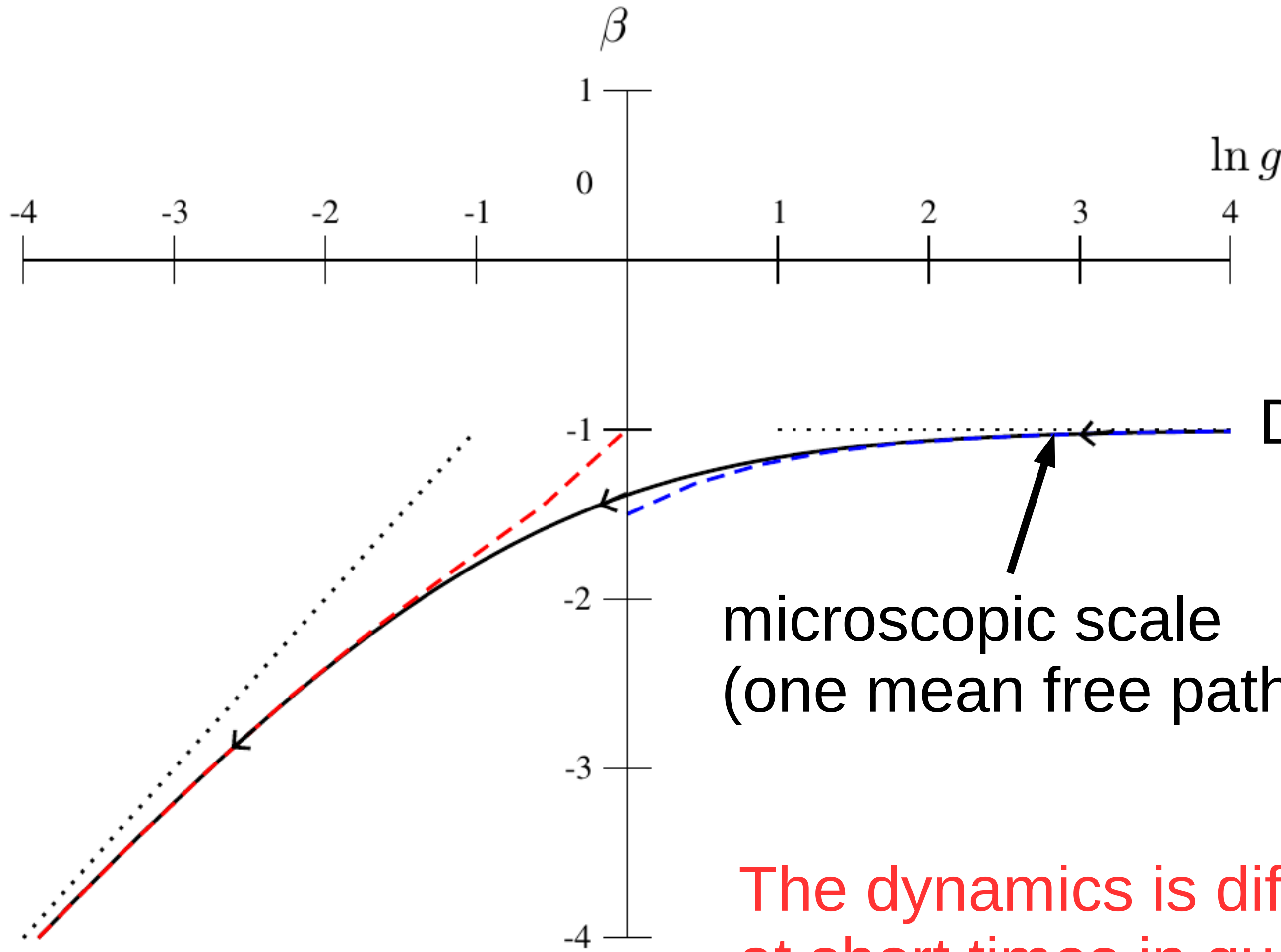
Diffusive regime

microscopic scale
(one mean free path)

The dynamics is never
diffusive in 1d systems!

Localized regime

Scaling function in 1d and quasi-1d



Diffusive regime

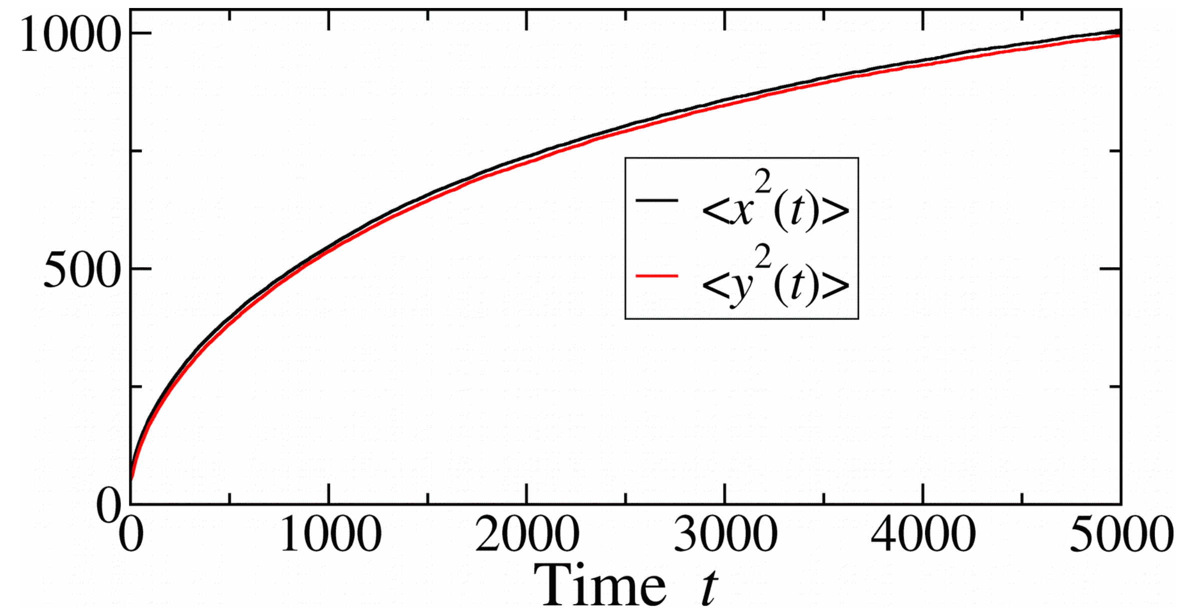
microscopic scale
(one mean free path)

The dynamics is diffusive
at short times in quasi-1d systems

Localized regime

2d Anderson localization takes a VERY long time!

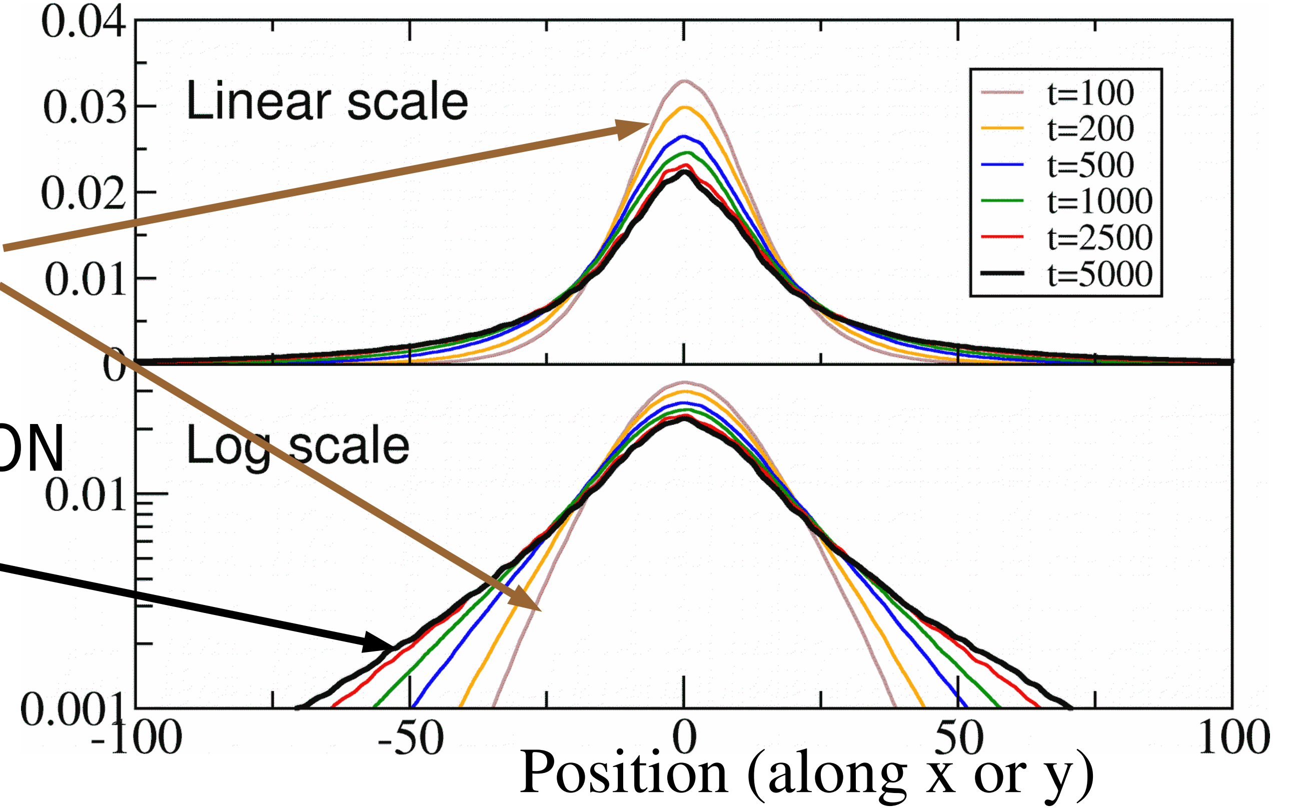
- On a long time scale:



- Average spatial density:

Gaussian profile at short time

ANDERSON LOCALIZATION
Approximately exponential profile at long time



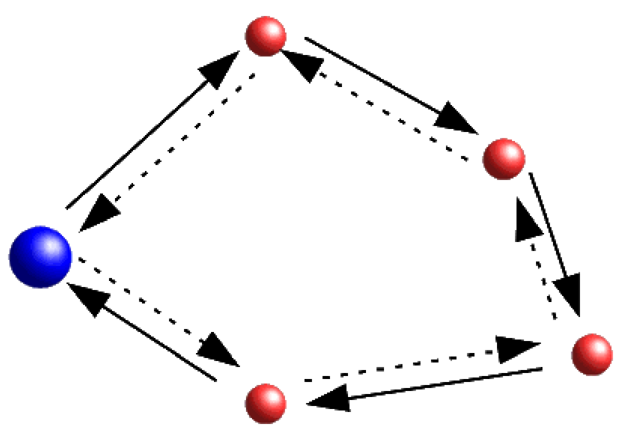
Anderson localization in 2d systems

- Dimensionless conductance in the "classical" diffusive regime

$$g = \frac{T_{\mathrm{H}}}{T_{\mathrm{Th}}} = h\ \rho(E)\ D\ L^{d-2} = \frac{k\ell}{2} \qquad \Rightarrow \beta(g) = 0$$

$$\rho(E) = \frac{m}{2\pi\hbar^2} \qquad D = \frac{v\ell}{2} = \frac{\hbar k\ell}{2m}$$

- For time-reversal invariant systems, the constructive interference between time-reversed paths increase the probability to come back at the initial point (enhanced return to origin) and decrease the conductance (weak localization)



$$g(L) \approx \frac{k\ell}{2}\left(1 - \frac{2}{\pi k\ell}\ln\frac{L}{\ell}\right)$$

$$\beta(g) \approx -\frac{1}{\pi g}$$

Scaling function in dimension *d*

- Localization is proved for sufficiently strong disorder (see lectures of A. Scardicchio), i.e. sufficiently small $k\ell$.
- (Over)simple model: an hyper-cube of size L is equivalent to L^{d-1} 1d resistances of length L in parallel.

$$\beta(\ln(g)) \approx d-1-(1+g)\ln(1+g^{-1})$$

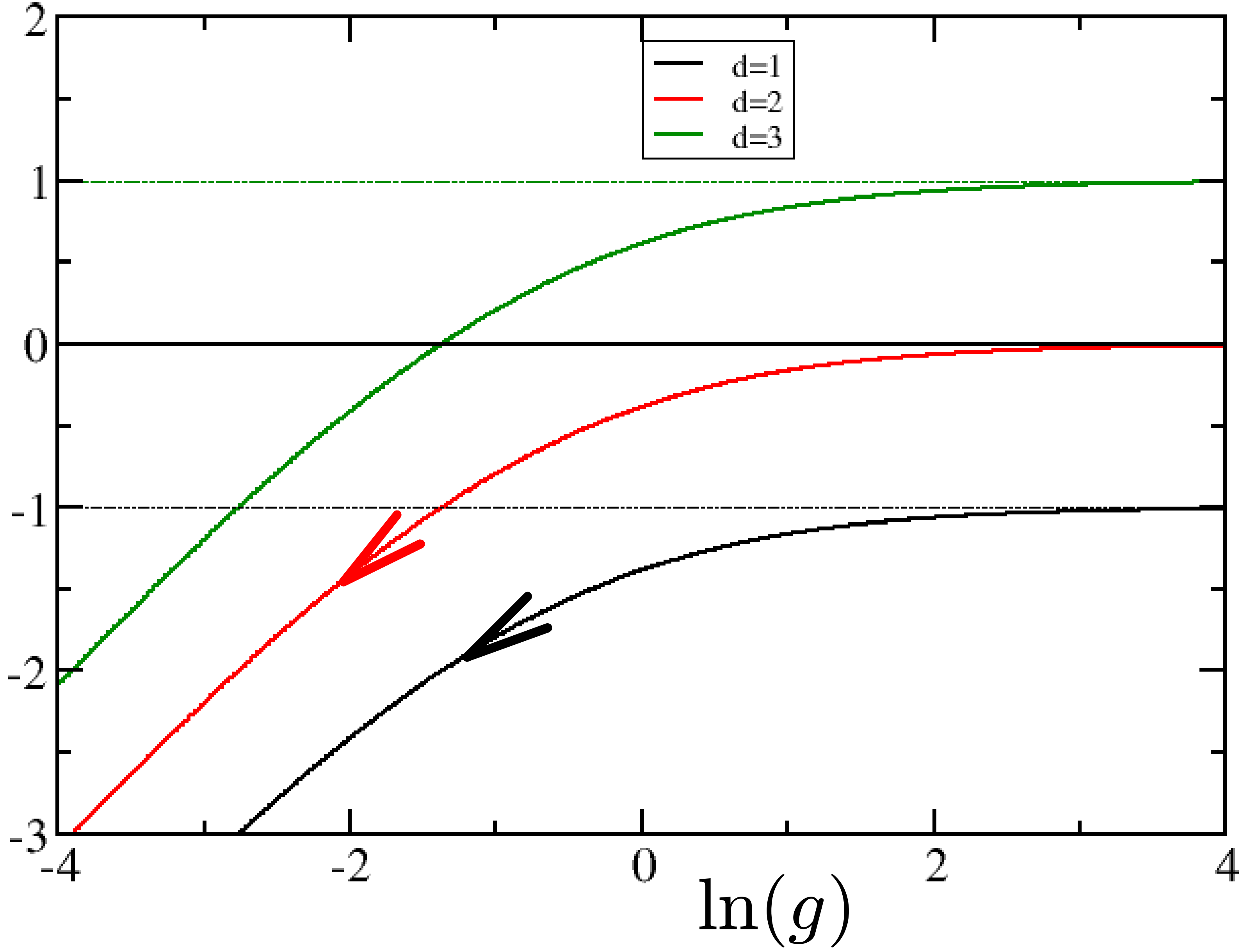
$$g \gg 1 \qquad \beta \approx d-2-\frac{1}{2}\frac{1}{g}$$

$$g \ll 1 \qquad \beta \approx \ln g$$

prefactor known to be wrong (weak localization correction)

Scaling function in dimension d

$$\beta(\ln(g)) = d - 1 - (1+g)\ln(1+g^{-1})$$



Prediction of the scaling theory

- Dimension 1: (almost) always localized.

  Localization length $\xi_{\rm loc} = 2\ell$ ℓ : mean free path.

- Dimension 2: marginally localized.

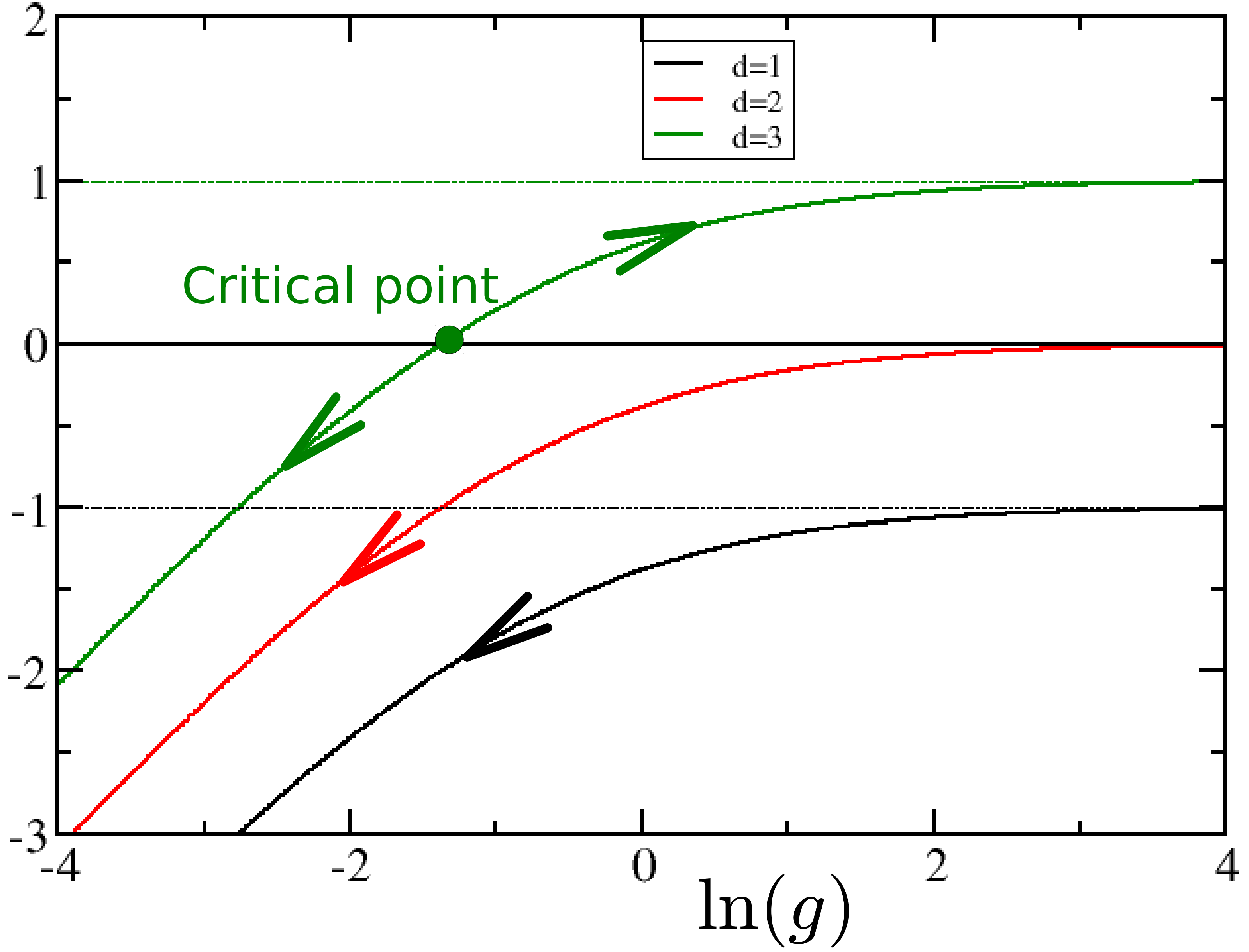
  Localization length $\xi_{\rm loc} \simeq \ell \exp\left(\frac{\pi k \ell}{2}\right)$

  Non-time reversal-invariant spinless systems: localization

  Time-reversal invariant half-integer spin systems: transition

Scaling function in dimension d

$$\beta(\ln(g)) = d - 1 - (1+g)\ln(1+g^{-1})$$



Anderson localization in 3d (and beyond)

- For weak disorder, the quantum motion is diffusive (weak localization) => metallic behaviour.
- For strong disorder, Anderson localization takes place => insulator.
- The metal-insulator transition (Anderson transition) takes place at the “mobility edge”. It is controlled by the $k\ell$ parameter. The critical point is approximately given by:

$$(k\ell)_\mathrm{c} \approx 1$$ (non universal) Ioffe-Regel criterion

- The Anderson transition is a second-order transition. On the insulating side, the localization length diverges like:

$$\xi_\mathrm{loc} \sim \frac{1}{[(k\ell)_\mathrm{c} - k\ell]^\nu} \quad \nu: \ \text{universal critical exponent}$$

- On the metallic side: $D \simeq [k\ell - (k\ell)_\mathrm{c}]^s \ \text{ with } \ s = (d-2)\nu$

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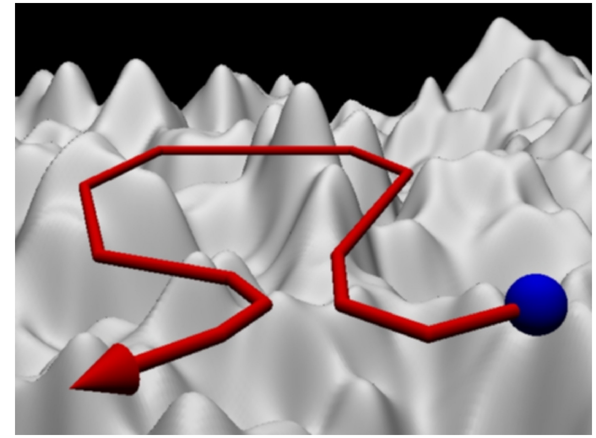
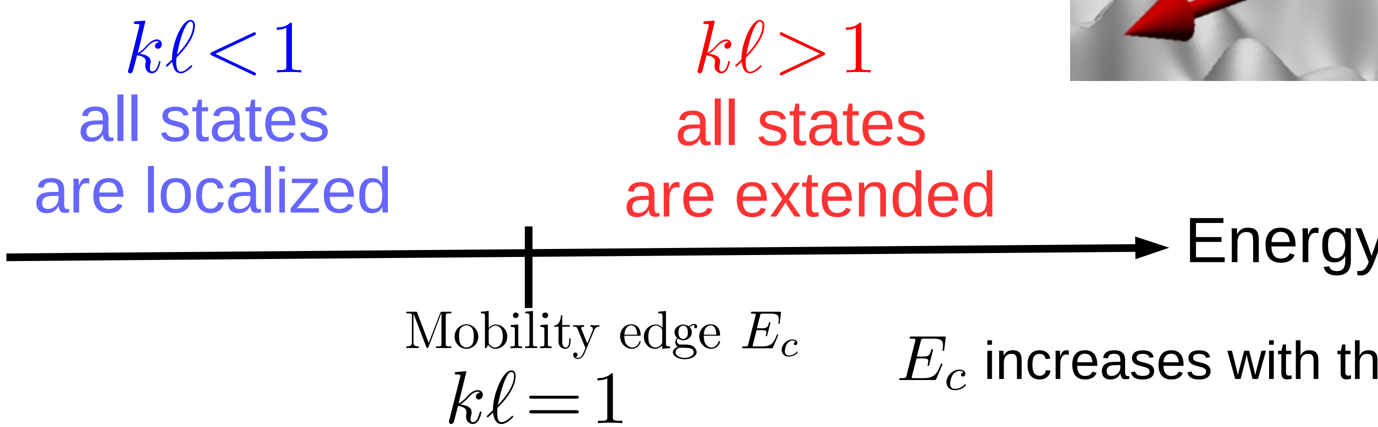
$$\xi_{\mathrm{loc}} \sim \frac{1}{[(k\ell)_{\mathrm{c}} - k\ell]^{\nu}} \quad \nu: \text{ universal critical exponent}$$

- On the metallic side: $D \simeq [k\ell - (k\ell)_{\mathrm{c}}]^{s}$ with $s = (d-2)\nu$
- Numerical results suggest $\nu \approx 1.57$ in 3d.
- A simple “mean field” approach also finds $\nu = 1$
- There are perturbative expansions in dimension $2+\epsilon$

Mobility edge in 3d (and higher dimensions)

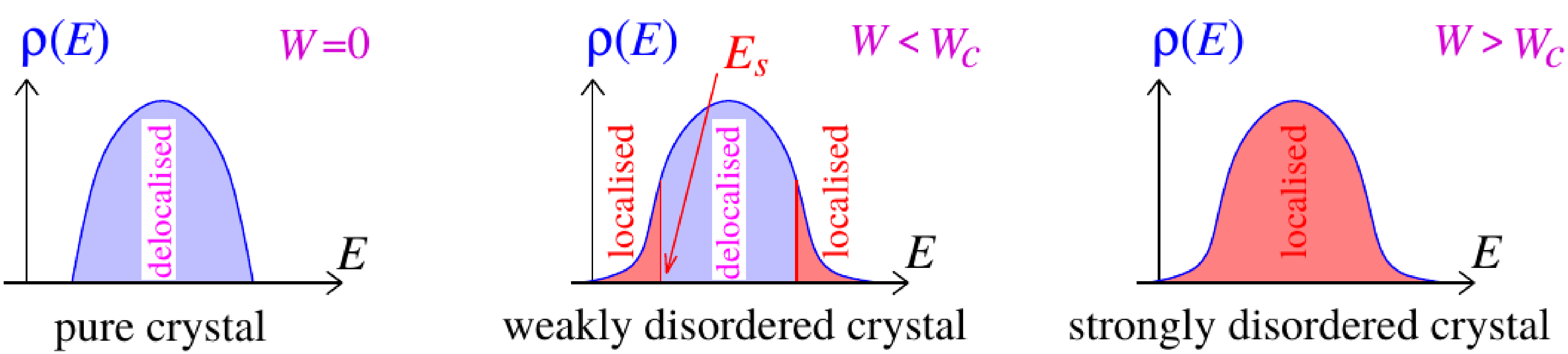
- Continuous system, the mean free path usually increases with energy.
- $k\ell = 1$ is reached at a given energy called the mobility edge.





E_c increases with the disorder strength

- For particles on a lattice (e.g. Anderson model):



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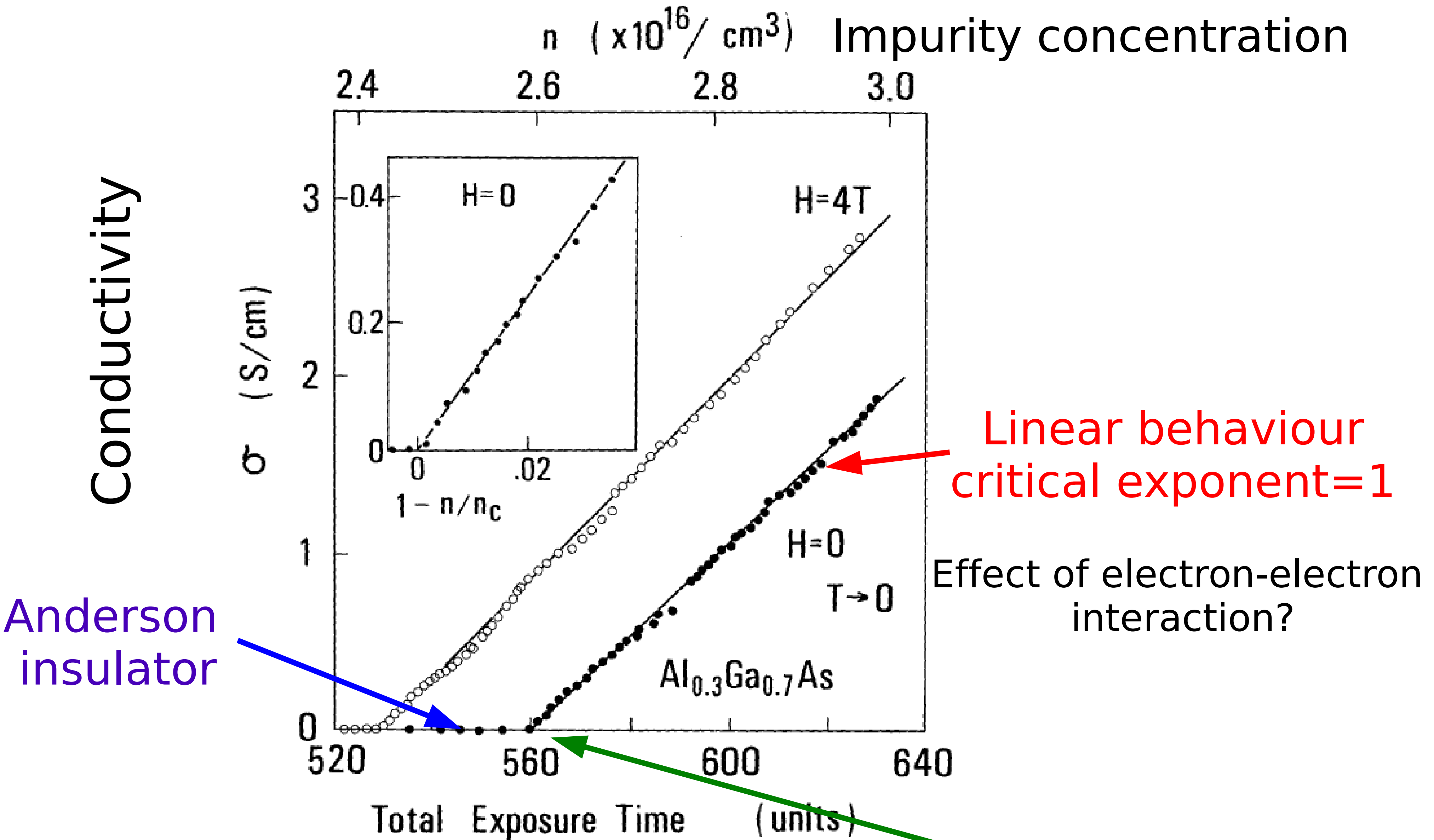
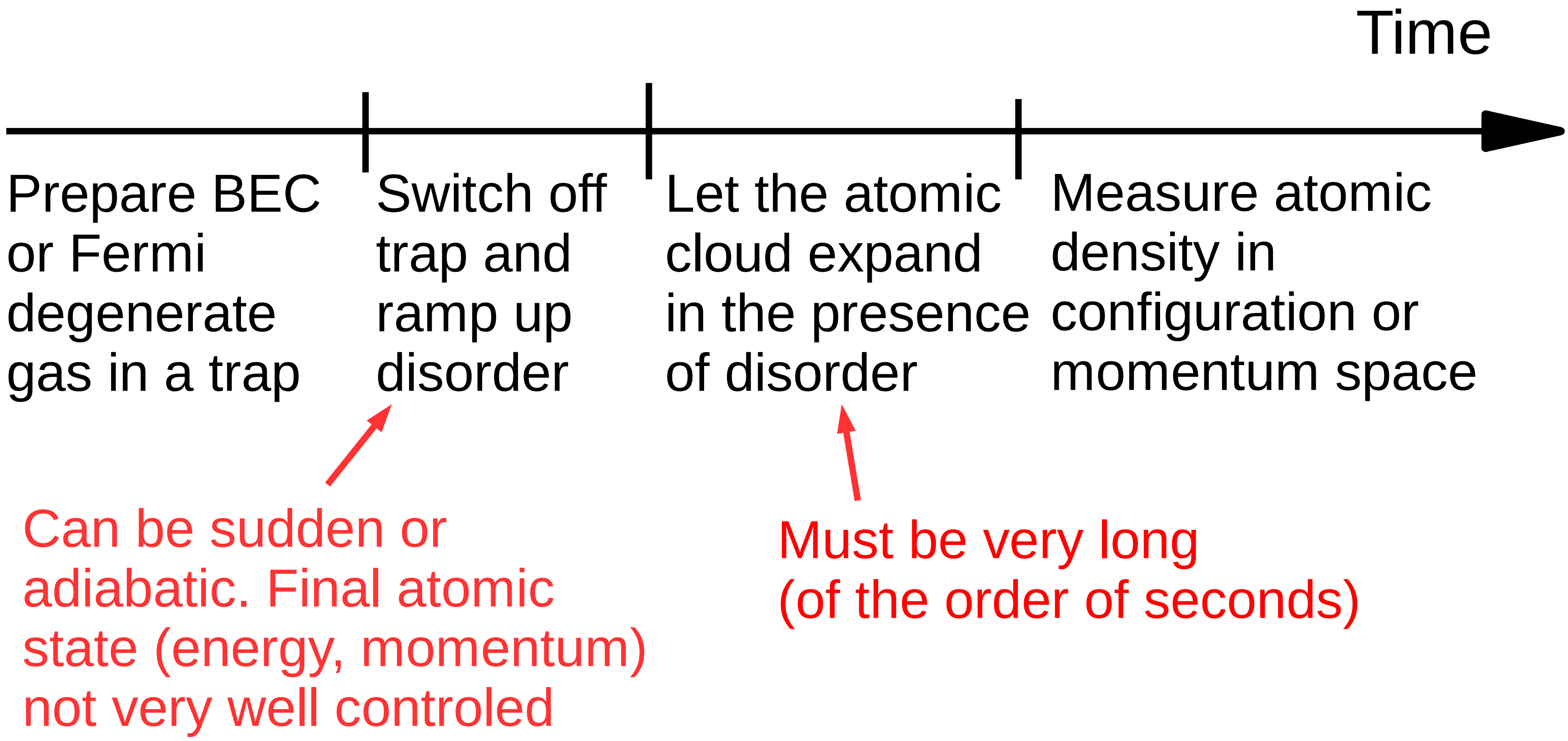
Metal-insulator transition d=3



S. Katsumoto et al, J. Phys. Soc. Japan, 56, 2259 (1987)

How to observe Anderson localization with atomic matter waves?

