Tensor networks: part II

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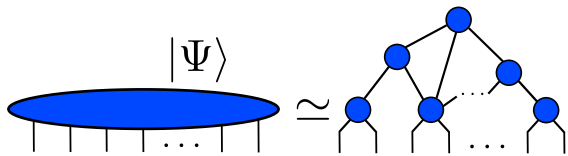
summary of lecture I

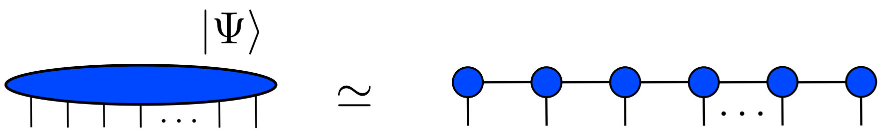
- divide and conquer

$$|\Psi\rangle = \sum_{i_1,\ldots,i_N=1}^{d} \Psi_{i_1\ldots i_N} |i_1 \ldots i_N\rangle$$



  - efficient description of quantum many-body wave-functions
  - no sign problem

- matrix product states



  - properties
  - applicability

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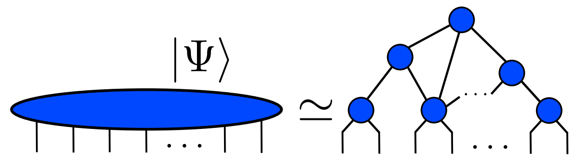

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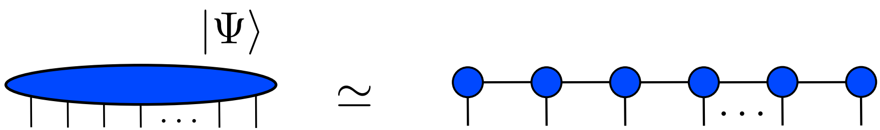

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MPS: correlations

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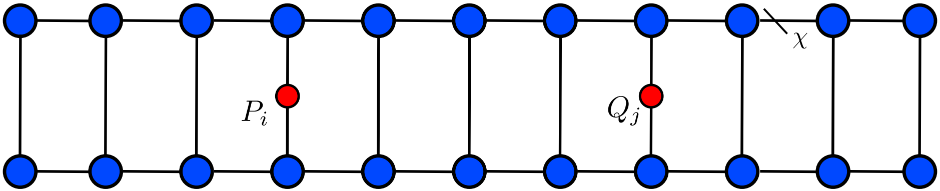

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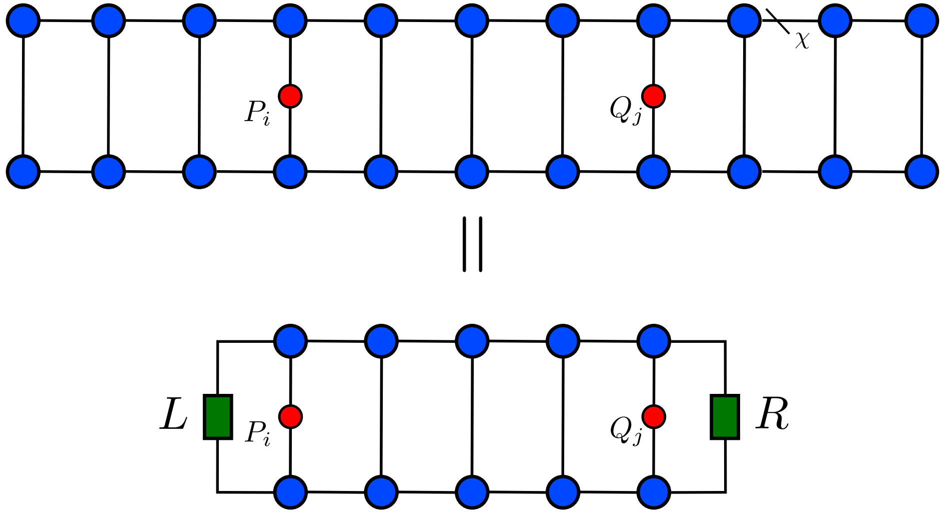

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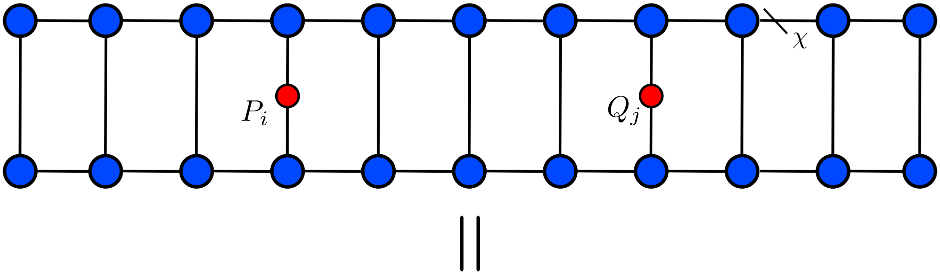
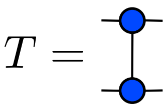
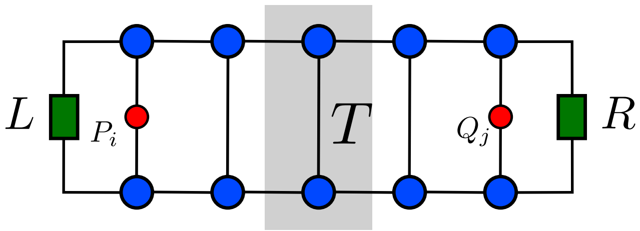


transfer matrix





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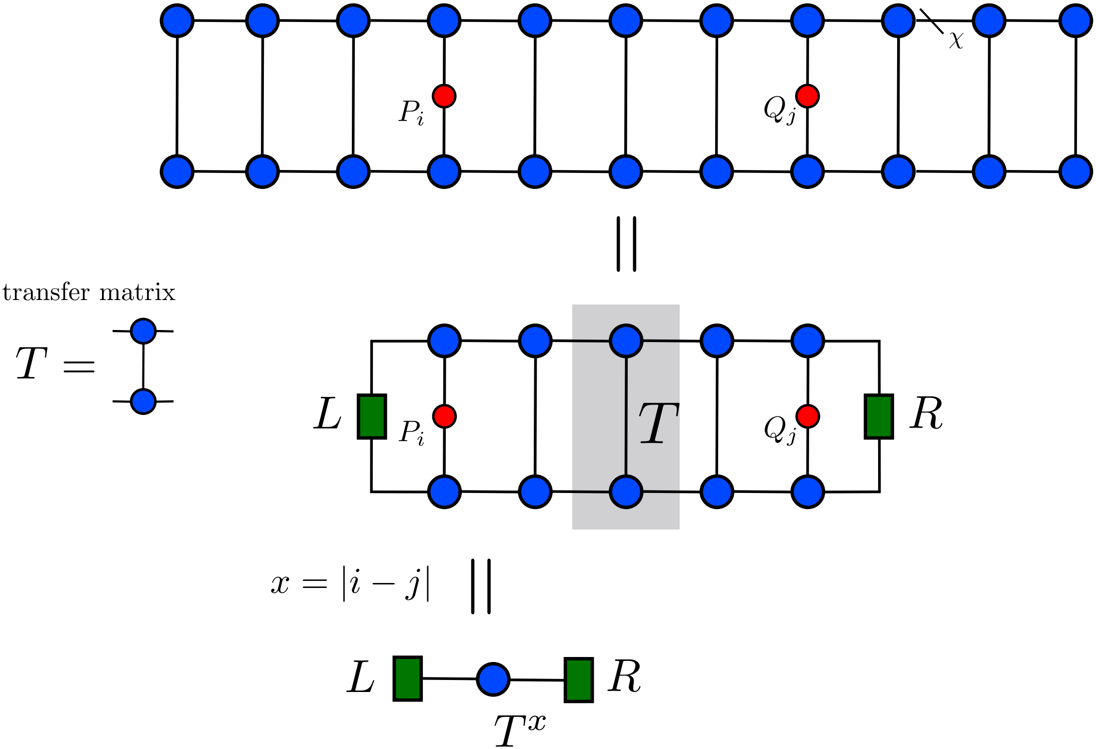

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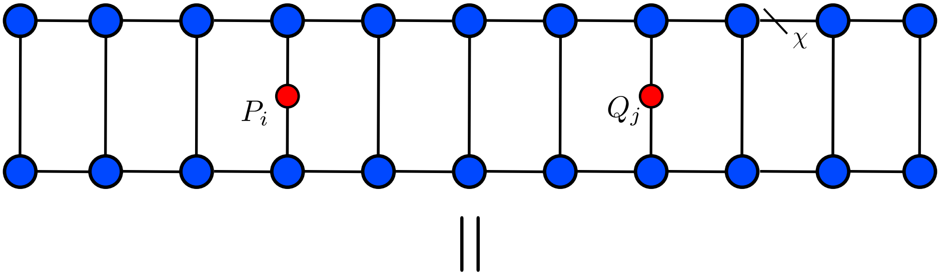

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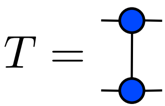
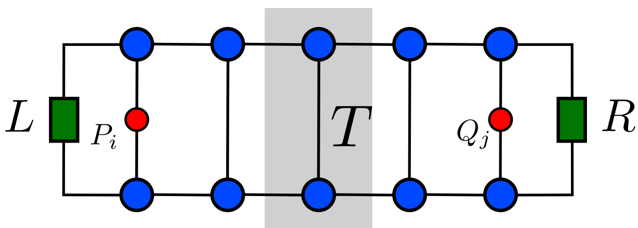
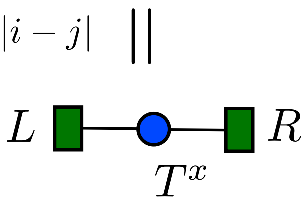




$x = |i - j|$



$$\sim \sum_{j=1}^{\chi^2} c_j (\lambda_j)^x = \sum_{j=1}^{\chi^2} c_j e^{-x \ln(1/\lambda_j)}$$

MPS: CORRELATIONS

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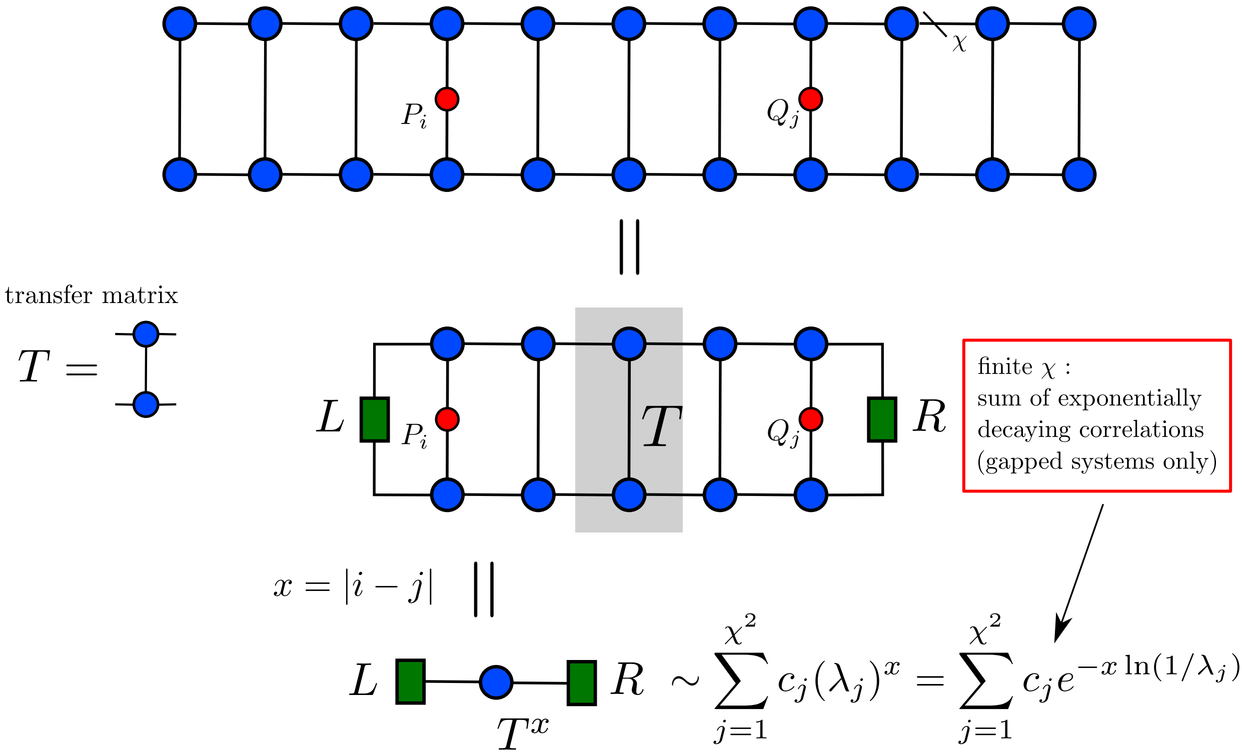


