

# Tensor networks: part II

Łukasz Cincio

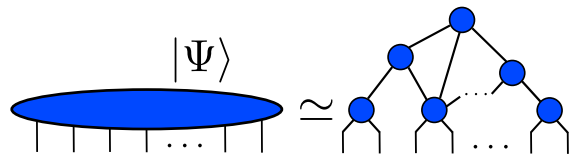


June 18, 2017

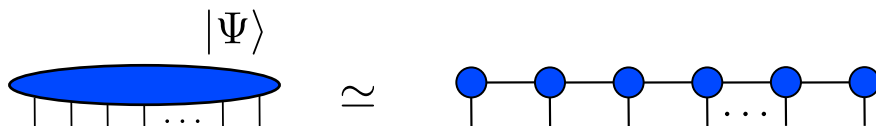
# SUMMARY OF LECTURE I

- divide and conquer

$$|\Psi\rangle = \sum_{i_1, \dots, i_N=1}^d \Psi_{i_1 \dots i_N} |i_1 \dots i_N\rangle$$



- efficient description of quantum many-body wave-functions
  - no sign problem
- 
- matrix product states

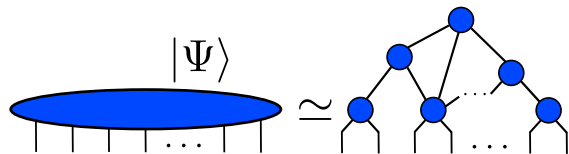


- properties
- applicability

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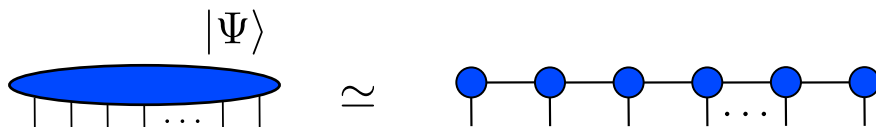
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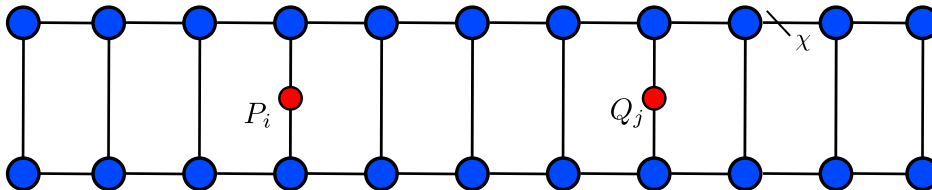
- properties
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# MPS: CORRELATIONS

- calculate correlator  $\langle \Psi | P_i Q_j | \Psi \rangle$

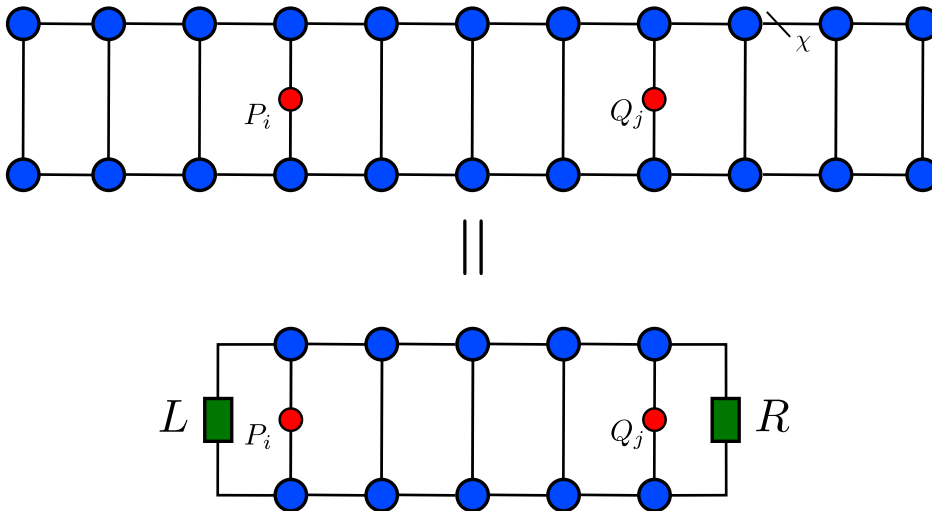
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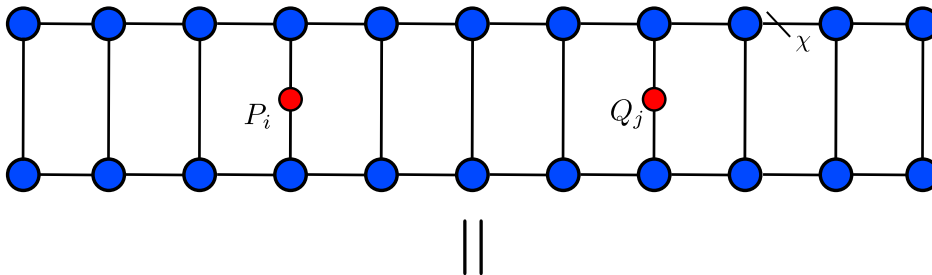
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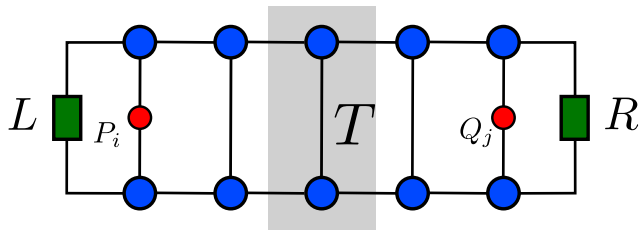
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||

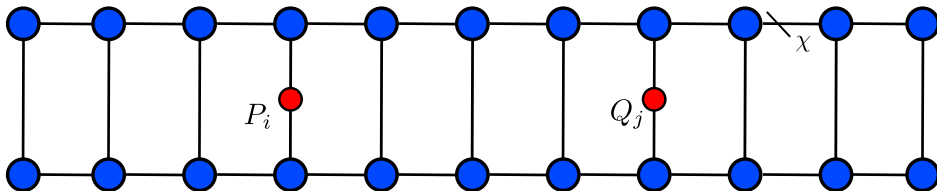
transfer matrix

$$T = \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \bullet \text{---} \end{array}$$



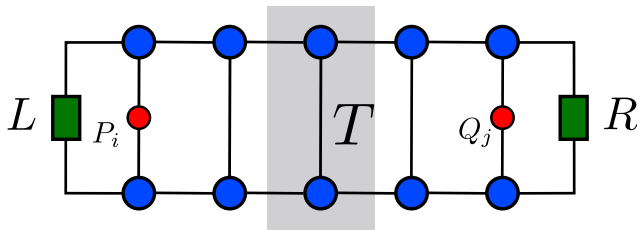
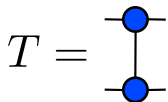
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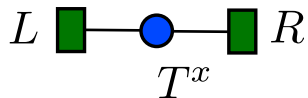


||

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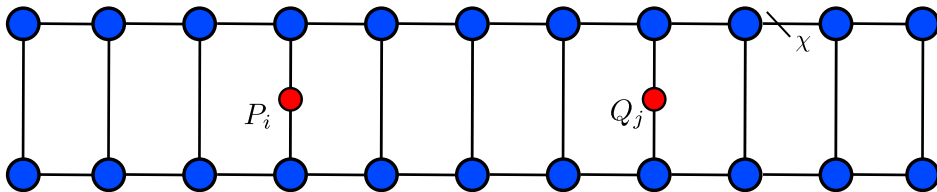
$x = |i - j|$  ||





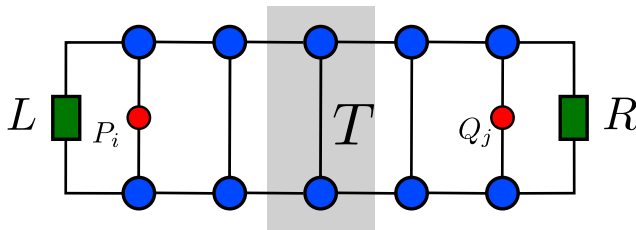
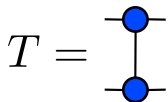
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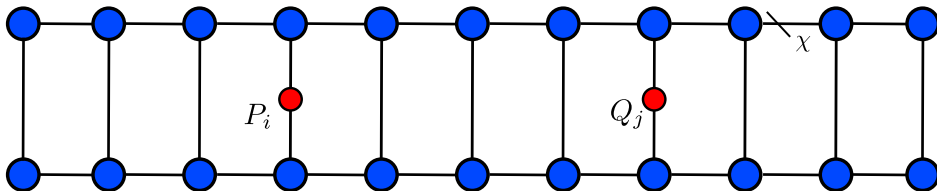
$x = |i - j|$  ||

$$L \text{ [green block]} \text{---} \text{ [blue node]} \text{---} \text{ [green block]} R \sim \sum_{j=1}^{\chi^2} c_j (\lambda_j)^x = \sum_{j=1}^{\chi^2} c_j e^{-x \ln(1/\lambda_j)}$$

$T^x$

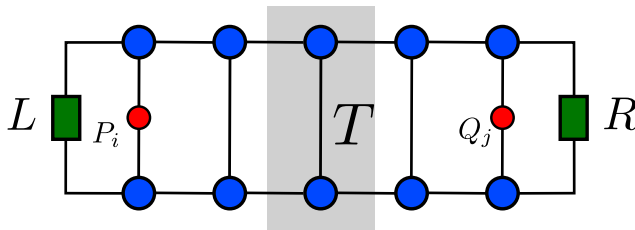
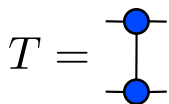
# MPS: CORRELATIONS

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||

transfer matrix



finite  $\chi$  :  
sum of exponentially  
decaying correlations  
(gapped systems only)

$x = |i - j|$  ||

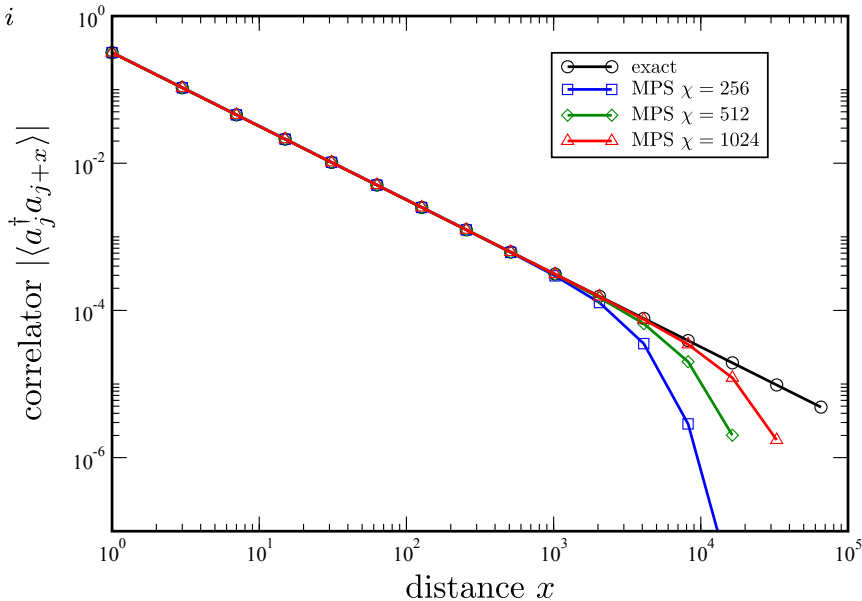
$$L \text{ [green box]} \text{---} T^x \text{ [blue circle]} \text{---} R \text{ [green box]} \sim \sum_{j=1}^{\chi^2} c_j (\lambda_j)^x = \sum_{j=1}^{\chi^2} c_j e^{-x \ln(1/\lambda_j)}$$

# MPS: LET'S TRY GAPLESS MODEL...

$$\mathcal{H} = \sum_i (X_i X_{i+1} + Y_i Y_{i+1}) \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