- Direct measurement of a “conductance” is difficult.
- Ioffe-Regel criterion $k\ell \simeq 1$ (k: atomic wavevector ℓ: mean-free path) requires correlation length in the sub micrometer range, in the three spatial directions.

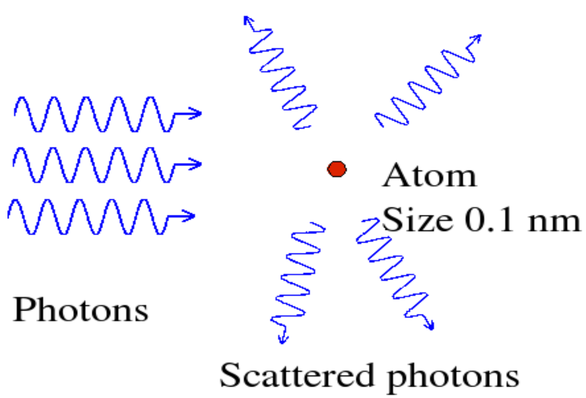


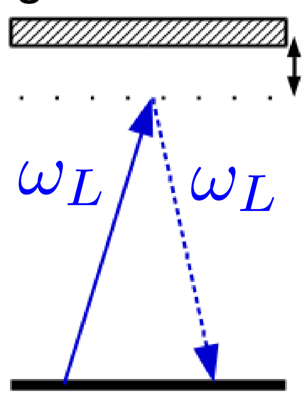
Quantum dynamics of the external motion of cold atoms

- Control of the dynamics with laser fields, magnetic fields, gravitational field.
- Orders of magnitude:
  - Velocity: cm/s
  - De Broglie wavelength: μm
  - Time: μs-ms

Very favorable!

- One-body (if sufficiently dilute) zero-temperature quantum dynamics with small decoherence. For dense gases, use Feshbach resonances to control atom-atom interaction.
- Interaction with quasi-resonant light





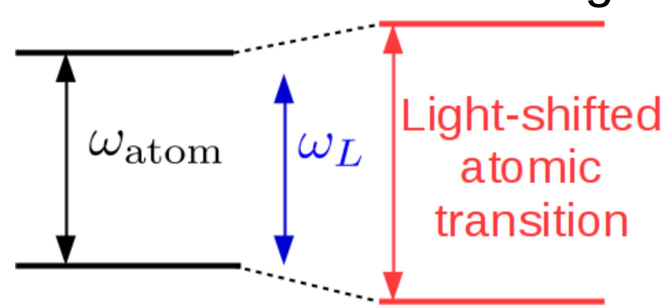
Excited state, width Γ

Detuning $\delta = \omega_L - \omega_{\text{atom}}$

Scattering amplitude $\propto \dfrac{1}{\delta + i\Gamma/2}$

Ground state

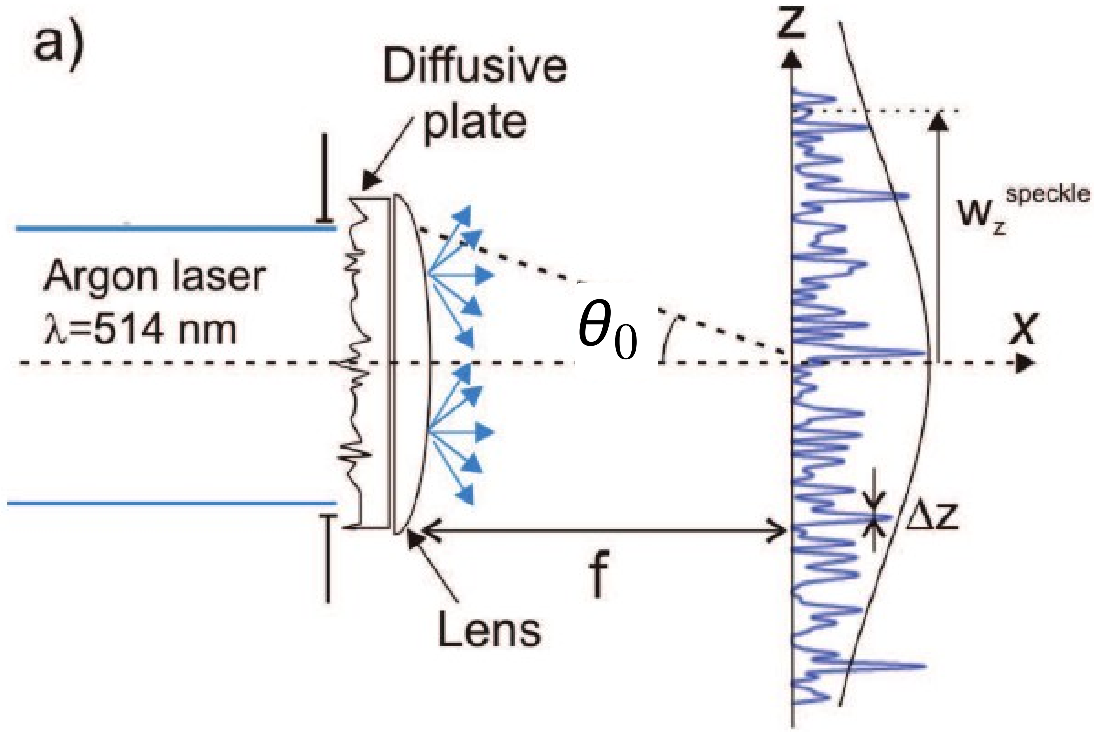
- "Real" scattering of photons kills phase coherence => use a large detuning
- Dominant effect: light-shift of the atomic transition proportional to the intensity I seen by the atom.



- Effective optical potential for the motion of the atom center of mass

Speckle optical potential (2D version)

- Speckle created by shining a laser on a diffusive plate:



Courtesy of V. Josse,
Institut d'Optique
(Palaiseau)

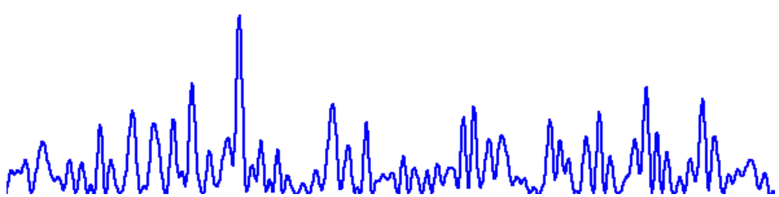
Speckle spot size

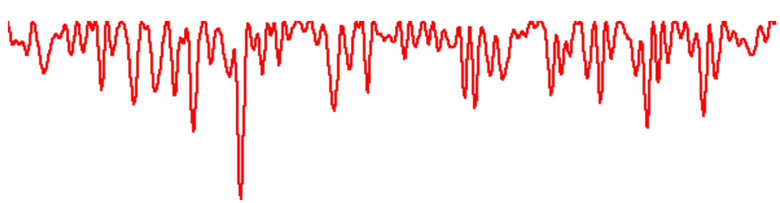
$$\sigma \approx \frac{\lambda}{\theta_0}$$

λ : laser wavelength

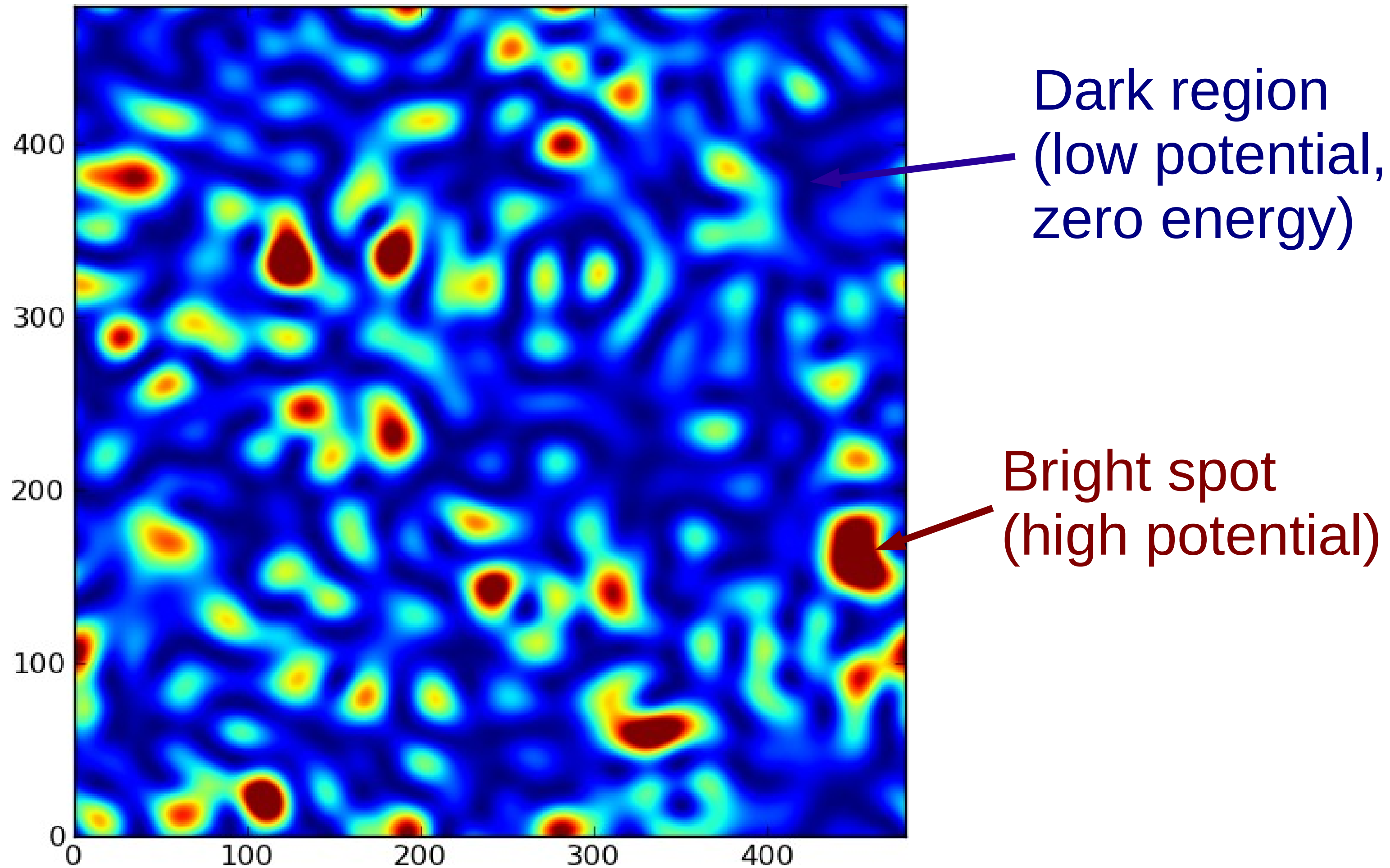
θ_0 : Numerical Aperture

- The speckle electric field is a (complex) random variable with Gaussian statistics. All correlation functions can be computed.
- Depending on the sign of the detuning, the optical potential is bounded either from above or from below





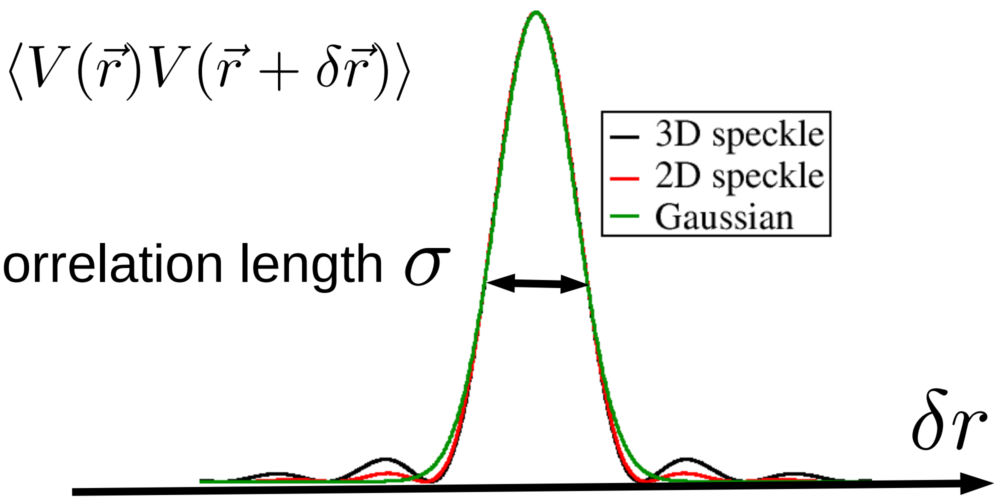
A typical realization of a 2D blue-detuned speckle potential



Distribution of potential value $P(V) = \frac{\exp(-V/V_0)}{V_0}\Theta(V)$

Rigorous low energy bound, no high energy bound

Spatial correlation function for speckle potential



$$\langle V(\vec{r})\rangle = V_0$$

$$\langle V(\vec{r'})V(\vec{r'}+\vec{r})\rangle = V_0^2\left(1+4\frac{J_1^2(\theta_0 k_{\rm L} r)}{\theta_0^2 k_{\rm L}^2 r^2}\right)$$

2D

$$\langle V(\vec{r'})V(\vec{r'}+\vec{r})\rangle = V_0^2\left(1+\frac{\sin^2 k_{\rm L} r}{k_{\rm L}^2 r^2}\right)$$

3D

* Important energy scales: energy of the atoms E
  potential strength V_0
  correlation energy $E_\sigma = \dfrac{\hbar^2}{m\sigma^2}$

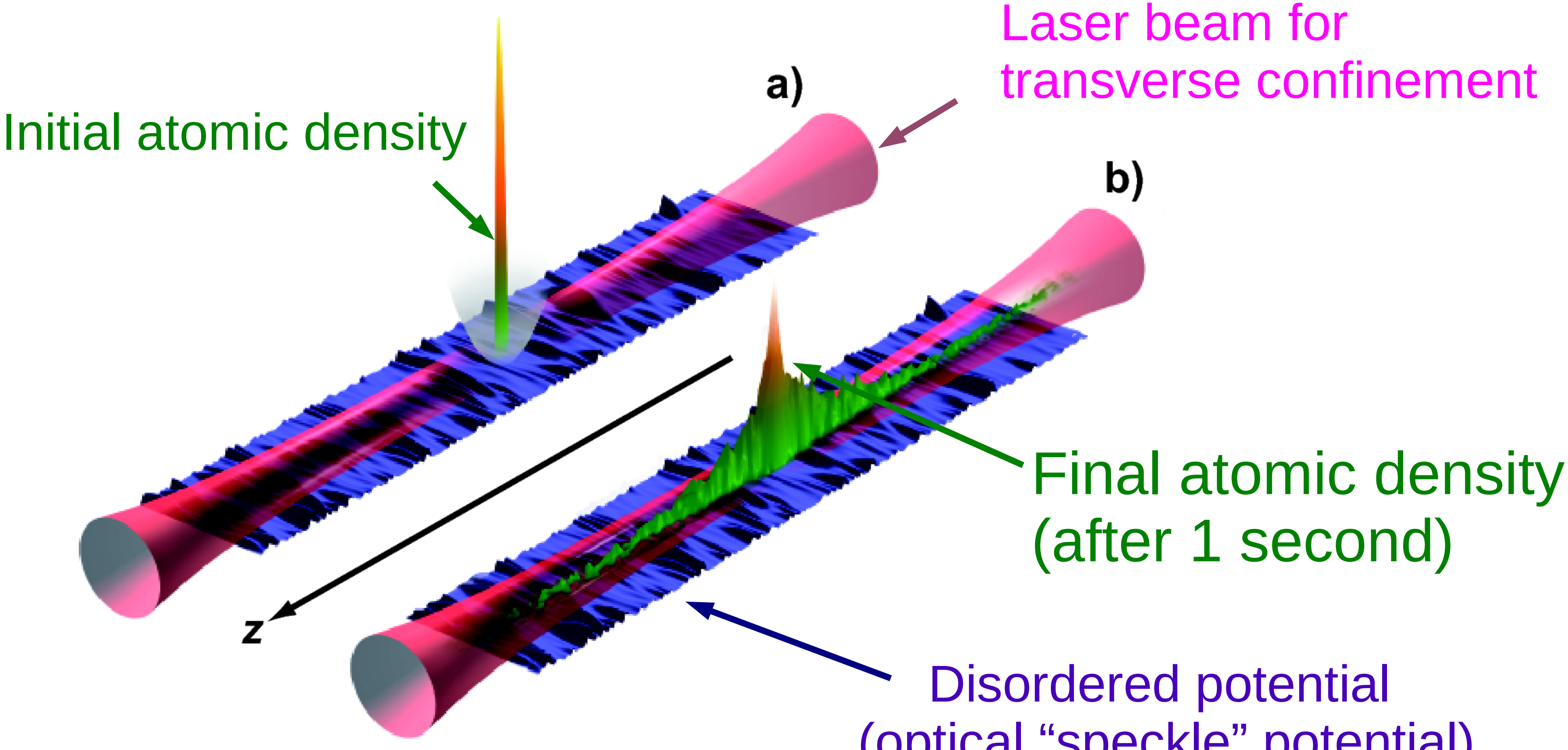
* When $E = E_\sigma$ the de Broglie wavelength is equal to σ

$V_0 \ll E_\sigma$ — "quantum" regime

$V_0 \gg E_\sigma$ — "classical" regime

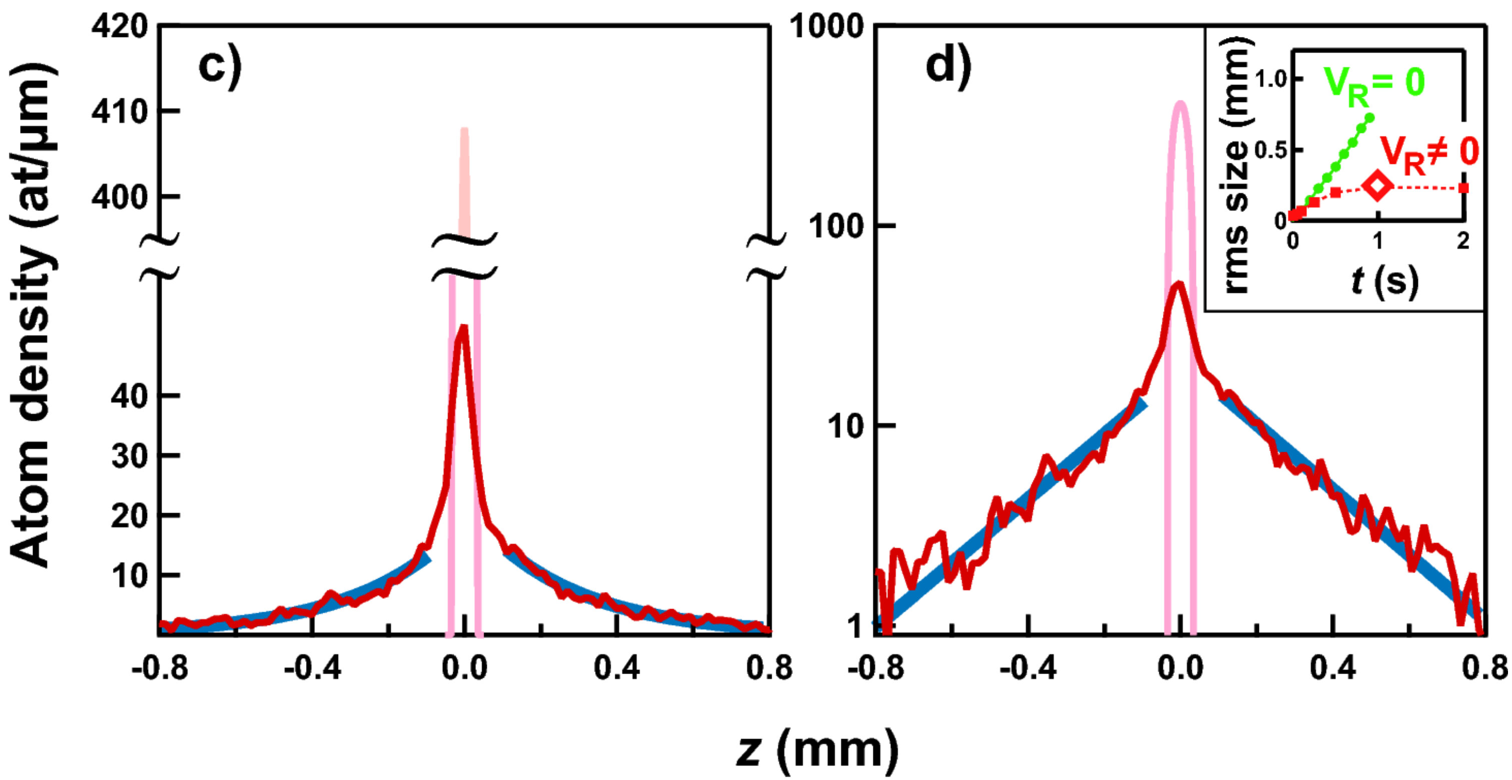
* Anderson localization is expected for $E \sim V_0 \sim E_\sigma$
* Weak disorder calculations (mean free path, diffusion constant...) are possible for $V_0^2 \ll E E_\sigma$, but not valid in the strong localization regime

1d Anderson localization of atomic matter waves



J. Billy et al, Institut d'Optique (Palaiseau, France), Nature, 453, 891 (2008)

1d Anderson localization of atomic matter waves

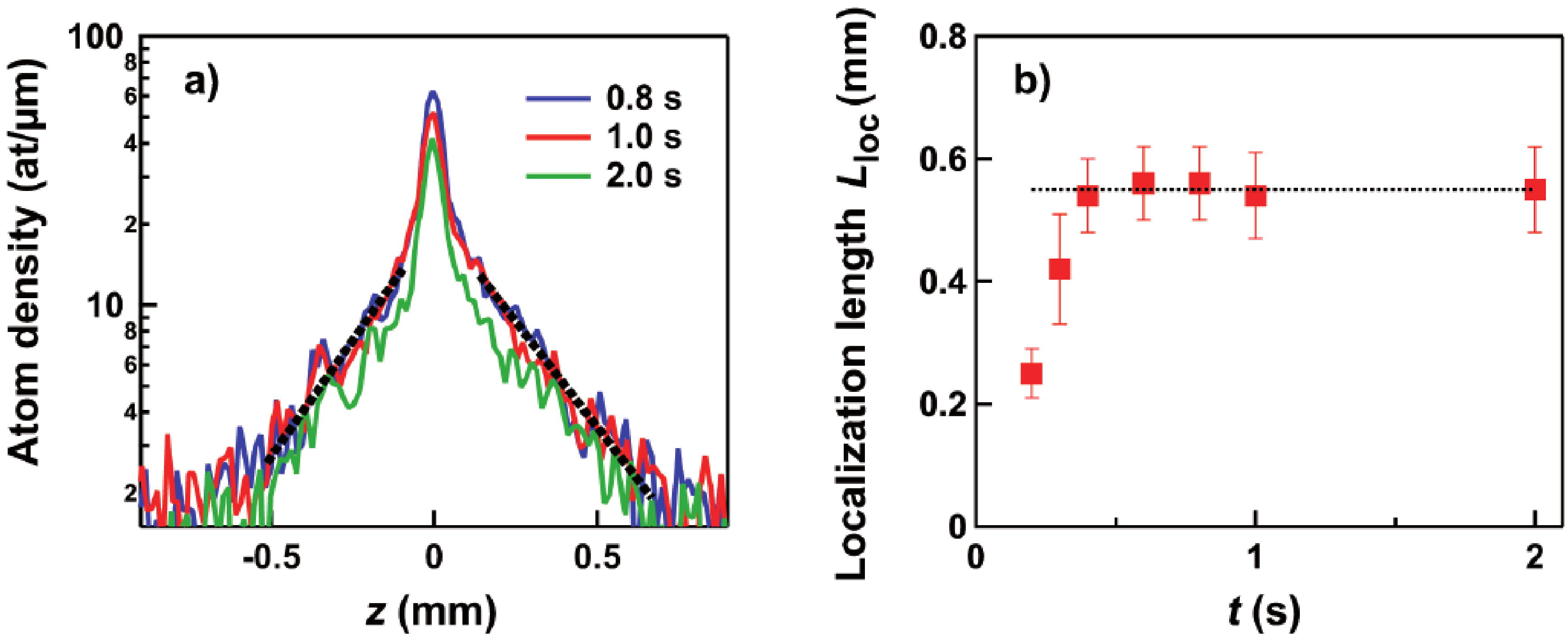


Log scale

The final atomic density shows exponential localization with localization length of few 100μm!

J. Billy et al, Institut d'Optique (Palaiseau, France), Nature, 453, 891 (2008)

Temporal dynamics for 1d Anderson localization of atomic matter waves



J. Billy et al, Institut d'Optique (Palaiseau, France), Nature, 453, 891 (2008)

1d Anderson localization of atomic matter waves

- Initial atomic cloud produced from dilution of a Bose-Einstein condensate => non mono-energetic initial wave-packet.
- All k atomic wavectors in a range $[-k_{max}, k_{max}]$ are populated => the long distance behaviour is dominated by k_{max}.
- Fluctuations are smoothed out by incoherent superposition of various k values.
- Lowest perturbation order (Born approximation) for the localization length:

$$\xi_{\mathrm{loc}} = \frac{\hbar^4 k_{\max}^2}{\pi m^2 V_0^2 \sigma (1 - k_{\max}\sigma)}$$

correlation length of the potential

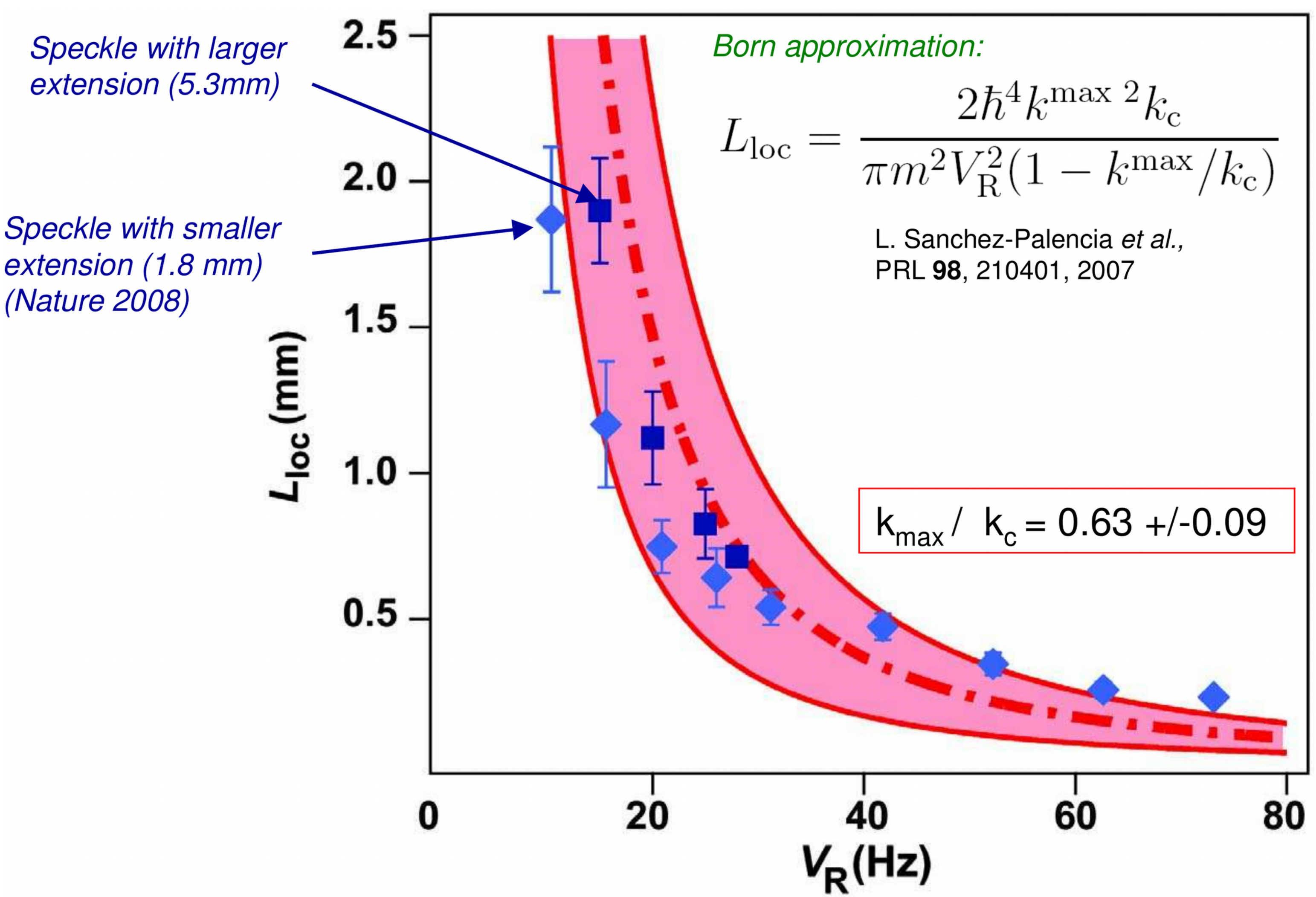
potential strength

- Observation of Anderson localization requires small $k_{\max}\sigma$
  - Very cold atoms
  - Very small speckle spots

$$\sigma = 0.26\mu\mathrm{m} \quad k_{\max}\sigma = 0.65$$

- Apparent mobility edge at $k_{\max}\sigma = 1$, but no real metal-insulator transition.

Comparison between the measured and calculated localization lengths

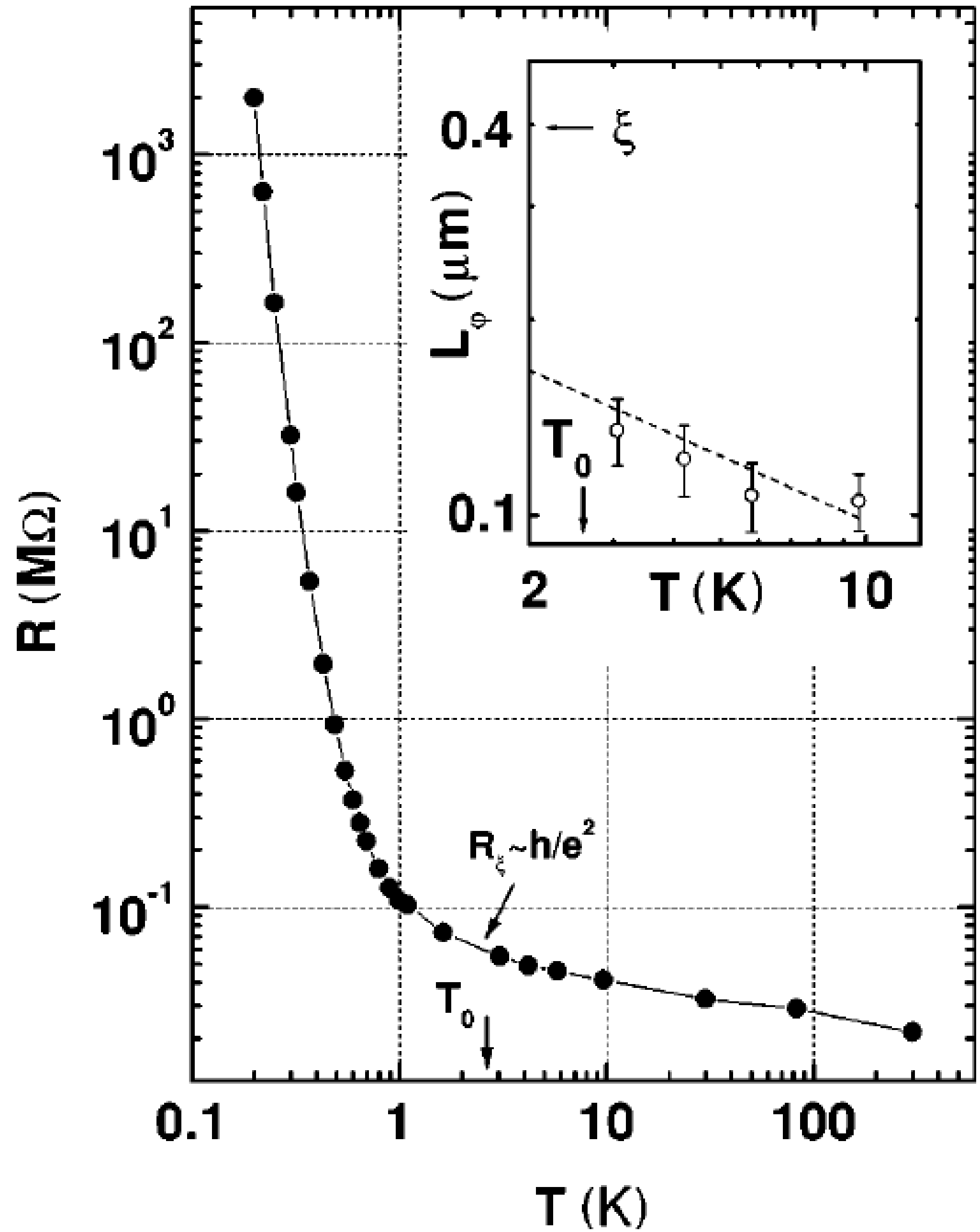


Disordered potential strength

J. Billy et al, Institut d'Optique (Palaiseau, France), Nature, 453, 891 (2008)

Anderson localization of electrons in a quasi-1d wire

- Measure resistance of a 1d wire vs. temperature
- When the phase coherence length becomes longer than the localization length => **exponential increase of the resistance**
- The phase coherence length is a smooth function of temperature



Y. Khavin et al, Phys. Rev. B 58, 8009 (1998)

Anderson localization of atomic matter waves in higher dimension?