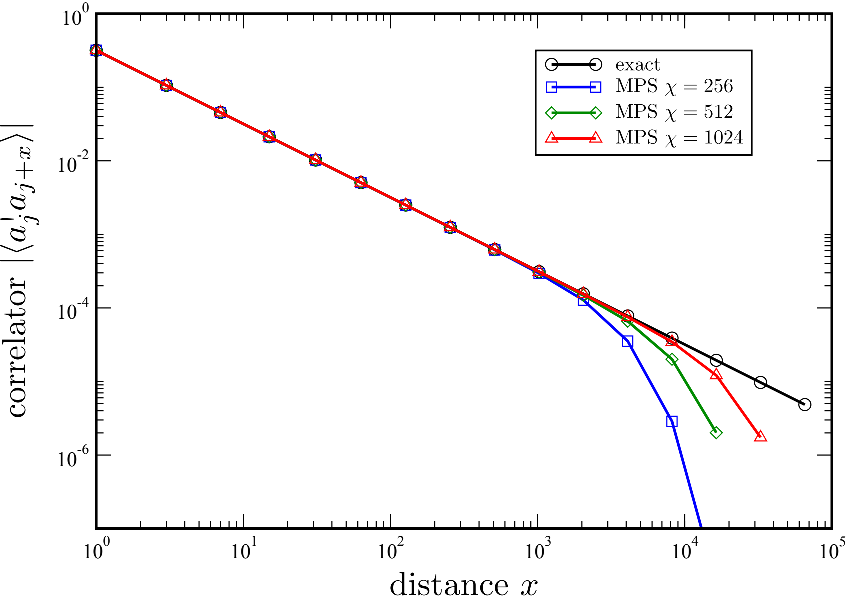
MPS: let's try gapless model...

$$\mathcal{H} = \sum_i (X_i X_{i+1} + Y_i Y_{i+1})$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

MPS: LET'S TRY GAPLESS MODEL...

$$\mathcal{H} = \sum_i (X_i X_{i+1} + Y_i Y_{i+1}) \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

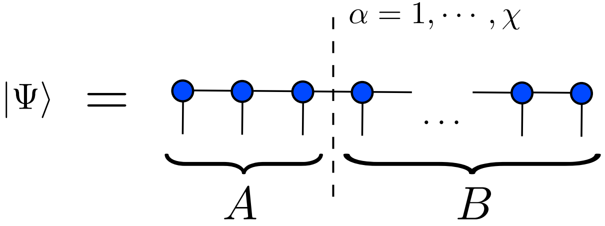


exact: $|\langle a_j^\dagger a_{j+x} \rangle| = \frac{1}{\pi x}$

$$a_j = \left(\prod_{k<j} Z_k\right) \frac{X_j - iY_j}{2}$$

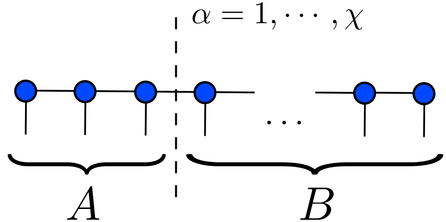
MPS: $|\langle a_j^\dagger a_{j+x} \rangle| = \sum_{j=1}^{\chi^2} c_j e^{-x \ln(1/\lambda_j)}$

MPS: ENTANGLEMENT ENTROPY

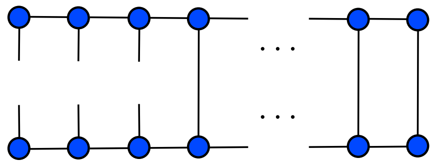


MPS: entanglement entropy

$|\Psi\rangle \;=$



$$\varrho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| =$$



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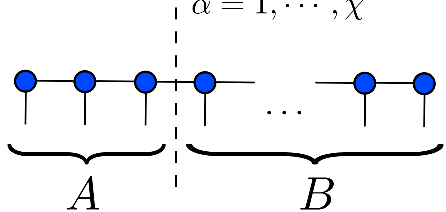
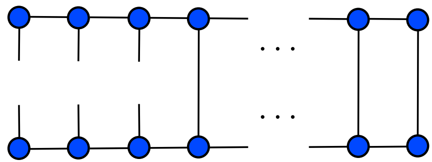
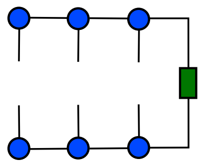
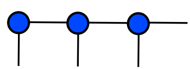
$|\Psi\rangle =$ 

$\varrho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| =$ 

$=$ 

$$\varrho_A^{\mathrm{MPS}} = W \varrho_A W^\dagger$$

where $W =$  $W : \mathbb{C}_\chi \to \mathbb{C}_d^{\otimes |A|}$

MPS: entanglement entropy

$$\varrho_A^{\text{MPS}} = W \varrho_A W^\dagger = \begin{pmatrix} 1/\chi & & \\ & \ddots & \\ & & 1/\chi \end{pmatrix} \qquad (\chi \times \chi \text{ matrix})$$

$$S(\varrho_A^{\text{MPS}}) = -\text{Tr}(\varrho_A^{\text{MPS}} \log \varrho_A^{\text{MPS}}) = \log \chi$$

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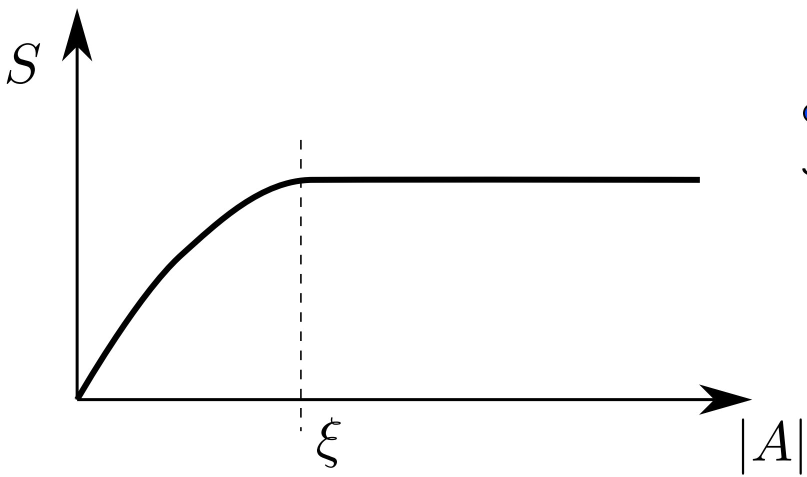

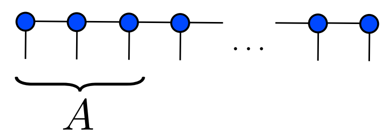

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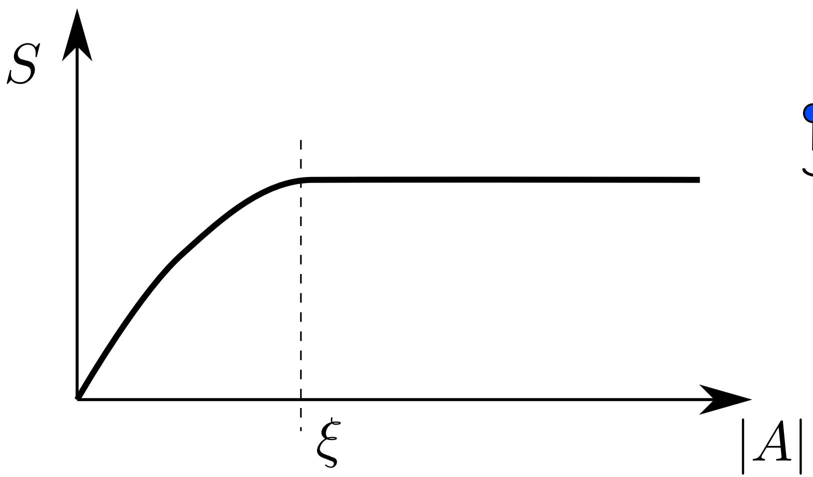
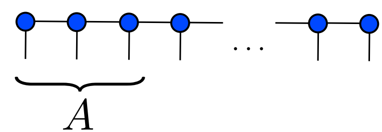


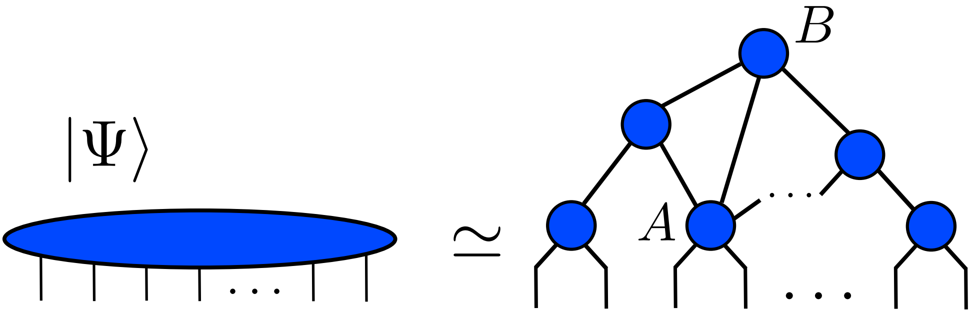


- area law

$$S \sim \partial A$$

GAUGE FREEDOM

- tensor network representation is not unique



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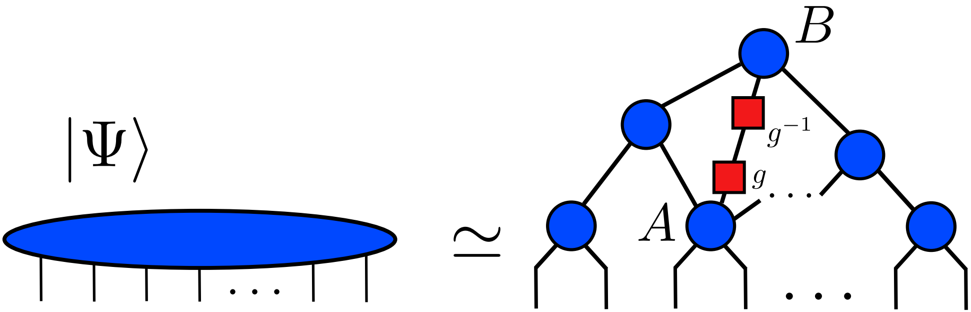

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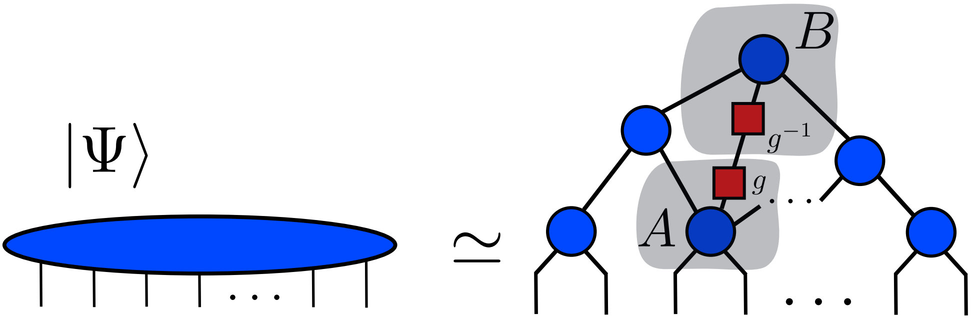