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$$a_j = \left( \prod_{k < j} Z_k \right) \frac{X_j - iY_j}{2}$$

$$\text{exact: } |\langle a_j^\dagger a_{j+x} \rangle| = \frac{1}{\pi x}$$

$$\text{MPS: } |\langle a_j^\dagger a_{j+x} \rangle| = \sum_{j=1}^{\chi^2} c_j e^{-x \ln(1/\lambda_j)}$$

# MPS: ENTANGLEMENT ENTROPY

$$|\Psi\rangle = \underbrace{\begin{array}{c} \bullet \\ | \\ \text{---} \\ \bullet \\ | \\ \text{---} \\ \bullet \\ | \\ \text{---} \\ \bullet \end{array}}_A \text{---} \underbrace{\begin{array}{c} \bullet \\ | \\ \text{---} \\ \dots \\ \text{---} \\ \bullet \\ | \\ \text{---} \\ \bullet \end{array}}_B$$

$\alpha = 1, \dots, \chi$

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$$\rho_A = \text{Tr}_B |\Psi\rangle\langle\Psi| = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \\ | \quad | \quad | \quad | \quad \dots \quad | \quad | \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \\ | \quad | \quad | \quad | \quad \dots \quad | \quad | \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \\ | \quad | \quad | \quad | \quad \dots \quad | \quad | \end{array}$$

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$\alpha = 1, \dots, \chi$

$$\varrho_A = \text{Tr}_B |\Psi\rangle\langle\Psi| =$$

$$=$$

$$\varrho_A^{\text{MPS}} = W \varrho_A W^\dagger$$

where  $W = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$       $W: \mathbb{C}_\chi \rightarrow \mathbb{C}_d^{\otimes |A|}$

# MPS: ENTANGLEMENT ENTROPY

$$\varrho_A^{\text{MPS}} = W \varrho_A W^\dagger = \begin{pmatrix} 1/\chi & & \\ & \ddots & \\ & & 1/\chi \end{pmatrix} \quad (\chi \times \chi \text{ matrix})$$

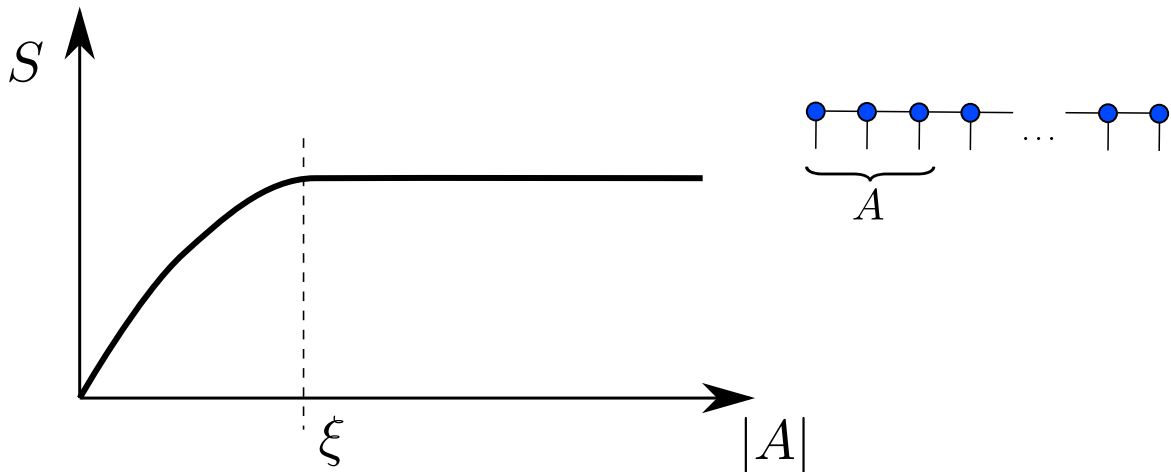
$$S(\varrho_A^{\text{MPS}}) = -\text{Tr}(\varrho_A^{\text{MPS}} \log \varrho_A^{\text{MPS}}) = \log \chi$$



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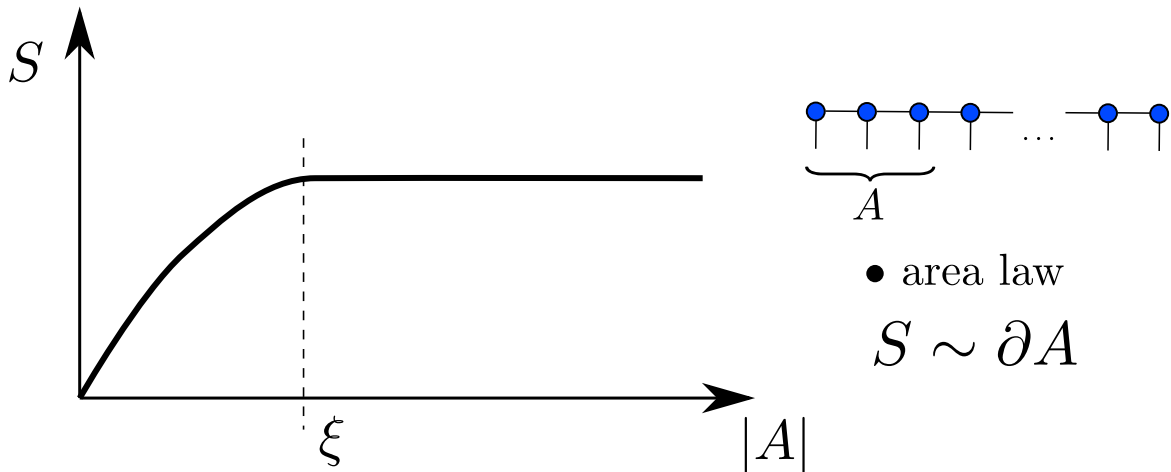
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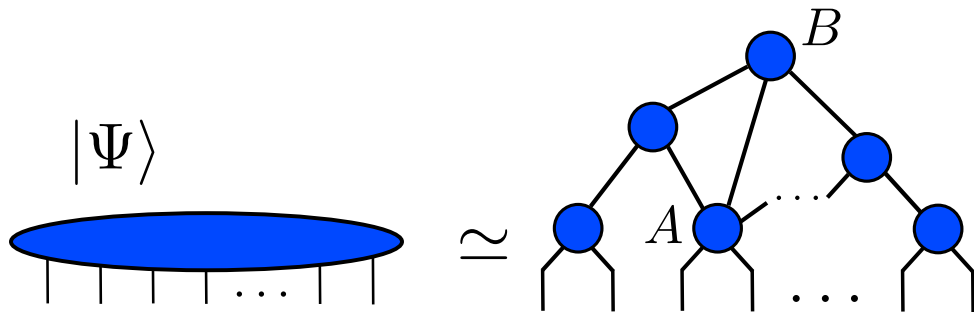
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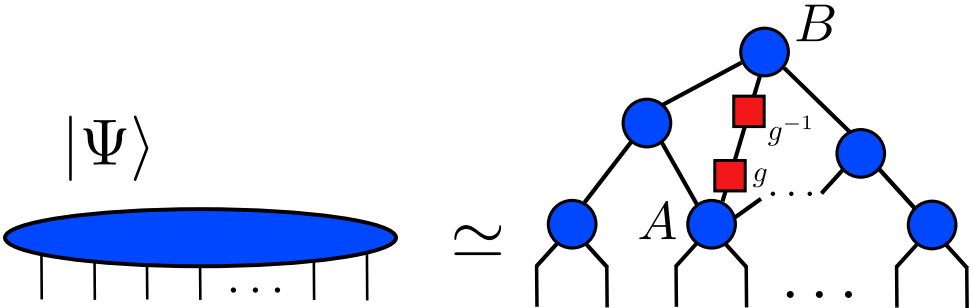
# GAUGE FREEDOM

- tensor network representation is not unique



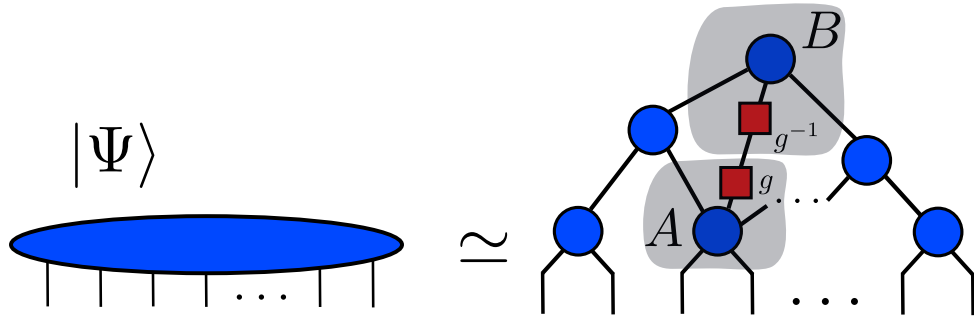
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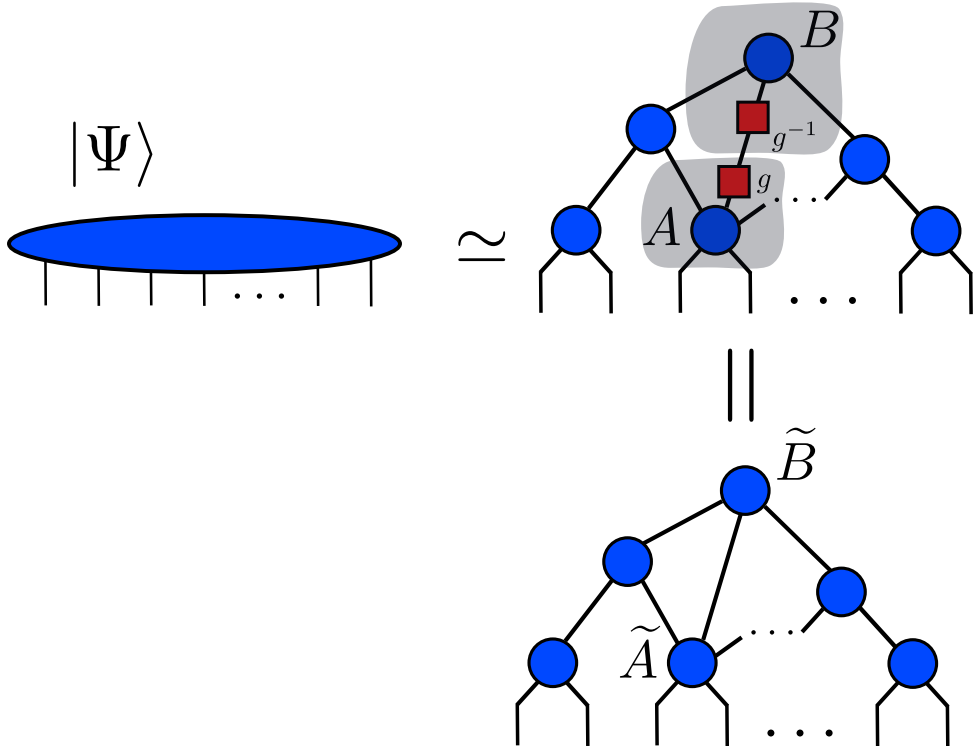
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# FIXING GAUGE: MPS

- canonical form of the MPS

$$|\Psi\rangle = \begin{array}{ccccccc} & A & B & & & & \\ & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet \\ & | & | & | & | & & | & | \\ \hline & & & & & & & \end{array}$$

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consider matrix  $L$

$$L \left[ \begin{array}{c} \blacksquare \end{array} \right] \equiv \begin{array}{c} A \\ \bullet \\ | \\ \bullet \\ A^* \end{array} = \begin{array}{c} \left[ \begin{array}{c} \bullet \\ \bullet \end{array} \right] \\ U \\ s \\ U^\dagger \end{array}$$

insert  $\mathbb{I} = gg^{-1}$  with  $g = U^\dagger s^{-1/2}$

$$|\Psi\rangle = \begin{array}{ccccccc} & A & B & & & & \\ & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet \\ & | & | & | & | & & | & | \end{array}$$