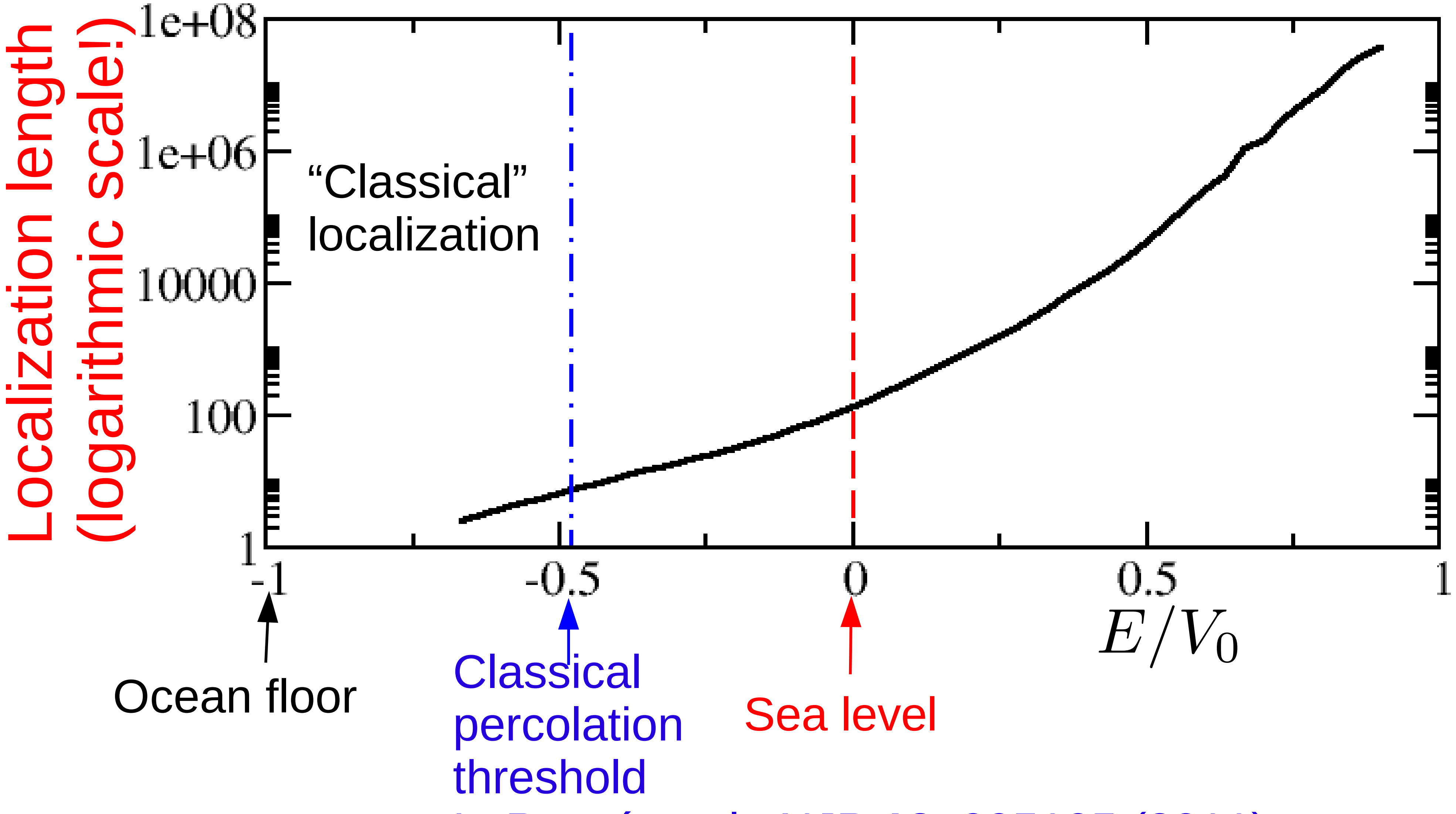
- In 2 dimensions: $\xi_{\text{loc}} \simeq \ell \exp\left(\frac{\pi k \ell}{2}\right)$ $\quad \ell : \text{mean free path}$
- Practical observation of Anderson localization requires very strong scattering:

$$k\ell \approx \text{few units}$$
$$k\sigma \approx 1$$

  - Very cold atoms
  - Powerful laser
  - State of the art 2d (or, even worse, 3d) speckle pattern

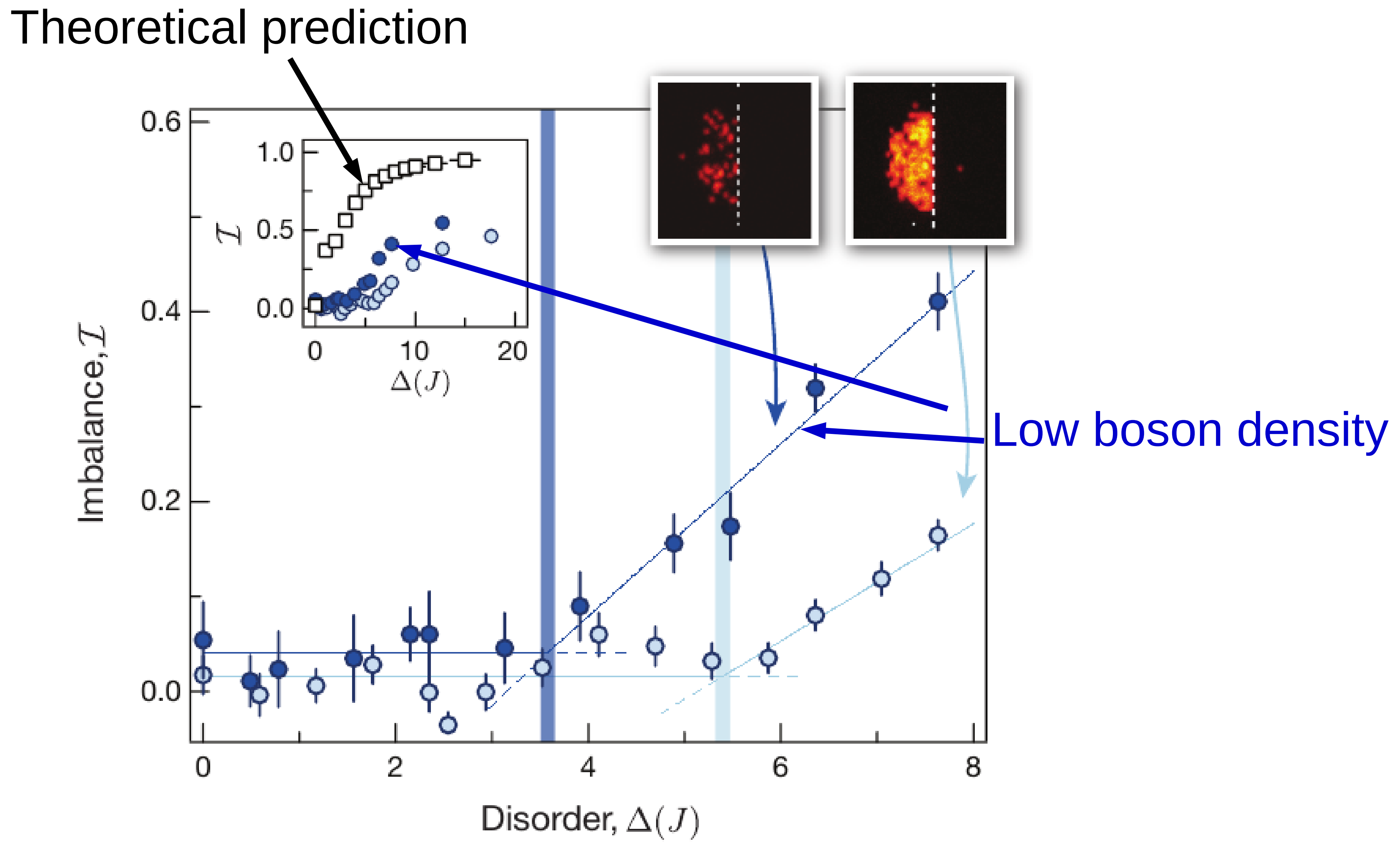
Localization length in a 2D blue speckle potential

$$E/E_\sigma = 1.5 \quad (k\sigma = \sqrt{3})$$



L. Pezzé et al., NJP **13**, 095105 (2011)

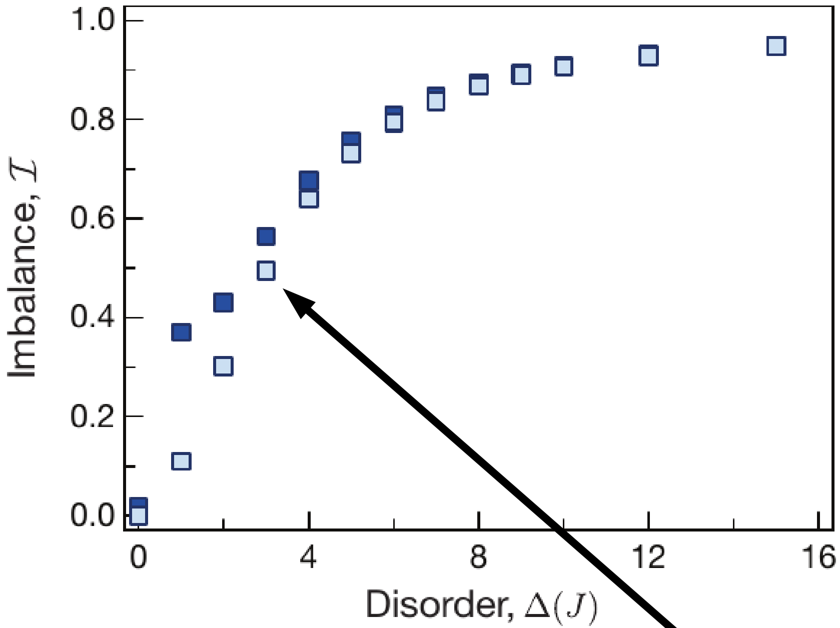
Anderson localization in 2d



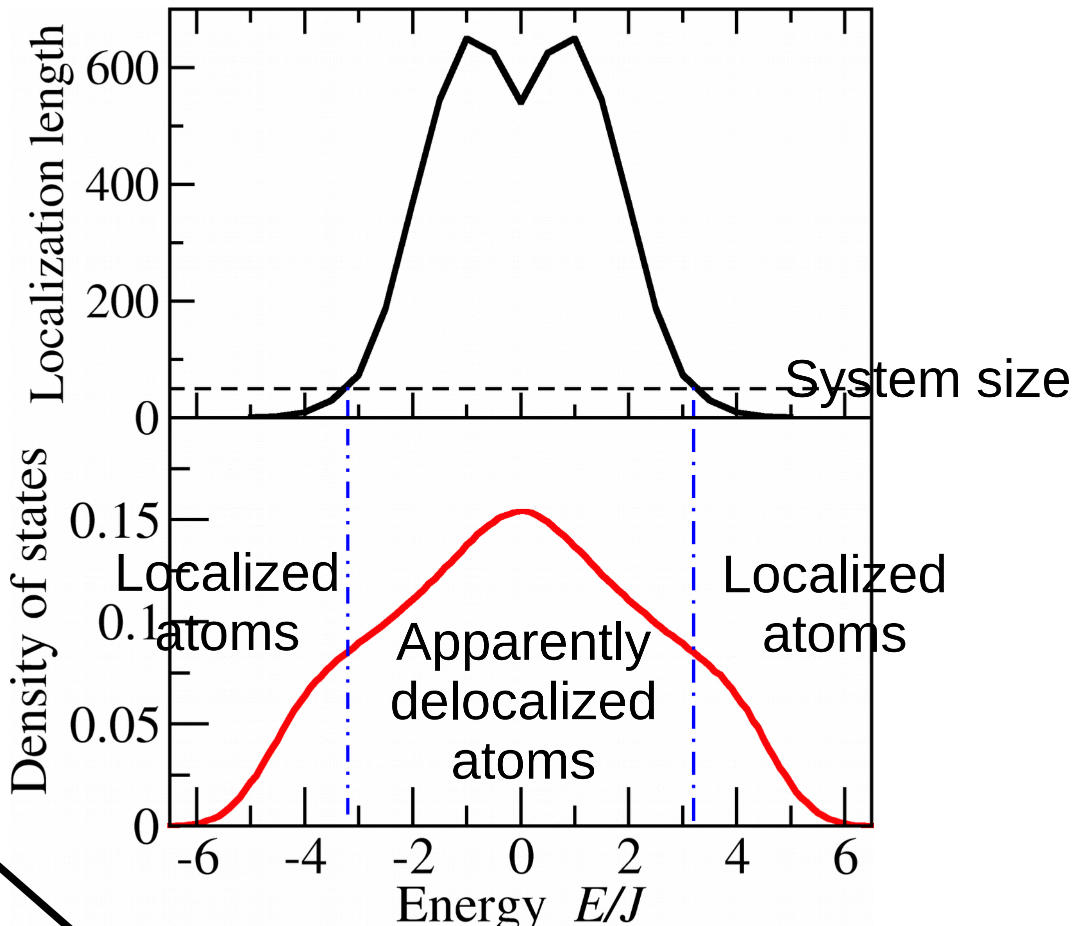
Choi et al, Science, 352, 1547 (2016)

Anderson localization in 2d

- Numerical results without interactions



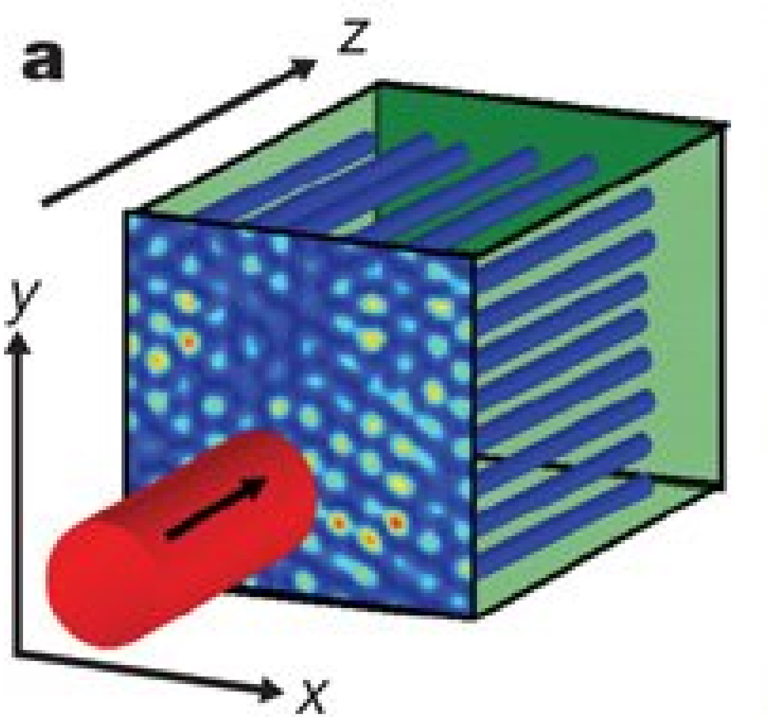
Choi et al, Science, 352, 1547 (2016)

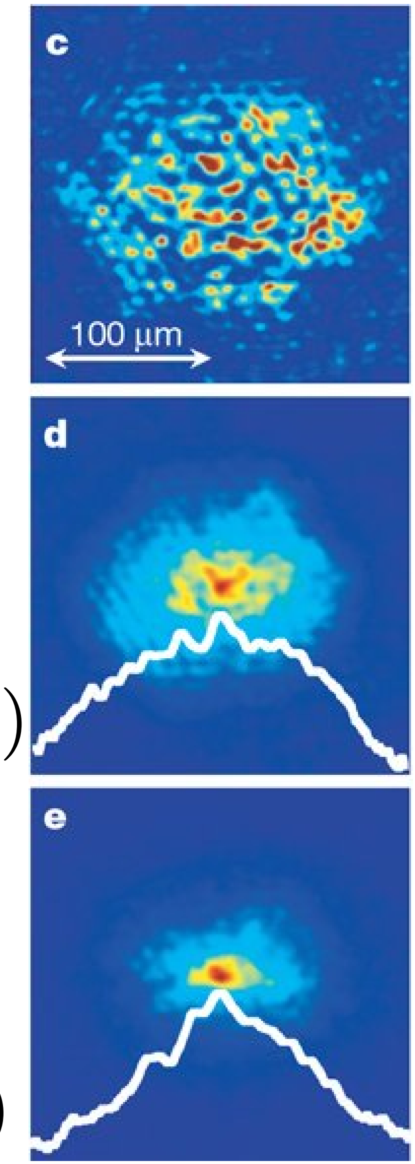


Coexistence of atoms with various energies and vastly different localization lengths

Numerical calculation for $\Delta/J \approx 3$

Anderson localization in 2d photonic lattices





Ballistic regime
(no disorder)

Diffusive regime
(small disorder)

$\ln(|\psi(x)|^2)$

Gaussian shape

Localized regime
(large disorder)

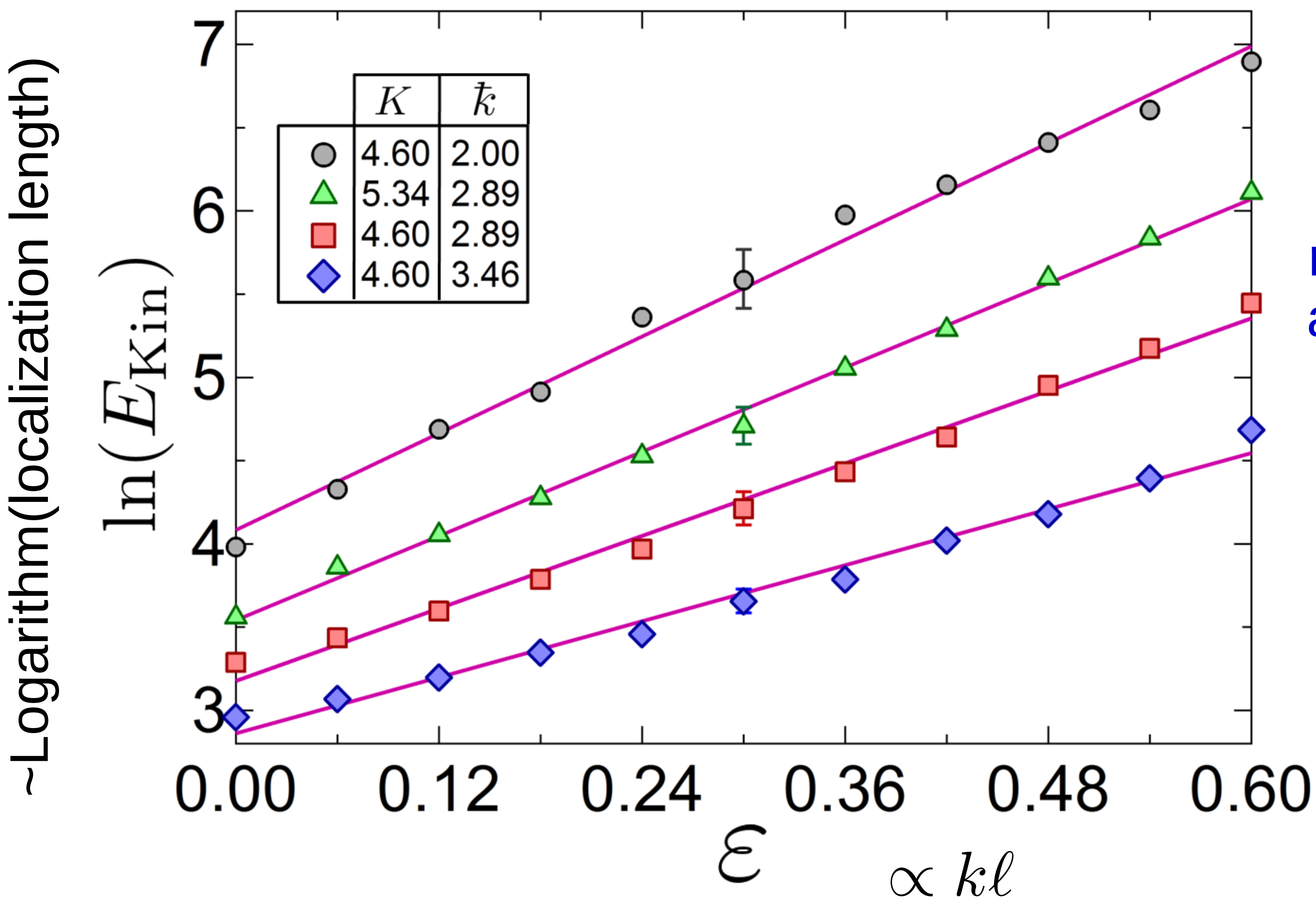
$\ln(|\psi(x)|^2)$

Exponential shape

T. Schwartz et al, Nature, 446, 52 (2007)

Two-dimensional Anderson localization

- Exponential dependence of the localization length with $k\ell$



Experimental results on the atomic kicked rotor

I. Manai et al., Phys. Rev. Lett. 115, 240603 (2015), arXiv:1504.04987

Anderson localization of atomic matter waves in higher dimension?

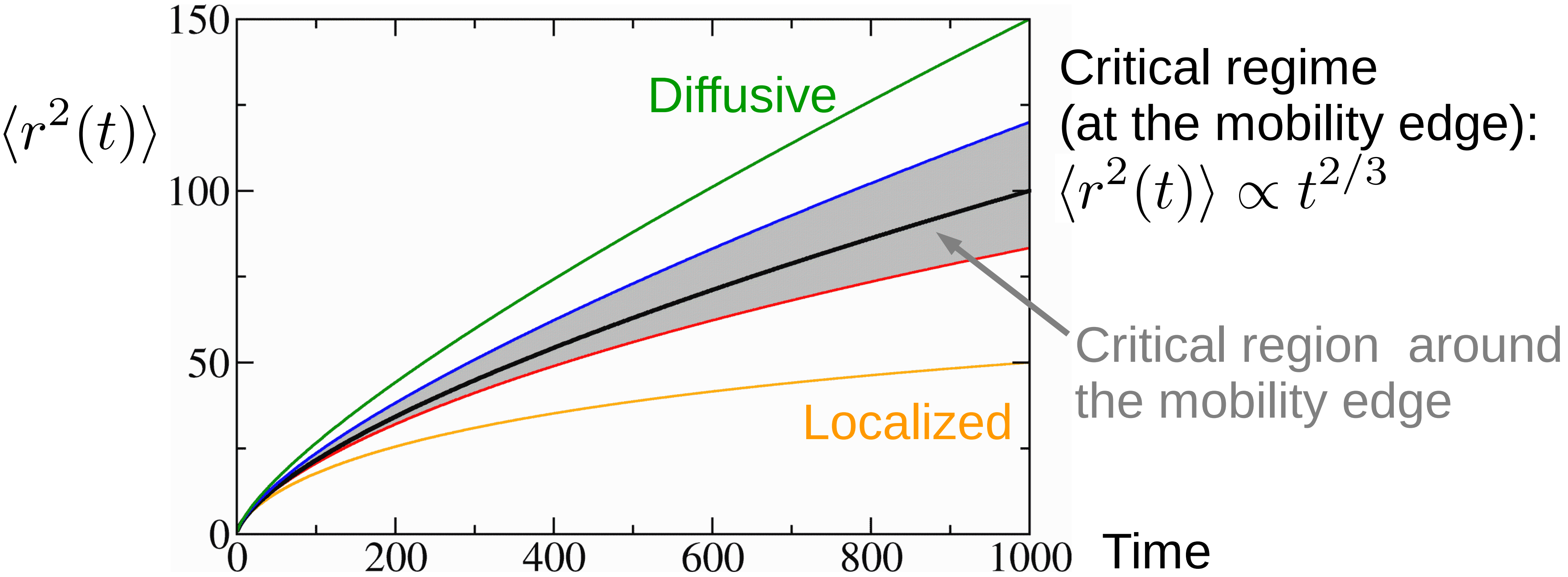
- In 2 dimensions: $\xi_{\mathrm{loc}} \simeq \ell \exp\left(\frac{\pi k \ell}{2}\right)$ $\quad \ell : \mathrm{mean\ free\ path}$
- In 3 dimensions, Ioffe-Regel criterion: $(k\ell)_{\mathrm{c}} = 1$
- Practical observation of Anderson localization requires very strong scattering:

$$k\ell \approx \mathrm{few\ units}$$
$$k\sigma \approx 1$$

  - Very cold atoms
  - Powerful laser
  - State of the art 2d (or, even worse, 3d) speckle pattern

Temporal dynamics in configuration space

- The asymptotically localized density in configuration space is reached very slowly (in any dimension).
- 3D dynamics close to the mobility edge is "critical" at short time => enormously long time to determine whether the dynamics is diffusive or localized.



in configuration space

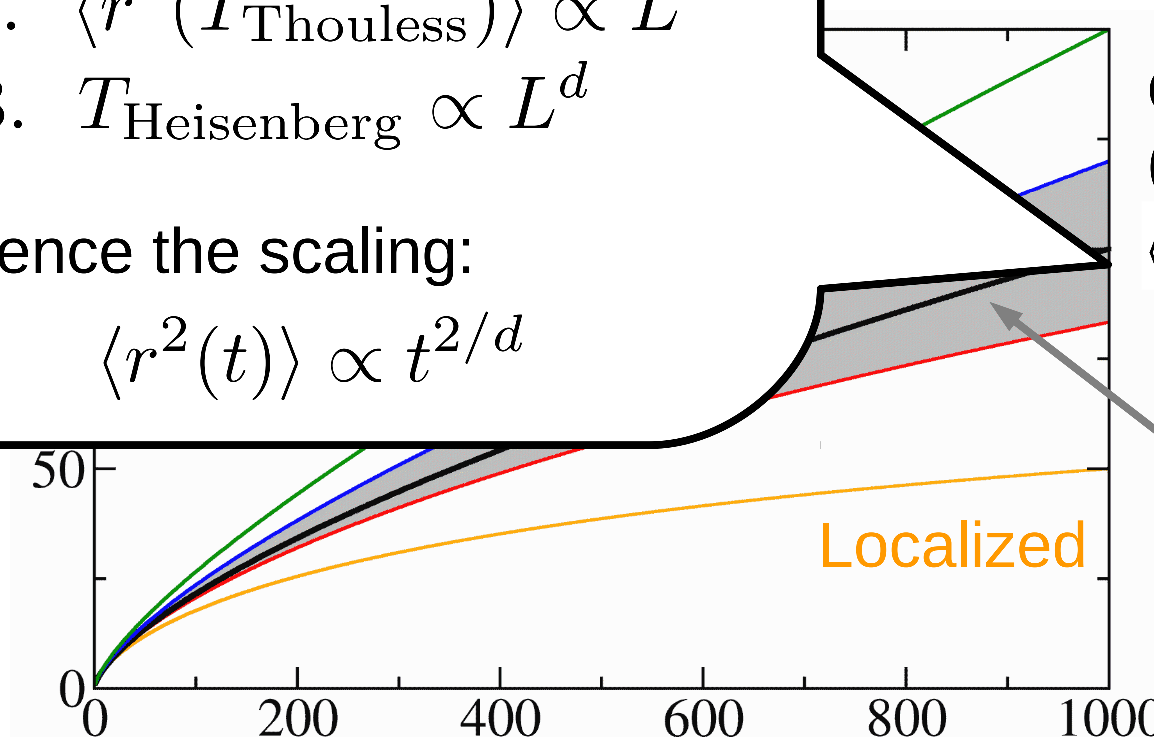
density in configuration space
dimension).

bility edge is “critical” at short
to determine whether the
ized.

Why $t^{2/3}$?

- At the critical point:
  1. $T_{\text{Heisenberg}} \propto T_{\text{Thouless}}$
  2. $\langle r^2(T_{\text{Thouless}})\rangle \propto L^2$
  3. $T_{\text{Heisenberg}} \propto L^d$
- Hence the scaling:

$$\langle r^2(t)\rangle \propto t^{2/d}$$



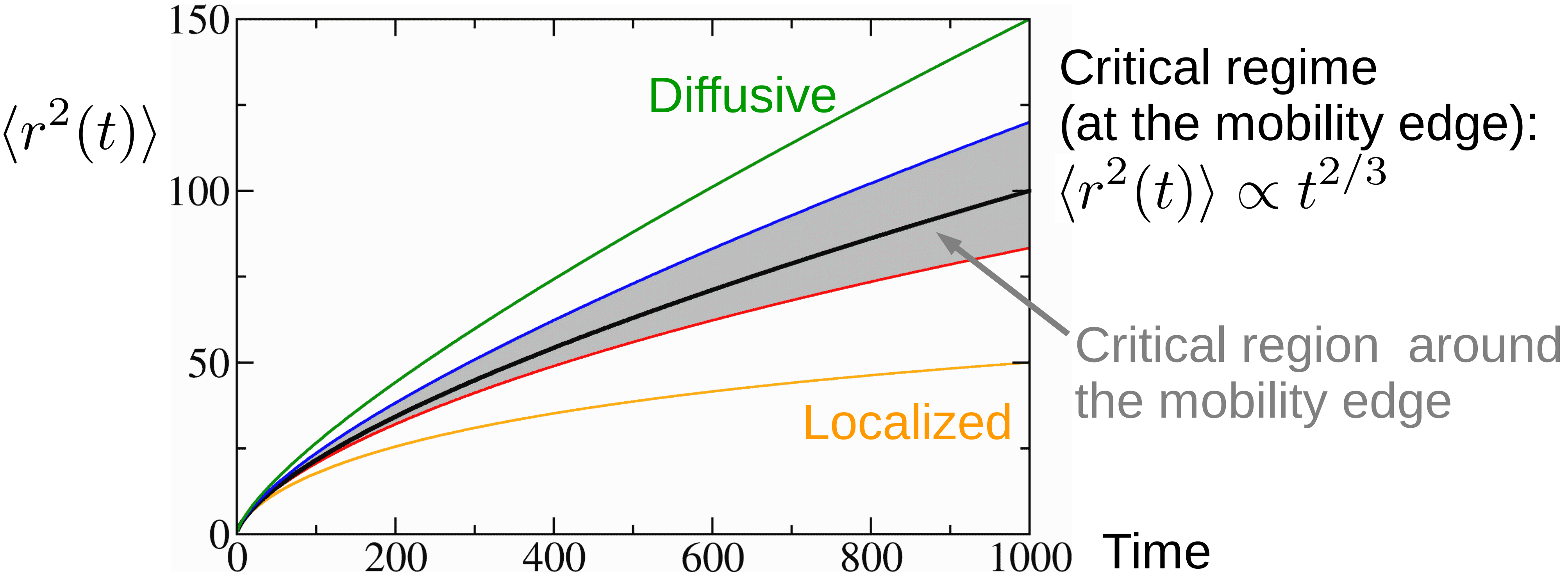
Time

Critical regime
(at the mobility edge):
$\langle r^2(t)\rangle \propto t^{2/3}$

Critical region around
the mobility edge

Temporal dynamics in configuration space

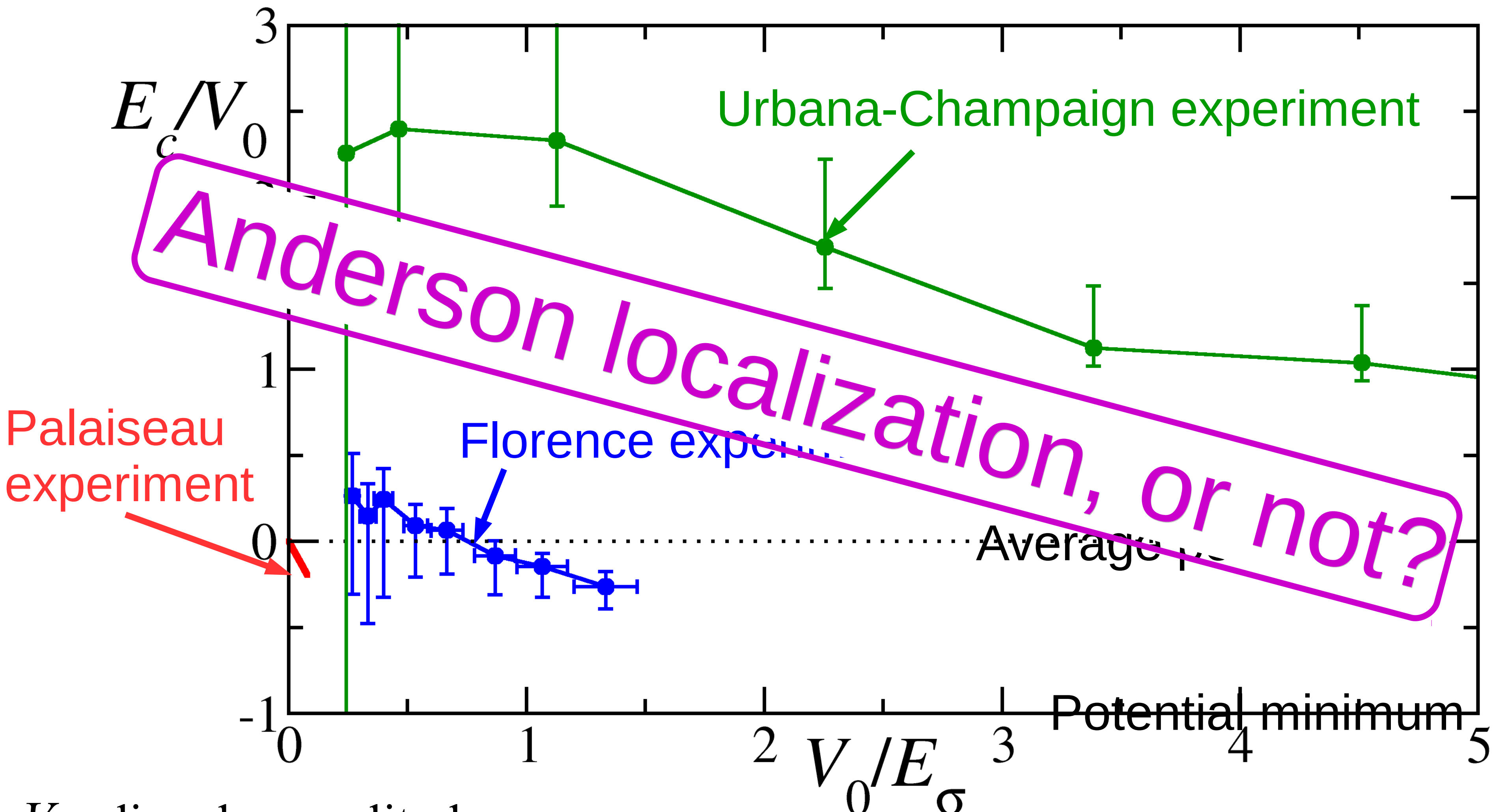
- The asymptotically localized density in configuration space is reached very slowly (in any dimension).
- 3D dynamics close to the mobility edge is “critical” at short time => enormously long time to determine whether the dynamics is diffusive or localized.