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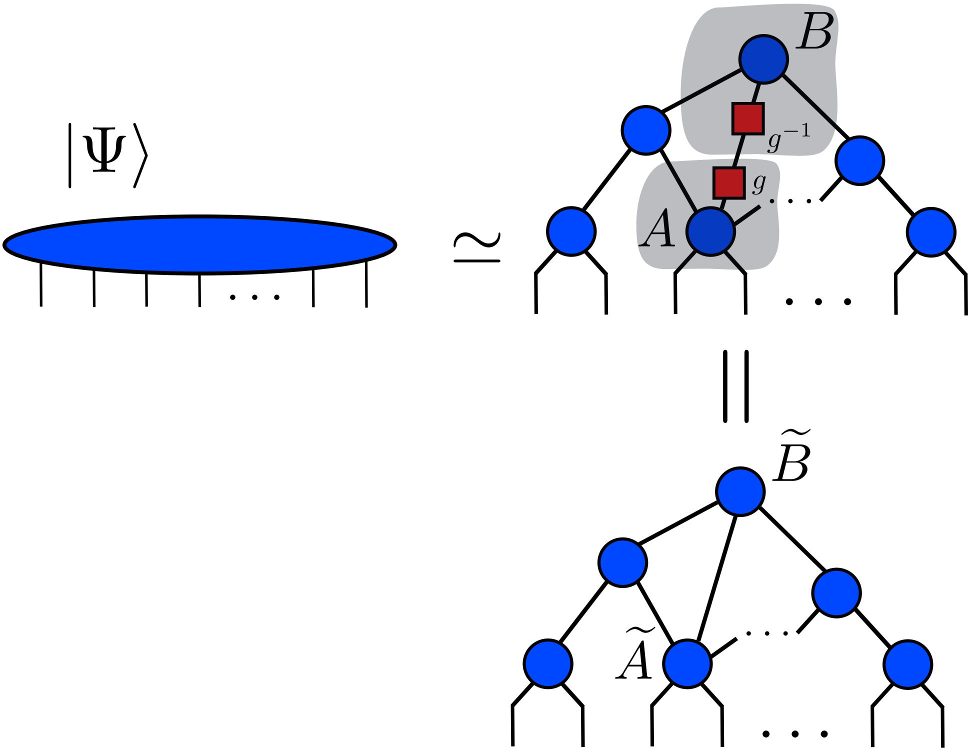
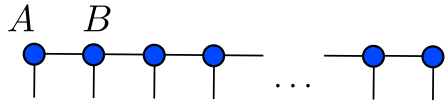


FIXING GAUGE: MPS

- canonical form of the MPS

$|\Psi\rangle =$ 

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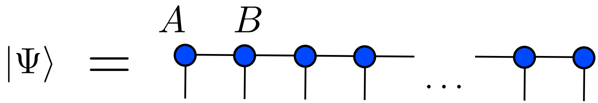
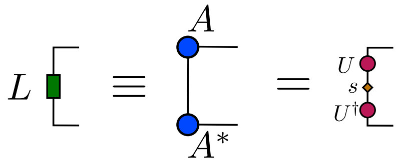
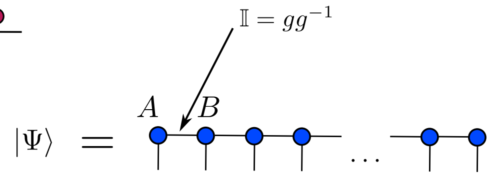


consider matrix L



insert $\mathbb{I} = gg^{-1}$ with $g = U^\dagger s^{-1/2}$



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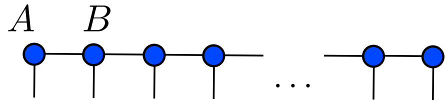
$|\Psi\rangle =$ 

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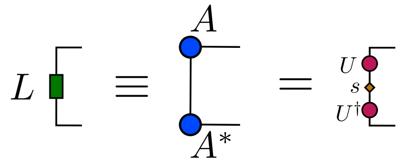

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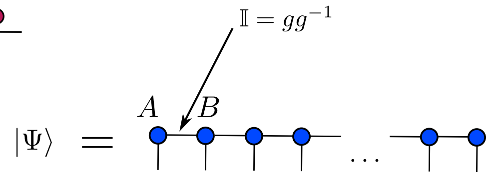
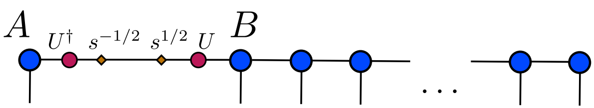


$|\Psi\rangle =$ 

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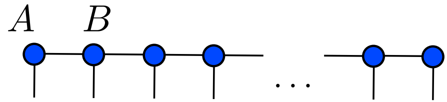
$|\Psi\rangle =$ 

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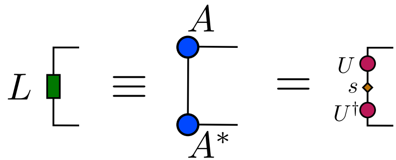

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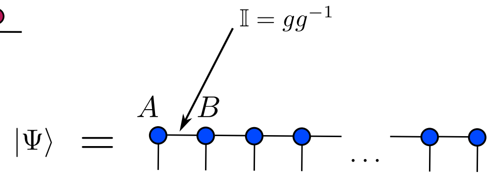
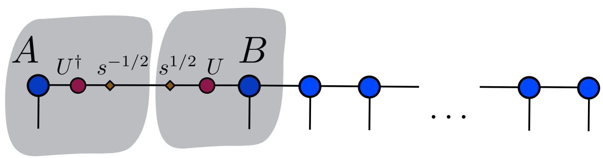


$|\Psi\rangle =$ 

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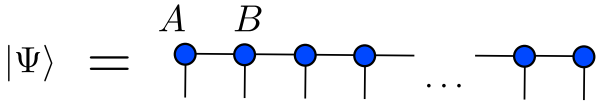

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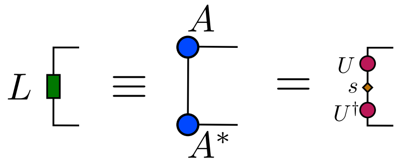

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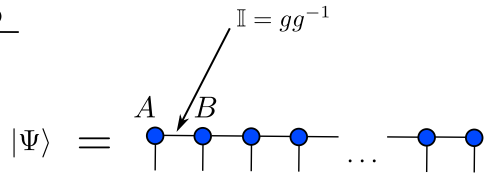

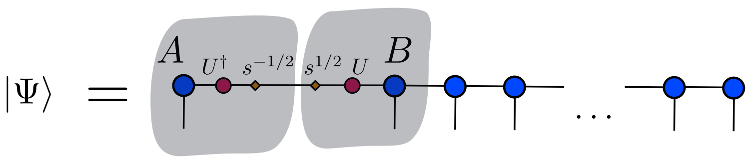

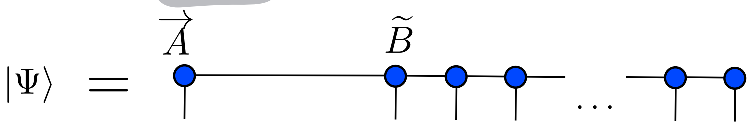

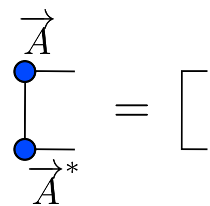

$\overrightarrow{A}$ — right-isometric

FIXING GAUGE: MPS

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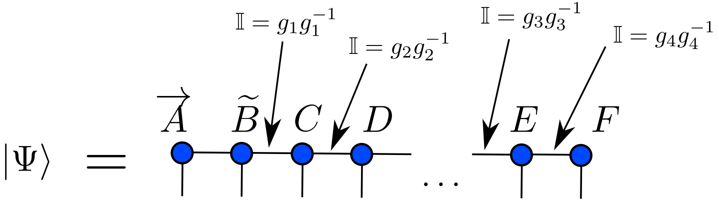
similarly:



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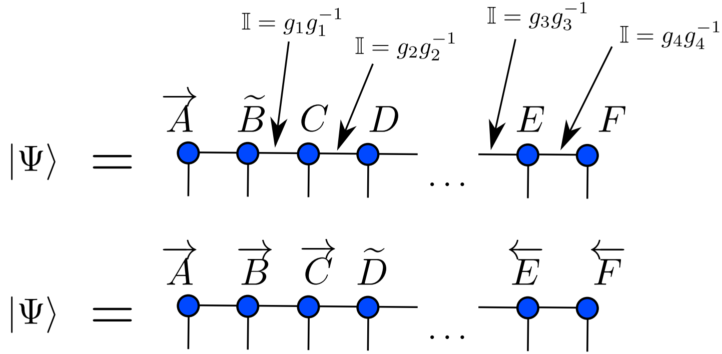
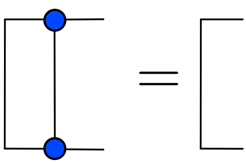
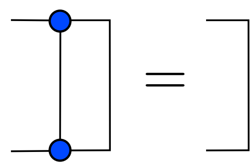


$\overrightarrow{B}, \overrightarrow{C}$ — right-isometric

$\overleftarrow{E}, \overleftarrow{F}$ — left-isometric



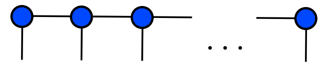


- provides numerical stability

MPS for a ground state

- find the ground state of $\mathcal{H} = \sum_i h^{[i,i+1]}$
- treat MPS as a variational ansatz and minimize the total energy of the system

$$E = \frac{\langle \Psi | \mathcal{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

$|\Psi\rangle =$ 

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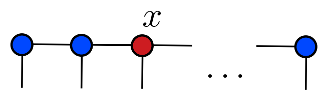

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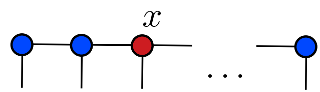

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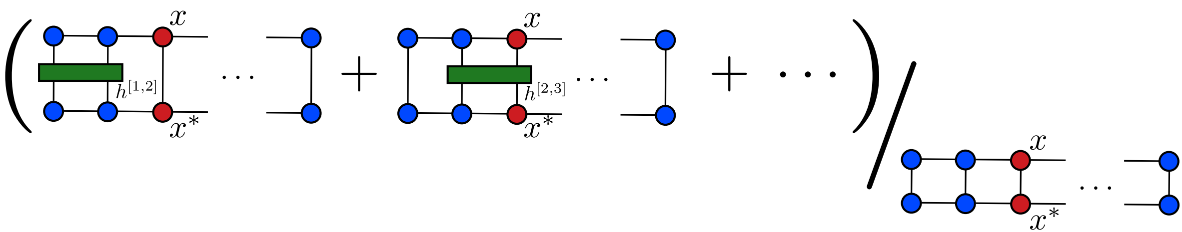

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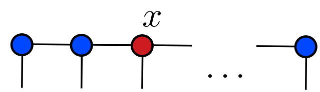
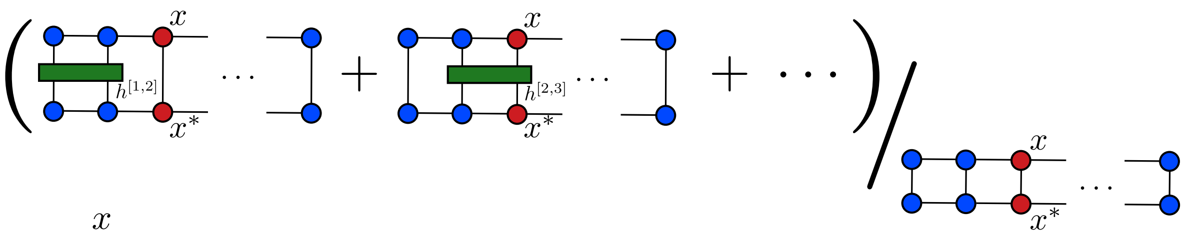
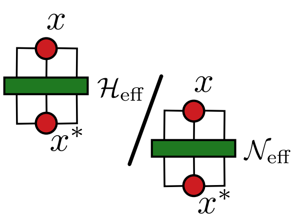
$|\Psi\rangle =$ 

$E =$ 

$E =$ 

$$= \frac{\mathbf{x}^\dagger \mathcal{H}_{\text{eff}} \mathbf{x}}{\mathbf{x}^\dagger \mathcal{N}_{\text{eff}} \mathbf{x}}$$

MPS FOR A GROUND STATE

- find the ground state of $\mathcal{H} = \sum_i h^{[i,i+1]}$
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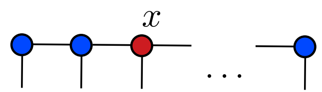

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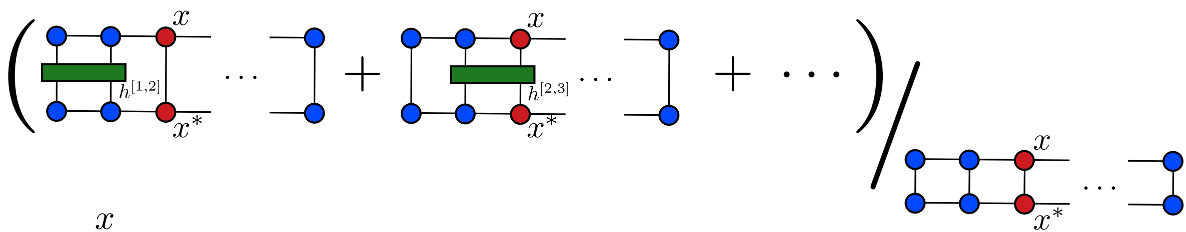