$\mathbb{I} = gg^{-1}$   
 $\swarrow$





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$$L \left[ \begin{array}{c} \color{green}\blacksquare \end{array} \right] \equiv \begin{array}{c} A \\ \bullet \\ | \\ \bullet \\ A^* \end{array} = \begin{array}{c} \color{red}\blacksquare \\ U \\ s \\ \color{red}\blacklozenge \\ U^\dagger \\ \color{red}\blacksquare \end{array}$$

insert  $\mathbb{I} = gg^{-1}$  with  $g = U^\dagger s^{-1/2}$

$$|\Psi\rangle = \begin{array}{ccccccc} & A & B & & & & \\ & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet \\ & | & | & | & | & & | & | \\ & & & & & & & \end{array}$$

$\mathbb{I} = gg^{-1}$  (arrow pointing to the  $B$  site)

$$|\Psi\rangle = \begin{array}{ccccccc} & A & & & B & & & \\ & \bullet & \color{red}\bullet & \color{orange}\blacklozenge & \color{red}\bullet & \bullet & \bullet & \dots & \bullet & \bullet \\ & | & | & | & | & | & | & & | & | \\ & & & & & & & & & \end{array}$$

(The first two sites and the last two sites are shaded gray)

# FIXING GAUGE: MPS

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insert  $\mathbb{I} = gg^{-1}$  with  $g = U^\dagger s^{-1/2}$

$$|\Psi\rangle = \begin{array}{c} A \quad B \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \\ | \quad | \quad | \quad | \quad \dots \quad | \quad | \end{array}$$

$$|\Psi\rangle = \begin{array}{c} A \quad U^\dagger \quad s^{-1/2} \quad s^{1/2} \quad U \quad B \\ \bullet \quad \color{red}{\bullet} \quad \color{orange}{\blacklozenge} \quad \color{orange}{\blacklozenge} \quad \color{red}{\bullet} \quad \bullet \\ | \quad | \quad | \quad | \quad | \quad | \quad | \quad | \quad \dots \quad | \quad | \end{array}$$

$$|\Psi\rangle = \begin{array}{c} \vec{A} \quad \vec{B} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \\ | \quad | \quad | \quad | \quad \dots \quad | \quad | \end{array}$$

$$\begin{array}{c} \vec{A} \\ \bullet \\ | \\ \bullet \\ \vec{A}^* \end{array} = \left[ \right.$$

$\vec{A}$  — right-isometric

# FIXING GAUGE: MPS

- canonical form of the MPS

similarly:

$$|\Psi\rangle = \begin{array}{c} \vec{A} \quad \tilde{B} \quad C \quad D \quad \dots \quad E \quad F \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \mathbb{I} = g_1 g_1^{-1} \quad \mathbb{I} = g_2 g_2^{-1} \quad \mathbb{I} = g_3 g_3^{-1} \quad \mathbb{I} = g_4 g_4^{-1} \end{array}$$

# FIXING GAUGE: MPS

- canonical form of the MPS

similarly:

$$\begin{aligned}
 |\Psi\rangle &= \begin{array}{c} \vec{A} \quad \tilde{B} \quad C \quad D \quad \dots \quad E \quad F \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \dots \quad \downarrow \quad \downarrow \\ \mathbb{I} = g_1 g_1^{-1} \quad \mathbb{I} = g_2 g_2^{-1} \quad \mathbb{I} = g_3 g_3^{-1} \quad \mathbb{I} = g_4 g_4^{-1} \end{array} \\
 |\Psi\rangle &= \begin{array}{c} \vec{A} \quad \vec{B} \quad \vec{C} \quad \tilde{D} \quad \dots \quad \overleftarrow{E} \quad \overleftarrow{F} \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \dots \quad \downarrow \quad \downarrow \end{array}
 \end{aligned}$$

$\vec{B}, \vec{C}$  — right-isometric

$\overleftarrow{E}, \overleftarrow{F}$  — left-isometric

$$\begin{array}{c} \bullet \\ | \\ | \\ \bullet \end{array} = \left[ \right.$$

$$\begin{array}{c} \bullet \\ | \\ | \\ \bullet \end{array} = \left. \right]$$

- provides numerical stability

# MPS FOR A GROUND STATE

- find the ground state of  $\mathcal{H} = \sum_i h^{[i,i+1]}$
- treat MPS as a variational ansatz and minimize the total energy of the system

$$E = \frac{\langle \Psi | \mathcal{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \quad |\Psi\rangle = \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet$$

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$$E = \left( \begin{array}{c} \begin{array}{ccccccc} \bullet & \bullet & x & & & & \\ & & | & & & & \\ \bullet & - & \bullet & - & \bullet & - & \dots & - & \bullet \\ & & | & & & & & & | \\ \bullet & \bullet & x^* & & & & \\ & & | & & & & \\ & & & h^{[1,2]} & & & \end{array} \dots & + & \begin{array}{ccccccc} \bullet & \bullet & x & & & & \\ & & | & & & & \\ \bullet & - & \bullet & - & \bullet & - & \dots & - & \bullet \\ & & | & & & & & & | \\ \bullet & \bullet & x^* & & & & \\ & & | & & & & \\ & & & h^{[2,3]} & & & \end{array} \dots & + & \dots \end{array} \right) / \begin{array}{ccccccc} \bullet & \bullet & x & & & & \\ & & | & & & & \\ \bullet & - & \bullet & - & \bullet & - & \dots & - & \bullet \\ & & | & & & & & & | \\ \bullet & \bullet & x^* & & & & \\ & & | & & & & \\ & & & h^{[2,3]} & & & \end{array}$$



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$$E = \frac{\langle \Psi | \mathcal{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \quad |\Psi\rangle = \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}} \text{---} \dots \text{---} \text{blue}$$

$$E = \left( \begin{array}{c} \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}} \\ \text{---} \text{---} \text{---} \\ \text{blue} \text{---} \text{blue} \text{---} \overset{x^*}{\text{red}} \end{array} \text{---} \dots \text{---} \text{blue} \text{---} \text{blue} \text{---} \text{blue} + \begin{array}{c} \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}} \\ \text{---} \text{---} \text{---} \\ \text{blue} \text{---} \text{blue} \text{---} \overset{x^*}{\text{red}} \end{array} \text{---} \dots \text{---} \text{blue} \text{---} \text{blue} \text{---} \text{blue} + \dots \right) / \begin{array}{c} \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}} \\ \text{---} \text{---} \text{---} \\ \text{blue} \text{---} \text{blue} \text{---} \overset{x^*}{\text{red}} \end{array} \text{---} \dots \text{---} \text{blue} \text{---} \text{blue} \text{---} \text{blue}$$

$$E = \frac{\text{blue} \text{---} \overset{x}{\text{red}} \text{---} \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}}}{\text{blue} \text{---} \overset{x}{\text{red}} \text{---} \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}}} \mathcal{H}_{\text{eff}} / \frac{\text{blue} \text{---} \overset{x}{\text{red}} \text{---} \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}}}{\text{blue} \text{---} \overset{x}{\text{red}} \text{---} \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}}} \mathcal{N}_{\text{eff}} = \frac{\mathbf{x}^\dagger \mathcal{H}_{\text{eff}} \mathbf{x}}{\mathbf{x}^\dagger \mathcal{N}_{\text{eff}} \mathbf{x}}$$

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$$E = \frac{\langle \Psi | \mathcal{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \quad |\Psi\rangle = \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}} \text{---} \dots \text{---} \text{blue}$$

$$E = \left( \begin{array}{c} \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}} \\ \text{---} \text{green} \text{---} \\ \text{blue} \text{---} \text{blue} \text{---} \underset{x^*}{\text{red}} \end{array} \dots + \begin{array}{c} \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}} \\ \text{---} \text{green} \text{---} \\ \text{blue} \text{---} \text{blue} \text{---} \underset{x^*}{\text{red}} \end{array} \dots + \dots \right) / \begin{array}{c} \text{blue} \text{---} \text{blue} \text{---} \overset{x}{\text{red}} \\ \text{---} \text{green} \text{---} \\ \text{blue} \text{---} \text{blue} \text{---} \underset{x^*}{\text{red}} \end{array} \dots$$

$$E = \frac{\begin{array}{c} \text{blue} \text{---} \overset{x}{\text{red}} \\ \text{---} \text{green} \text{---} \\ \text{blue} \text{---} \underset{x^*}{\text{red}} \end{array} \mathcal{H}_{\text{eff}}}{\begin{array}{c} \text{blue} \text{---} \overset{x}{\text{red}} \\ \text{---} \text{green} \text{---} \\ \text{blue} \text{---} \underset{x^*}{\text{red}} \end{array} \mathcal{N}_{\text{eff}}} = \frac{\mathbf{x}^\dagger \mathcal{H}_{\text{eff}} \mathbf{x}}{\mathbf{x}^\dagger \mathcal{N}_{\text{eff}} \mathbf{x}}$$

$$\frac{\partial E}{\partial \mathbf{x}^\dagger} = 0 \longrightarrow \boxed{\mathcal{H}_{\text{eff}} \mathbf{x} = E_0 \cdot \mathcal{N}_{\text{eff}} \mathbf{x}} \longrightarrow \mathbf{x} \text{ --- solution of a generalized eigenvalue problem}$$

# MPS FOR A GROUND STATE

- find the ground state of  $\mathcal{H} = \sum_i h^{[i,i+1]}$
- treat MPS as a variational ansatz and minimize the total energy of the system

$$E = \frac{\langle \Psi | \mathcal{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \quad |\Psi\rangle = \begin{array}{ccccccc} & & & x & & & \\ & & & \bullet & & \dots & \\ \bullet & - & \bullet & - & \bullet & - & \dots & - & \bullet \\ | & & | & & | & & & & | \end{array}$$

$$\frac{\partial E}{\partial \mathbf{x}^\dagger} = 0 \quad \longrightarrow \quad \boxed{\mathcal{H}_{\text{eff}} \mathbf{x} = E_0 \cdot \mathcal{N}_{\text{eff}} \mathbf{x}} \quad \longrightarrow \quad \mathbf{x} \text{ — solution of a generalized eigenvalue problem}$$

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- solution: use isometric tensors:  $|\Psi\rangle = \begin{array}{c} \overrightarrow{A} \quad \overrightarrow{B} \quad x \quad \dots \quad \overleftarrow{F} \\ \bullet \quad \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad | \quad \dots \quad | \end{array}$

# MPS FOR A GROUND STATE

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$$\bullet \mathcal{N}_{\text{eff}} = \begin{array}{c} \vec{A} \quad \vec{B} \\ \bullet \text{---} \bullet \\ | \quad | \\ \overleftarrow{A^*} \quad \overleftarrow{B^*} \end{array} \quad | \quad \begin{array}{c} \overleftarrow{D} \\ \bullet \\ | \\ \overleftarrow{D^*} \end{array} \quad \dots \quad \begin{array}{c} \overleftarrow{F} \\ \bullet \\ | \\ \overleftarrow{F^*} \end{array} = \left[ \begin{array}{c} | \\ | \\ \vdots \\ | \end{array} \right] = \mathbb{I}_{d \times d}$$

$$\longrightarrow \quad \boxed{\mathcal{H}_{\text{eff}} \mathbf{x} = E_0 \cdot \mathbf{x}} \quad \text{standard eigenvalue problem: numerically stable}$$

# SUMMARY: MATRIX PRODUCT STATES

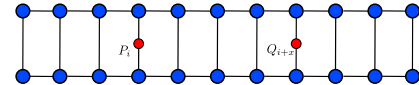
- intended for 1D systems
- extendable to infinite systems

$$|\Psi^{\text{inf}}\rangle = \cdots \text{---} \overset{A}{\bullet} \text{---} \overset{A}{\bullet} \text{---} \overset{A}{\bullet} \text{---} \overset{A}{\bullet} \text{---} \overset{A}{\bullet} \text{---} \overset{A}{\bullet} \text{---} \cdots$$

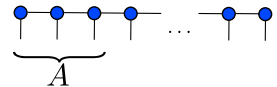
- accurately captures ground states of local gapped Hamiltonians

$$\mathcal{H} = \sum_i h^{[i, i+1]}$$

- finite correlations:  $\langle \Psi | P_i Q_{i+x} | \Psi \rangle \sim e^{-x/\xi}$