Critical regime (at the mobility edge): $\langle r^2(t) \rangle \propto t^{2/3}$

Where is the 3d mobility edge in a speckle optical potential?

- Three different experiments give very different results...

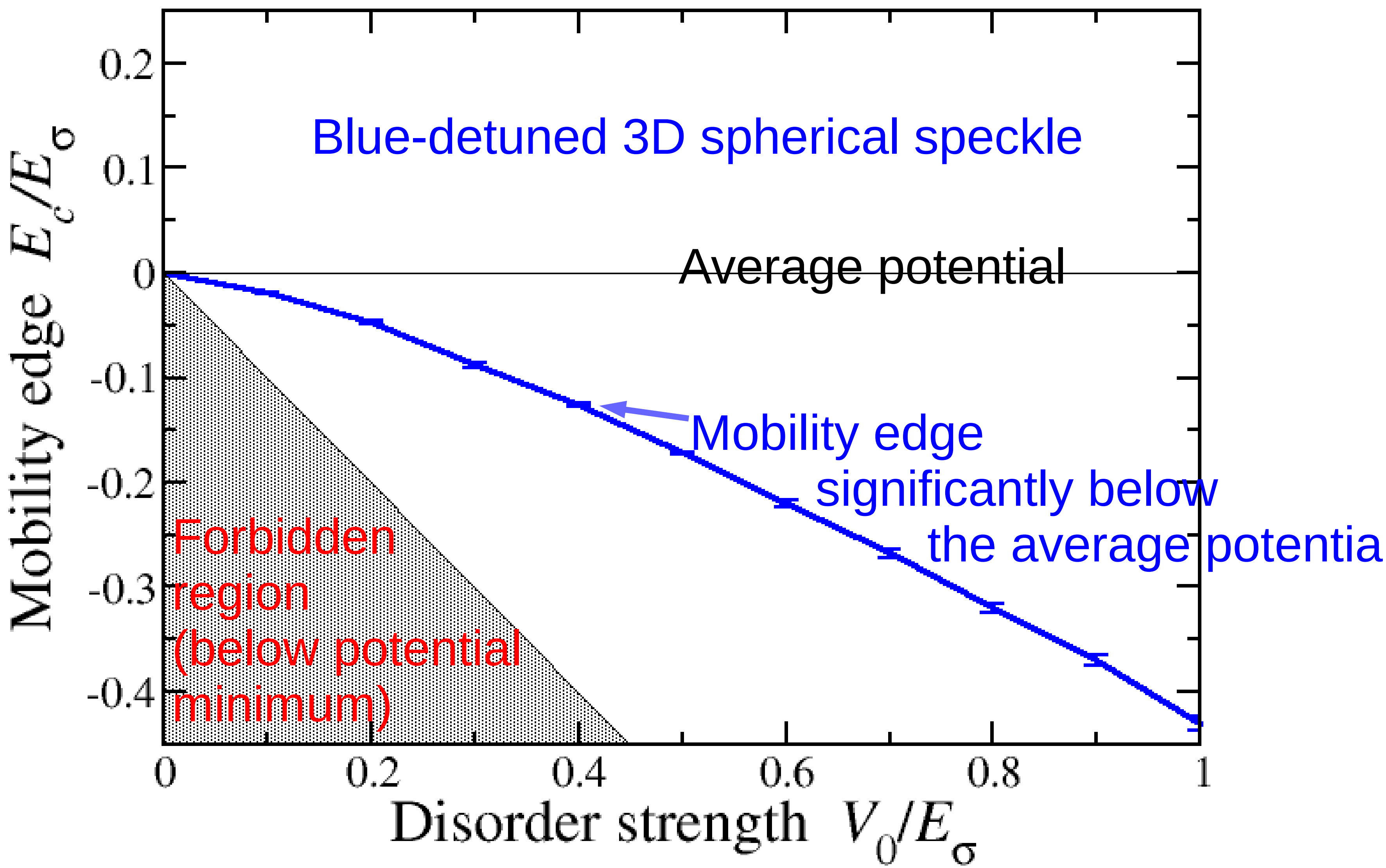


V_0: disorder amplitude

E_c: mobility edge

$$E_\sigma = \frac{\hbar^2}{m\sigma^2} : \text{correlation energy}$$

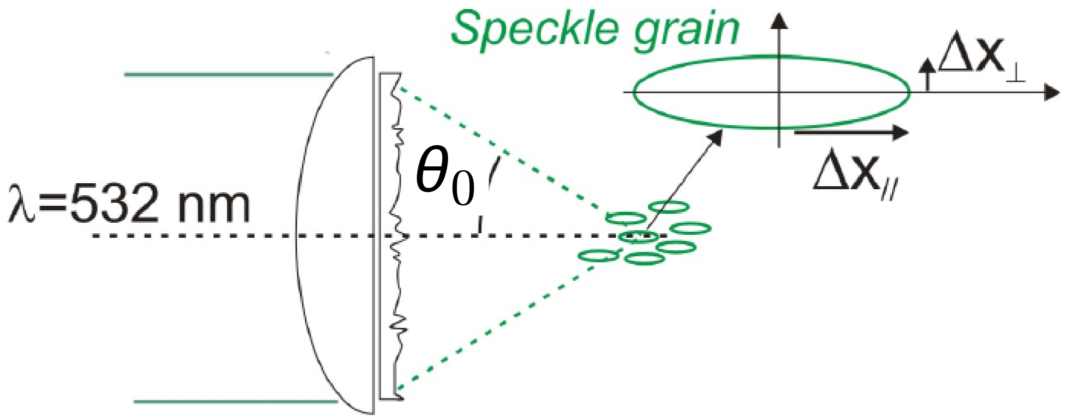
Numerical results for the mobility edge



Can be semi-quantitatively predicted using
self-consistent theory of localization...

Anisotropic correlated speckle potential

- 3D speckle has anisotropic correlation functions, depending on the numerical aperture NA of the laser beam



courtesy V. Josse

Speckle spot size in the transverse direction:

$$\Delta \mathrm{x}_\perp = \sigma_\perp \approx \frac{\lambda}{\theta_0}$$

Speckle spot size in the longitudinal direction *x*:

$$\Delta \mathrm{x}_\parallel = \sigma_\parallel \approx \frac{\lambda}{{\theta_0}^2} \qquad \frac{\sigma_\parallel}{\sigma_\perp} \approx 5 - 10$$

λ : laser wavelength $\theta_0 =$ NA: Numerical Aperture

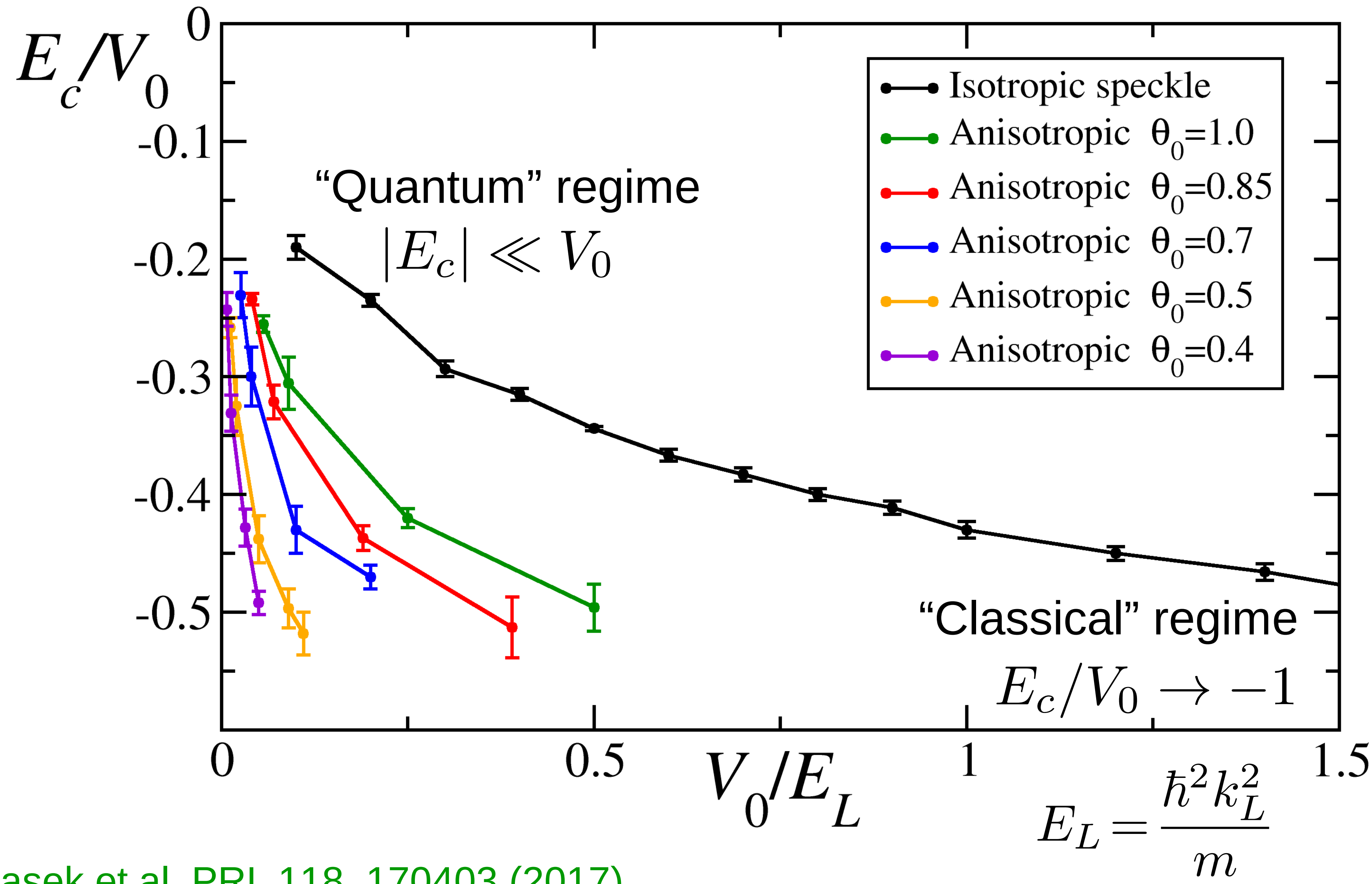
- Crucial question: what is the relevant energy scale (correlation energy)?

$$\frac{1}{\sigma_\perp^2}? \qquad \frac{1}{\sigma_\parallel^2}?$$ some other combination?

- Define an energy scale independent of θ_0: $$E_L = \frac{\hbar^2 k_L^2}{m}$$ $$k_L = \frac{2\pi}{\lambda}$$ Laser wave-vector

Mobility edge E_c for an anisotropic speckle

- E_c is negative (below average potential V_0) for a blue-detuned speckle
- Strongly depends on θ_0.



Pasek et al, PRL 118, 170403 (2017)

What is the correlation energy for anisotropic disorder?

- In the vicinity of a potential minimum, the "typical" potential is:

$$V = V_0 \left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2} \right)$$

- Because the random disorder couples the 3 directions, the ground state energy in this potential is IRRELEVANT.
- What matters is the geometry of the energy shell.
- The relevant parameter is thus $\sqrt[3]{\sigma_x \sigma_y \sigma_z}$
- Same scaling obtained when computing the weak localization correction (minimum size of the closed loops).
- Correlation energy:

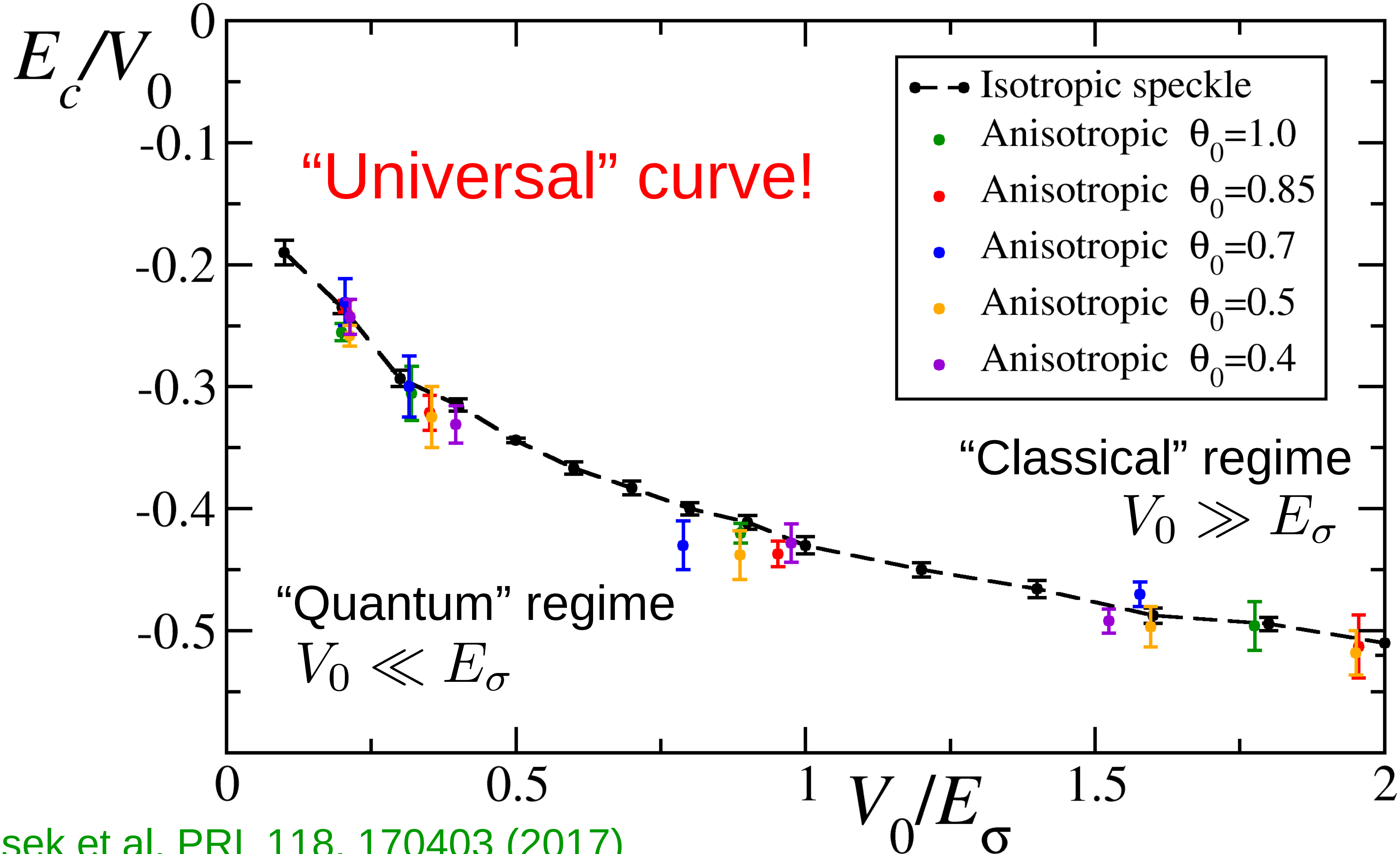
$$E_\sigma = \frac{\hbar^2}{m(\sigma_x \sigma_y \sigma_z)^{2/3}}$$

Approximate scaling property for the mobility edge

- Define a proper "correlation" energy:

$$E_\sigma = \frac{1}{(\sigma_\perp^2 \sigma_\parallel)^{2/3}} \quad (\times \hbar^2/m)$$

$E_\sigma \propto \theta_0^{8/3}$ at small θ_0

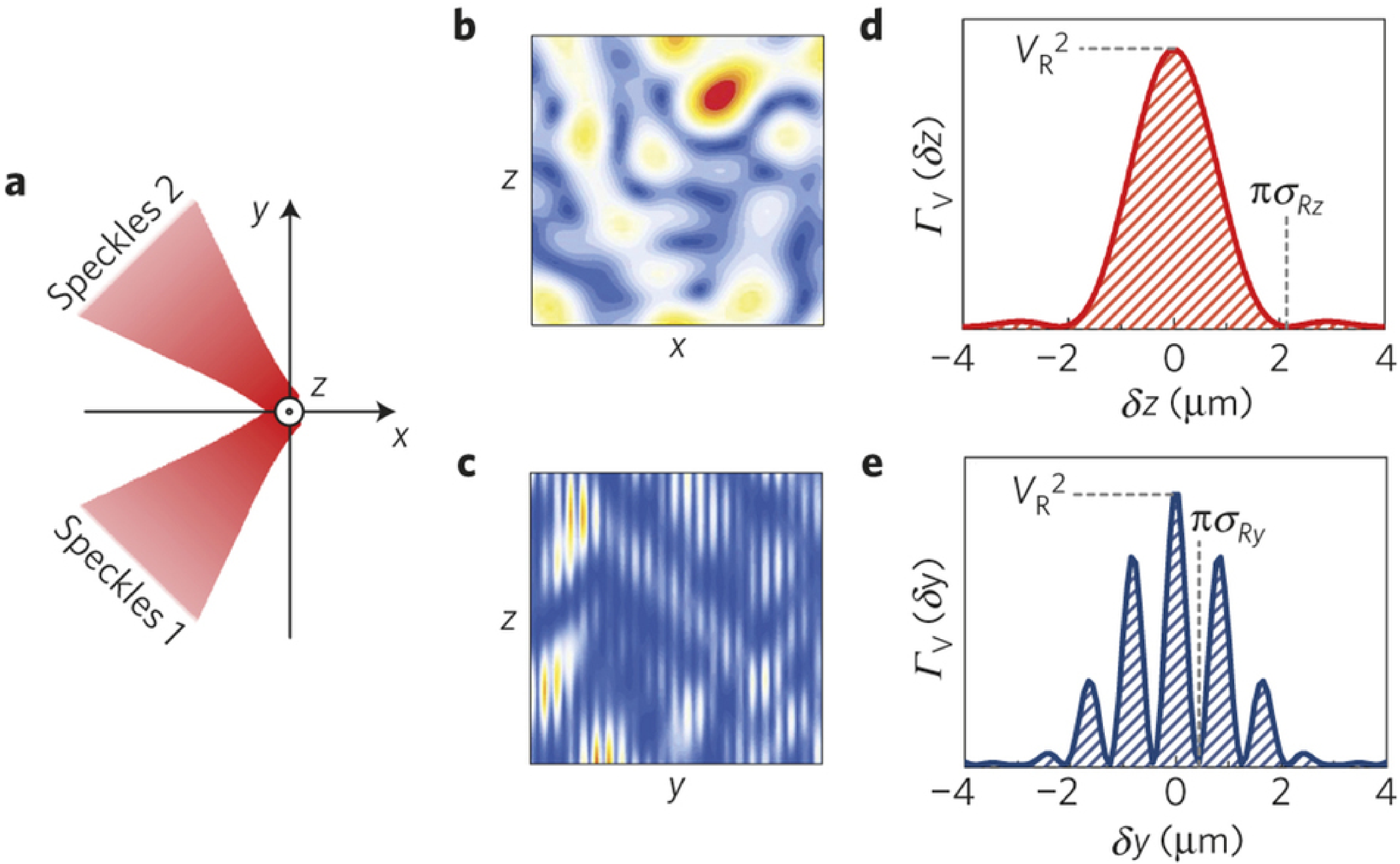


Pasek et al, PRL 118, 170403 (2017)

Two (coherent) crossed speckles

- Use two different interfering (coherent) crossed speckles

Disorder correlation functions



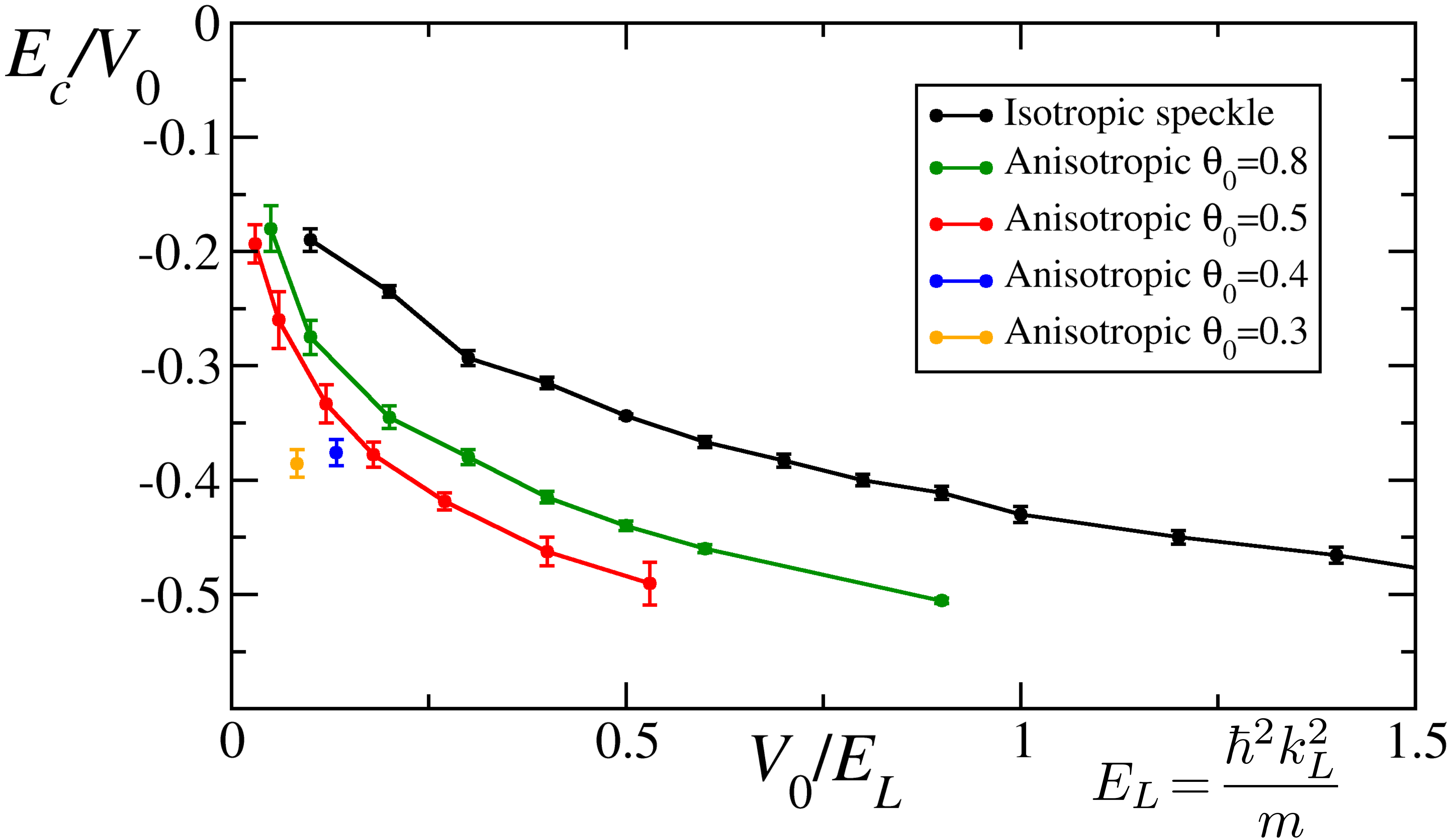
Short correlation length in all 3 directions!

Which scale along the *y* axis is relevant?

Shamelessly stolen from G. Semeghini et al., Nat. Phys. 11, 554 (2015)

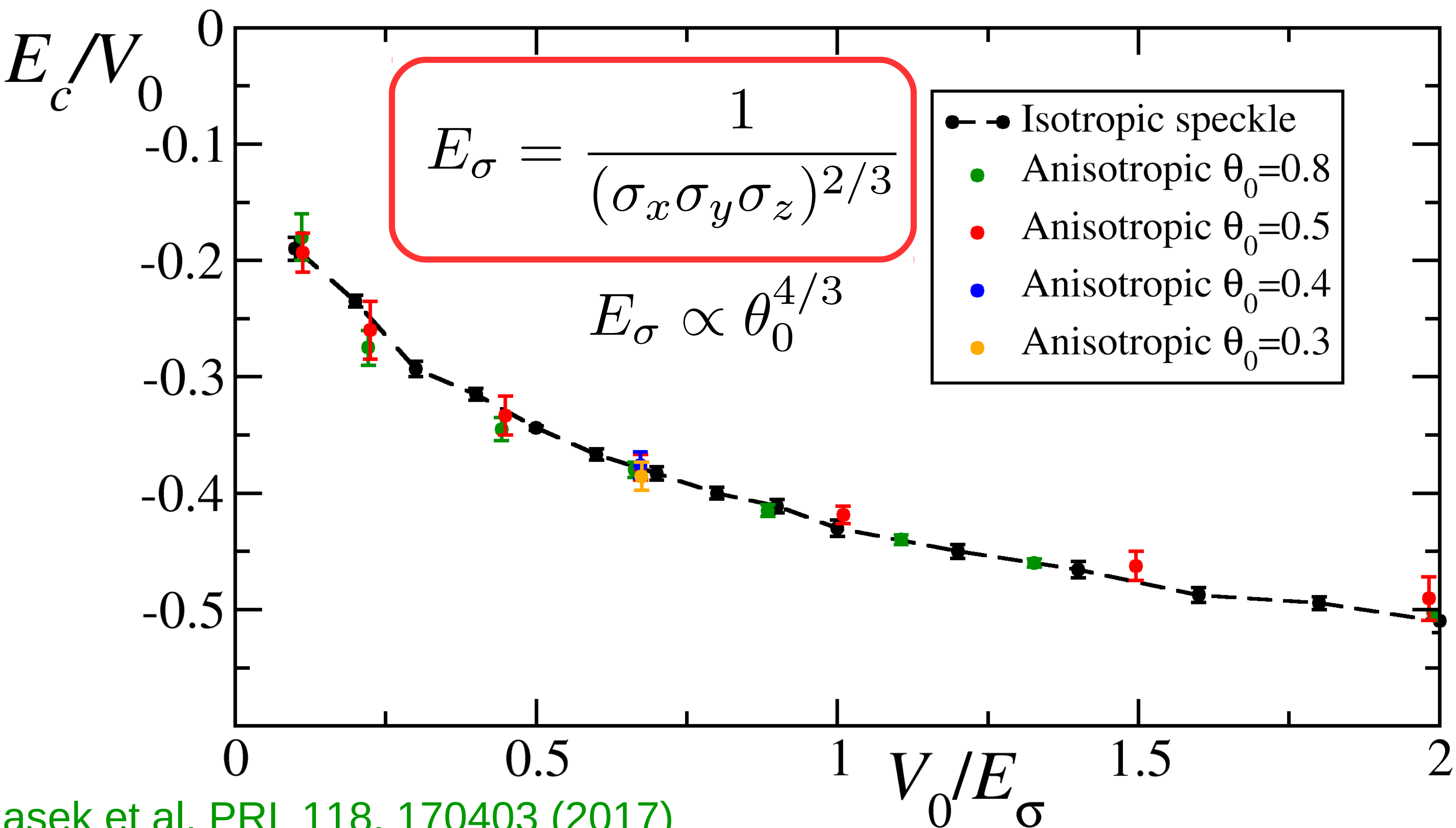
Mobility edge E_c for two crossed speckles

- The relevant energy scale is smaller than for isotropic speckle, but larger than for a single anisotropic speckle
- The mobility edge E_c is always negative for a blue-detuned speckle
- Again, strong dependence with the numerical aperture θ_0



Mobility edge E_c for two crossed speckles

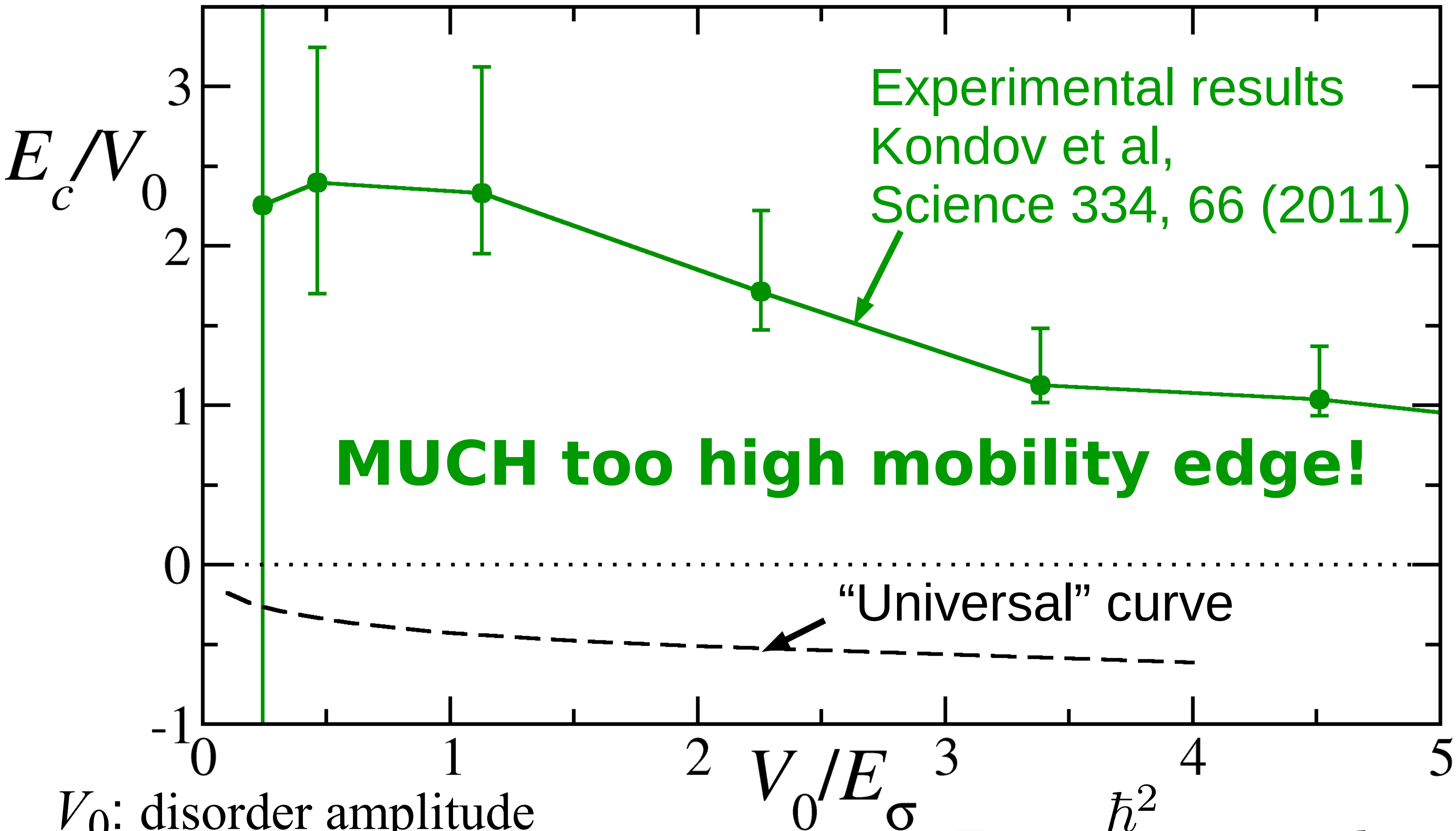
- The relevant energy scale is smaller than for isotropic speckle, but larger than for a single anisotropic speckle
- The mobility edge E_c is always negative for a blue-detuned speckle
- Again, approximate scaling for the relevant correlation energy:



Pasek et al, PRL 118, 170403 (2017)

Experimental results on the mobility edge

- 1. Group of B. De Marco (Urbana Champaign). Very anisotropic single speckle. Relatively well defined energy (fermionic atoms). Duration of the experiment too short to detect slow diffusion above the mobility edge.