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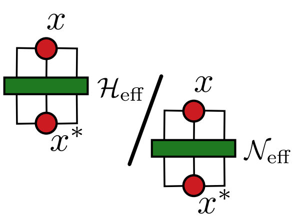

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$\mathbf{x}$ — solution of a generalized eigenvalue problem

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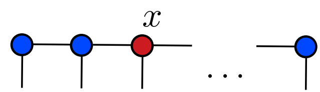

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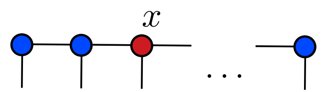

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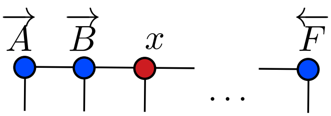

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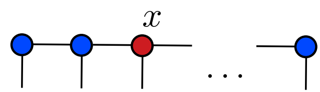

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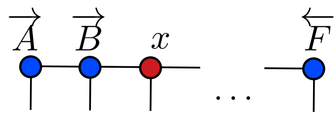
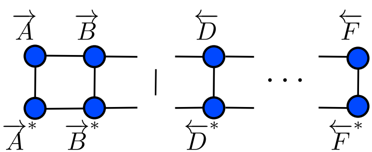
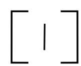


- $\mathcal{N}_{\text{eff}} =$



$=$



$= \mathbb{I}_{d\chi^2}$

$$\longrightarrow \boxed{\mathcal{H}_{\text{eff}} \mathbf{x} = E_0 \cdot \mathbf{x}}$$

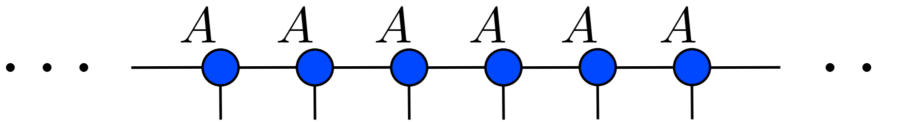
standard eigenvalue problem: numerically stable

SUMMARY: MATRIX PRODUCT STATES

- intended for 1D systems

- extendable to infinite systems

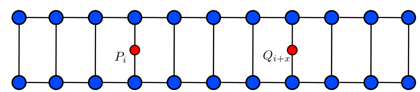
$|\Psi^{\text{inf}}\rangle =$



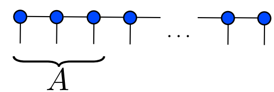
- accurately captures ground states of local gapped Hamiltonians

$$\mathcal{H} = \sum_i h^{[i,i+1]}$$

- finite correlations: $\langle \Psi | P_i Q_{i+x} | \Psi \rangle \sim e^{-x/\xi}$

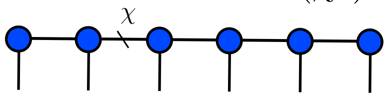


- area law: $S \sim \partial A \; (= \text{const})$

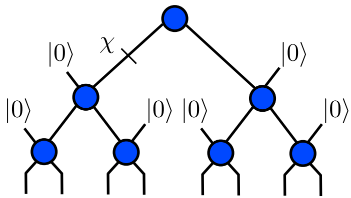


EXAMPLES

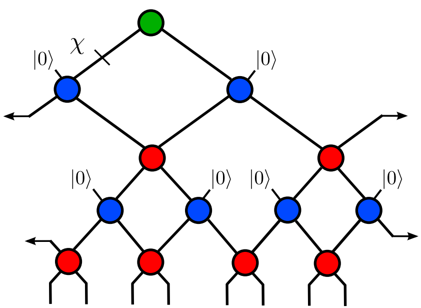
- MPS $\mathcal{O}(\chi^3)$



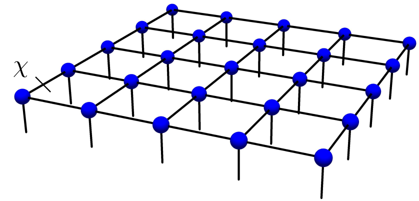
- TTN $\mathcal{O}(\chi^4)$



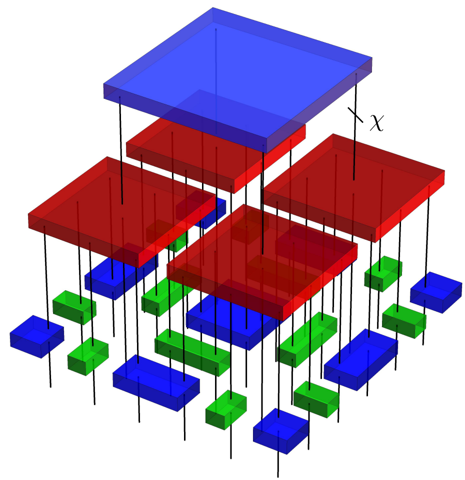
- 1D MERA $\mathcal{O}(\chi^9)$



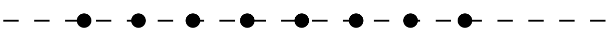
- PEPS $\mathcal{O}(\chi^{10\ldots12})$



- 2D MERA $\mathcal{O}(\chi^{16})$



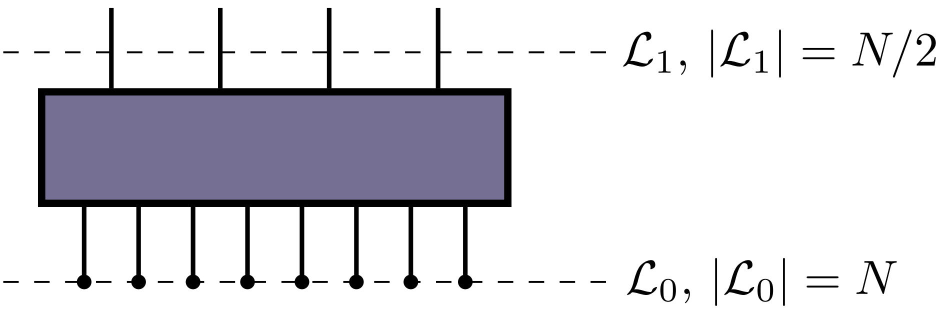
MERA



$\mathcal{L}_0, \ |\mathcal{L}_0| = N$

- coarse-graining procedure

MERA



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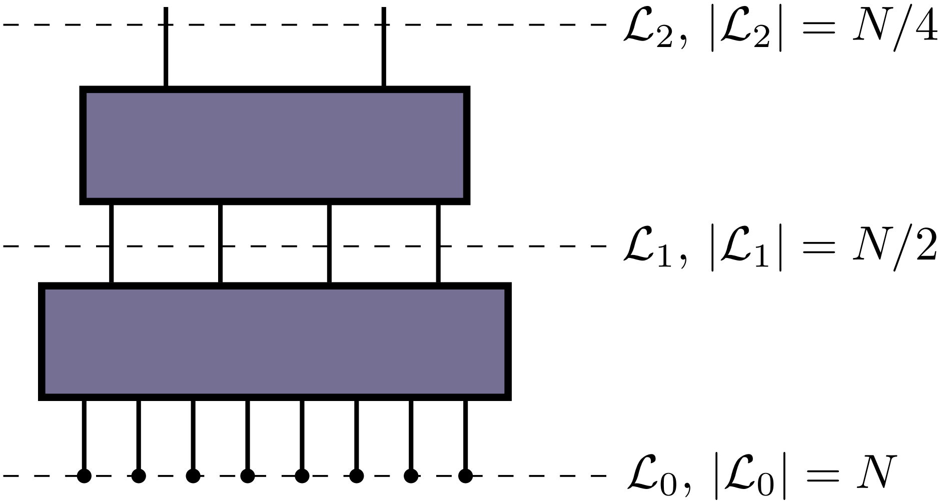

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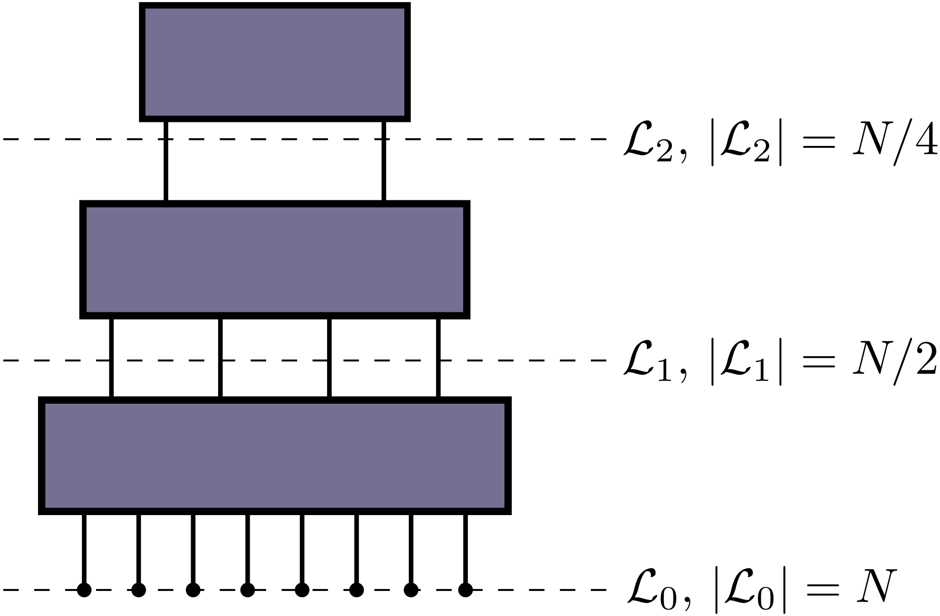

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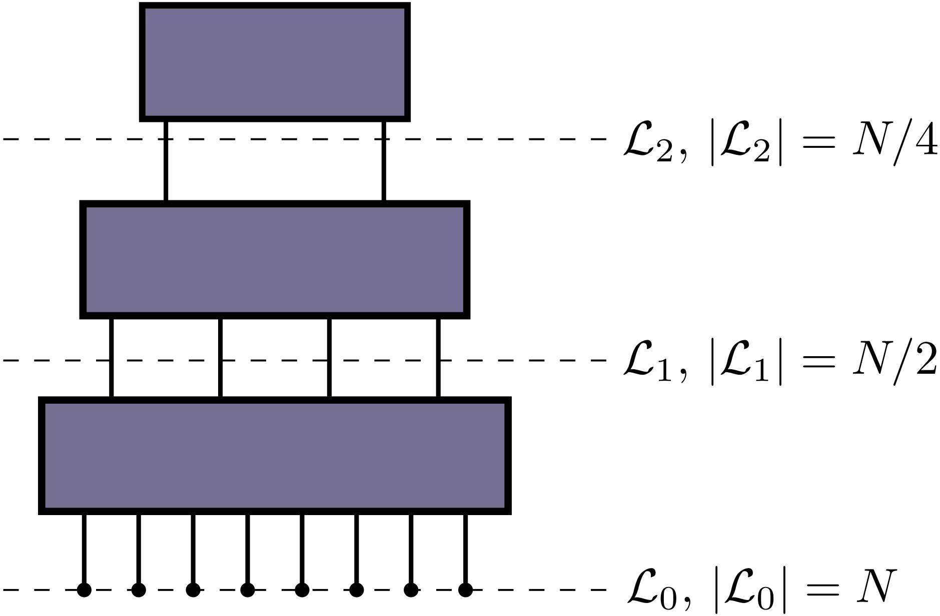

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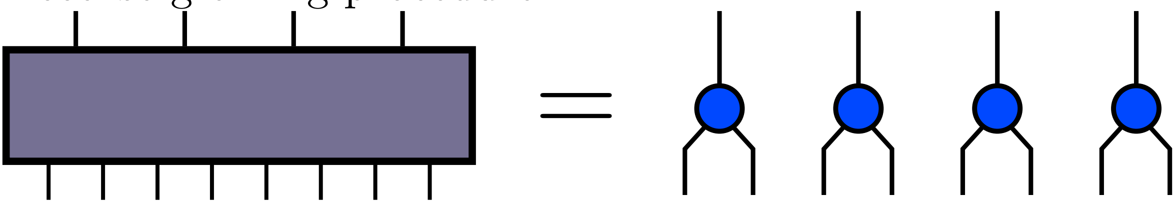