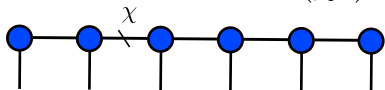
- area law:  $S \sim \partial A (= \text{const})$



# EXAMPLES

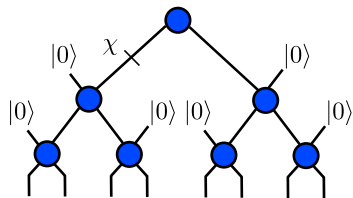
- MPS

$$\mathcal{O}(\chi^3)$$



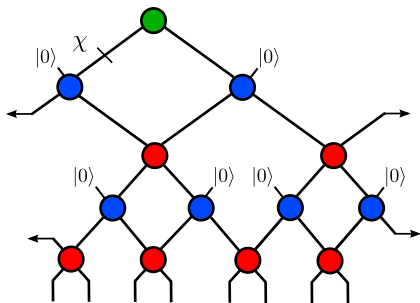
- TTN

$$\mathcal{O}(\chi^4)$$



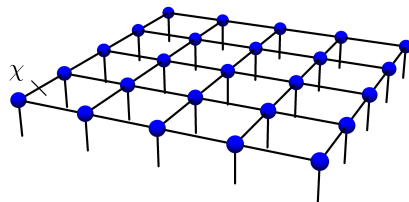
- 1D MERA

$$\mathcal{O}(\chi^9)$$



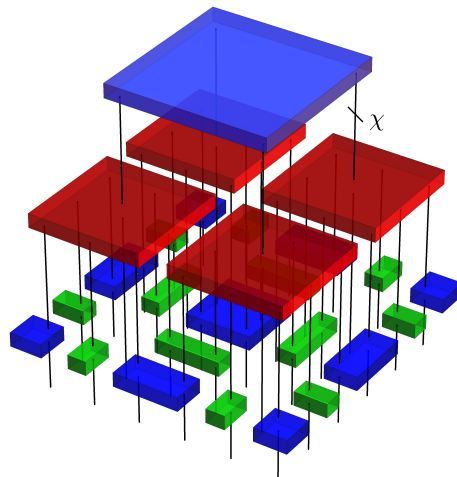
- PEPS

$$\mathcal{O}(\chi^{10\dots 12})$$



- 2D MERA

$$\mathcal{O}(\chi^{16})$$



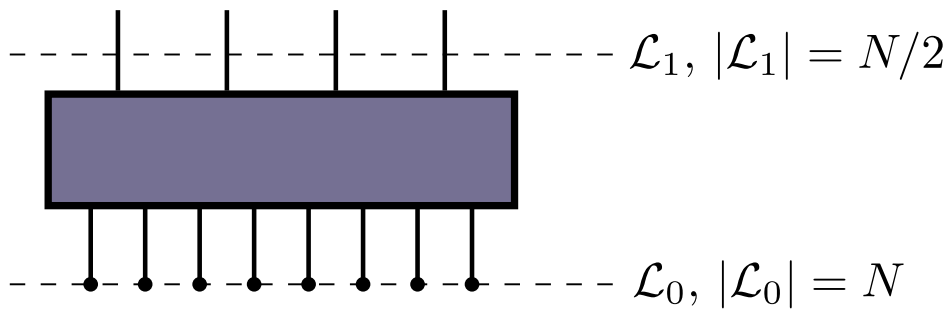
# MERA

---●---●---●---●---●---●---●---●---  $\mathcal{L}_0, |\mathcal{L}_0| = N$

- coarse-graining procedure

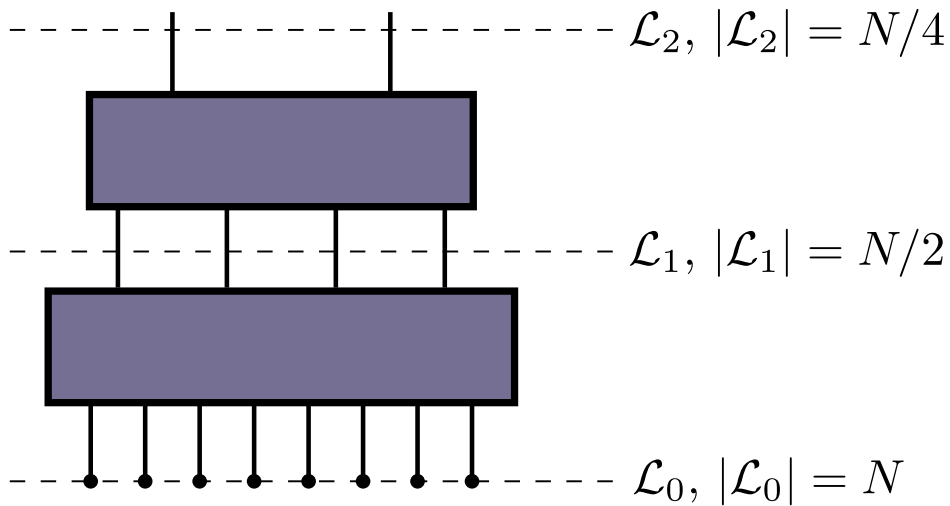


# MERA



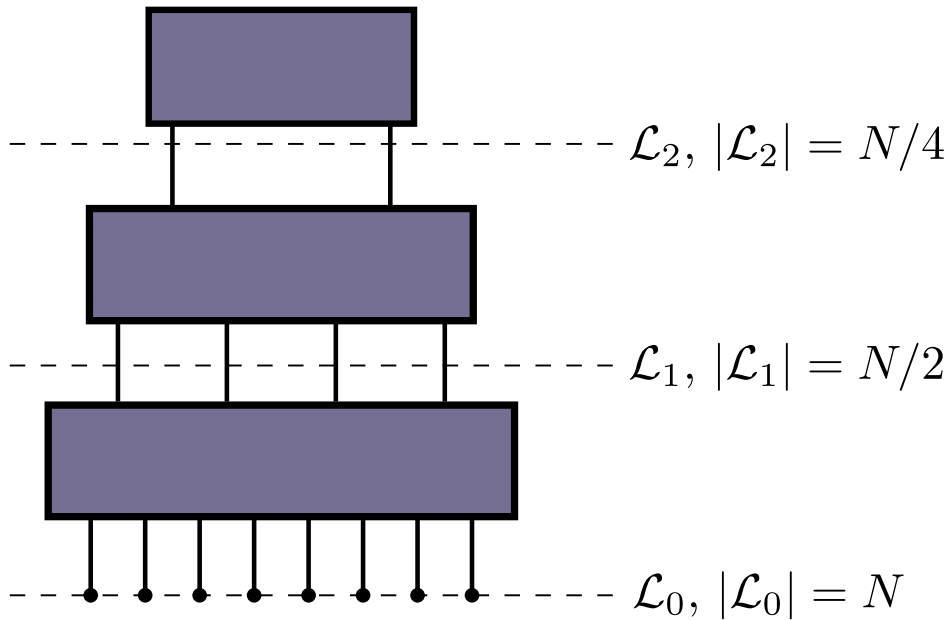
- coarse-graining procedure

# MERA



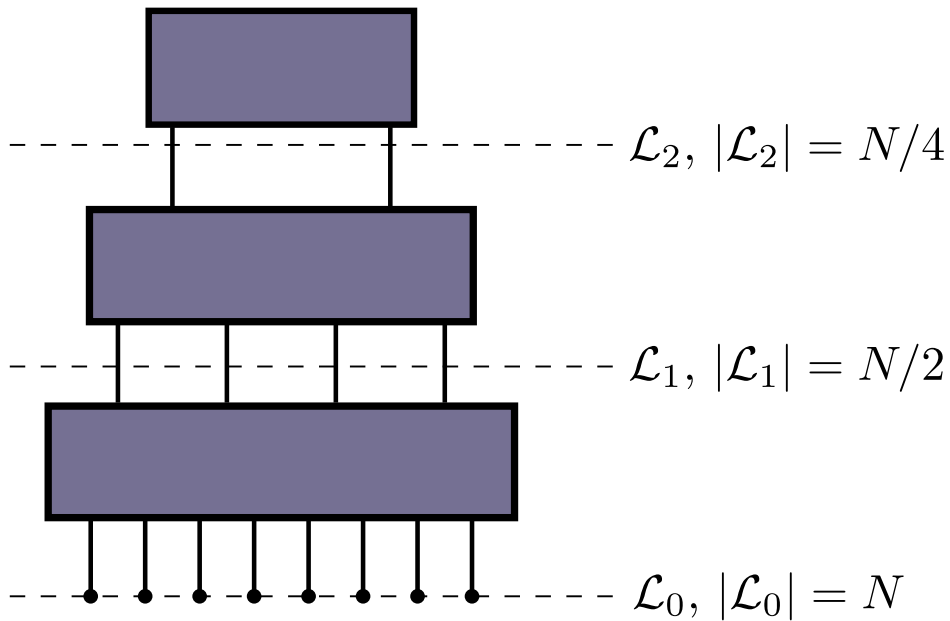
- coarse-graining procedure

# MERA

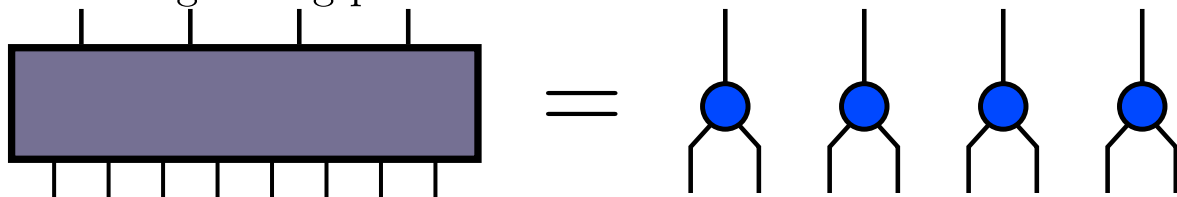


- coarse-graining procedure

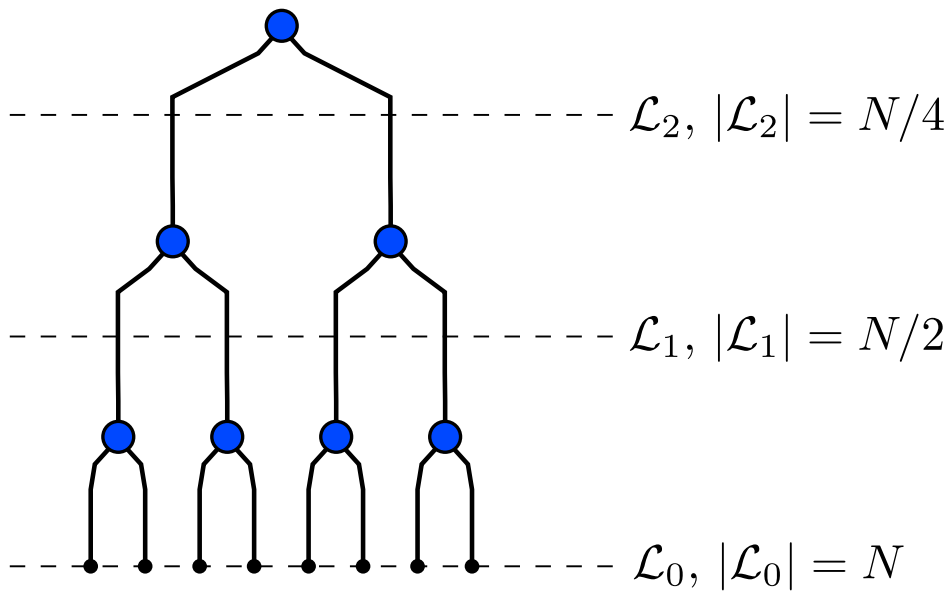
# MERA



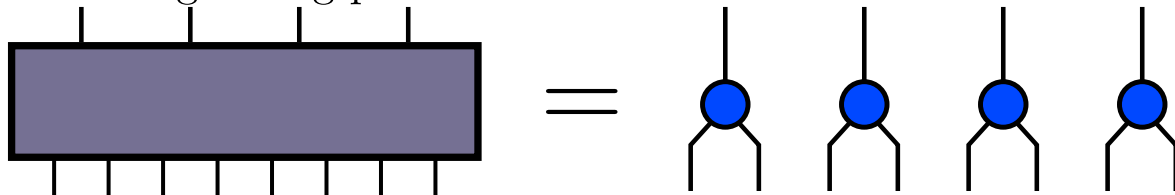
● coarse-graining procedure



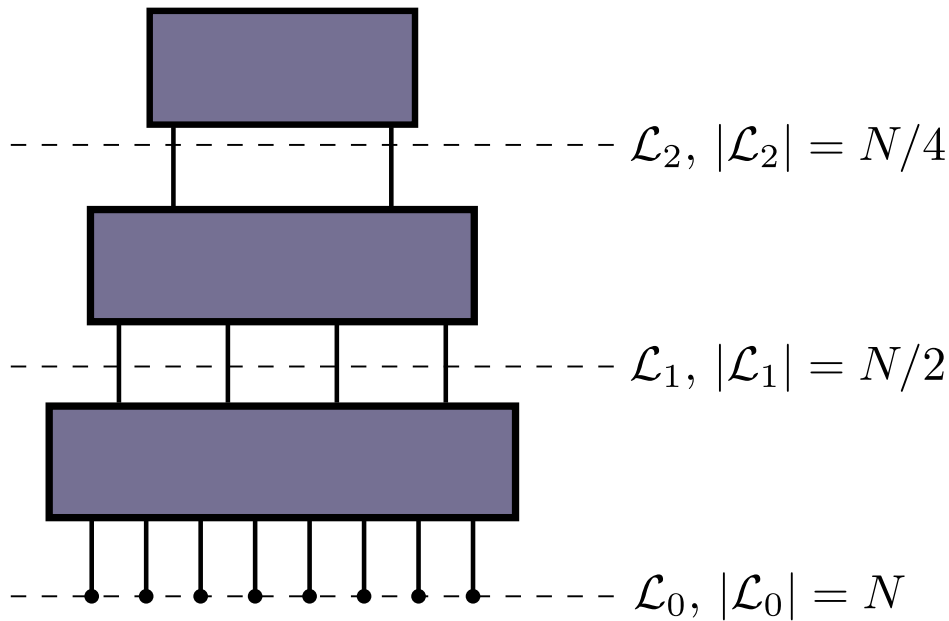
# MERA



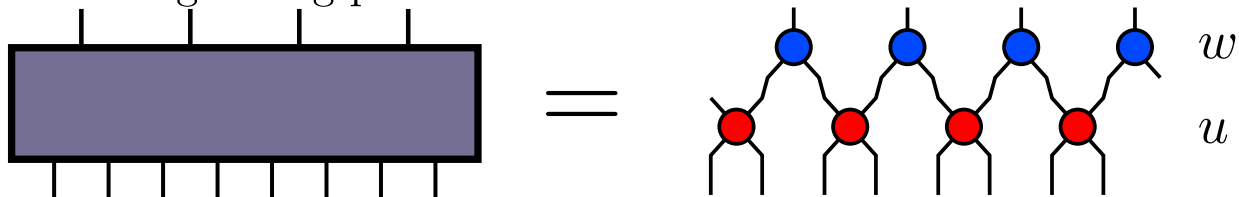
● coarse-graining procedure



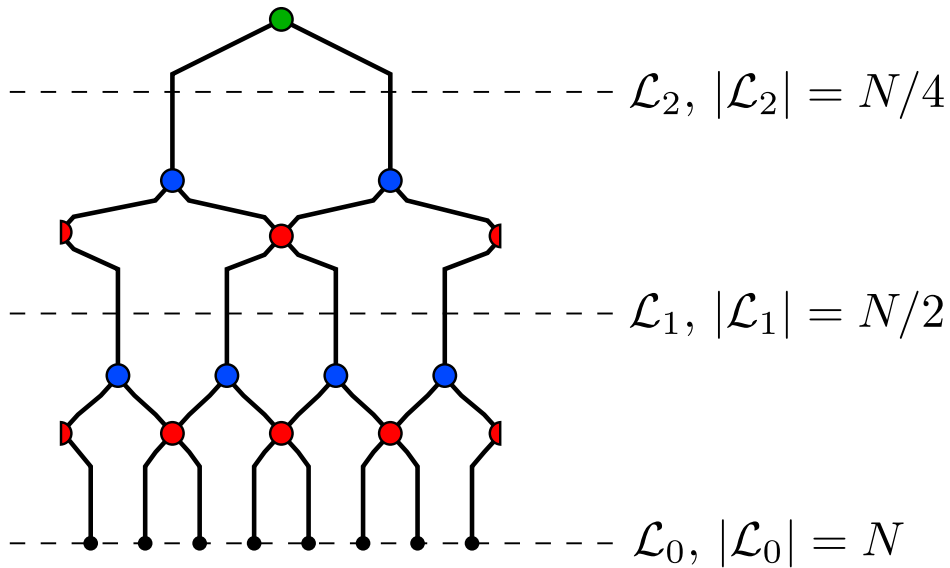
# MERA



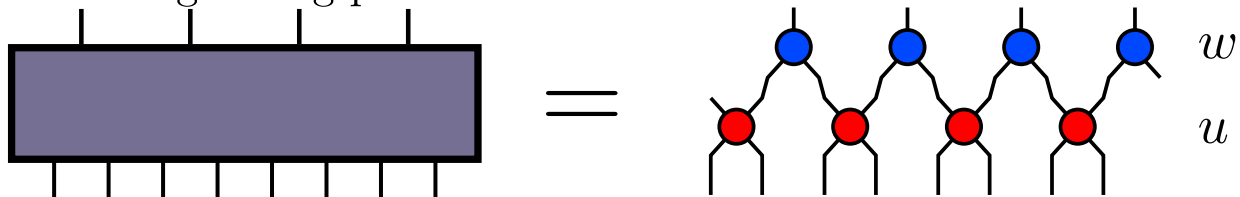
• coarse-graining procedure