V_0: disorder amplitude

E_c: mobility edge

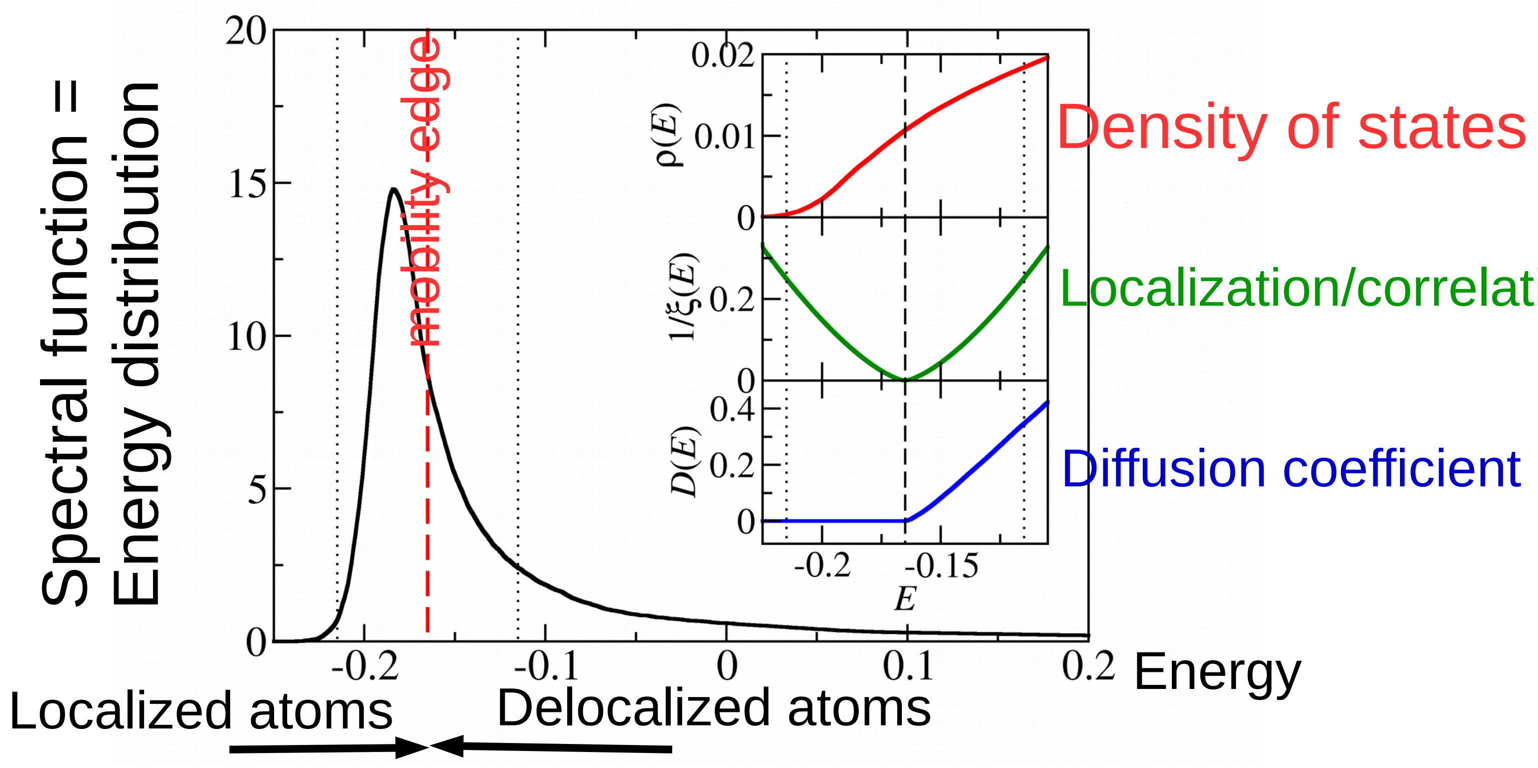
$$E_\sigma = \frac{\hbar^2}{m\sigma^2} \text{ : correlation energy}$$

Experimental results on the mobility edge

- 2. Group of V. Josse (Palaiseau). Two crossed speckles, rather low fraction of atoms below the mobility edge (energy distribution is not directly measured).
  - Start with atom with *k*=0, but disorder is branched abruptly
  - The energy distribution is given by the spectral function at *k*=0

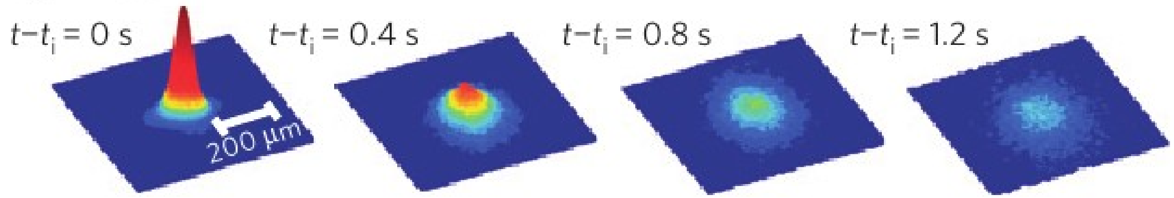
$$A(E,k) = \overline{\langle k|\delta(H-E)|k\rangle}$$

  - The spectral function is broad for strong disorder => atoms are excited on both sides of the mobility edge.



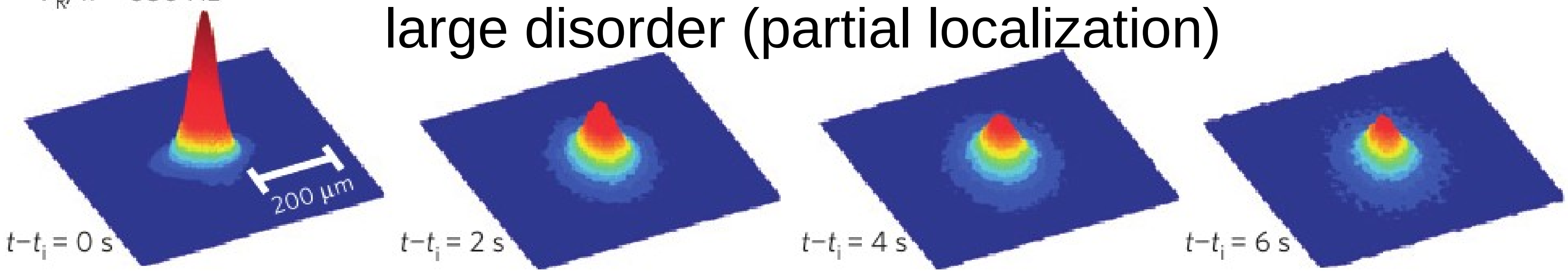
3d localization with ultra-cold bosons (Palaiseau)

small disorder (no localization)

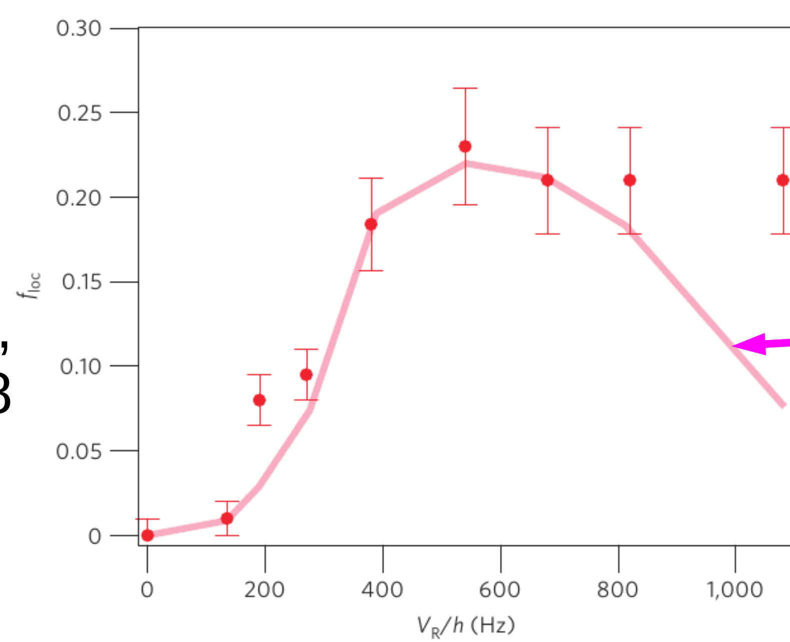


Temporal expansion of a localized wavepacket in the presence of disorder

large disorder (partial localization)



fraction of localized atoms



experiment

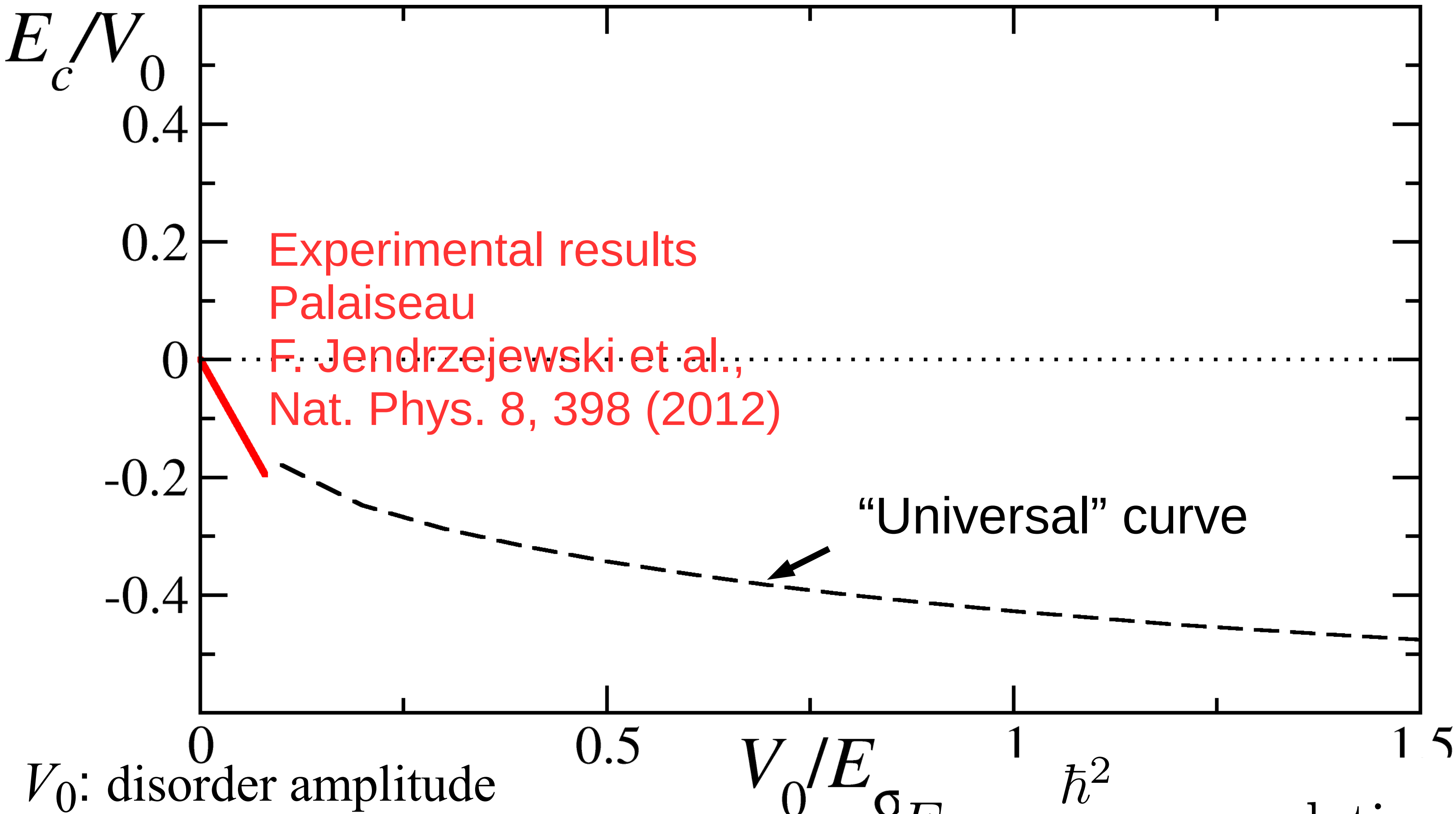
Self-consistent theory (corrected by “heuristic” shift)

potential strength

F. Jendrzejewski et al, Nature Physics **8**, 398 (2012)
Institut d'Optique (Palaiseau)

Experimental results on the mobility edge

- 2. Group of V. Josse (Palaiseau). Two crossed speckles, rather low fraction of atoms below the mobility edge (energy distribution is not directly measured). Measured mobility edge below zero, in decent agreement. Mostly in the “quantum” regime.



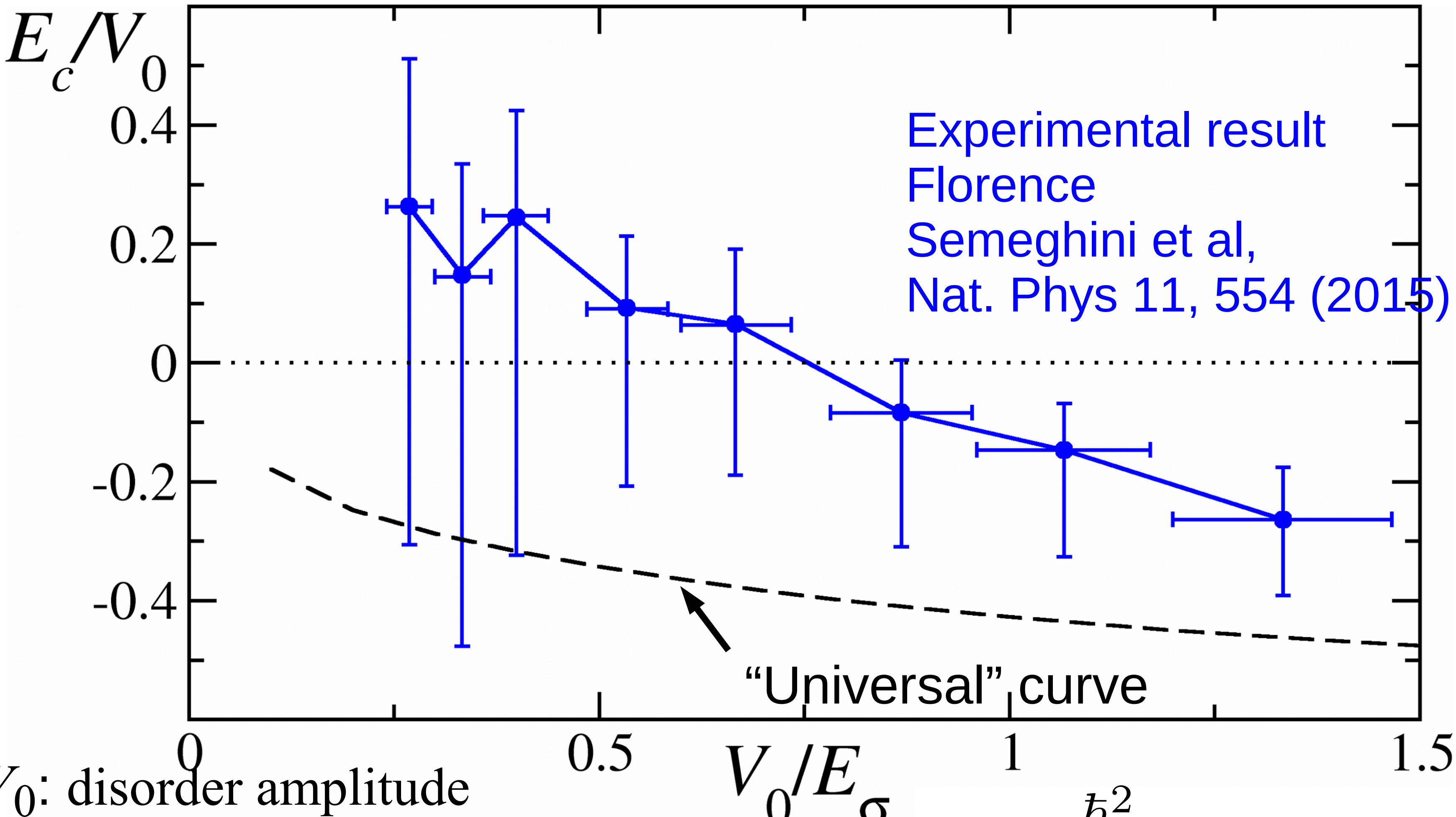
V_0: disorder amplitude

E_c: mobility edge

$E_\sigma = \dfrac{\hbar^2}{m\sigma^2}$: correlation energy

Experimental results on the mobility edge

- 3. Group of G. Modugno (Florence).Two crossed speckles, large localized fraction, thanks to control of atom-atom interactions. Qualitative behavior of the mobility edge with V_0 is not too bad. Mobility edge is quantitatively too high.



V_0: disorder amplitude

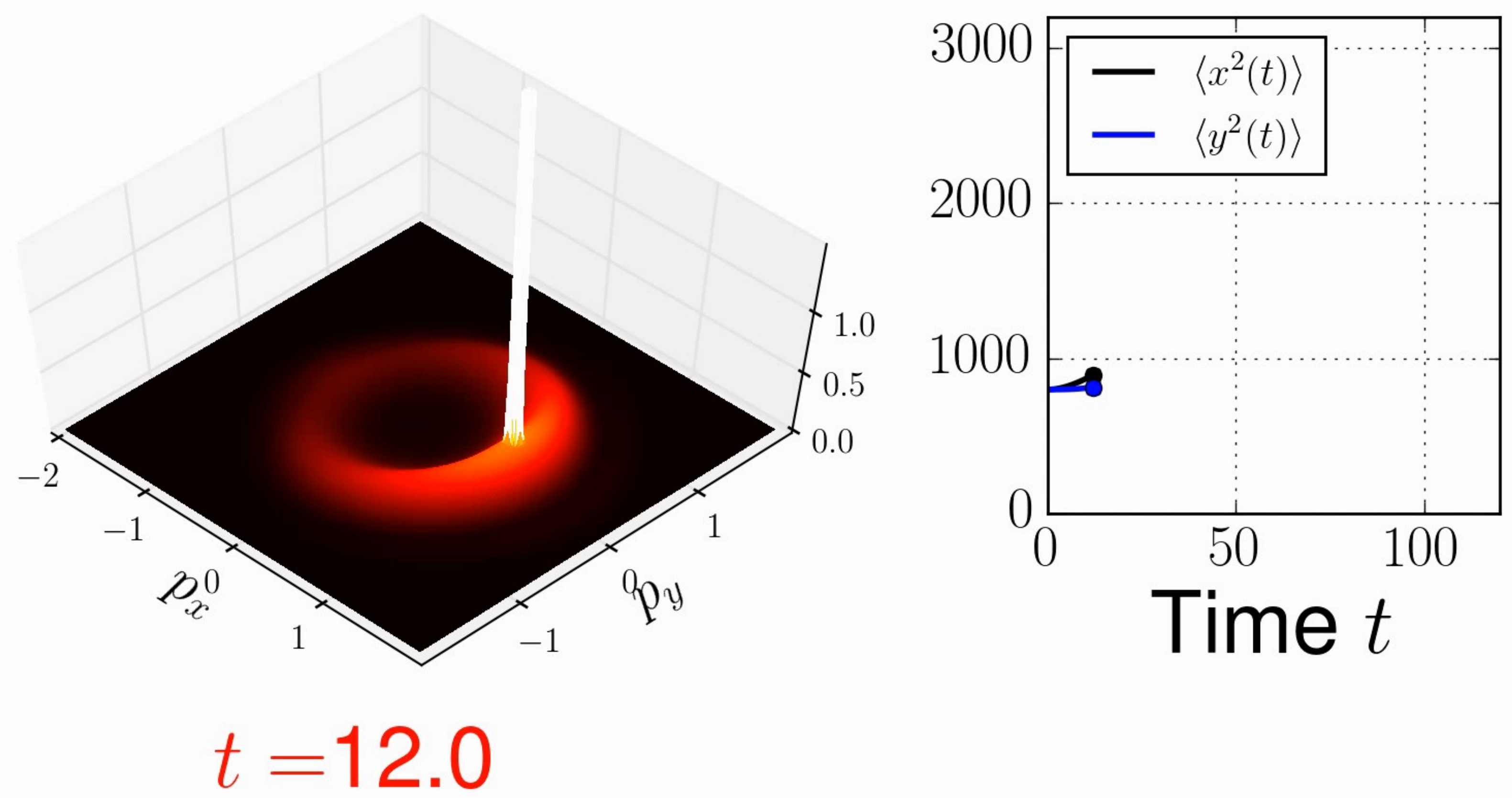
E_c: mobility edge

$$E_\sigma = \frac{\hbar^2}{m\sigma^2} : \text{correlation energy}$$

Pasek et al, PRL 118, 170403 (2017)

Temporal dynamics in momentum space

- Start from a wavepacket with non-zero initial velocity.
- Weak disorder: scattering by disorder to different direction, but with roughly the same velocity => isotropization of the momentum distribution

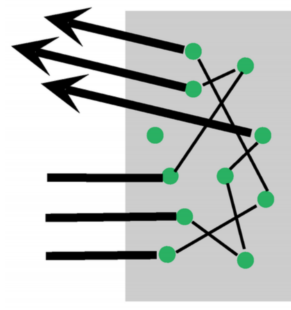


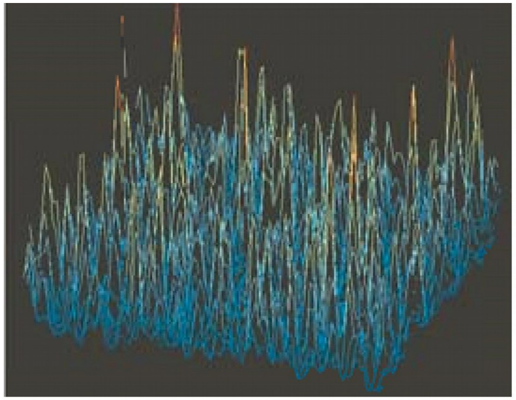
Initial momentum $p_x = 1.0,\ p_y = 0.0$

$V_0/E_\sigma = 0.2$

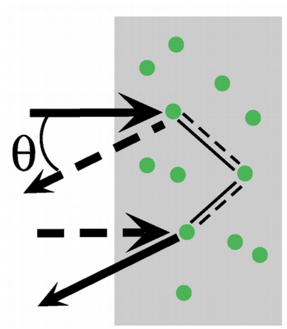
Coherent back-scattering (CBS) of light

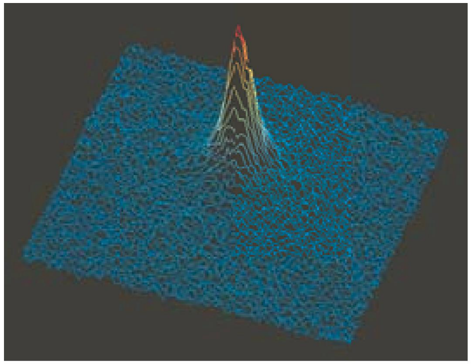
- In general, the interference between multiply scattered paths produces a random pattern => speckle



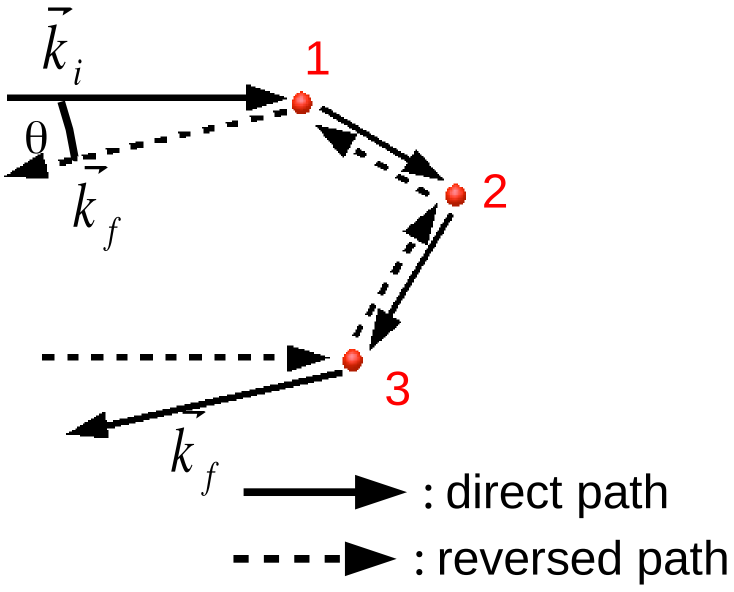


- When averaged over disorder realizations, the fluctuation are washed out, except in the backward direction => CBS





- The physics of CBS:



Phase difference between the two contributions:

$$\Delta\phi = (\vec{k}_i + \vec{k}_f).(\vec{r}_3 - \vec{r}_1) \simeq k\ell\theta$$

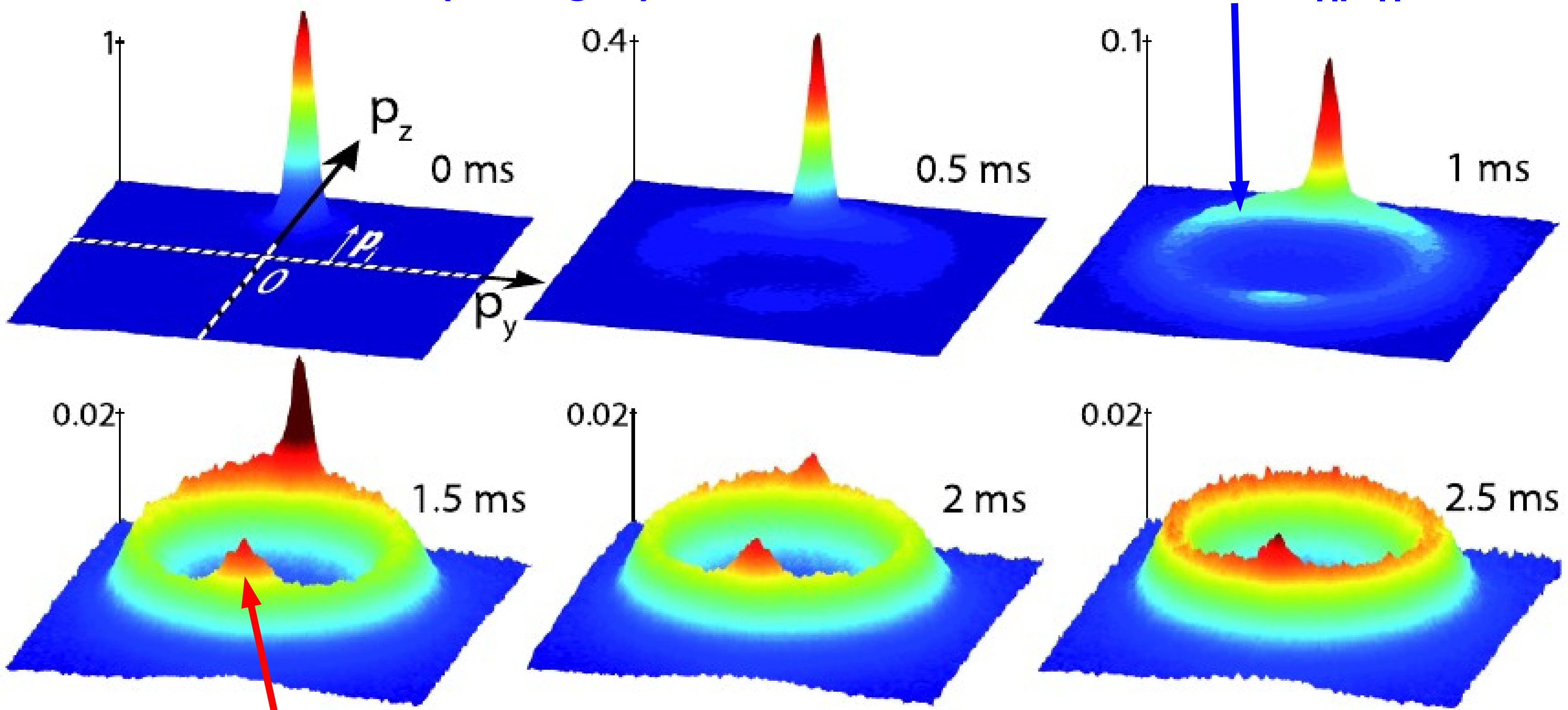
Constructive interference between any pair of reversed paths in the back-scattering direction => enhancement factor of the order of 2.