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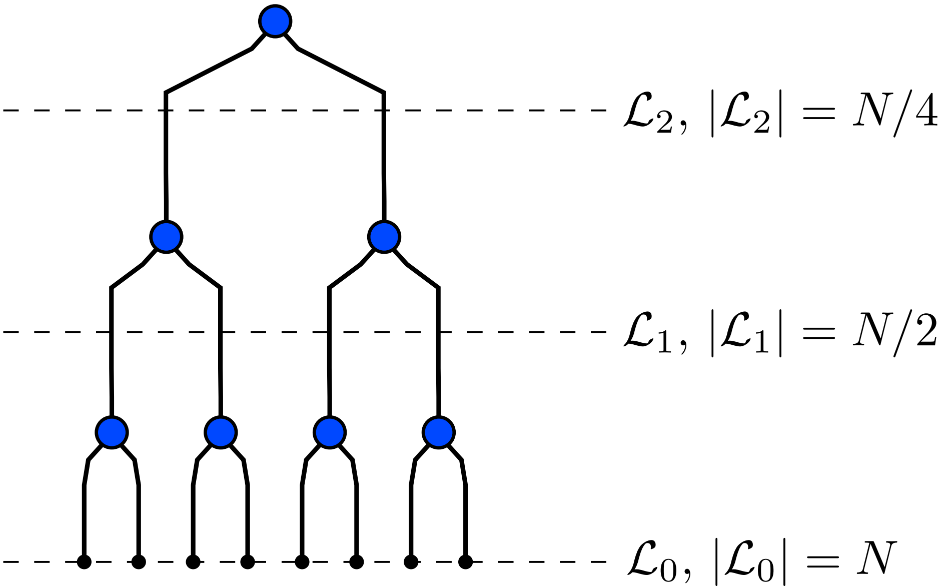

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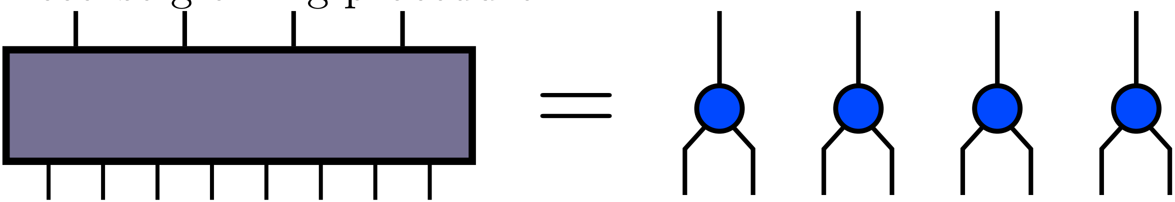

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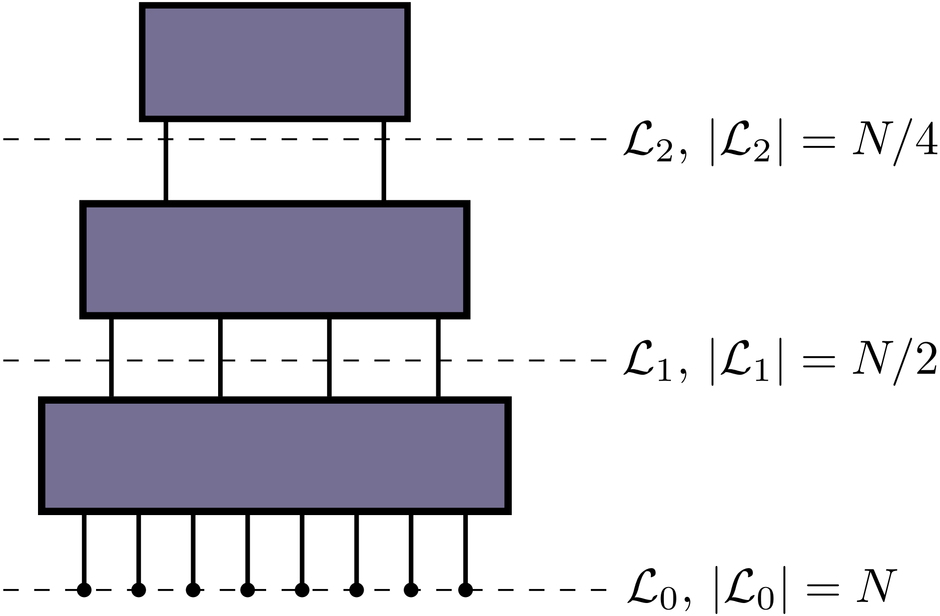

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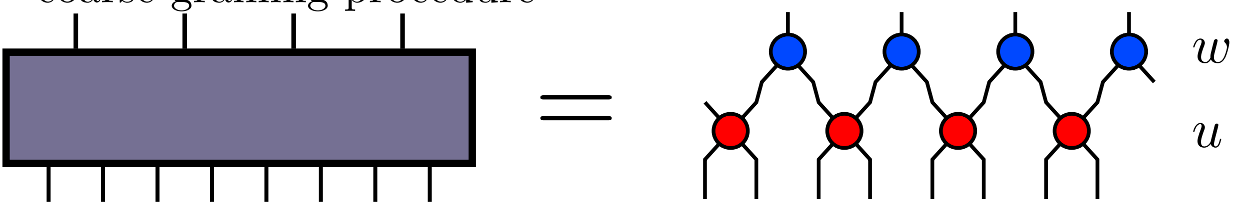

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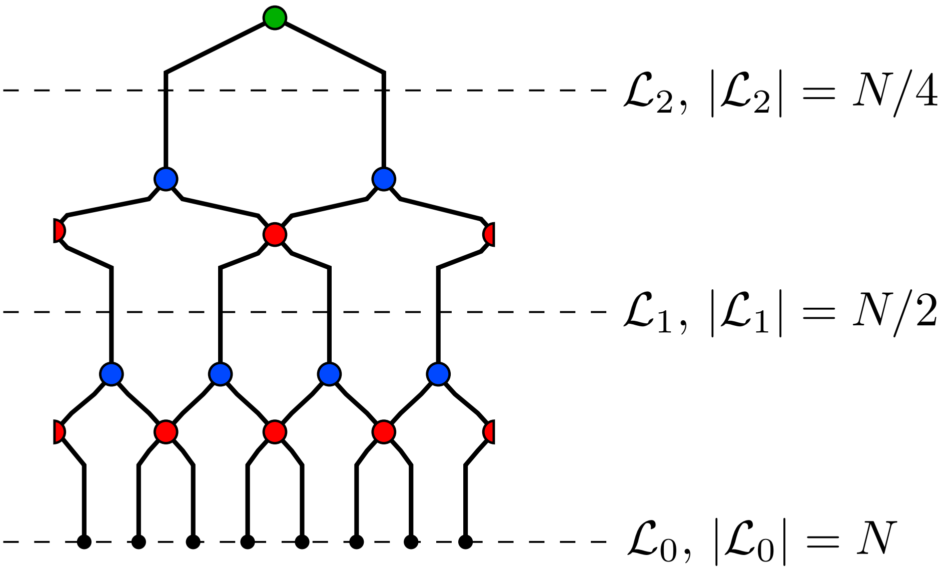

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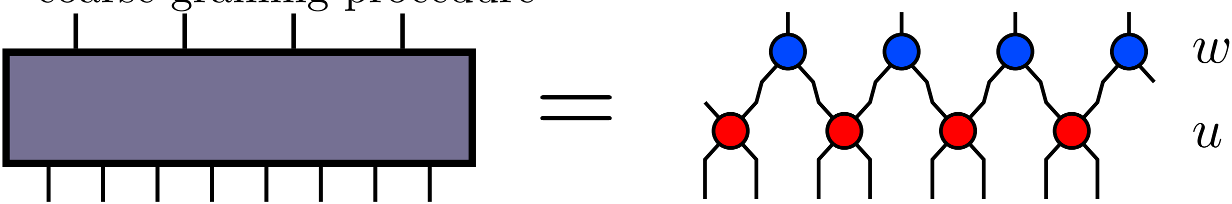

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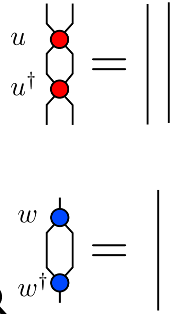

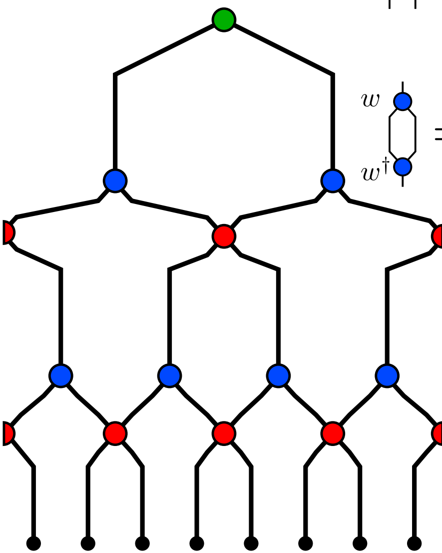

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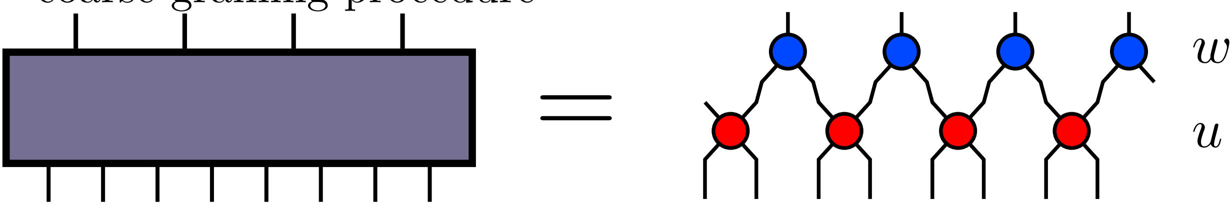

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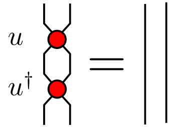

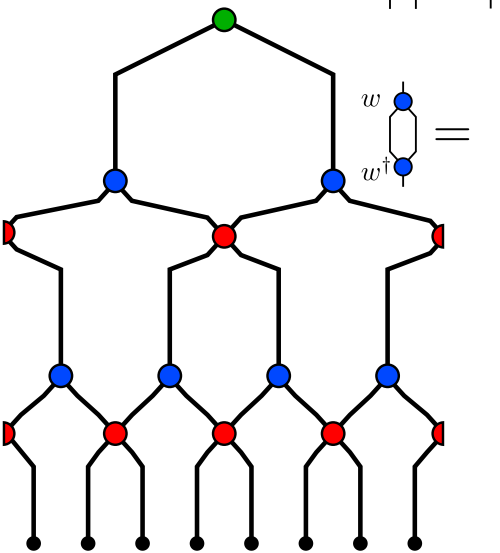

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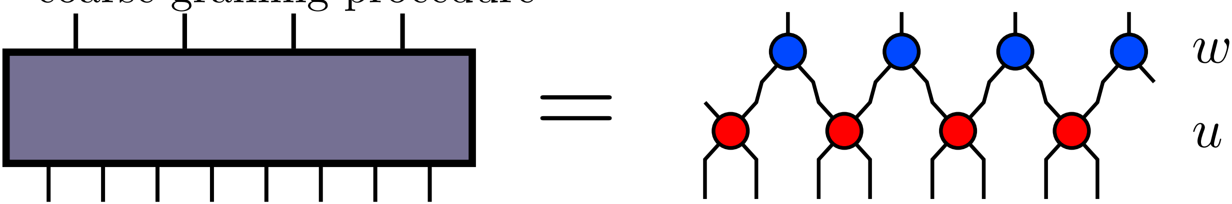
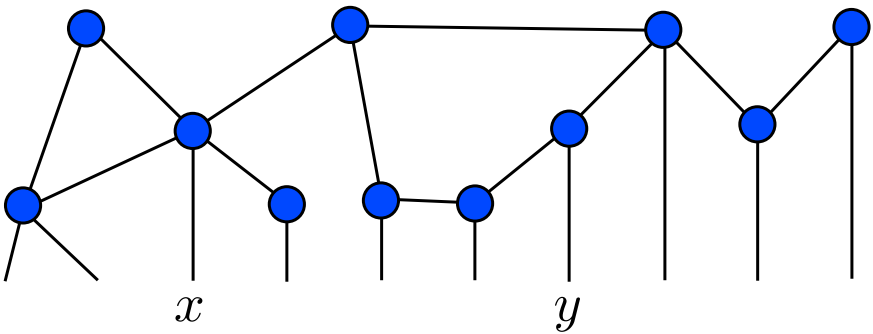