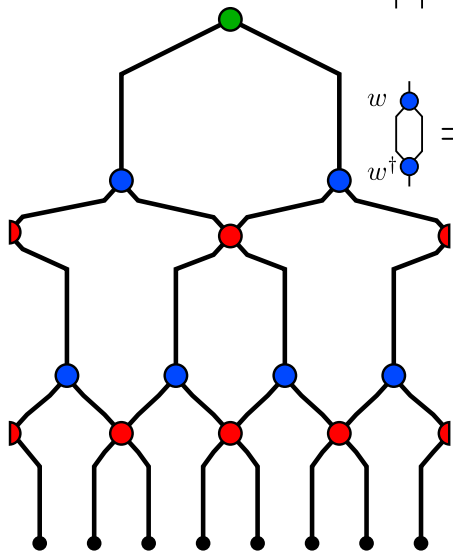
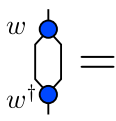
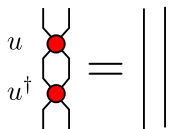
# MERA



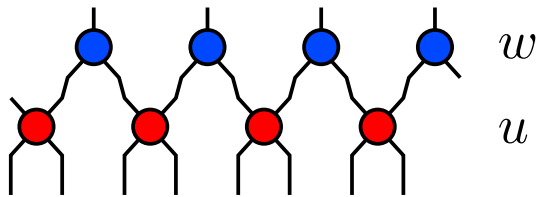
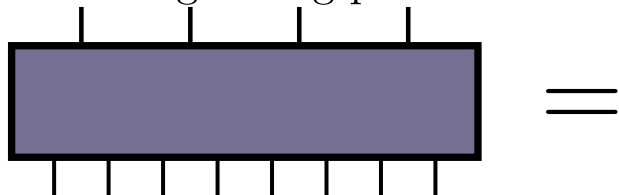
● coarse-graining procedure



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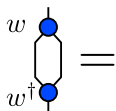
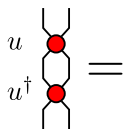
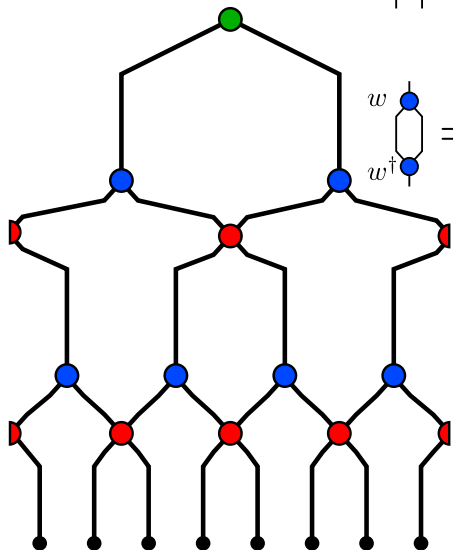


• coarse-graining procedure





# MERA



G. Vidal, **PRL 99**, 220405 (2007)

P. Corboz, G. Vidal, **PRB 80**, 165129 (2009)

L. Cincio, J. Dziarmaga, M. M. Rams, **PRL 100**, 240603 (2008)

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J. Haegeman, T. J. Osborne, H. Verschelde, F. Verstraete,  
**PRL 110**, 100402, (2013)

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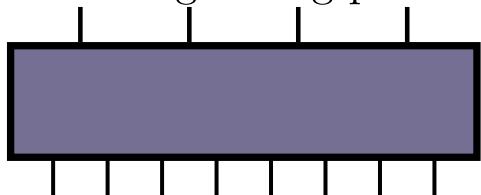
G. Evenbly, G. Vidal, **PRL 102**, 180406 (2009)

G. Evenbly, G. Vidal, **PRL 112**, 240502 (2014)

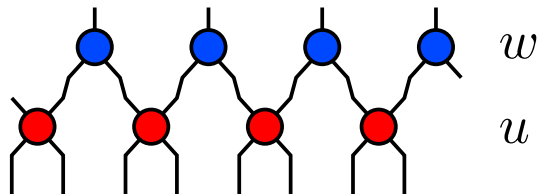
G. Evenbly, S. R. White, **PRL 116**, 140403 (2016)

.....

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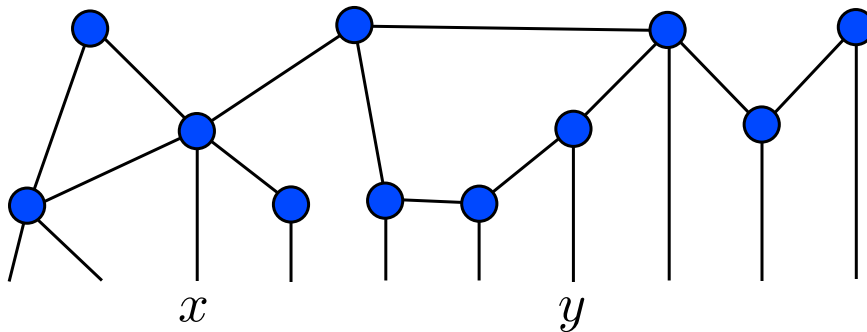


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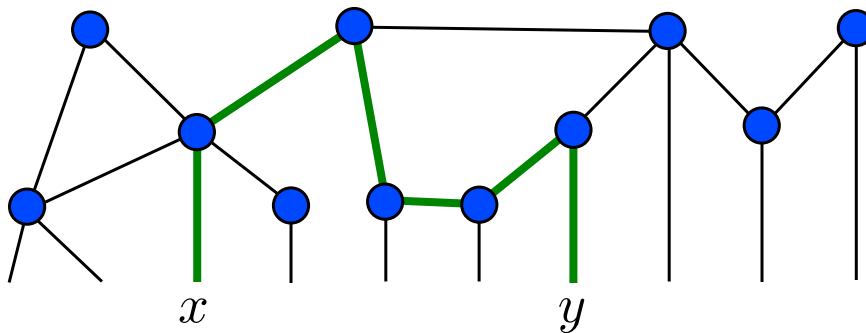
# CORRELATIONS IN TENSOR NETWORK STATES