Experimental observation of CBS with cold atoms (2D)

- Experiment at Institut d'Optique (Palaiseau), weak disorder
- One measures the momentum space distribution $\overline{|\psi(p)|^2}$

Initial momentum distribution (along *z*)

Scattering modifies *p*, at fixed ||*p*||



Coherent Back-scattering of atomic matter waves (signature of quantum transport preserving interference)

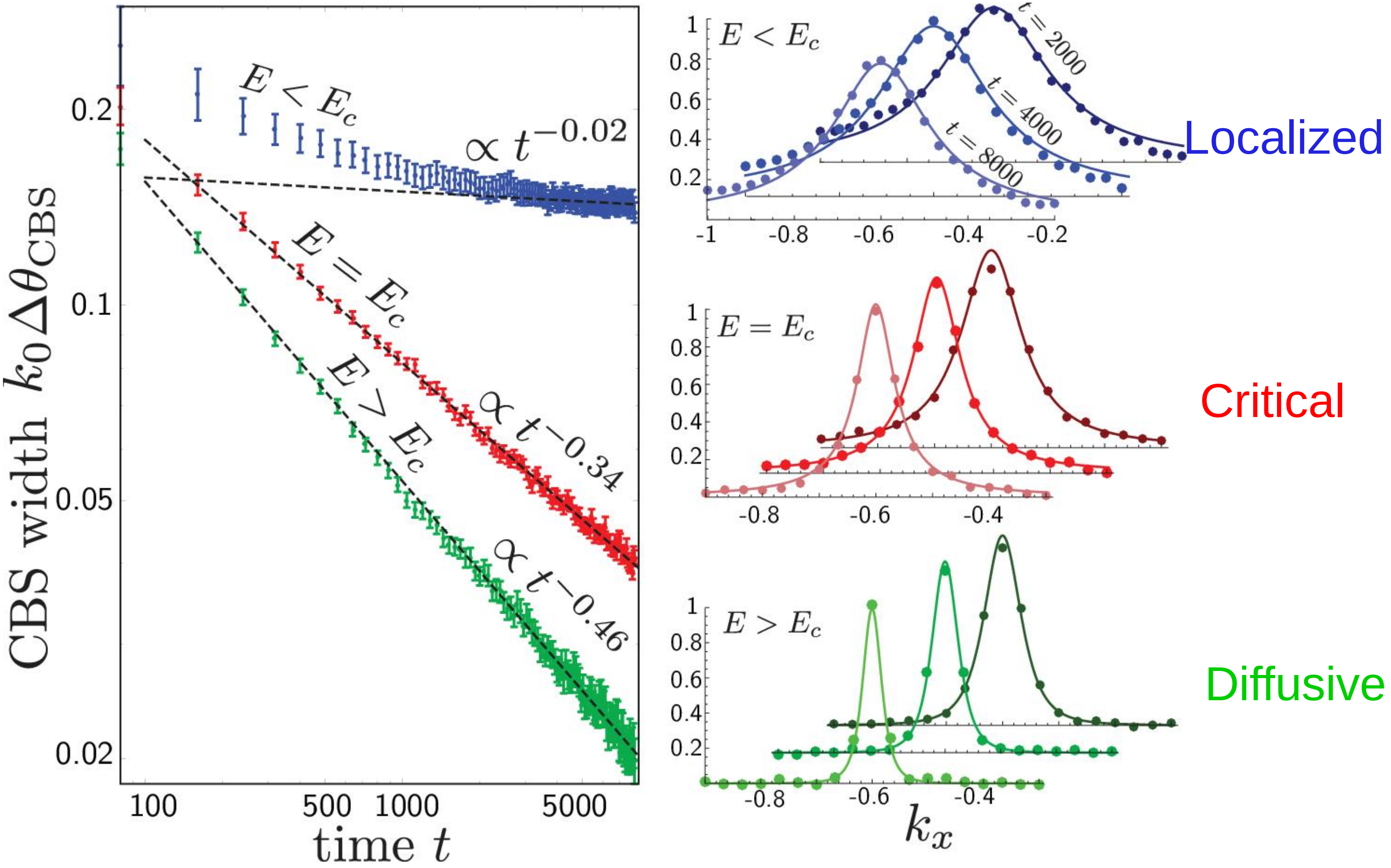
F. Jendrzejewski et al, PRL 109, 195302 (2012)

Coherent Back-Scattering as a probe of Anderson localization

- Measure the width of the CBS peak as a function of time:
  - Probe of the localized/diffusive dynamics in configuration space:
    - In the diffusive regime: $\Delta\theta_{CBS}(t) \propto 1/\sqrt{Dt}$
    - In the localized regime: $\Delta\theta_{CBS}(t) \propto 1/\xi$
    - At the mobility edge (critical point): $\Delta\theta_{CBS}(t) \propto 1/t^{1/3}$
  - It should be possible to measure it experimentally
- Numerical simulations using a 3D speckle potential:
  - Initial state: plane wave with wavevector $\vec{k}_0$
  - Propagate in the presence of disorder
  - Measure the (average) momentum distribution at time *t*
  - Measure the angular width of the CBS peak around $\vec{k} = -\vec{k}_0$

CBS as a smoking gun of 3D Anderson localization

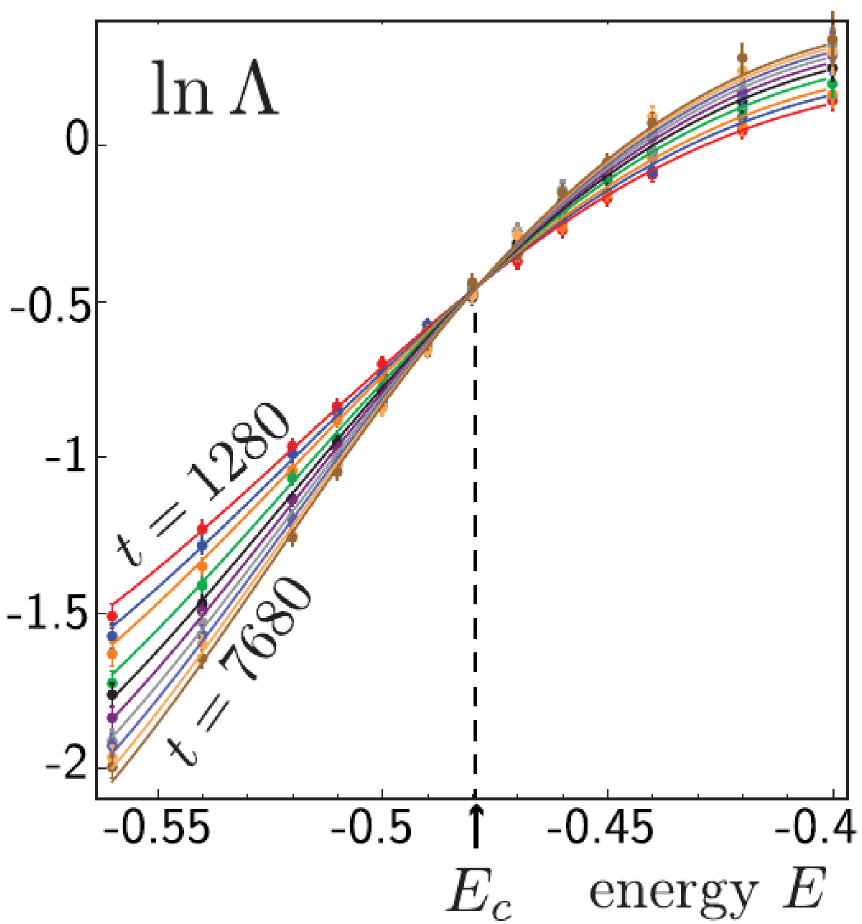
- Results of numerical simulations:

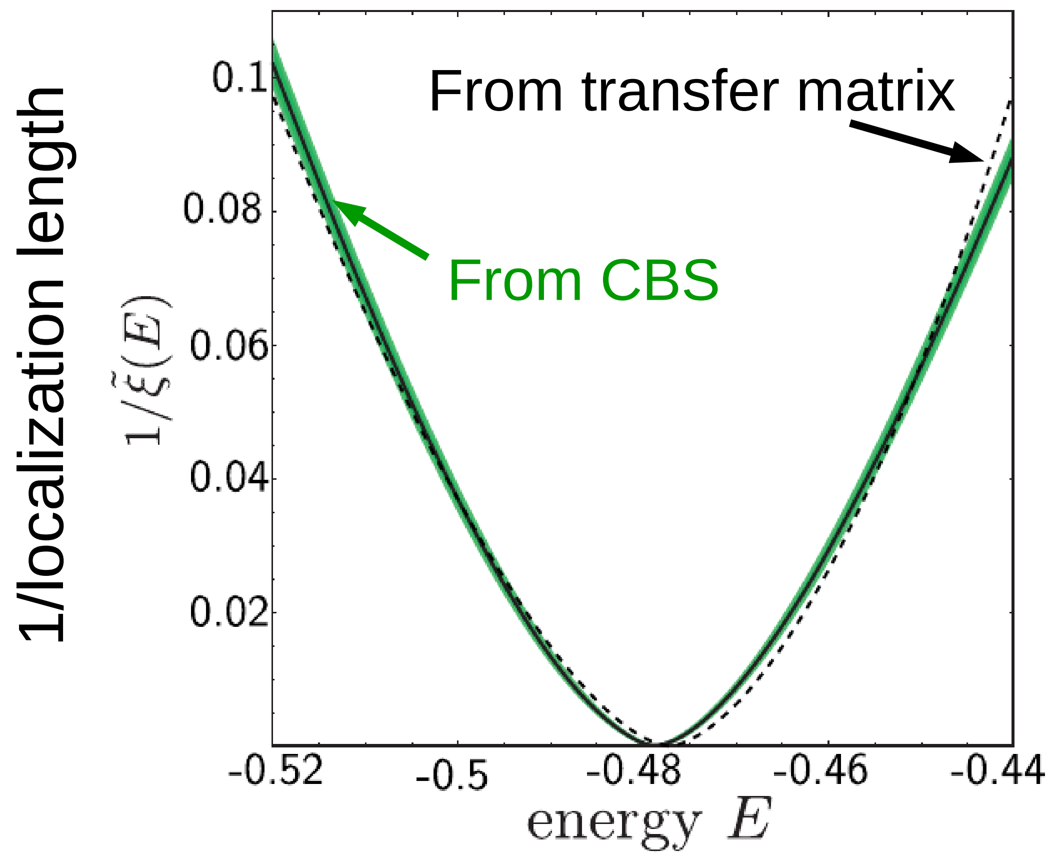


S. Ghosh et al., Phys. Rev. Lett. 115, 200602 (2015)

Scaling law for CBS

- Effective size of the system: $L(t,E) = \left[\frac{t}{2\pi\rho(E)}\right]^{1/3}$ $\rho(E)$: density of states
  (size where the Heisenberg time is *t*)
- Scaled quantity $\Lambda(t,E) = \frac{1}{L k_0 \Delta\theta_{\mathrm{CBS}}}$



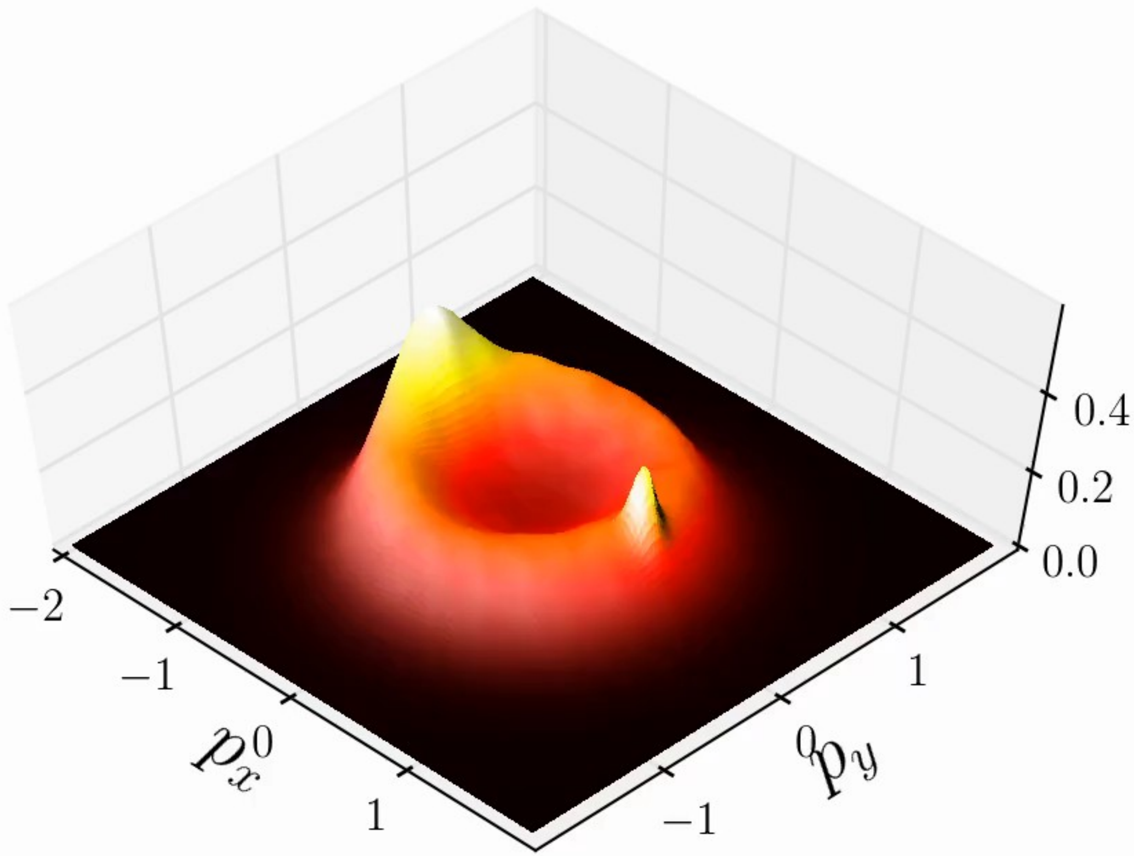


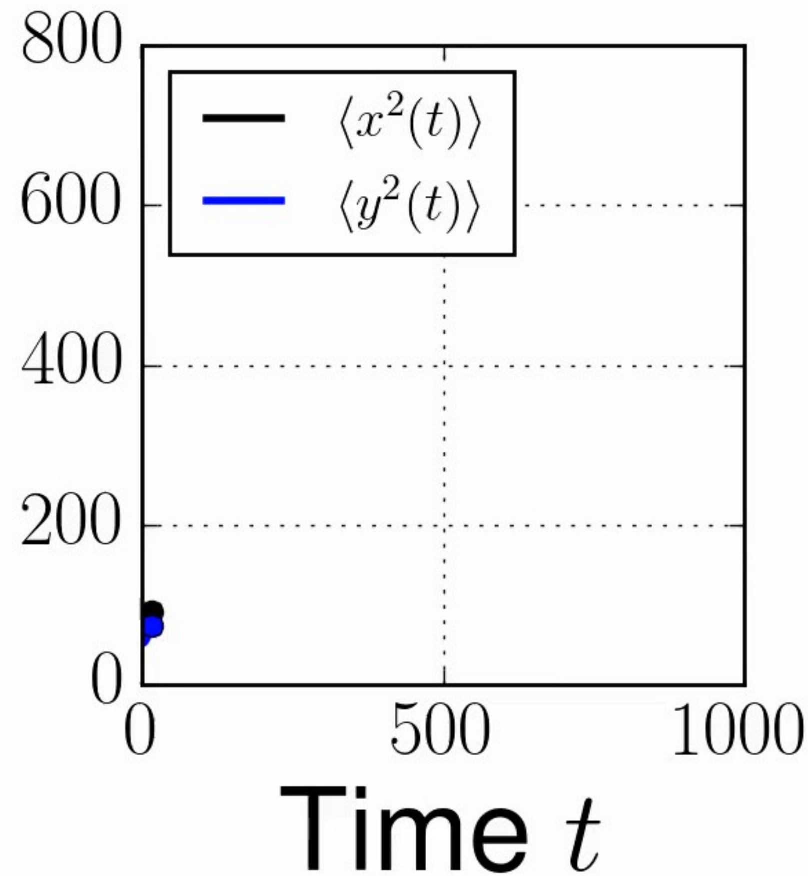
$E_c = -0.478 \pm 0.001 \quad \nu = 1.61 \pm 0.02$ from CBS data

$E_c = -0.477 \pm 0.001 \quad \nu = 1.62 \pm 0.03$ from transfer matrix data

Temporal dynamics in momentum space

- Rather strong disorder => 2D Anderson localization on a not-too-long time scale.





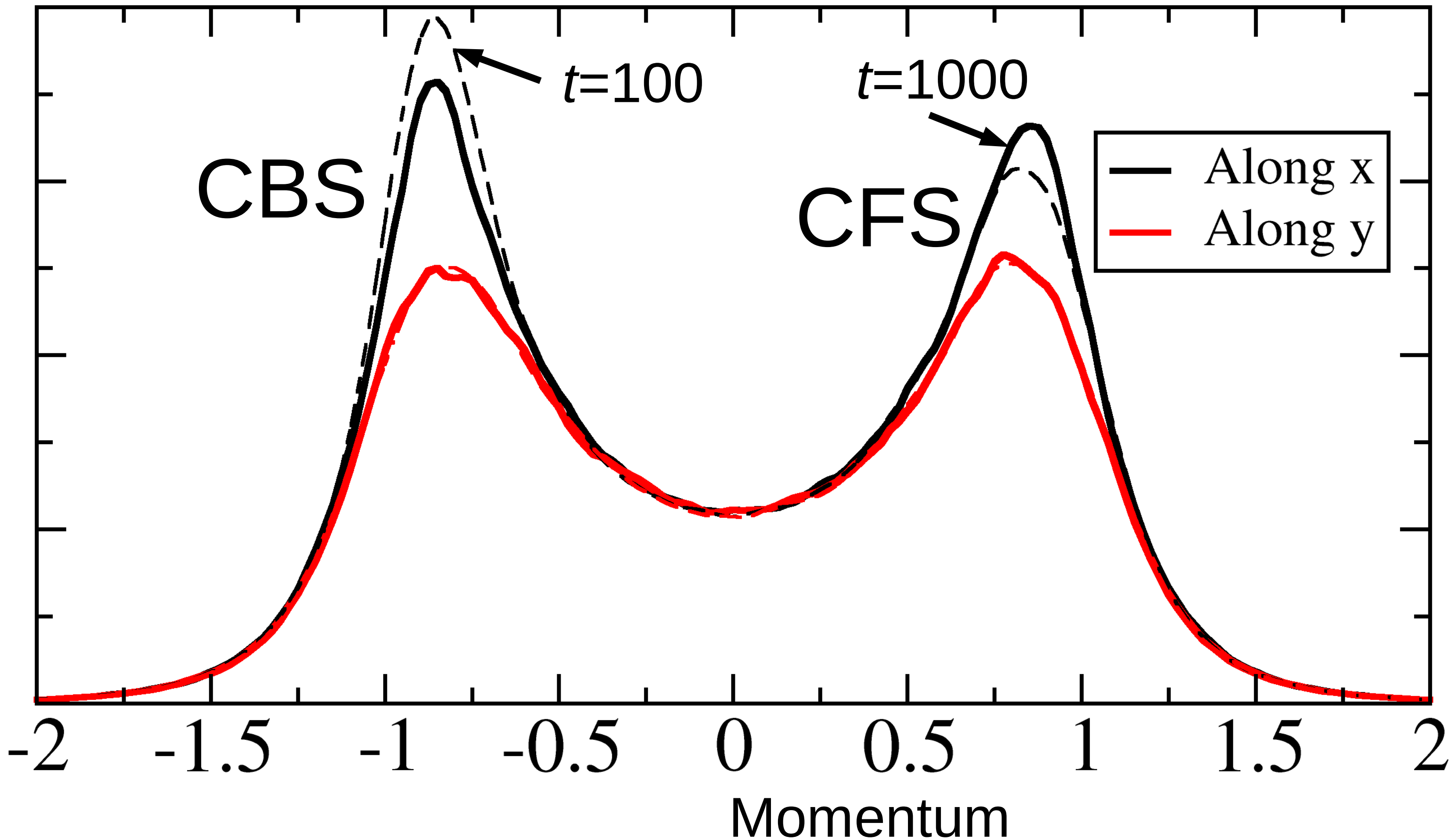
$t =$16.0

Initial momentum $p_x = 0.9, \; p_y = 0.0$

$V_0 / E_\sigma = 1$

Coherent back and forward scattering

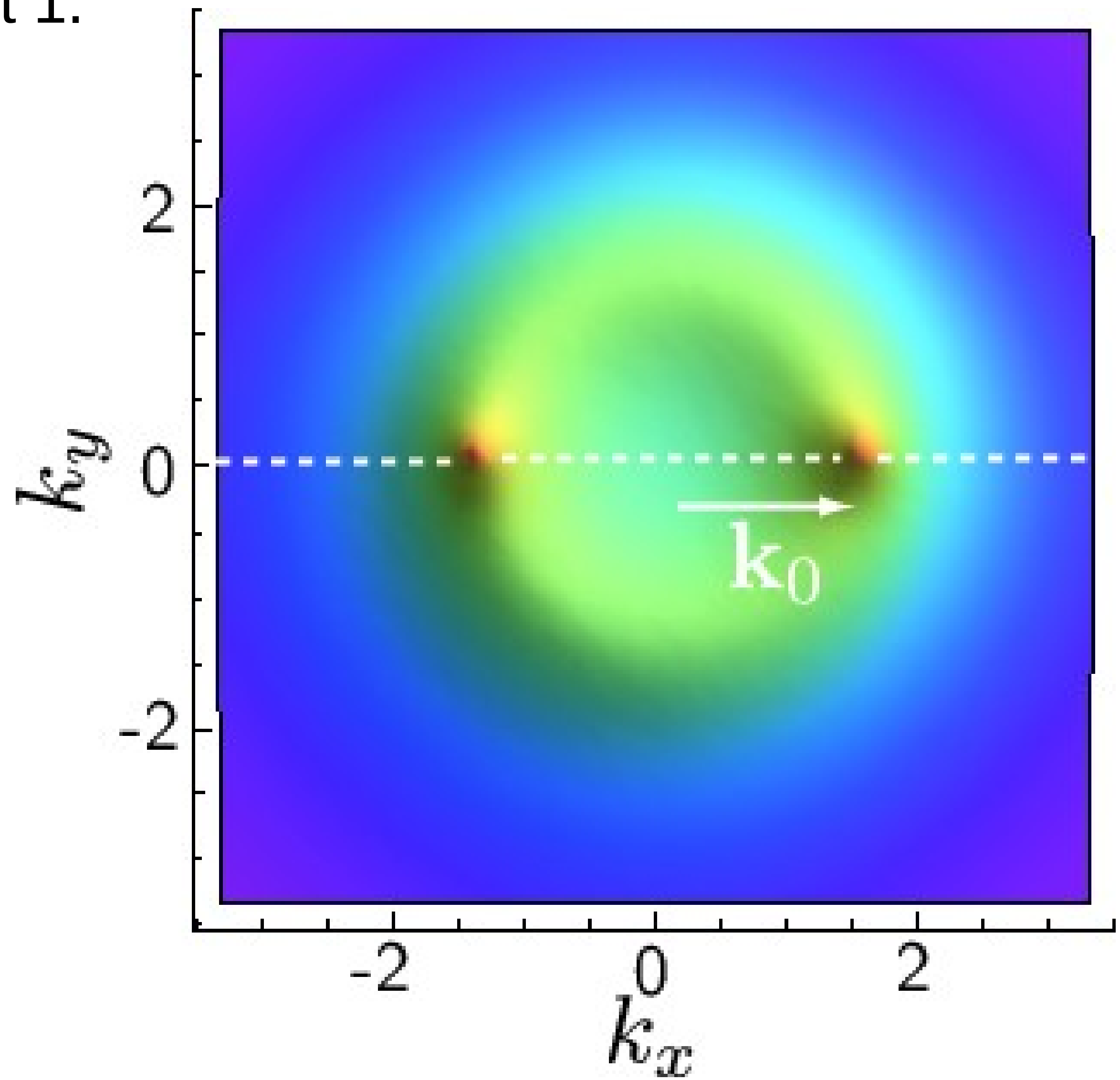
- Cuts of the momentum distribution along the *x* (// initial velocity) and *y* axes



Enhancement factor smaller than 2 because of the initial momentum dispersion

CBS and CFS twin peaks (2D)

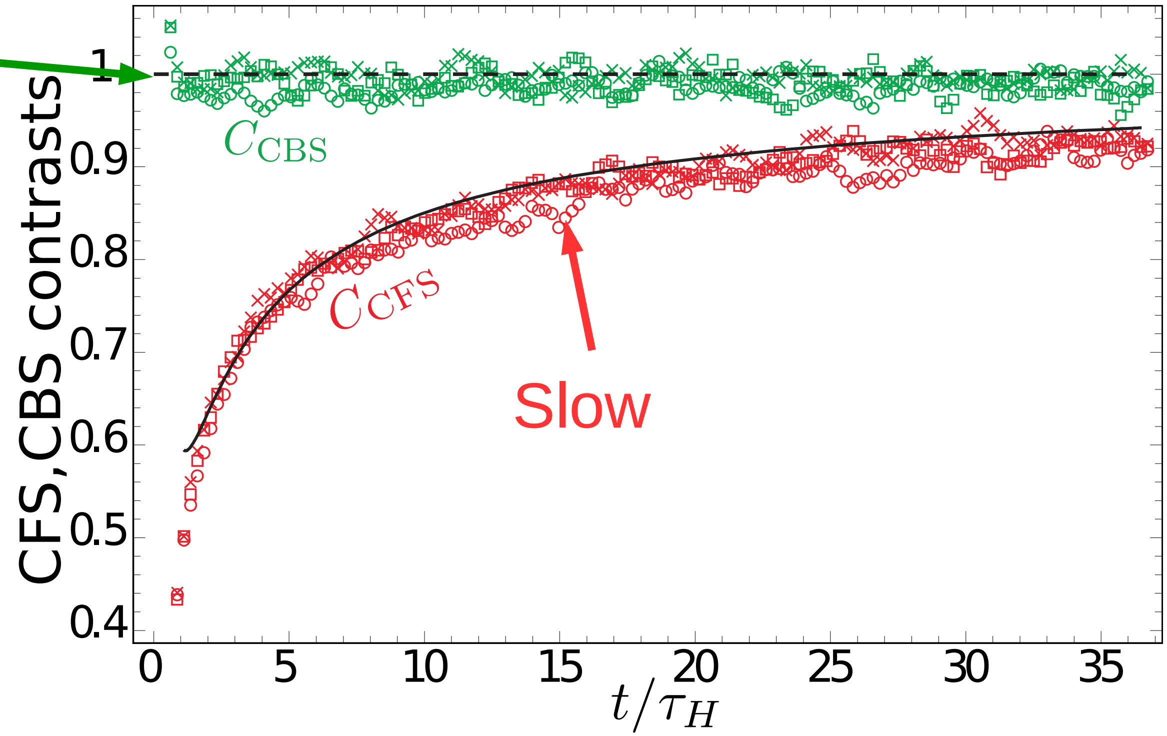
- Initial state: pure plane wave
- At long time, twin CBS and CFS peaks with enhancement factor 2, i.e. contrast 1.
- Width of the peaks: $1/\xi$

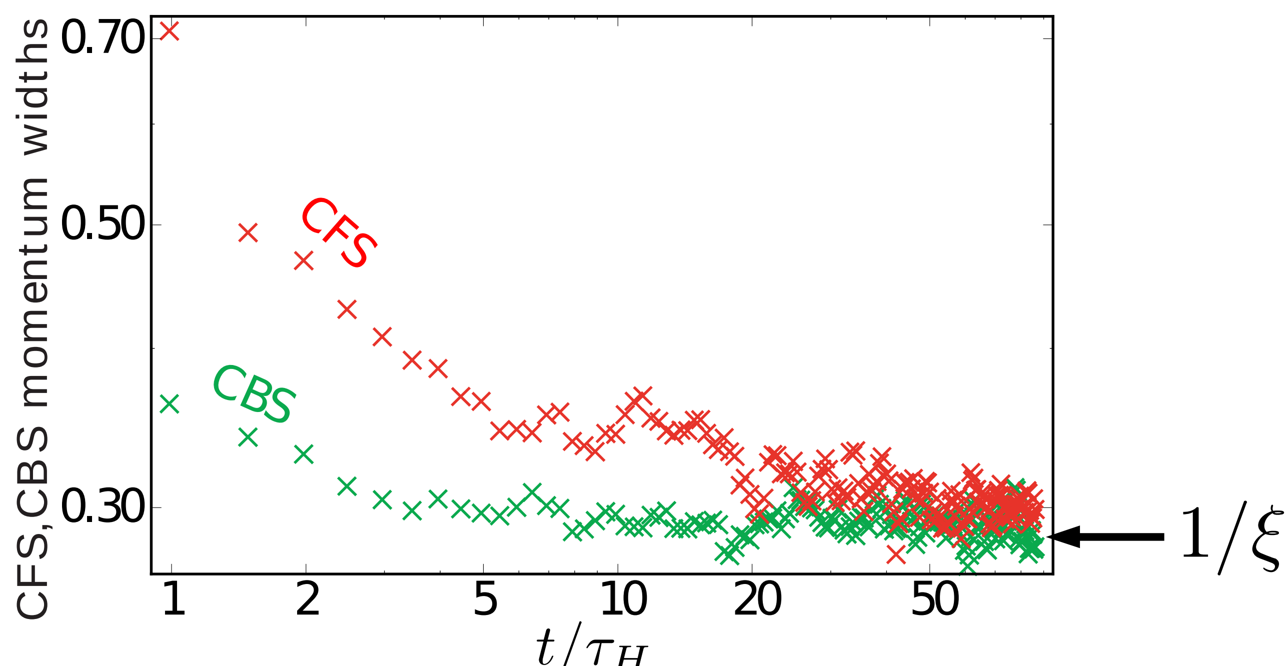


Contrasts and widths of the CBS/CFS peaks vs. time (2D)

Maximum CBS contrast after few scattering events

- Perfect contrast at long time
- Characteristic time scale for CFS is τ_H, the Heisenberg time
- Both peaks have the same width $1/\xi$ in momentum space at long time
- Perfect twin peaks at long time





Why are CBS and CFS twins?

- Solution of the Schrödinger equation: expansion on the eigenbasis of the (time-independent) Hamiltonian *H:*

$$|\psi(t)\rangle = \sum_i c_i \exp\left(-i\frac{E_i t}{\hbar}\right)|\phi_i\rangle$$

where $H|\phi_i\rangle = E_i|\phi_i\rangle$ and $c_i = \langle\phi_i|\psi(0)\rangle$

- Expectation value of any operator *O*:

$$\langle\psi(t)|O|\psi(t)\rangle = \sum_{i,j} c_i c_j^* \exp\left(-i\frac{(E_i - E_j)t}{\hbar}\right)\langle\phi_j|O|\phi_i\rangle$$

- In the long time limit, all non-diagonal terms are scrambled:

$$\langle\psi(t)|O|\psi(t)\rangle \approx \sum_i |c_i|^2 \langle\phi_i|O|\phi_i\rangle$$

- Take $|\psi(t=0)\rangle = |\vec{p}_0\rangle$ and $O = |\vec{p}\rangle\langle\vec{p}|$

$$|\psi(\vec{p},t)|^2 \approx \sum_i |\phi_i(\vec{p})|^2 |\phi_i(\vec{p}_0)|^2$$

Density-density correlator in momentum space

Why are CBS and CFS twins?

- In a disordered system, the momentum densities of eigenstates $|\phi_i(\vec{p})|^2$ have strong "speckle" fluctuations, with "speckle" spots of size $1/\xi_{\mathrm{loc}}$.
- Simple model (Random Matrix Theory): $\phi_i(\vec{p})$ is a complex random variable with Gaussian distribution => the CFS peak at $\vec{p}=\vec{p}_0$ has a contrast 1 and a width $1/\xi_{\mathrm{loc}}$.

$$|\psi(\vec{p},t)|^2 \approx \sum_i |\phi_i(\vec{p})|^2 |\phi_i(\vec{p}_0)|^2$$

- For a time-reversal invariant system localized in a finite region of space, all eigenfunctions $\phi_i(\vec{r})$ can be chosen REAL =>

$$\phi_i(-\vec{p}) = \phi_i^*(\vec{p})$$

- Thus: $|\psi(\vec{p},t)|^2 \approx |\psi(-\vec{p},t)|^2$

CBS and CFS are exact twins in the infinite time limit!

N.B.: If time-reversal invariance is broken, CBS disappears but CFS survives.

- Important question: how long does it take?

Characteristic time for the onset of CFS

$$\langle \psi(t)|O|\psi(t)\rangle = \sum_{i,j} c_i c_j^* \exp\left(-i\frac{(E_i - E_j)t}{\hbar}\right)\langle \phi_j|O|\phi_i\rangle$$

- How long does it take to scramble all non-zero phases?
- Only states within one localization volume of size ξ contribute to the dynamics.
  - The density of such states is $\xi^d \rho(E)$ where $\rho(E)$ is the density of states per unit volume.
  - All phases are scrambled when:

$$t \gg \tau_H = 2\pi\rho(E)\xi^d$$

Heisenberg time

- Confirmed by diagrammatic expansions at short time
- Exactly solvable in 1D (Micklitz, Müller, Altland, PRL '14)
- Approximate asymptotic expression known at long time
- Diffusive regime: $\tau_H \to \infty$, no CFS...