CORRELATIONS IN TENSOR NETWORK STATES

- define **distance** within tensor network



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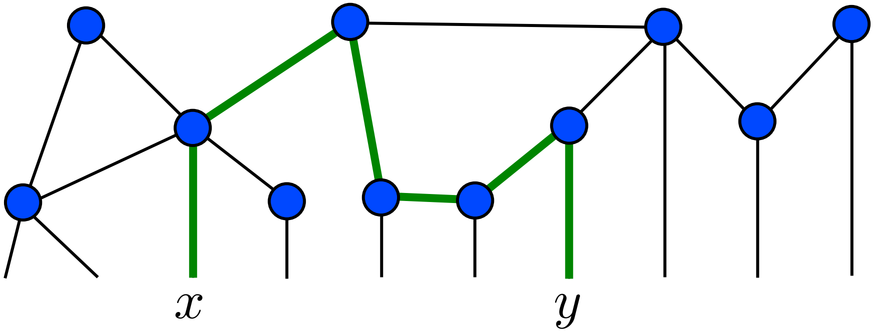

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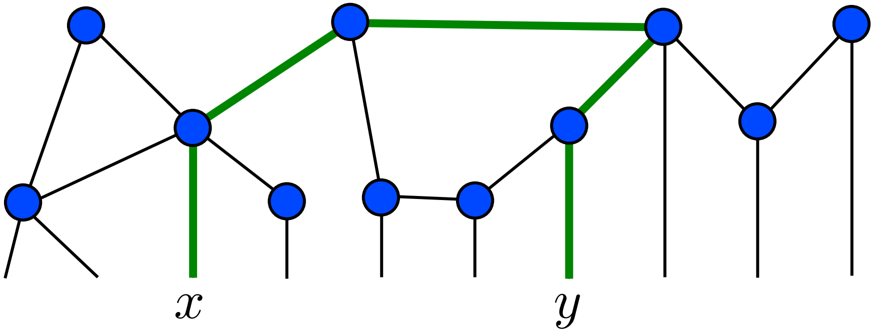

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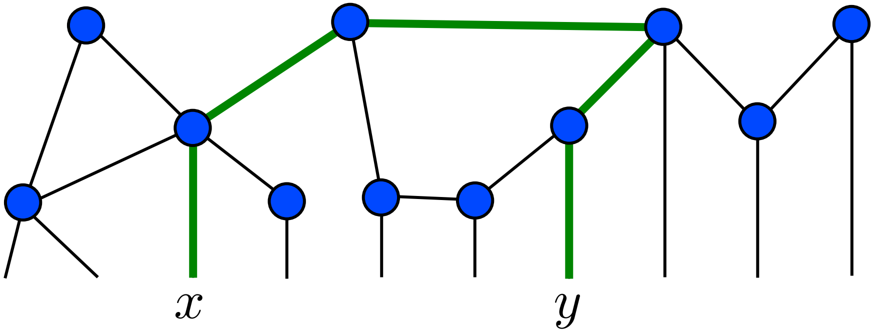

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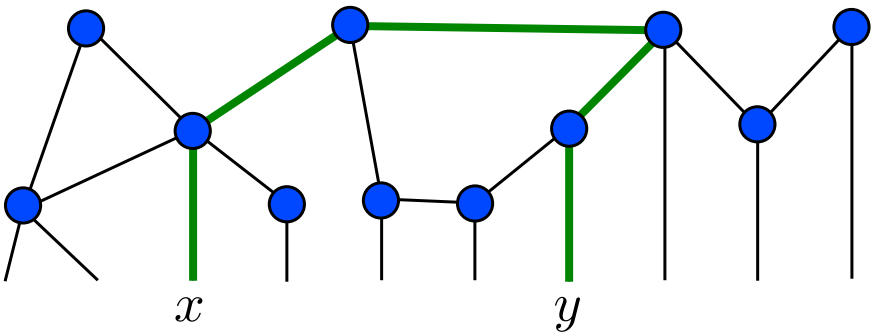

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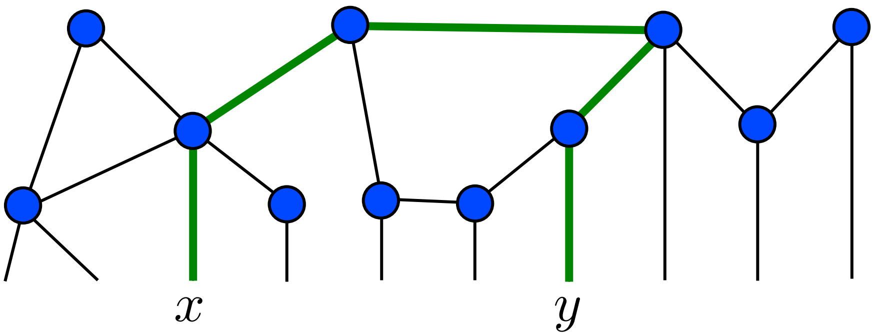

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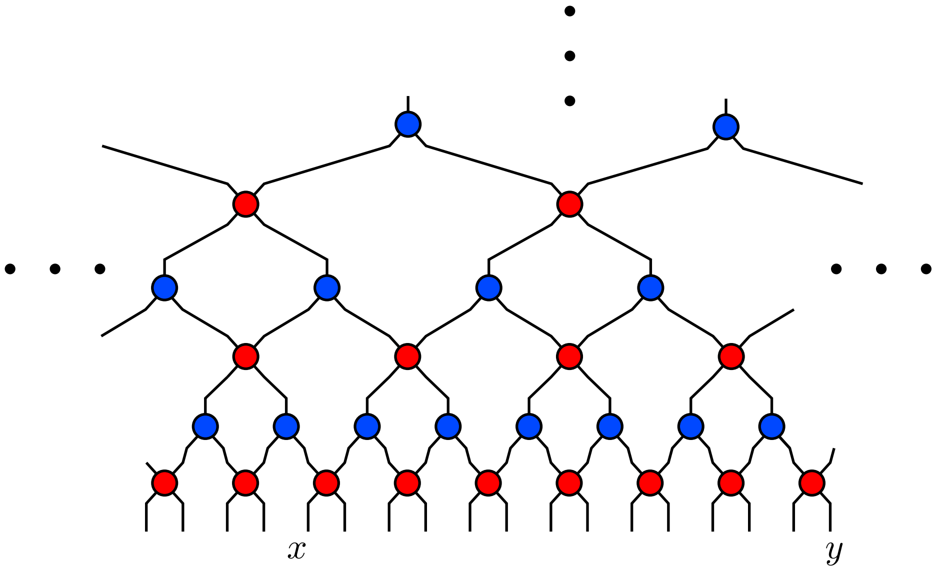
MERA: correlations



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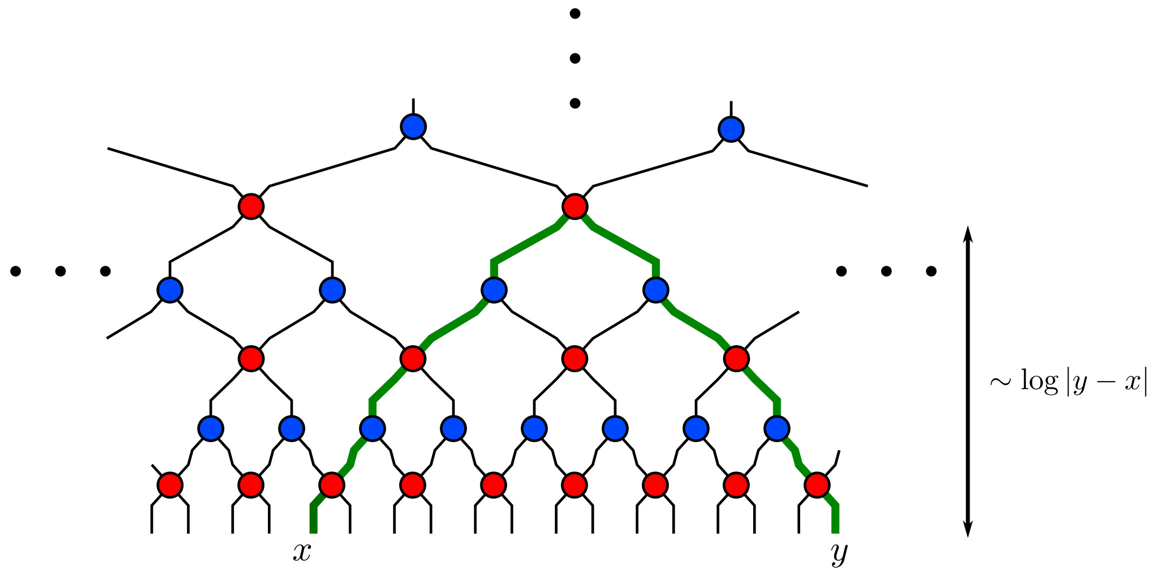

- distance in MERA: $D(x,y) \sim \log|y-x|$
- correlations in MERA: $C(x,y) = e^{-D(x,y)\cdot\ln(1/\lambda)}$

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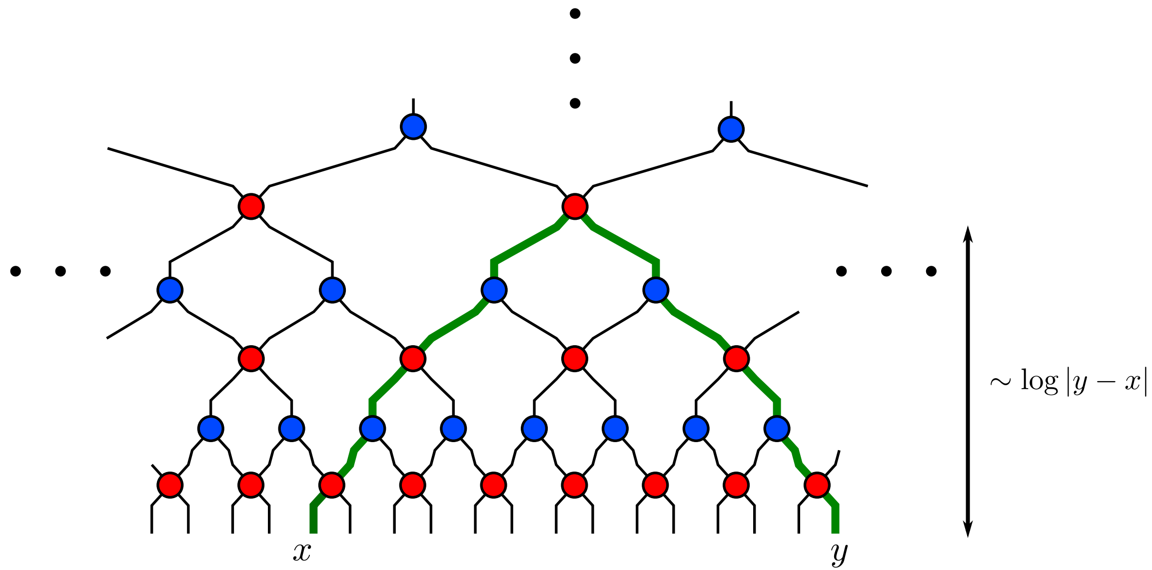

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$$= |y-x|^{-b}$$

for some $b > 0$

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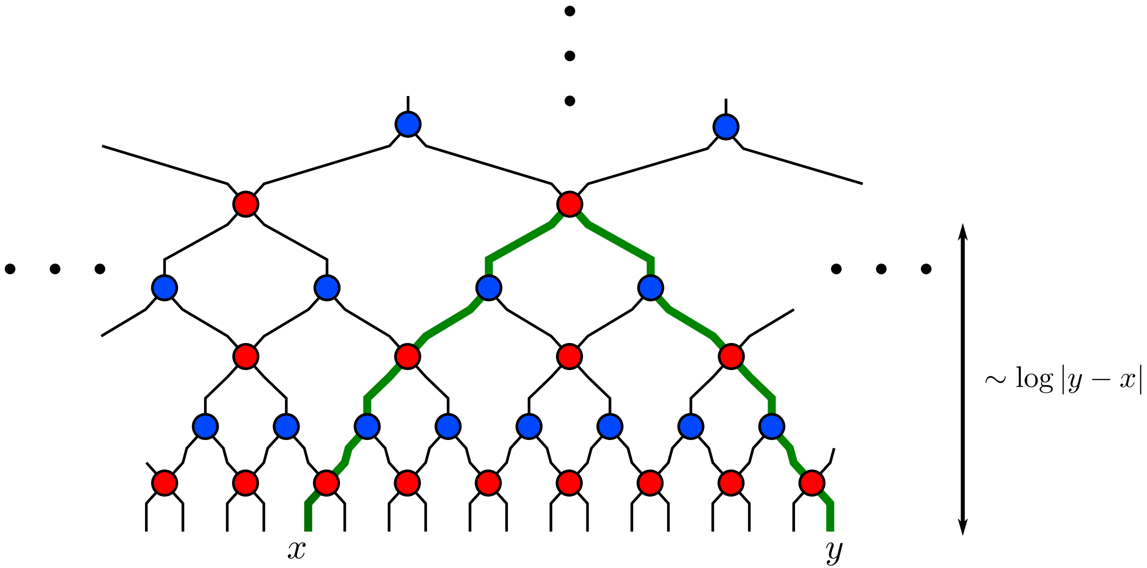
MERA: CORRELATIONS