- define **distance** within tensor network



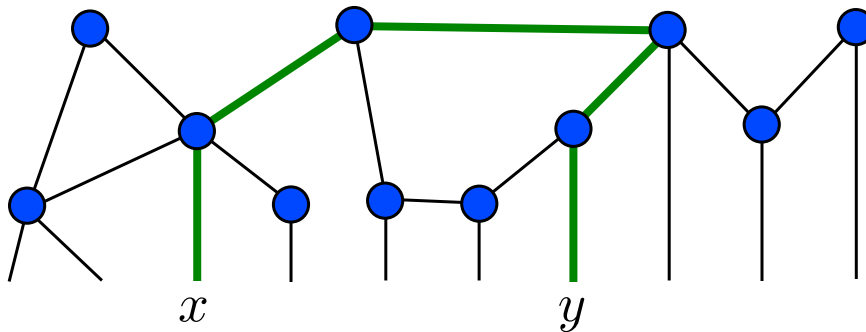
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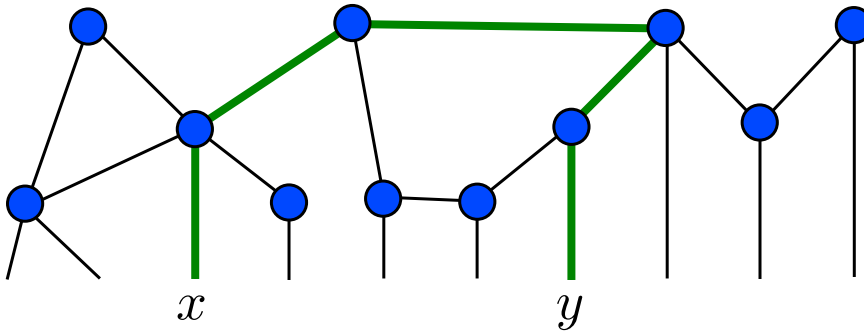
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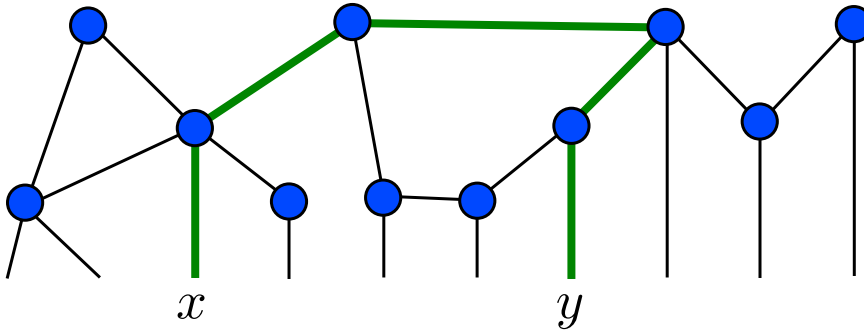


- **distance** between  $x$  and  $y$  – length of the shortest path

$$D(x, y) = 4$$

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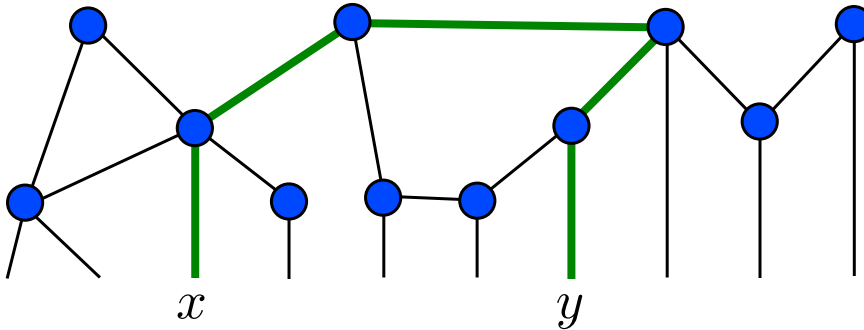
$$D(x, y) = 4$$

- correlations between  $x$  and  $y$

$$C(x, y) = v_L^\dagger \cdot T^{D(x, y)} \cdot v_R$$

# CORRELATIONS IN TENSOR NETWORK STATES

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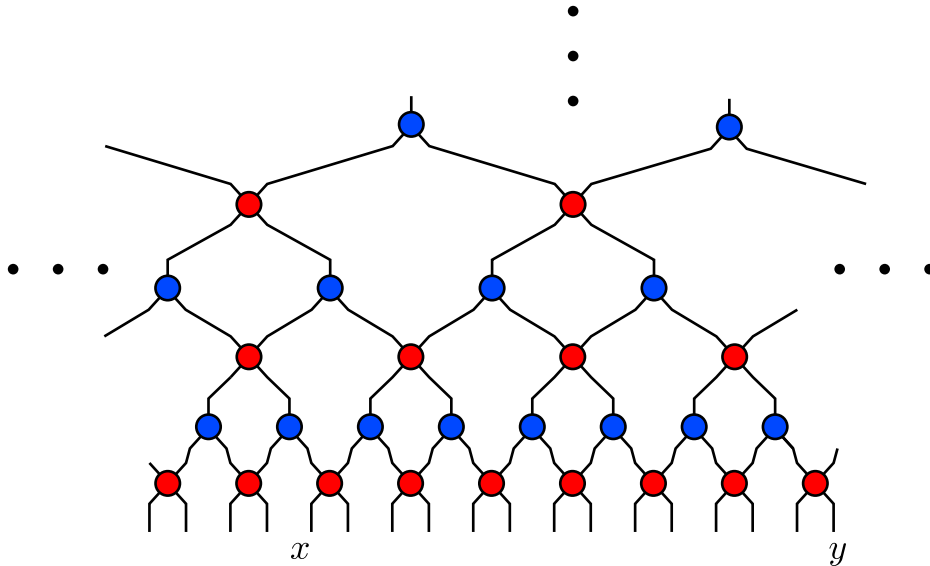
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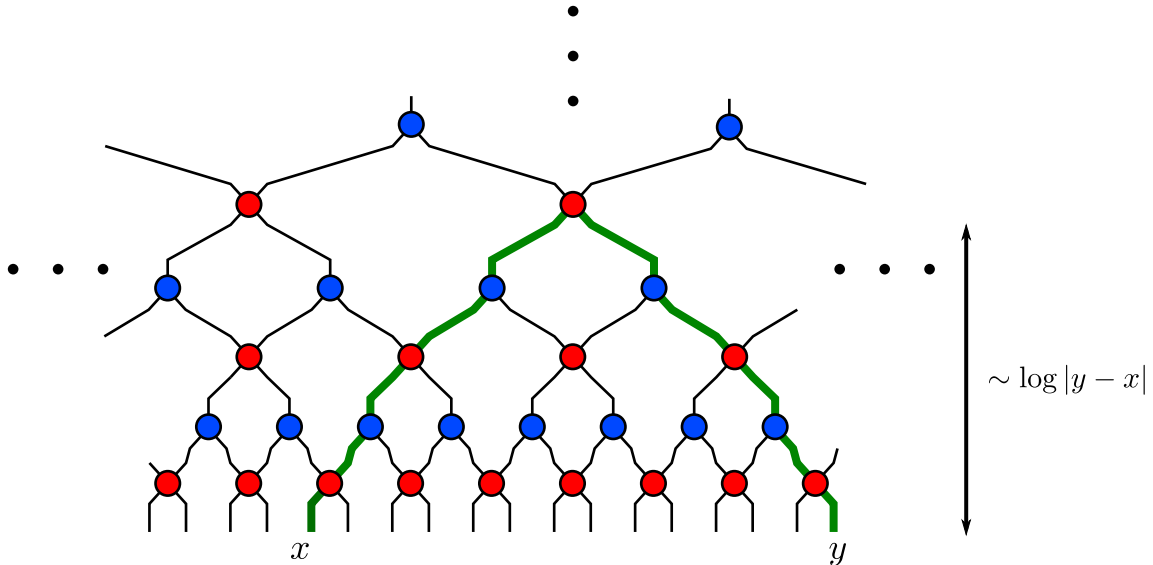
$$C(x, y) = v_L^\dagger \cdot T^{D(x, y)} \cdot v_R \sim \lambda^{D(x, y)} = e^{-D(x, y) \cdot \ln(1/\lambda)}$$

# MERA: CORRELATIONS



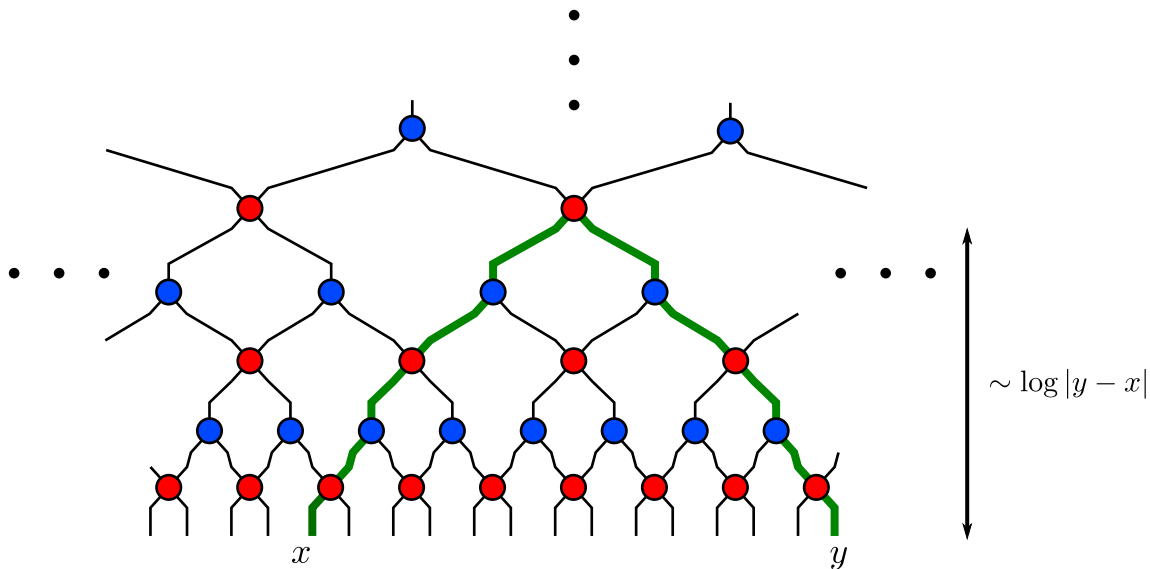


# MERA: CORRELATIONS



- distance in MERA:  $D(x, y) \sim \log |y - x|$
- correlations in MERA:  $C(x, y) = e^{-D(x, y) \cdot \ln(1/\lambda)}$

# MERA: CORRELATIONS



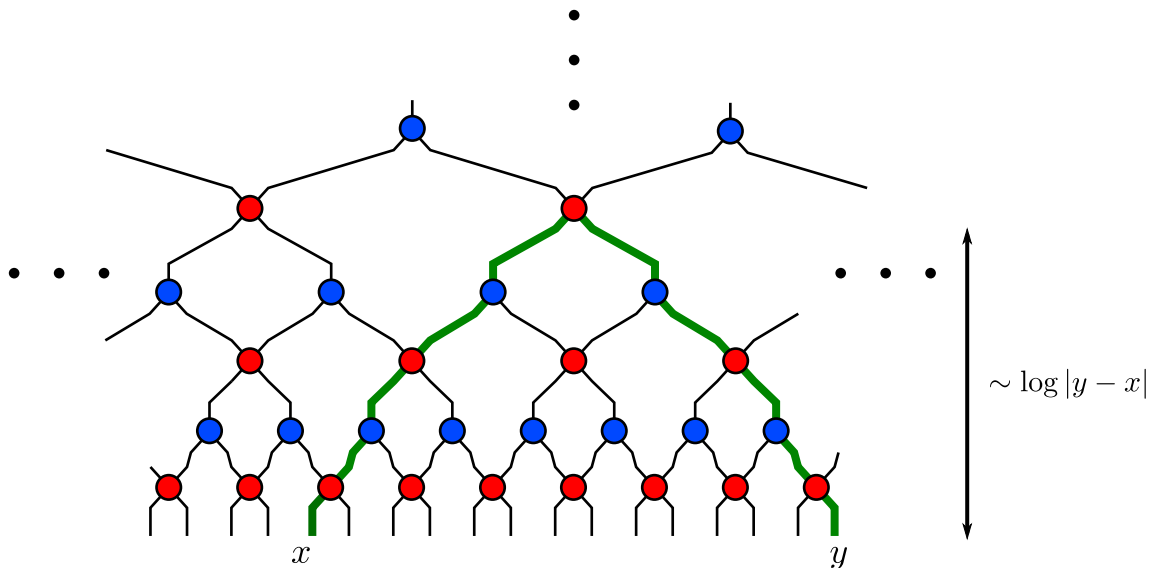
- distance in MERA:  $D(x, y) \sim \log |y - x|$