Contrasts and widths of the CBS/CFS peaks vs. time (2D)

Maximum CBS contrast after few scattering events

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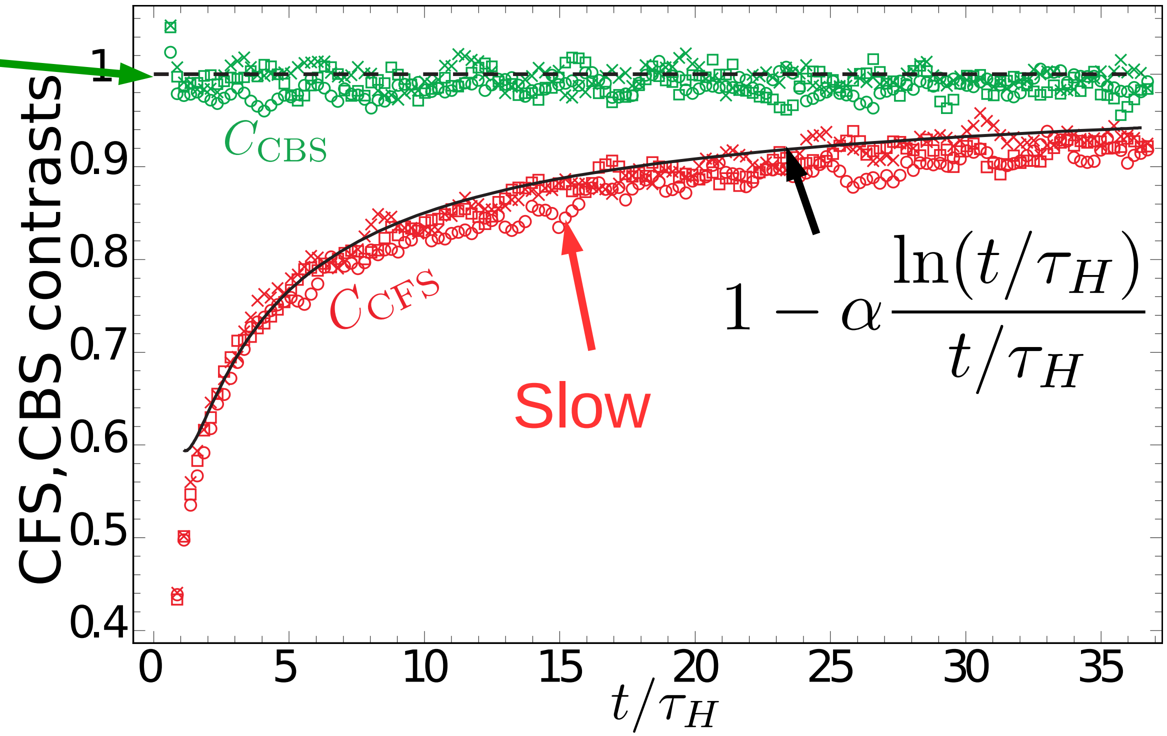
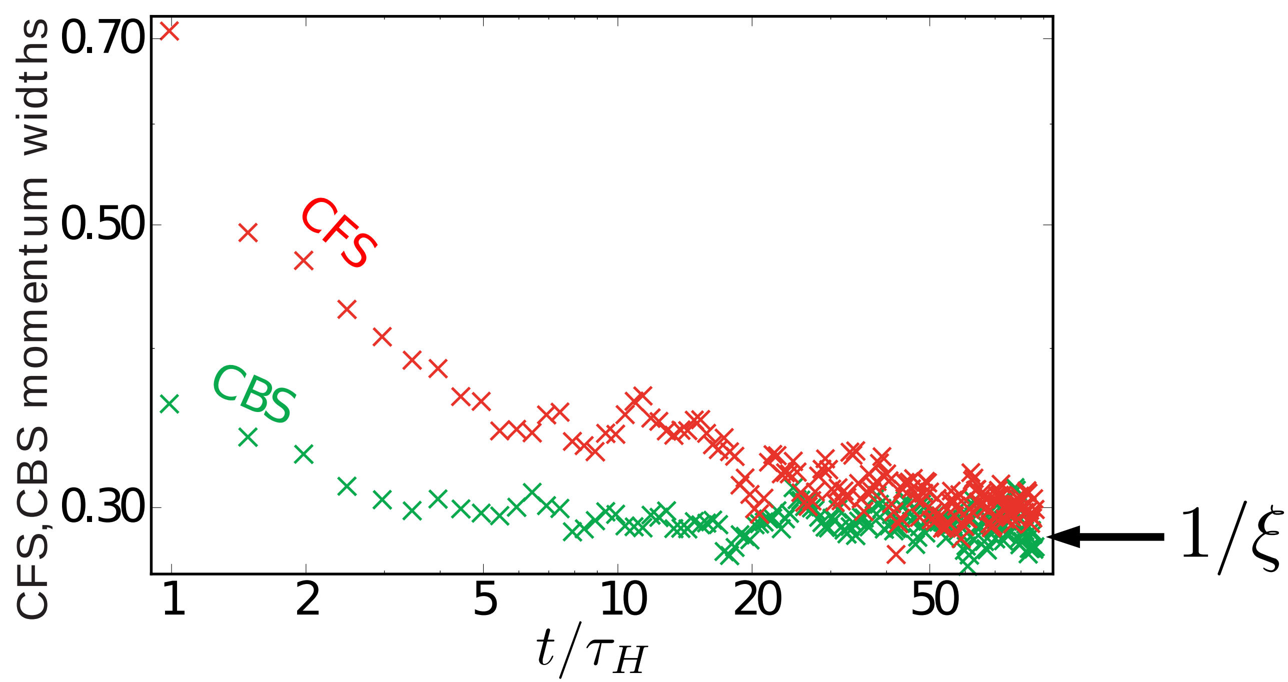


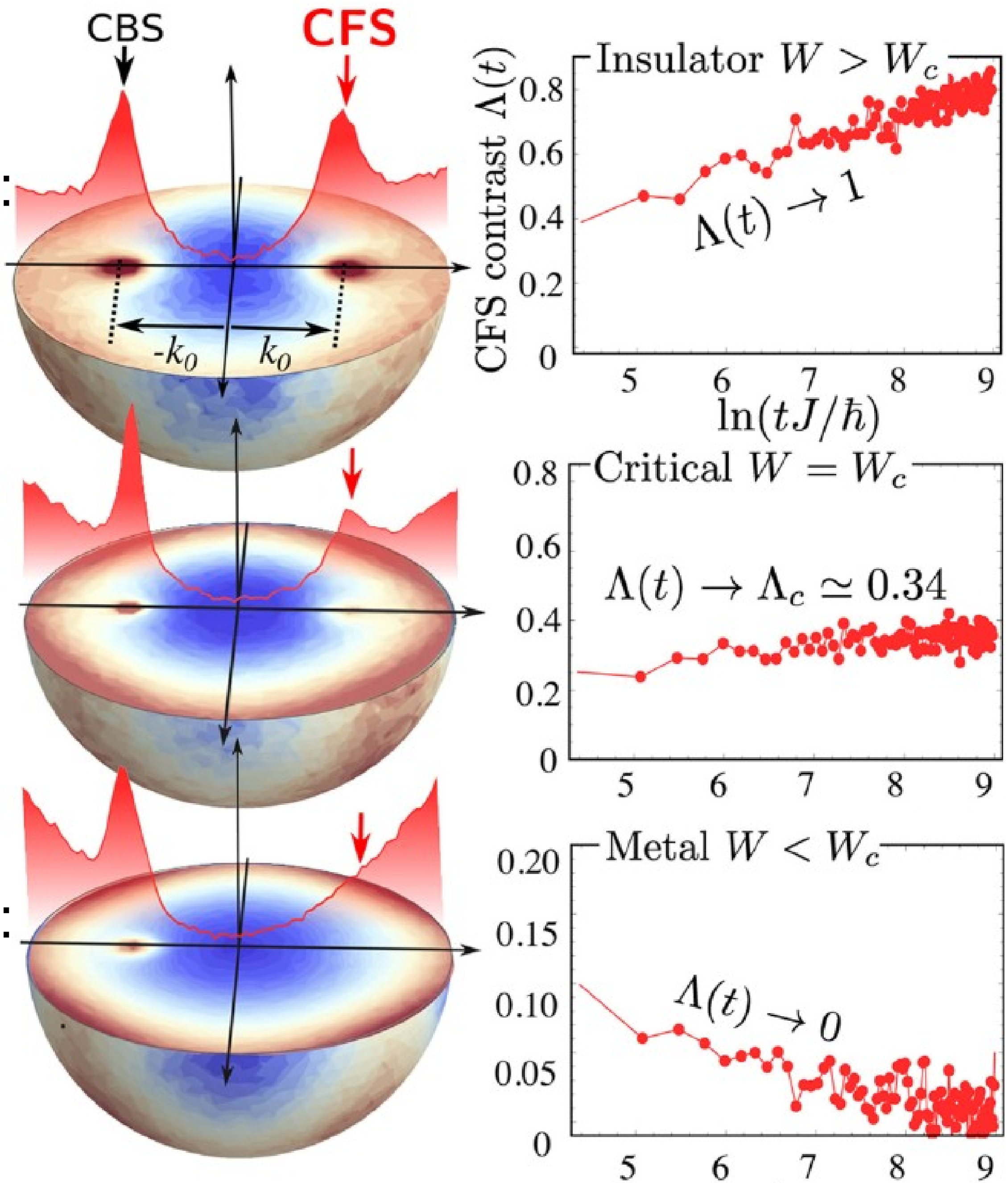


Coherent Forward Scattering in 3D

Localized regime:
CFS is present
and twin of CBS

Critical regime

Diffusive regime:
no CFS!

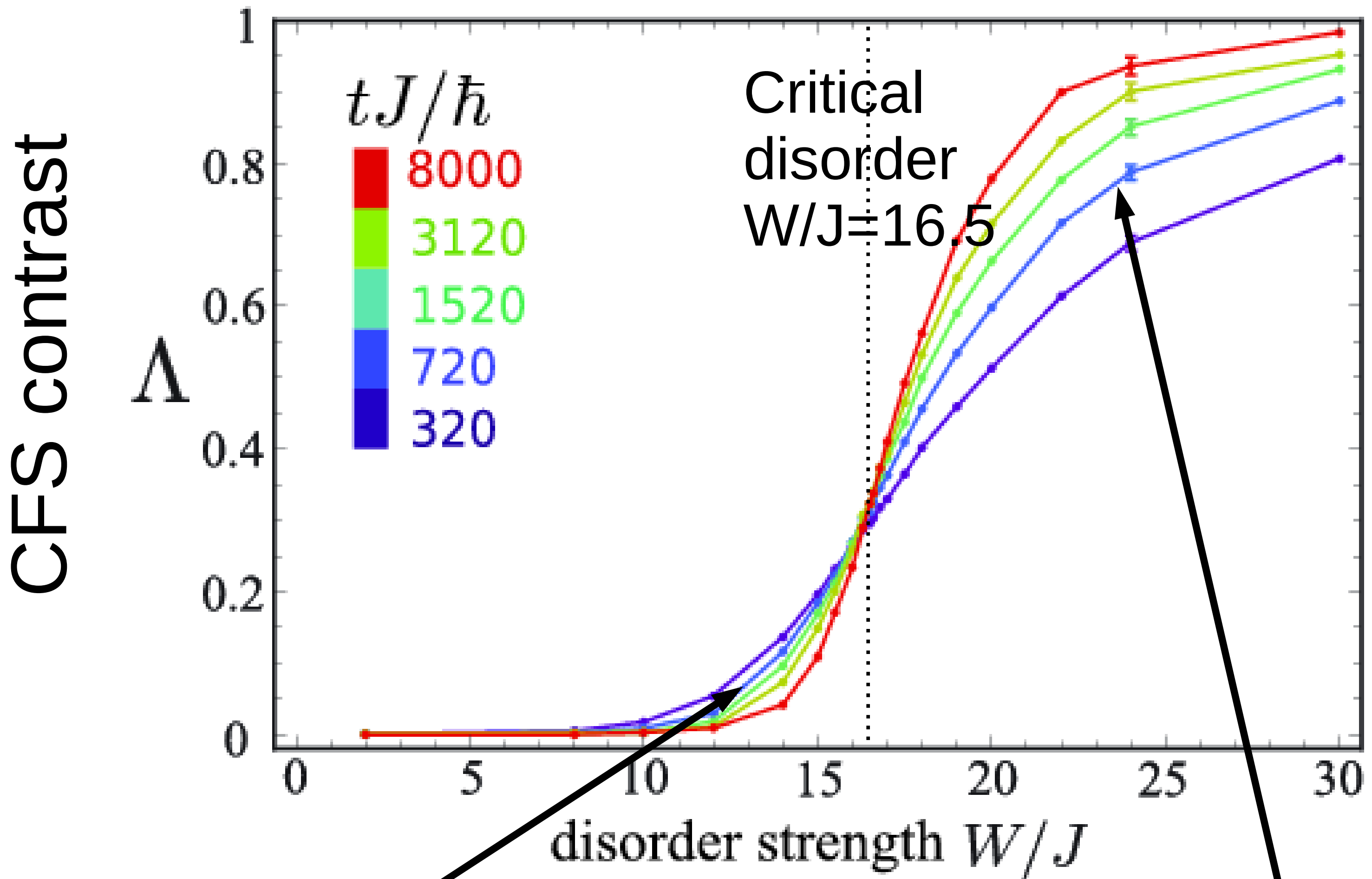


3D Anderson model

S. Ghosh et al, PRA 95, 041602 (2017)

CFS is a smoking gun of Anderson localization!

CFS contrast for the 3D Anderson model:
an effective order parameter of the Anderson transition



Diffusive regime: $\Lambda(t) \propto \frac{1}{\sqrt{t}}$

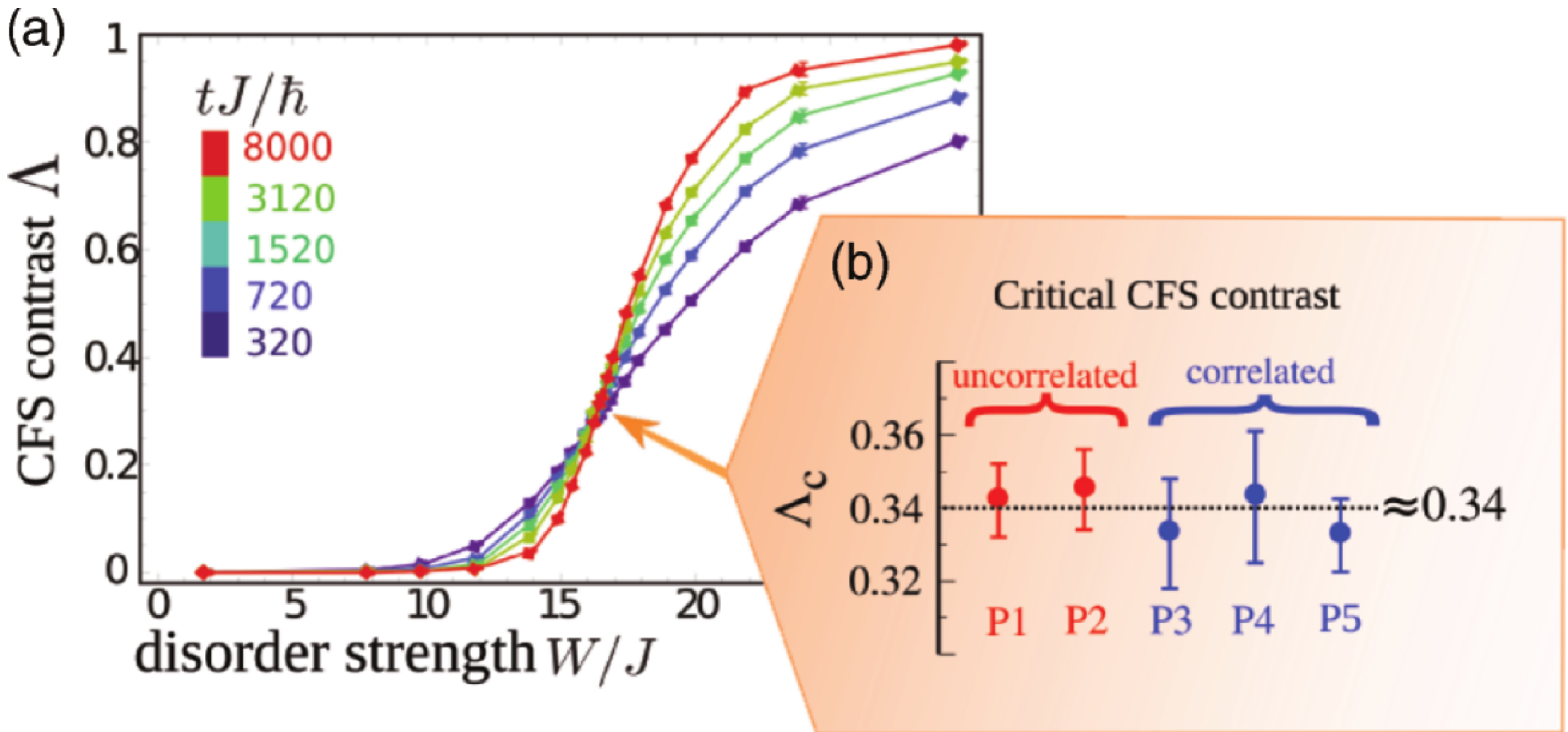
Localized regime:

$$\Lambda(W,t) \approx 1 - \alpha \frac{\ln^2(\eta t/\tau_H)}{(t/\tau_H)}$$

S. Ghosh et al,
PRA 95, 041602 (2017)

CFS at the critical point

- The contrast of the CFS peak at criticality is universal (identical for various disorder models)

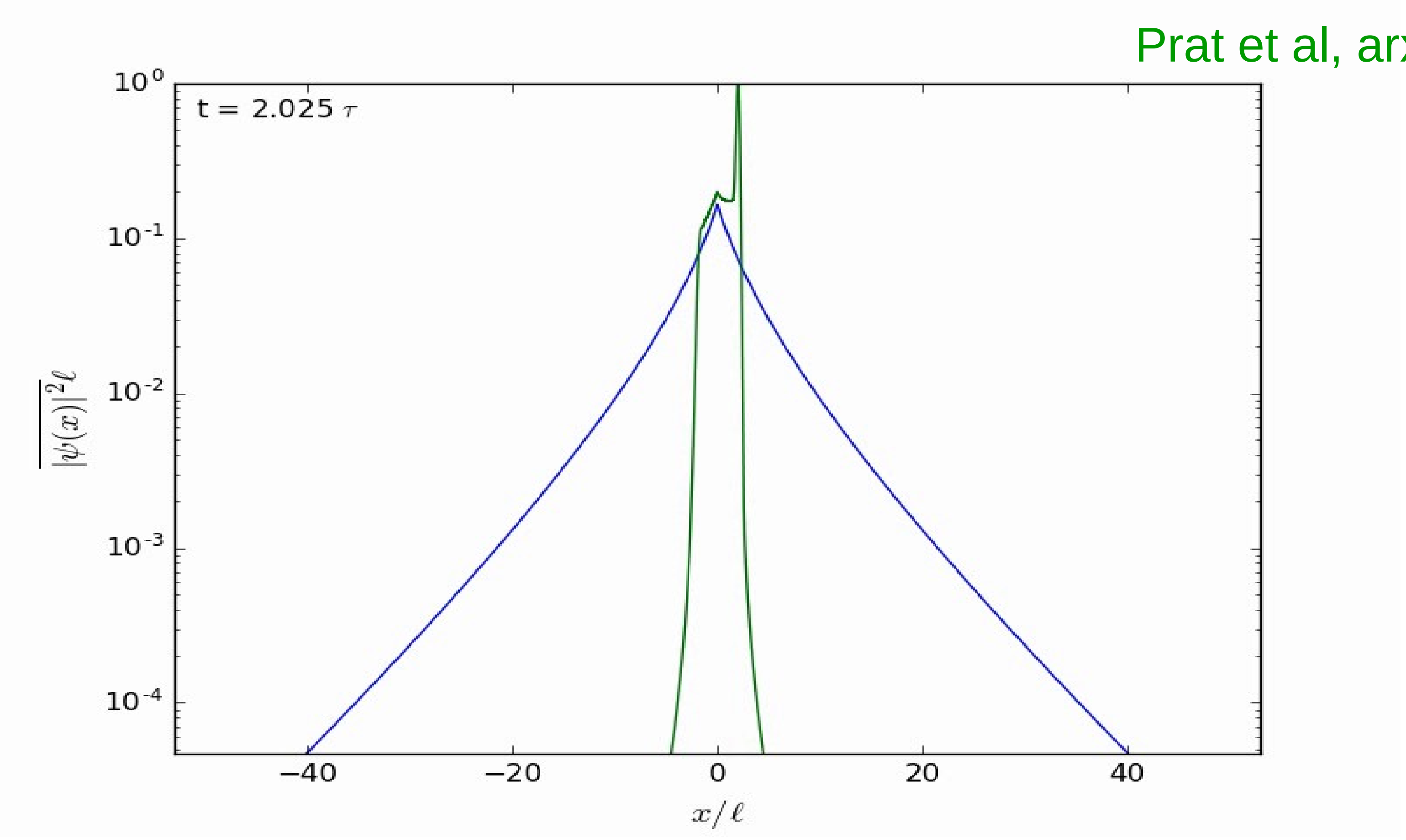


- Conjecture: it is directly related to statistical properties of the energy spectrum close to the critical point, itself related to the multifractal dimension D_1. Work in progress...

Quantum boomerang
or
“Get back to where you once belonged”

- Exactly solvable in 1D for weak disorder.
- Start from a localized initial wavepacket with non-zero velocity.
- At infinite time, same symmetric distribution in configuration space irrespective of initial velocity!

Prat et al, arxiv:1704:05241



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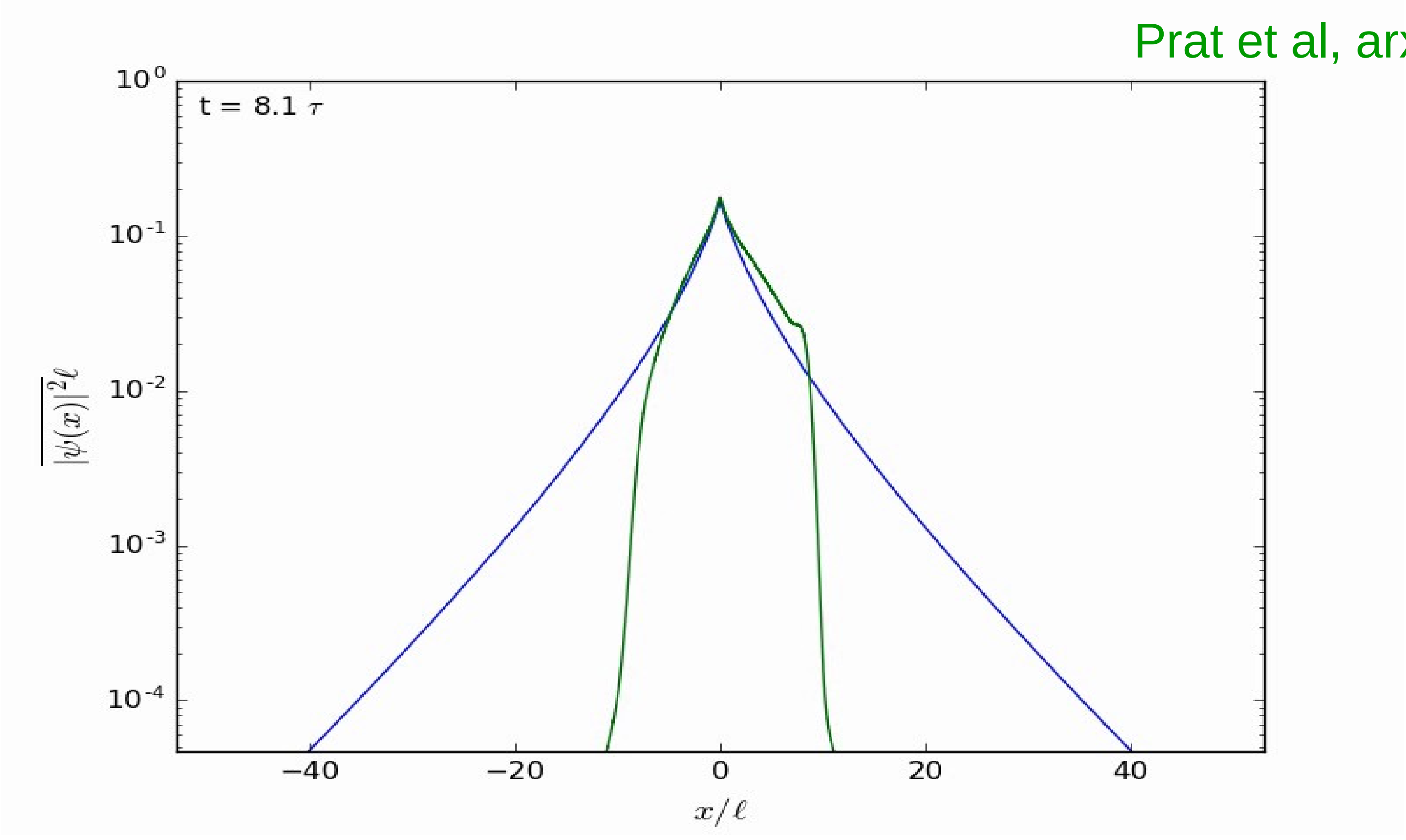


Quantum boomerang

- Using the Berezinskii/DMPK method, we derive the long-time behavior:

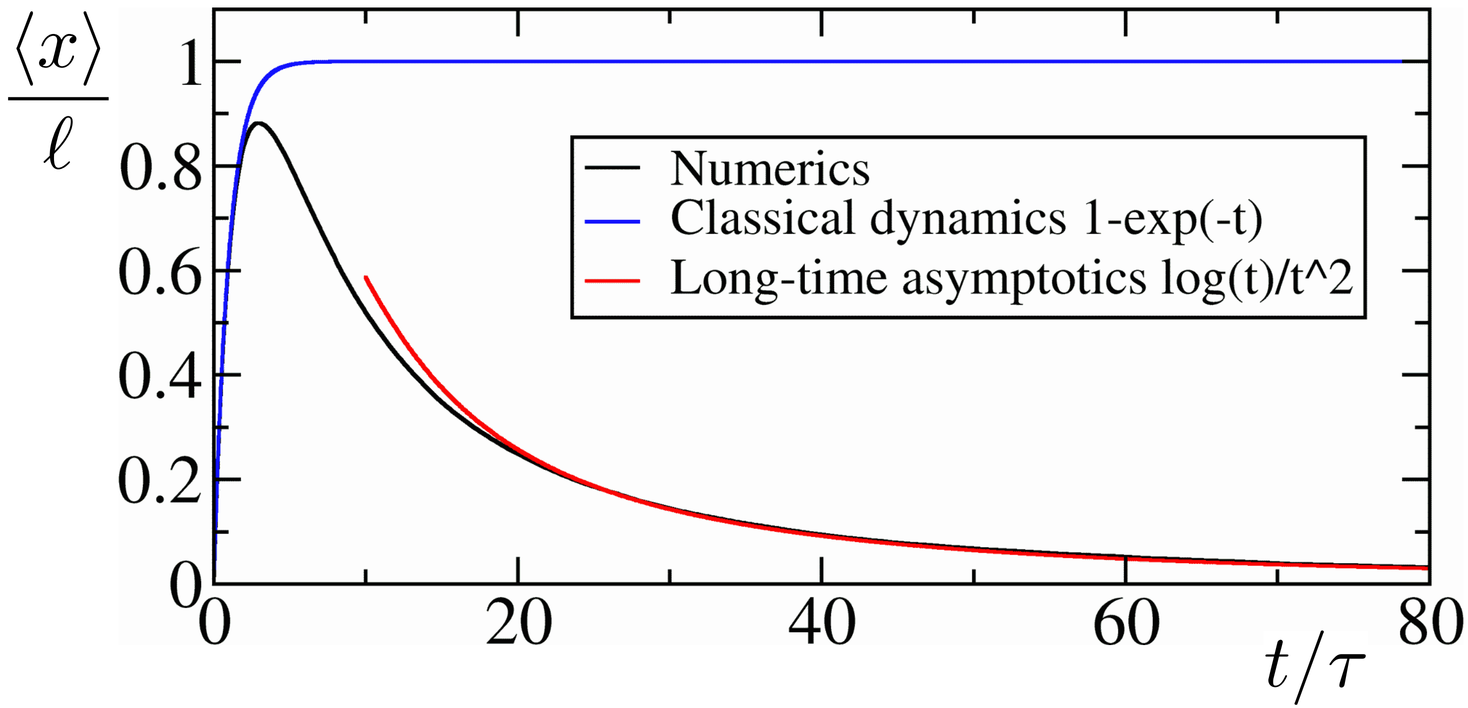
$$\frac{\langle x(t)\rangle}{\ell} \approx \frac{\log(t/4\tau)}{(t/\tau)^2}$$

ℓ : mean free path $\quad \tau = \ell/\sqrt{2mE}$: scattering time

- Classical dynamics does NOT return to the origin:

$$\frac{\langle x(t)\rangle}{\ell} = 1 - \mathrm{e}^{-t/\tau}$$

Prat et al, arxiv:1704:05241



Return to the origin is a smoking gun of coherent evolution!

Why does a wavepacket return to its origin?

- Start from the long time limit of:

$$\langle\psi(t)|O|\psi(t)\rangle \approx \sum_i |\langle\phi_i|\psi_0\rangle|^2 \langle\phi_i|O|\phi_i\rangle$$

- Choose $O = x$ and $|\psi_0\rangle$ as a wavepacket with initial momentum k_0 .

$$\langle x(t\to\infty)\rangle \approx \sum_i |\langle\phi_i|\psi_0\rangle|^2 \langle\phi_i|x|\phi_i\rangle$$

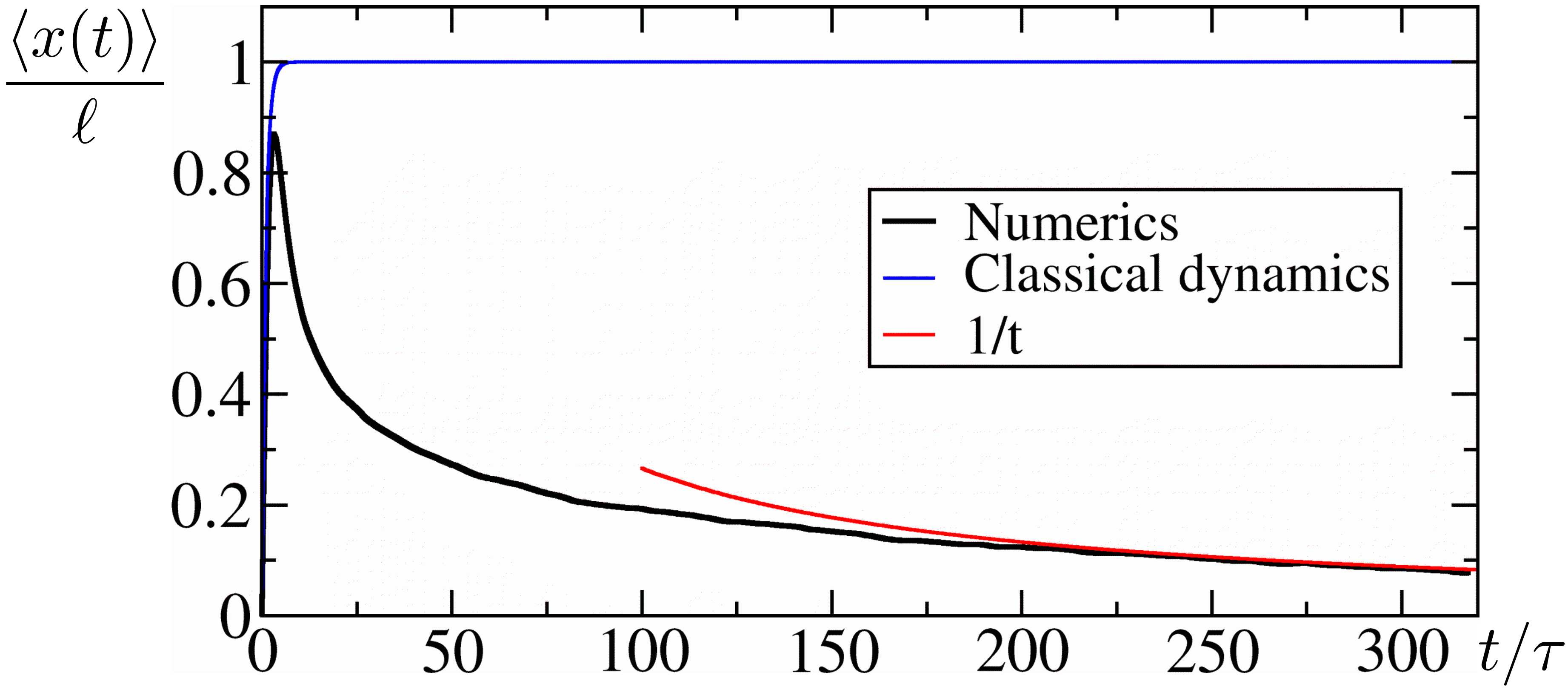
- Expression unchanged when time is reversed $k_0 \to -k_0$
- When averaging over disorder realizations, statistical invariance by translation and parity is restored =>

$$\overline{\langle x(t\to\infty)\rangle} = 0$$

- Characteristic time scale: Heisenberg time. Truly phase coherent quantum effect.
- No simple perturbative/diagrammatic explanation (yet).
- Is not expected for a non-localized system.

Quantum boomerang in 2D

- No detailed theory yet.
- Numerics: return to origin on the time scale of the localization time.
- Even slower than in 1D, maybe like 1/*t*.



Localization time around 200*τ*

Prat et al, arxiv:1704:05241

Quantum boomerang in 3D

?

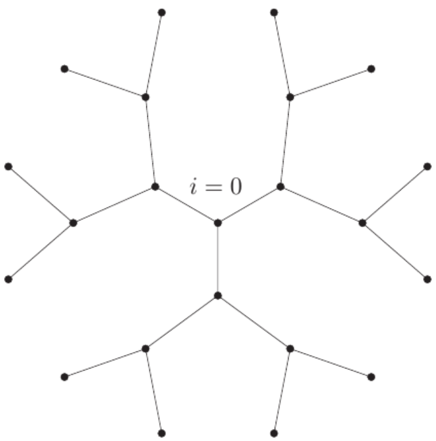
- Expected to be a smoking gun of Anderson localization.
- Work in progress...

Alternative signatures of Anderson localization for many-body systems?

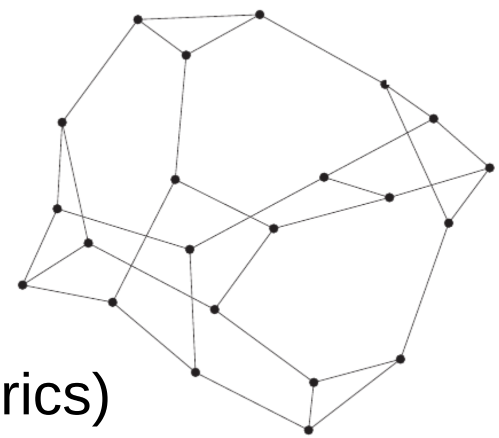
- Coherent back-scattering is a good probe of phase coherence. Could be observable on many-body systems. Scaling of the CBS peak near the many-body mobility edge is unknown.
- Coherent Forward Scattering is a **signature** of one-body localization. Not yet observed with cold atomic in disordered potentials (recent observation of the kicked rotor…). Existence/relevance/properties for many-body systems is a completely open question.
- Quantum boomerang is another signature, known to be affected by decoherence. Unknown behavior in the many-body regime.