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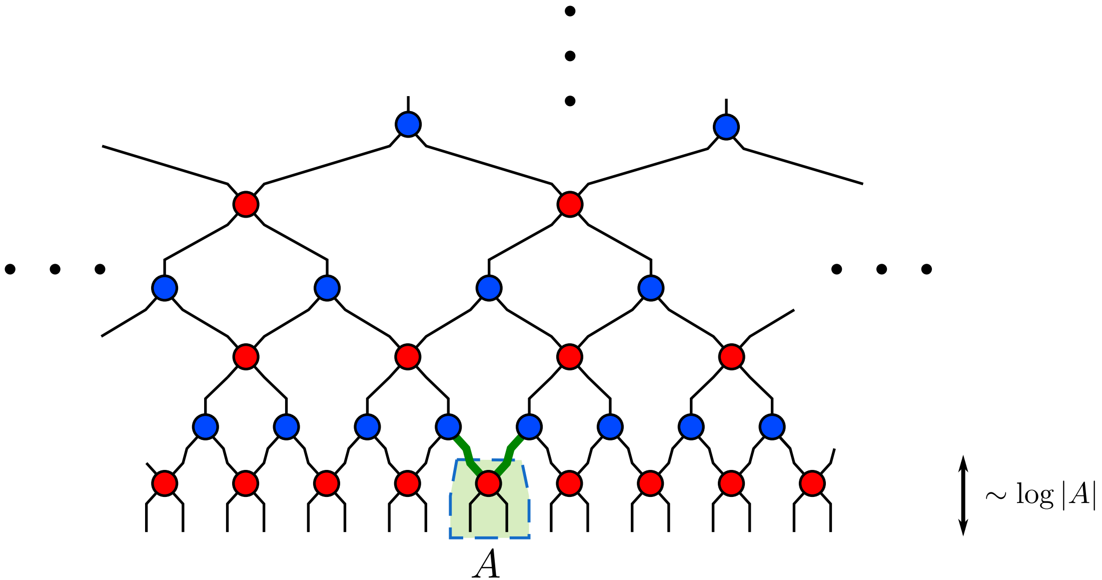
MERA: ENTANGLEMENT ENTROPY



- entanglement entropy of a region A
- connectivity between region A and the rest of the wave-function
- $S(\varrho_A) \leq \log(\chi) \cdot (\sharp \text{ crossings}) \sim \log |A|$

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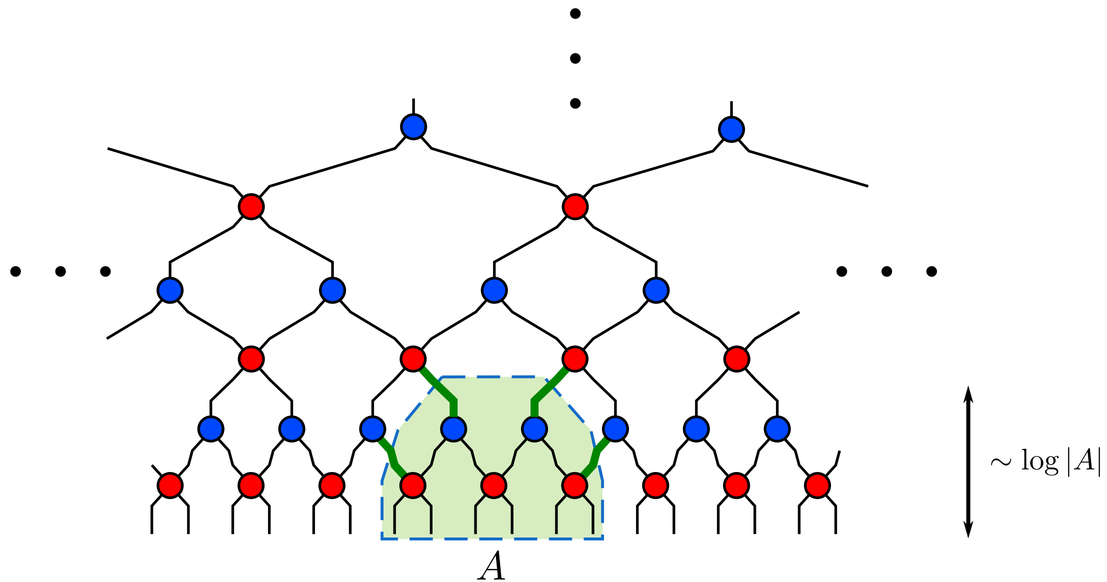

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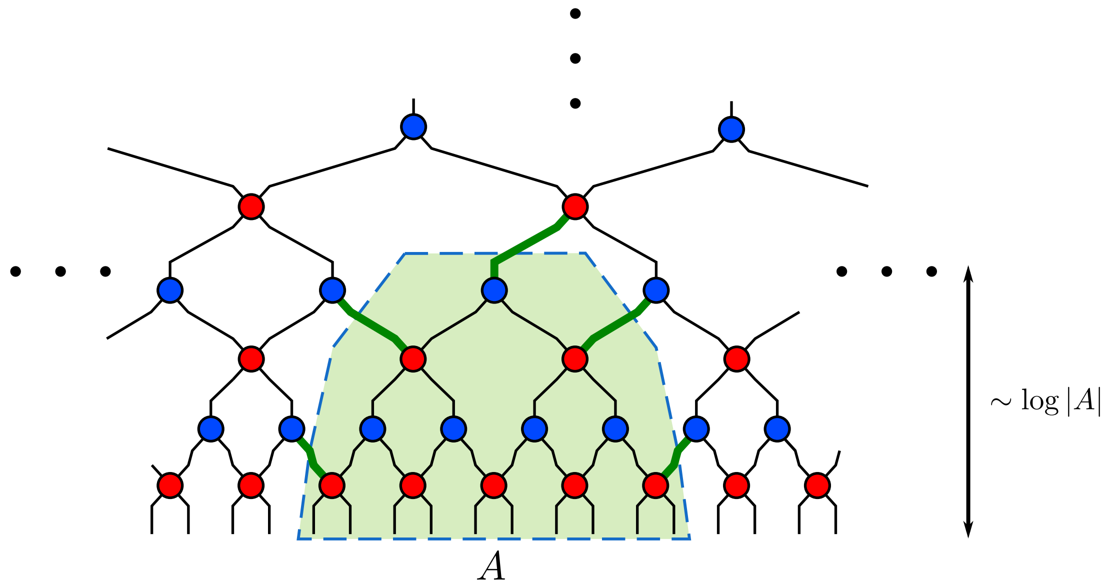

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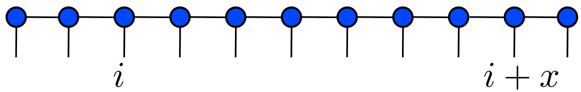
- $S(\varrho_A) \leq \log(\chi) \cdot (\sharp \text{ crossings}) \sim \log|A|$

MPS vs MERA: correlations

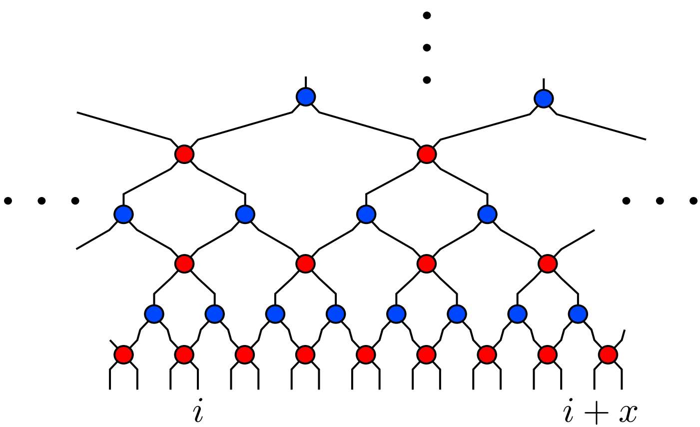


$$\langle \Psi | P_i Q_{i+x} | \Psi \rangle \sim e^{-x/\xi}$$

exponential correlations

(gapped)

vs.

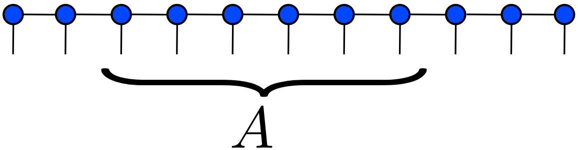


$$\langle \Psi | P_i Q_{i+x} | \Psi \rangle \sim x^{-b}$$

polynomial correlations

(gapless)

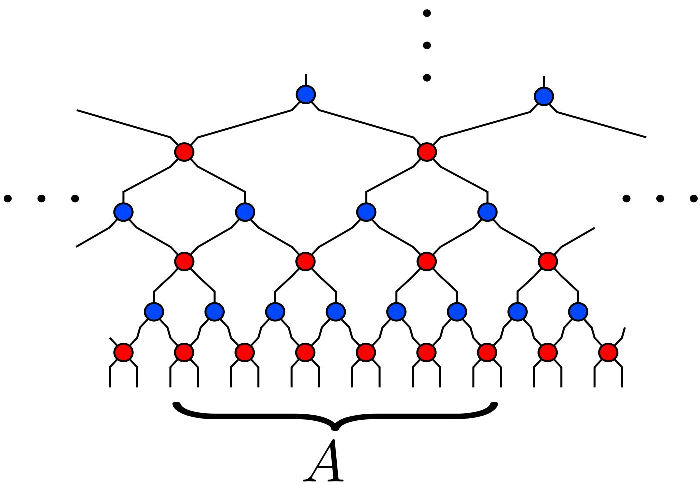
MPS vs MERA: entanglement entropy



$S(\varrho_A) \sim \text{const}$

(gapped)

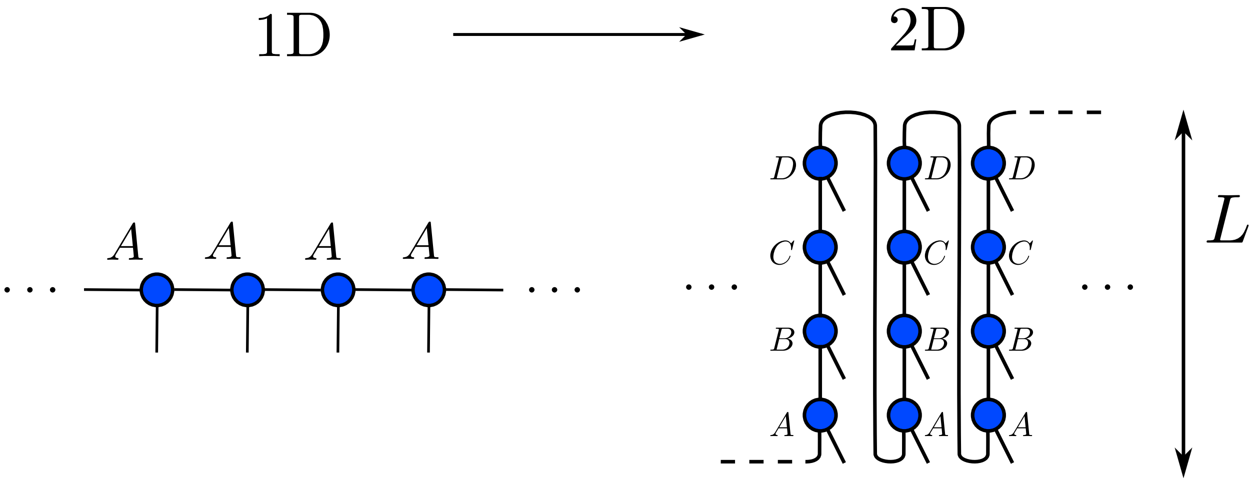
vs.