- correlations in MERA:  $C(x, y) = e^{-D(x, y) \cdot \ln(1/\lambda)} = e^{-a \log |y - x|}$

$$= |y - x|^{-b}$$

for some  $b > 0$

# MERA: CORRELATIONS



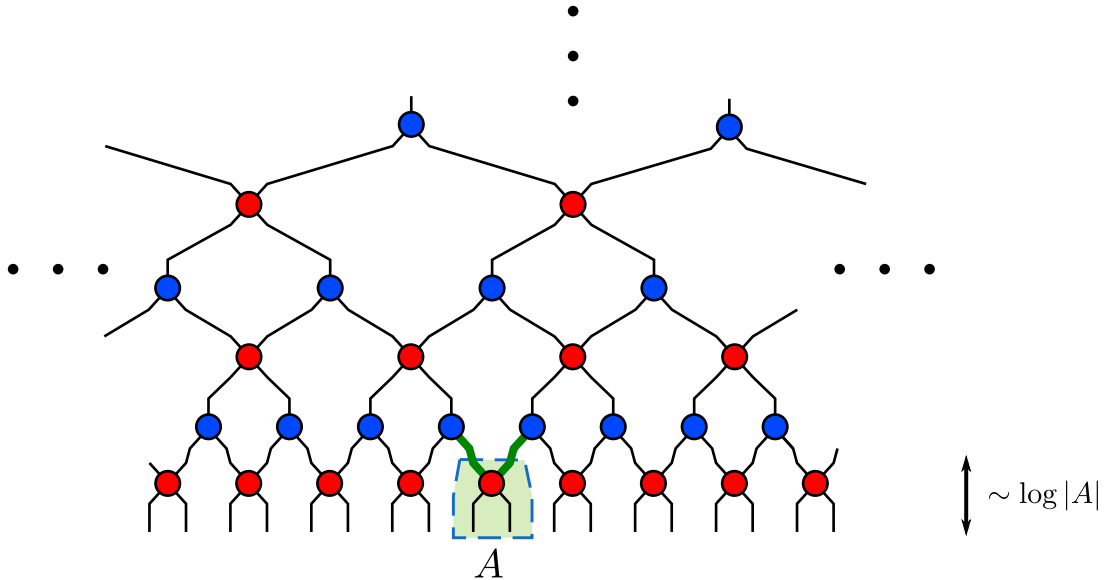
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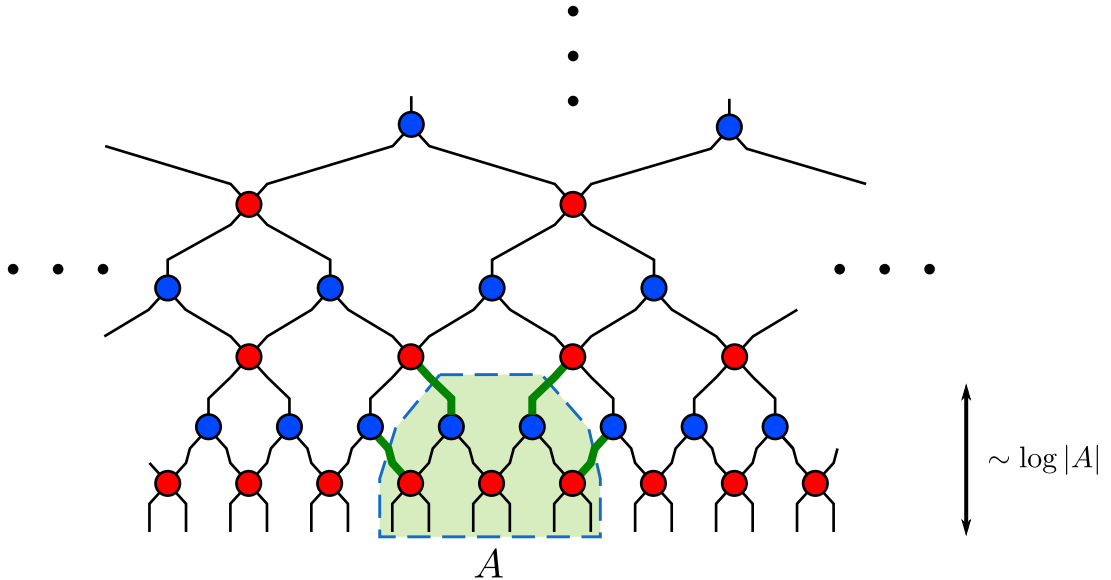
for some  $b > 0$

# MERA: ENTANGLEMENT ENTROPY



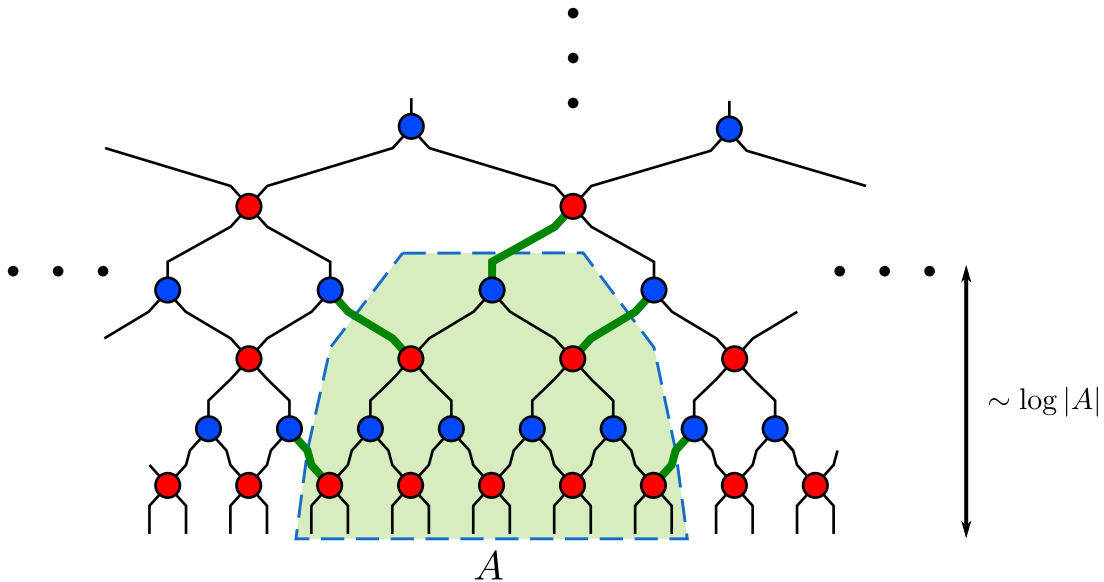
- entanglement entropy of a region  $A$
- connectivity between region  $A$  and the rest of the wave-function
- $S(\varrho_A) \leq \log(\chi) \cdot (\# \text{ crossings}) \sim \log |A|$

# MERA: ENTANGLEMENT ENTROPY



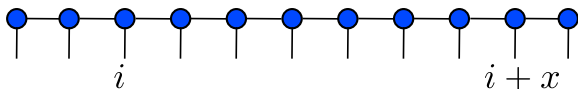
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# MPS vs MERA: CORRELATIONS

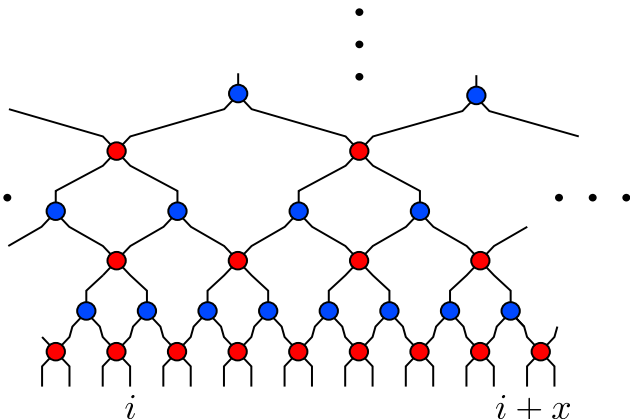


$$\langle \Psi | P_i Q_{i+x} | \Psi \rangle \sim e^{-x/\xi}$$

exponential correlations

(gapped)

VS.

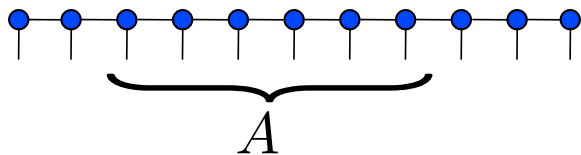


$$\langle \Psi | P_i Q_{i+x} | \Psi \rangle \sim x^{-b}$$

polynomial correlations

(gapless)

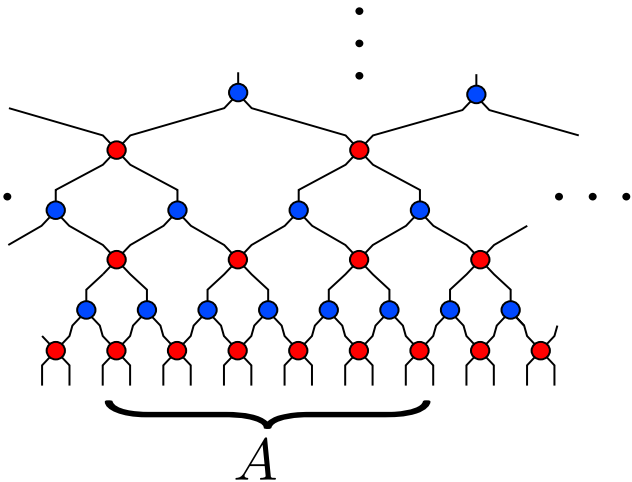
# MPS vs MERA: ENTANGLEMENT ENTROPY



$$S(\rho_A) \sim \text{const}$$

(gapped)

VS.



$$S(\rho_A) \sim \log |A|$$

(gapless)

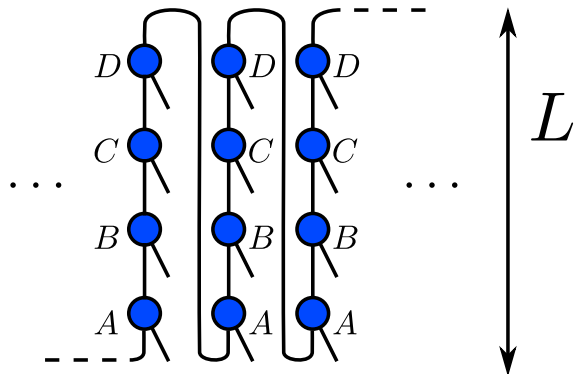
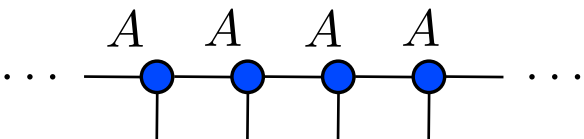