Towards many-body localization

- Anderson localization in high (>>3) dimensions
  - Critical disorder strength increases
  - Each site is coupled to more and more (2**d*) sites
  - Loops are less and less probable
  - The ergodic regime is further and further from the critical point
  - Upper critical dimension??
- Simpler model: Bethe lattice without loops
  - Localized phase for strong disorder
  - Non ergodic states below the mobility edge
  - Drawback: plenty of edge sites



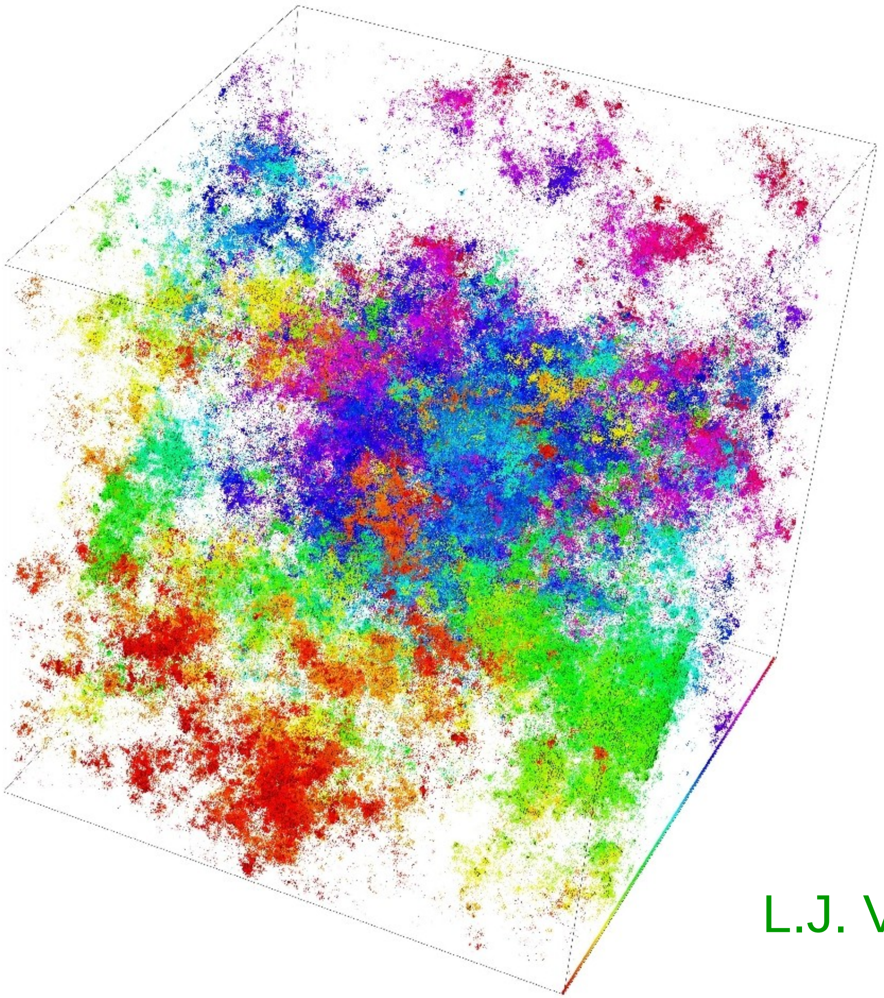
- Random regular graphs
  - Mainly long loops
  - No edge sites
  - Controversy on the existence of delocalized non-ergodic states (large finite-size effects in numerics)



K.S. Tikhonov et al, PRB 94, 220203(R) (2016)
B.L. Altshuler et al, PRL 94, 156601 (2017)
G. Biroli et al, arxiv:1211.7334
I. Garcia-Mata et al, arxiv:1609.05857

Beyond the average Green function : multifractality of the wavefunctions

- At the critical point, the eigenstates display strong fluctuations:
  - Regions where the wavefunction is exceptionally large;
  - Regions where the wavefunction is exceptionally small;



A typical eigenstate of the 3d Anderson model at the mobility edge

L.J. Vasquez et al. PRB 78, 195106 (2008)

Beyond the average Green function : multifractality of the wavefunctions

- At the critical point, the eigenstates display strong fluctuations:
  - Regions where the wavefunction is exceptionally large;
  - Regions where the wavefunction is exceptionally small;
- Can be quantitatively studied using the multifractality spectrum.
- Finite system of size *L* in dimension *d*.
  - In the diffusive regime, the average behaviour is: $|\psi(\mathbf{r})|^2 = \frac{1}{L^d}$
  - In the localized regime:

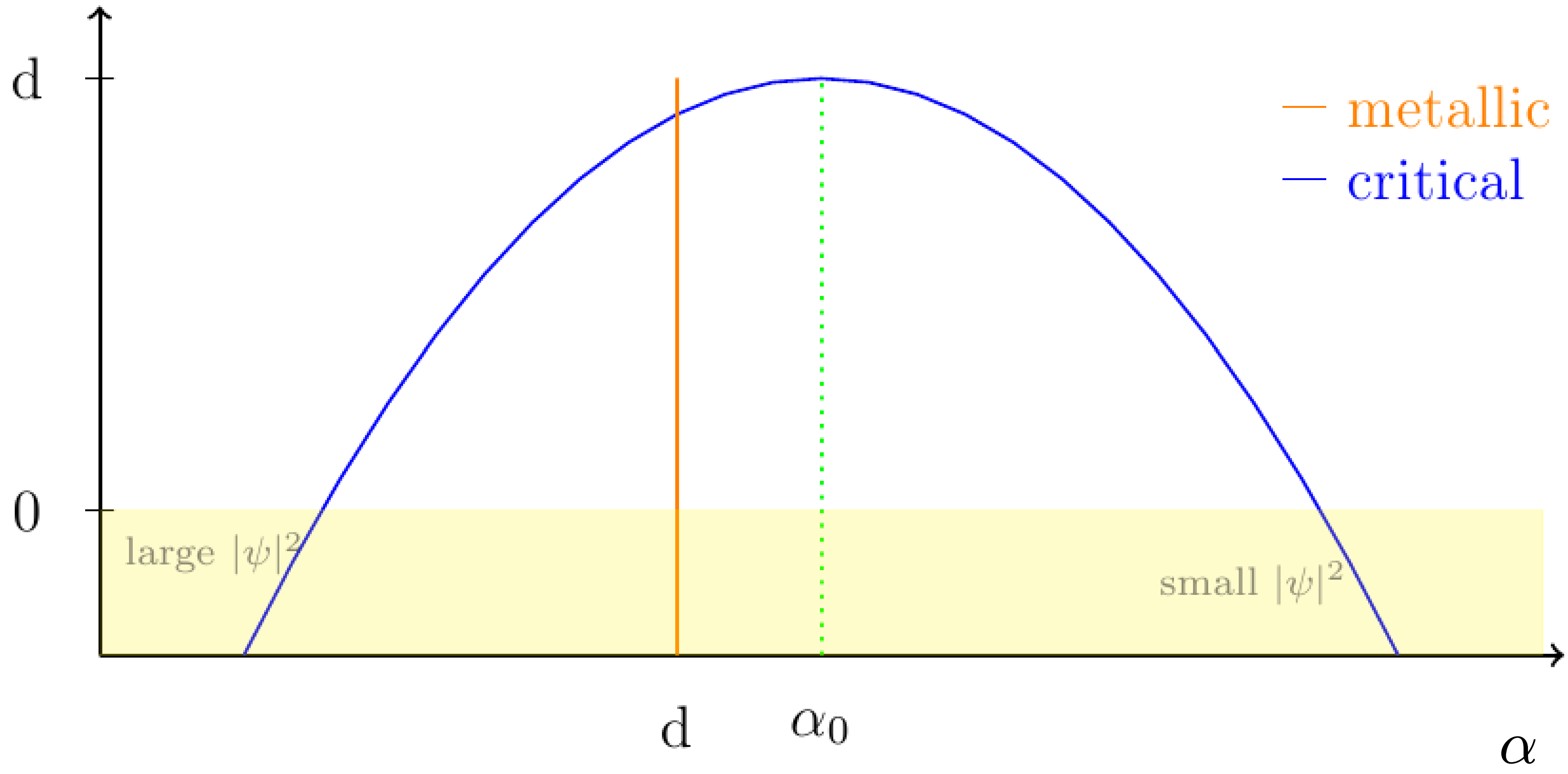
$$|\psi(\mathbf{r})|^2 \propto \frac{1}{\xi_{\mathrm{loc}}^d} \exp\left(-\frac{|\mathbf{r}-\mathbf{r}_0|}{\xi_{\mathrm{loc}}}\right)$$

- Multifractality spectrum $f(\alpha)$: measure of the regions of space where: $|\psi(\mathbf{r})|^2 \propto L^{-\alpha}$

- Directly related (Legendre transform) to the "mass exponent" $\tau(q)$ of the generalized Inverse Participation ratio:

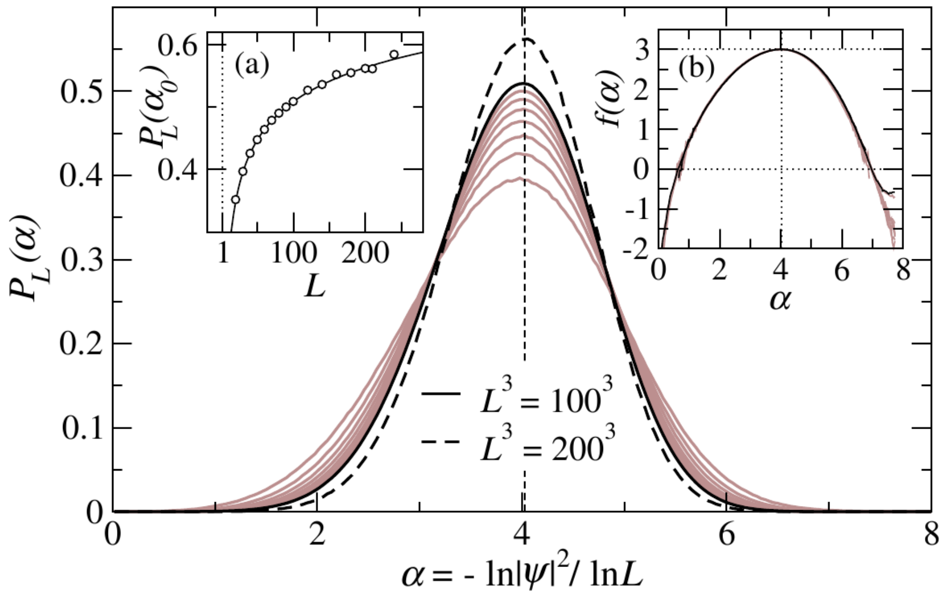
$$P_q = \int |\psi(\mathbf{r})|^{2q} \mathrm{d}^d\mathbf{r} = L^{-\tau(q)}$$

Multifractality spectrum

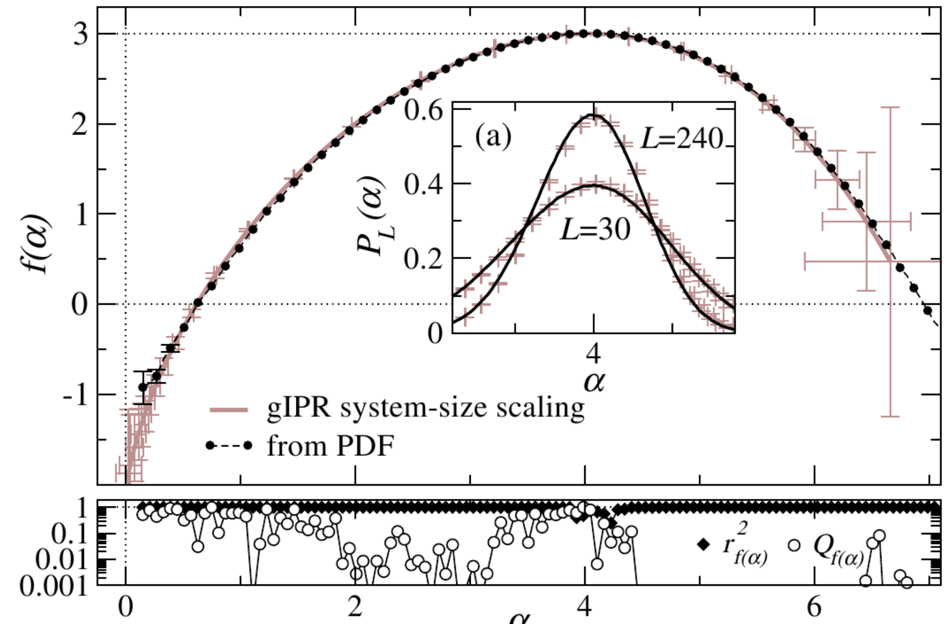


At the critical point, the whole curve scales with the system size

Multifractality spectrum for the 3d Anderson model at the critical point



Fixed system size *L*



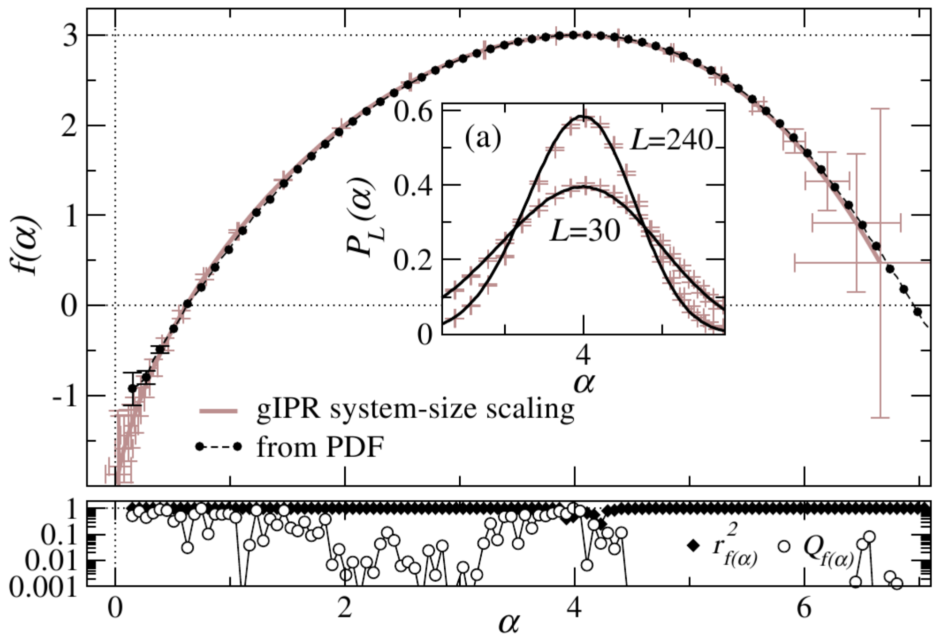
Rescaled data for various sizes

Eigenstates of the critical 3d Anderson model

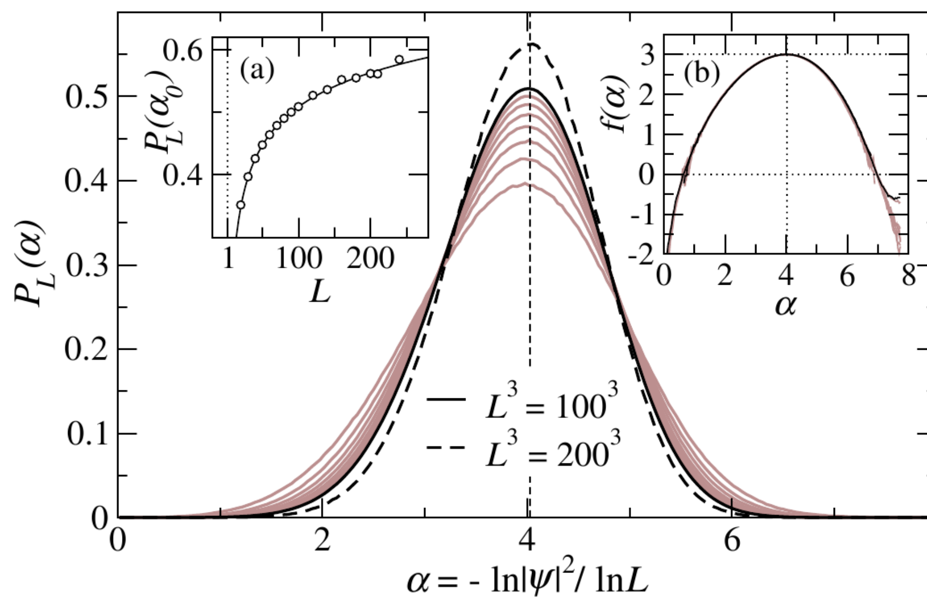
Rodriguez et al, PRL 102, 106406 (2009)

Multifractality spectrum of the 3d Anderson model at the critical point

- Eigenstates of the critical Anderson model [Rodriguez et al, PRL 102, 106406 (2009)]



Fixed system size *L*

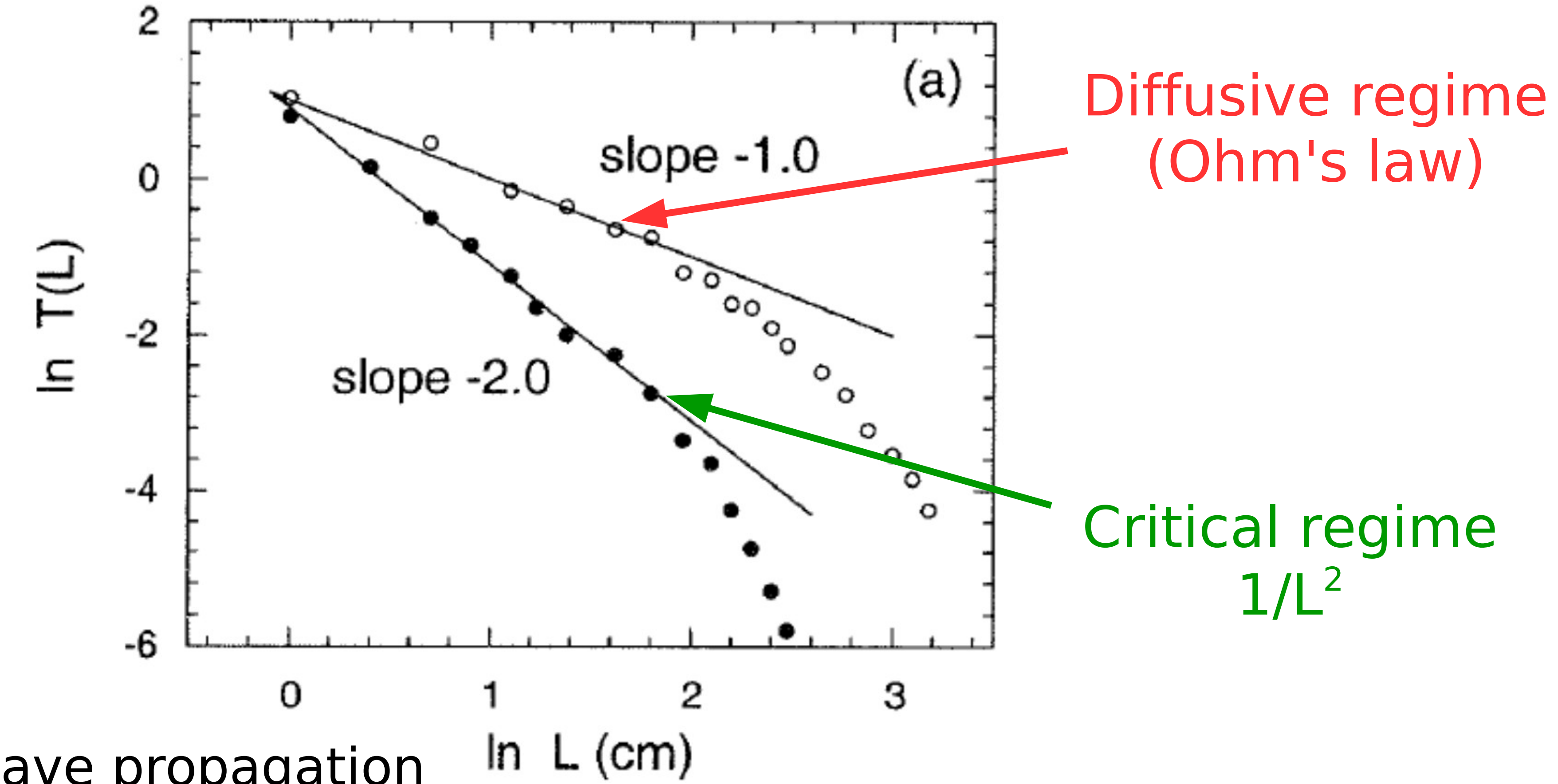


Rescaled data for various sizes

- The most probable value of $|\psi(r|^2$ is $L^{-\alpha_0}$ with $\alpha_0 = 4.027$
- Expansions in dimension 2+ε predicts $\alpha_0 = 4$
- With increasing dimension, the multifractality spectrum becomes broader and less parabolic
- A similar behavior could exist for many-body states at the critical point.

Anderson localization of electromagnetic waves

- All claims of experimental observation of Anderson localization of light have been withdrawn, see S.E. Skipetrov and J. Page, NJP 18, 021101 (2016)
- Observation with microwaves (transmission and fluctuations):

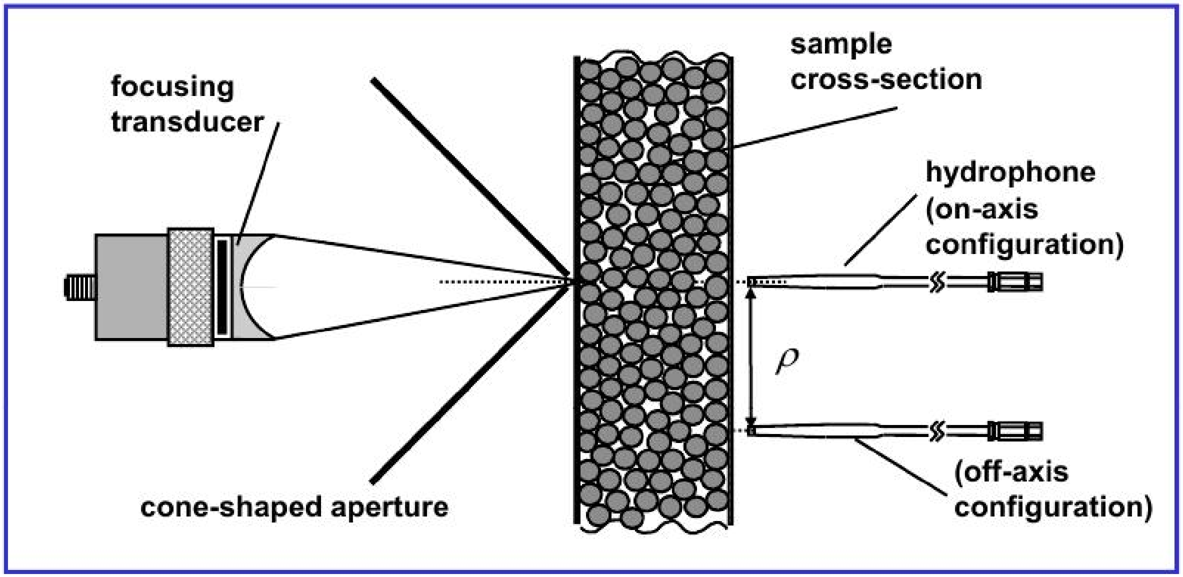


Microwave propagation in a mixture of Teflon and Aluminium spheres

N. Garcia and A.Z. Genack, PRL, 66, 1850 (1991)

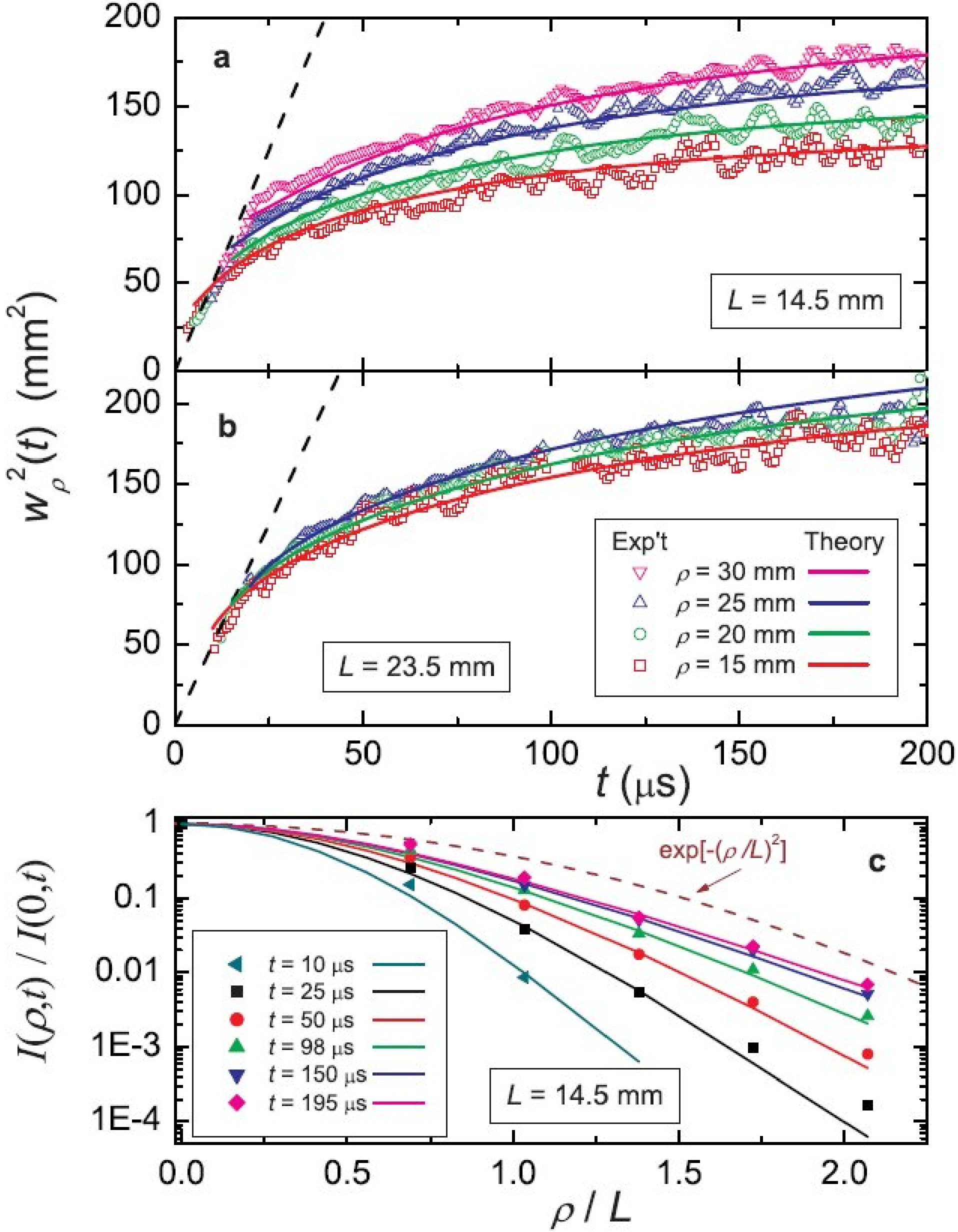
Anderson localization of acoustic waves

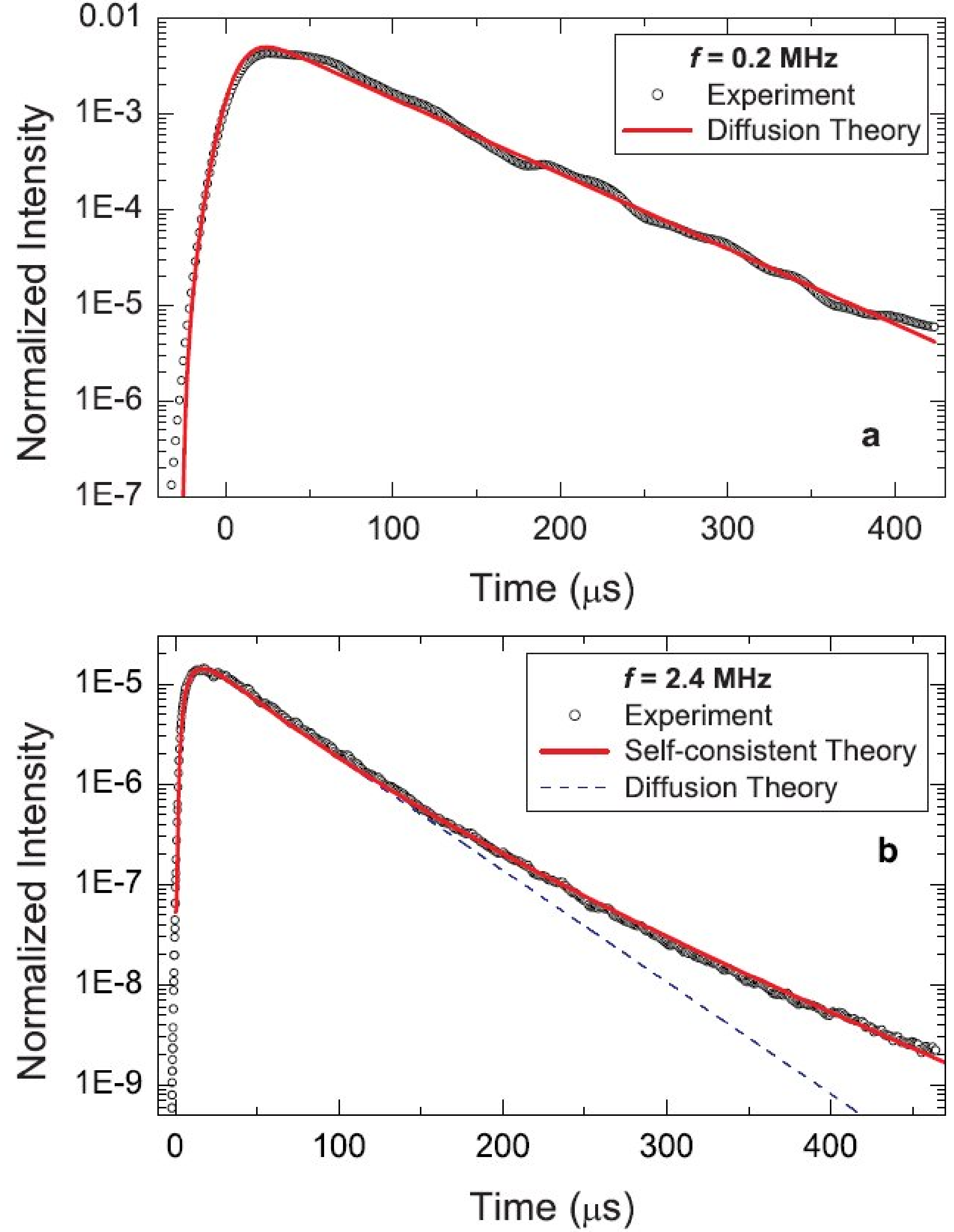
- Packed (disordered) aluminium beads
- Inject acoustic wave at a given point



- Look at the spatial profile of the transmitted intensity
  - In the diffusive regime, expect a Gaussian profile (even in the presence of absorption!)
  - Theory uses a position-dependent diffusion coefficient (B. v. Tiggelen et al, LPMMC Grenoble)
  - Experiment in the group of J. H. Page (Winnipeg)

Anderson localization of acoustic waves





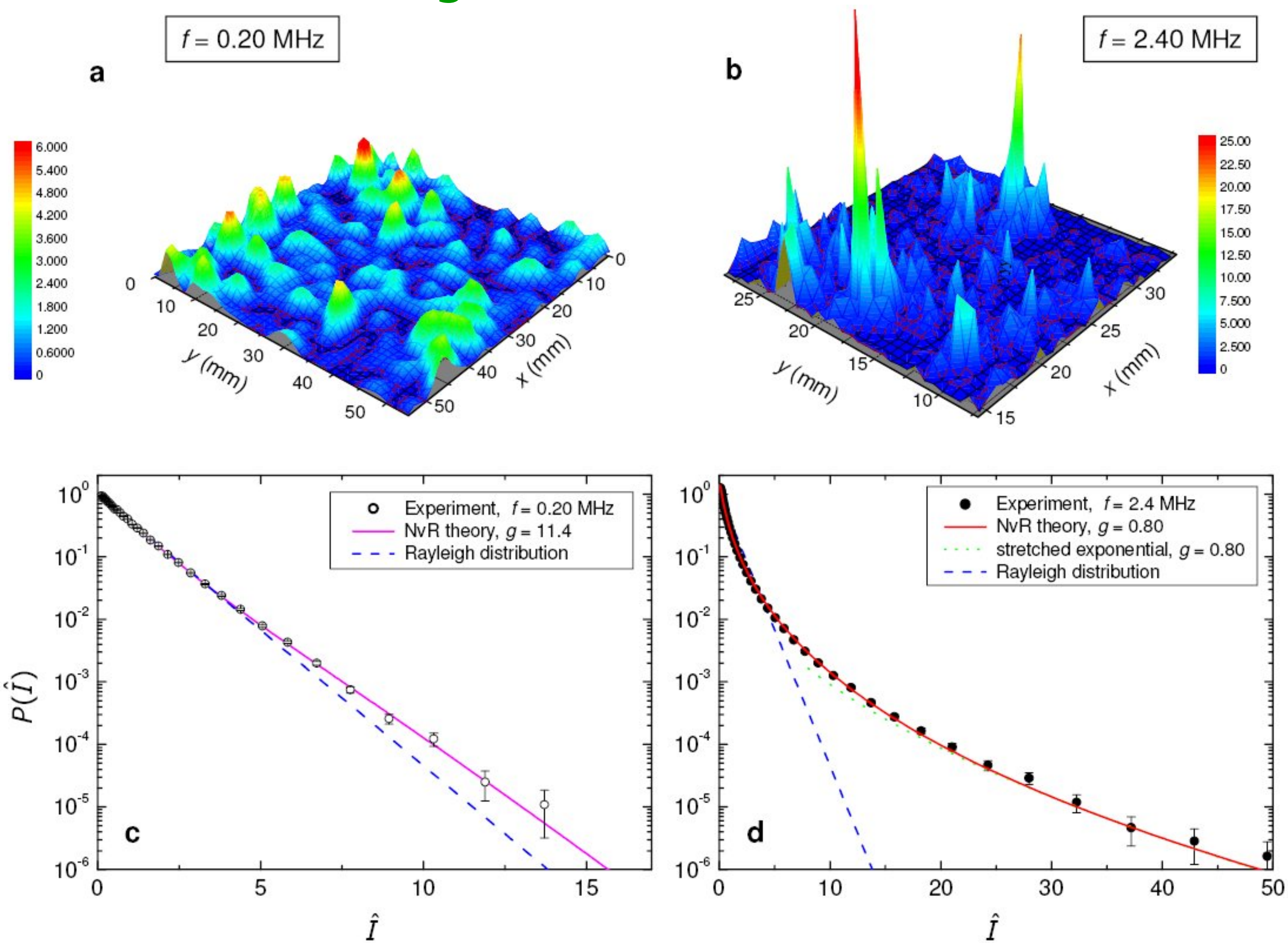
Spatial profile on the outgoing face

Total transmitted intensity

H. Hu et al, Nat. Phys. 4, 495 (2008), group of J. H. Page (Winnipeg)

Anderson localization of acoustic waves: fluctuations

Diffusive regime — Localized regime



H. Hu et al, Nat. Phys. 4, 495 (2008), group of J. H. Page (Winnipeg)

See also multifractality of the intensity distribution:
S. Faez et al, PRL 103, 155703 (2009)