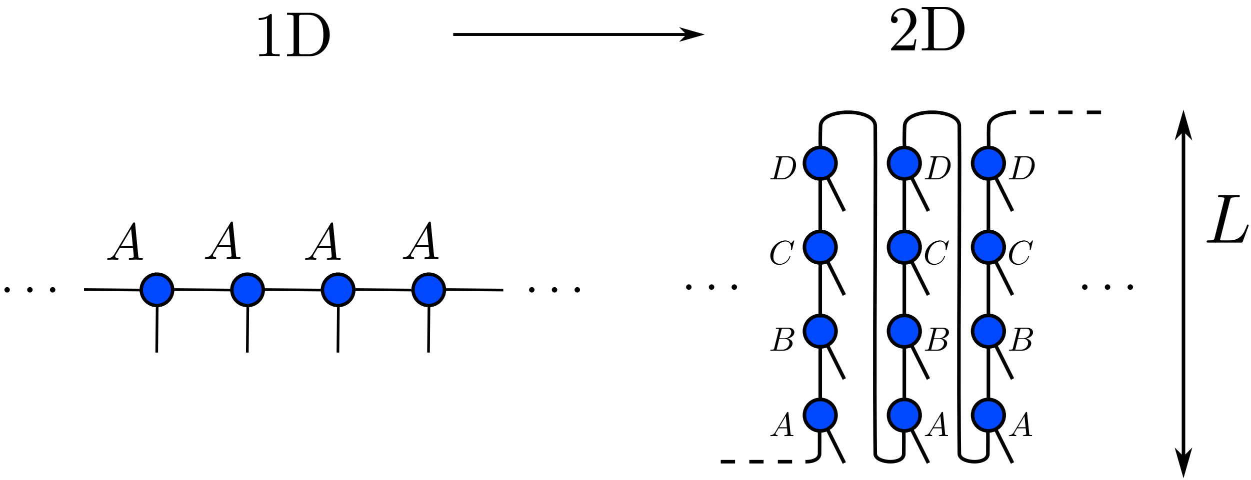
$S(\varrho_A) \sim \log |A|$

(gapless)

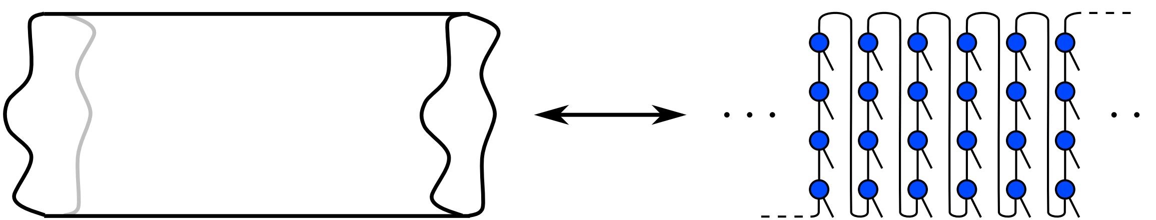
MPS for 2D systems



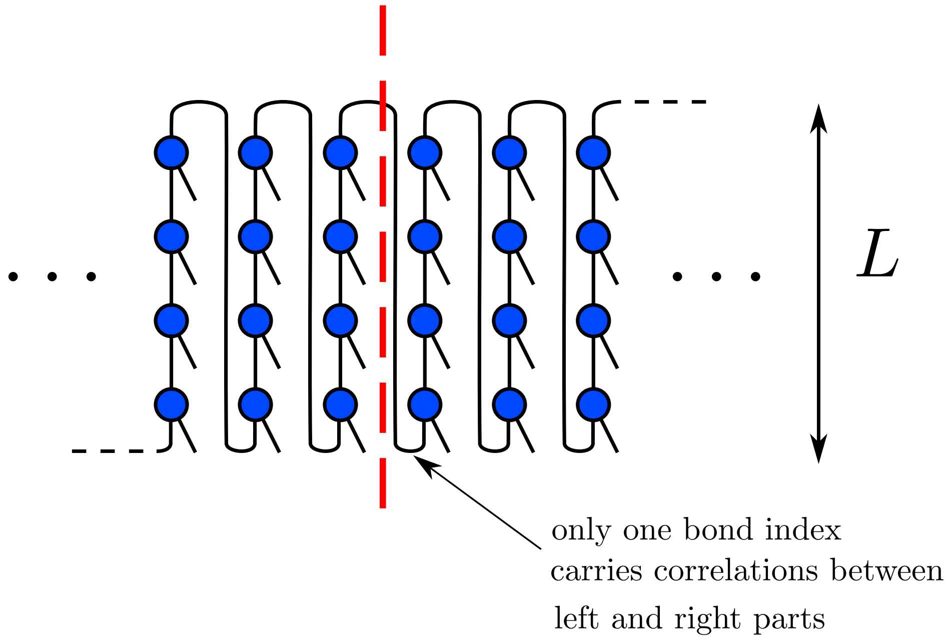
MPS for 2D systems



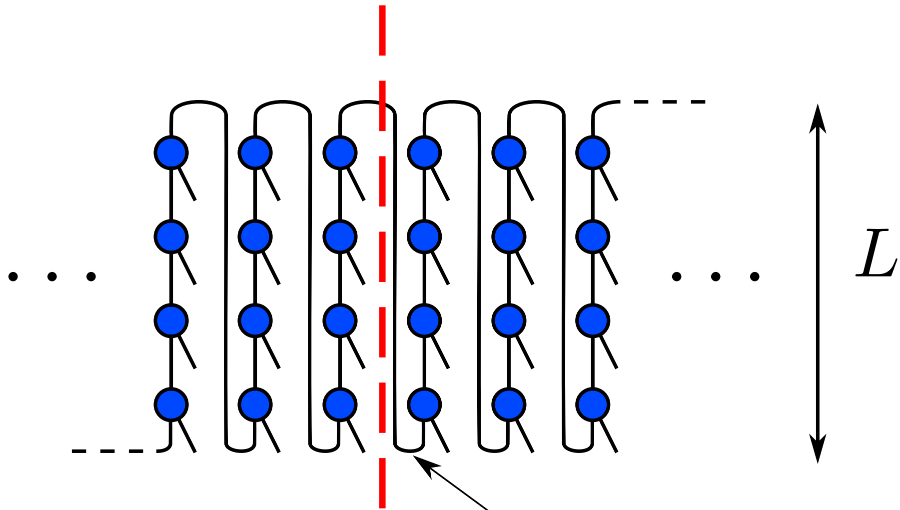
- applicable to infinite cylinders



MPS for 2D systems: limitations



MPS for 2D systems: limitations

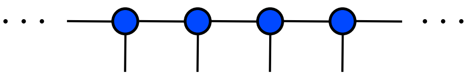


- cost: $\mathcal{O}(e^L)$
- works if $\xi_{\max} \ll L$
- in practice: $L \sim 10 \dots 12$

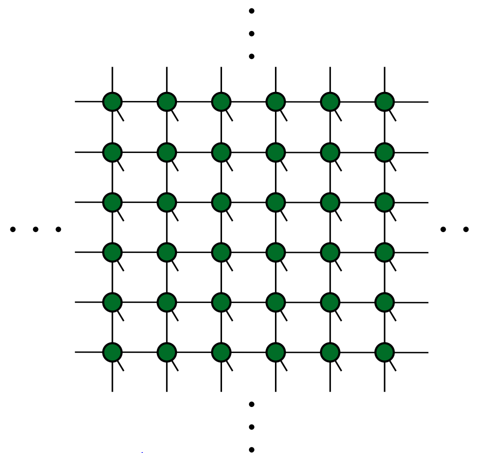
Projected Entangled Pair States

1D ⟶ 2D

MPS



PEPS



F. Verstraete, J. I. Cirac, **cond-mat/0407066**

F. Verstraete, M. M. Wolf, D. Perez-Garcia, J. I. Cirac, **PRL 06, 220601 (2006)**

P. Corboz, S. R. White, G. Vidal, M. Troyer, **PRB 84, 041108 (2011)**

P. Czarnik, J. Dziarmaga, **PRB 92, 035152 (2015)**

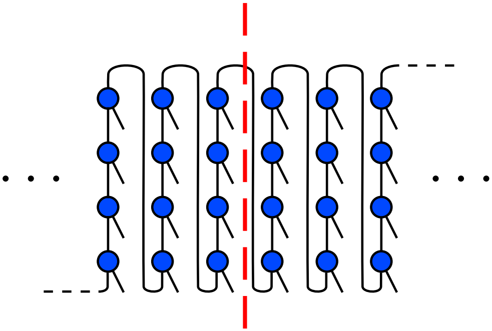
J. Jordan, R. Orús, G. Vidal, F. Verstraete, J. I. Cirac, **PRL 101, 250602 (2008)**

P. Corboz, **PRB 94, 035133 (2016)**

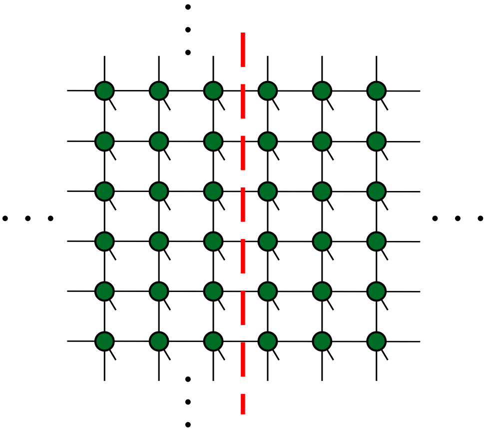
... ...

MPS in 2D vs PEPS

- correlations



only one bond index
carries correlations between
left and right parts



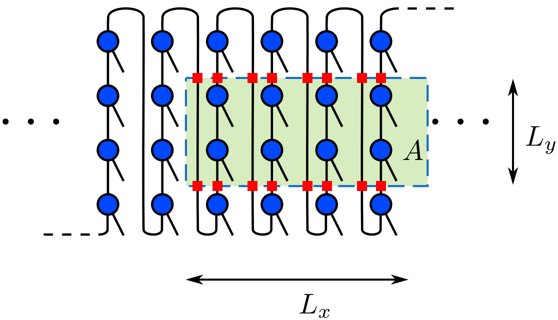
one bond index per every
correlation across the boundary

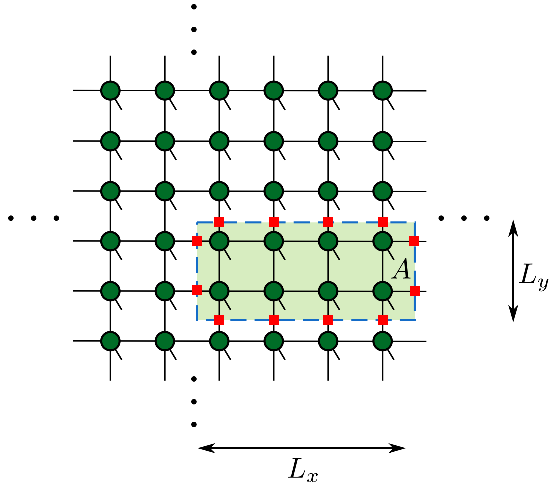
inefficient

efficient

MPS in 2D vs PEPS

- entanglement entropy





$$S \not\sim |\partial A|$$

(scales only in L_x)

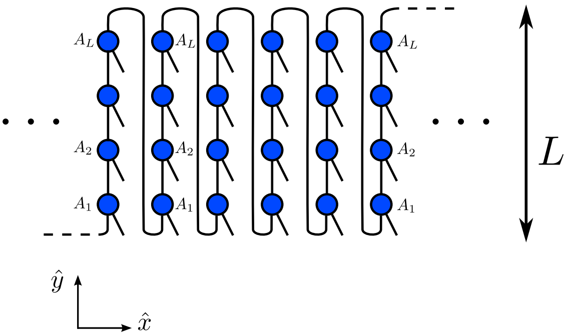
inefficient

$$S \sim |\partial A|$$

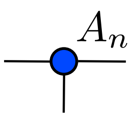
efficient

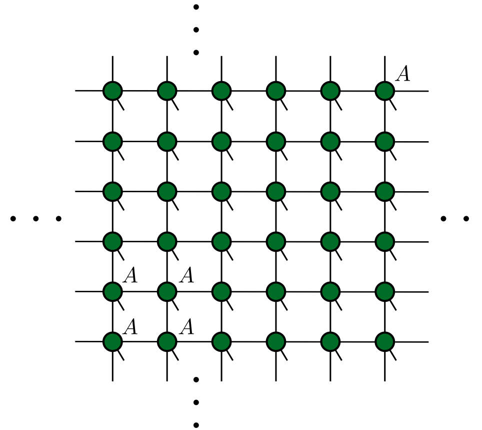
MPS in 2D vs PEPS

- translation invariance



- no (explicit) translation invariance in $\hat{y}$ direction
  (recovered through optimization)
- the state described by L tensors





- explicit translation invariance
- the state described by a single tensor