# MPS FOR 2D SYSTEMS

1D



2D

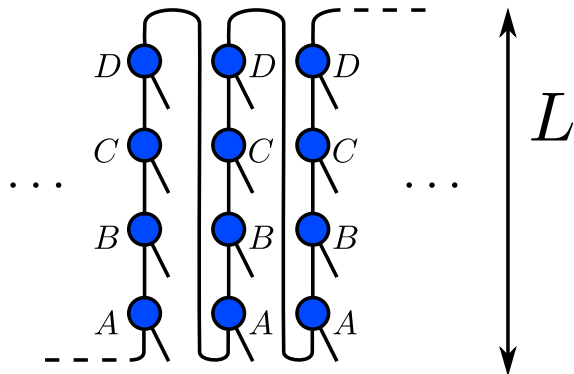
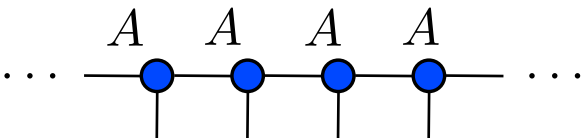


# MPS FOR 2D SYSTEMS

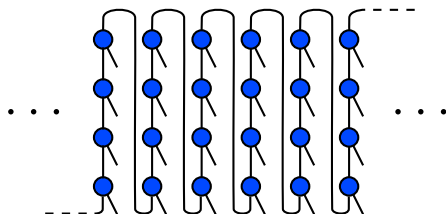
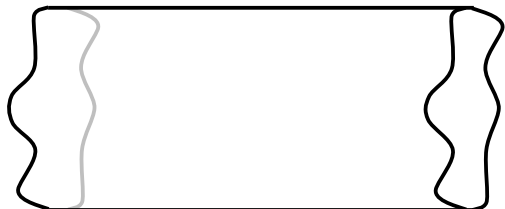
1D



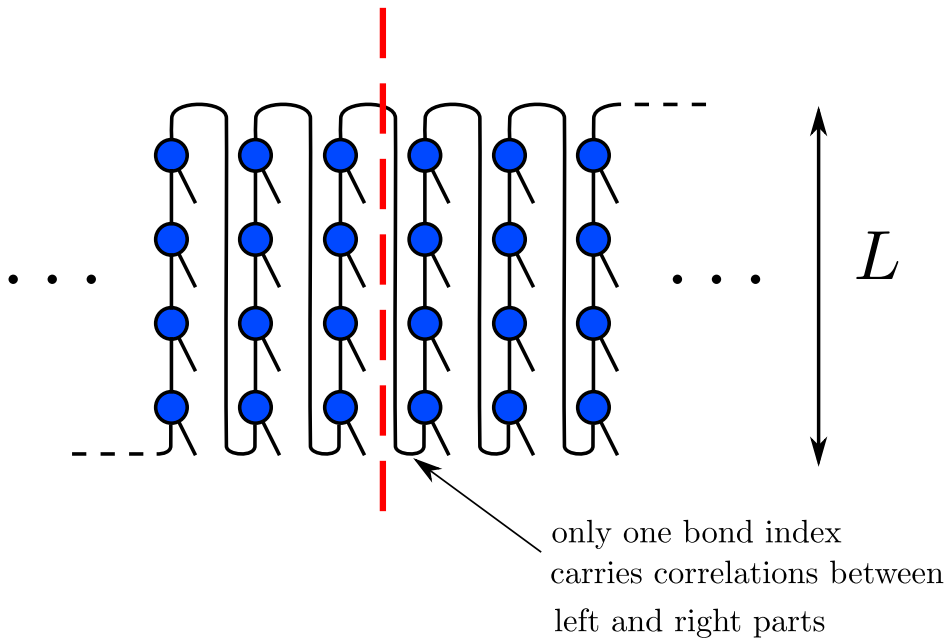
2D



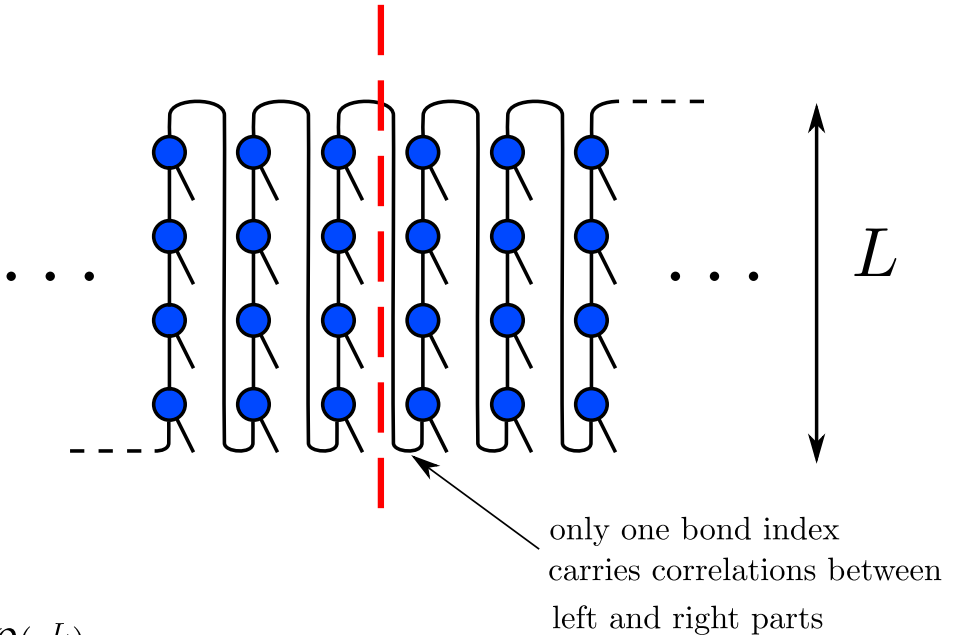
- applicable to infinite cylinders



# MPS FOR 2D SYSTEMS: LIMITATIONS



# MPS FOR 2D SYSTEMS: LIMITATIONS



- cost:  $\mathcal{O}(e^L)$
- works if  $\xi_{\max} \ll L$
- in practice:  $L \sim 10 \dots 12$

# PROJECTED ENTANGLED PAIR STATES

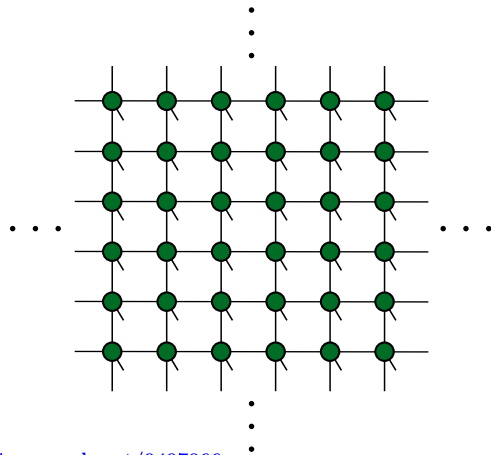
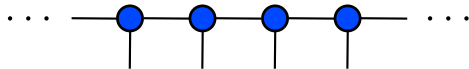
1D



2D

MPS

PEPS



F. Verstraete, J. I. Cirac, [cond-mat/0407066](#)

F. Verstraete, M. M. Wolf, D. Perez-Garcia, J. I. Cirac, [PRL 06, 220601 \(2006\)](#)

P. Corboz, S. R. White, G. Vidal, M. Troyer, [PRB 84, 041108 \(2011\)](#)

P. Czarnik, J. Dziarmaga, [PRB 92, 035152 \(2015\)](#)

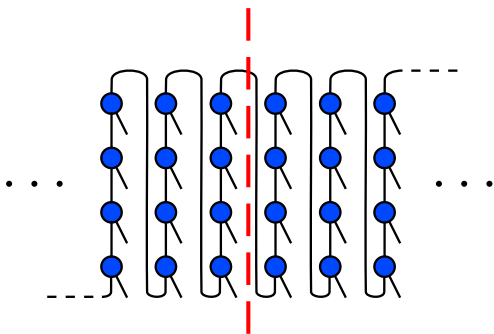
J. Jordan, R. Orús, G. Vidal, F. Verstraete, J. I. Cirac, [PRL 101, 250602 \(2008\)](#)

P. Corboz, [PRB 94, 035133 \(2016\)](#)

... ..

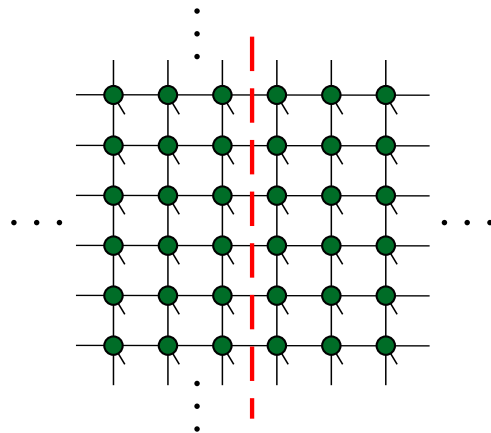
# MPS IN 2D VS PEPS

- correlations



only one bond index  
carries correlations between  
left and right parts

**inefficient**

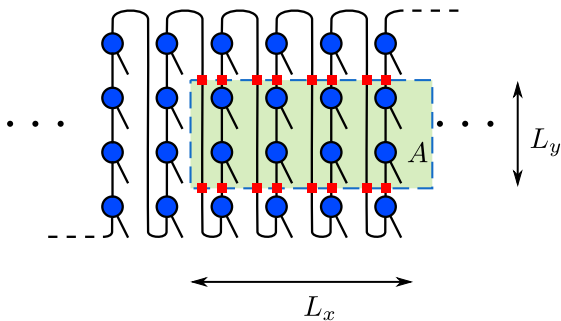


one bond index per every  
correlation across the boundary

**efficient**

# MPS IN 2D VS PEPS

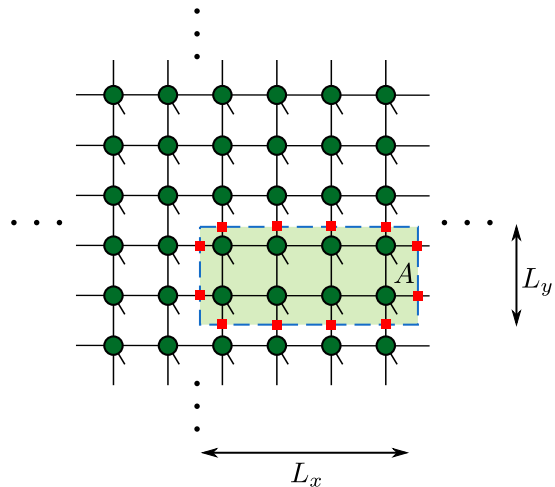
- entanglement entropy



$$S \not\sim |\partial A|$$

(scales only in  $L_x$ )

**inefficient**

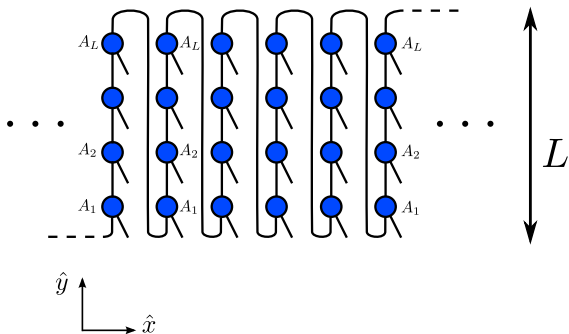


$$S \sim |\partial A|$$

**efficient**

# MPS IN 2D VS PEPS

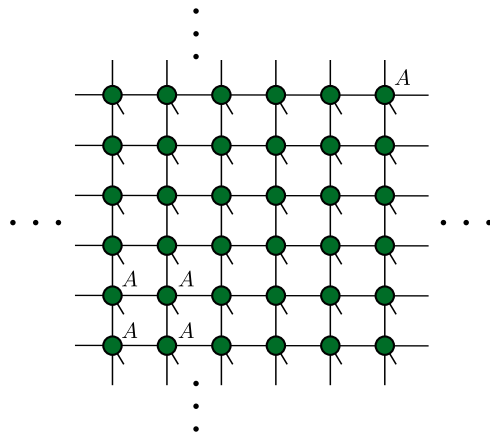
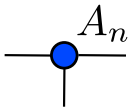
- translation invariance



- no (explicit) translation invariance in  $\hat{y}$  direction

(recovered through optimization)

- the state described by  $L$  tensors



- explicit translation invariance
- the state described by a single tensor

