Projective symmetry group analysis of spin liquid states

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1. Symmetry Groups in classical physics
2. Mean field theory of the Heisenberg model
3. Gauge symmetry and projective symmetry group equations

- In classical physics we classify the possible phases by symmetries of the system - if there is the change in symmetry, then we have phase transition
- The generic example is water-ice transition : **Symmetry Group (SG)** of liquid water is the $ISO(3)$ and of ice is one of the 3D space groups.
- The symmetry breaking is controlled by the **order parameter** (e.g. magnetization $\mathbf{M}$ in magnetic systems) - if $\mathbf{M} = 0$, the system is in disordered phase.
- **Spin liquids** are systems where the symmetry is not broken even when $T \to 0$, i.e. we can not use the SG classification in that case.

Consider the Heisenberg Hamiltonian on the 2D triangular lattice:

$$H = J \sum_{<ij>} \mathbf{S}_i \cdot \mathbf{S}_j, \quad J > 0.$$

Let's try to do the standard Hartree-Fock decoupling:

$$(\mathbf{S}_i - \langle \mathbf{S}_i \rangle) \cdot (\mathbf{S}_j - \langle \mathbf{S}_j \rangle) \approx 0 \Rightarrow$$

$$H_{MF} = J \sum_{<ij>} (\langle \mathbf{S_i} \rangle \cdot \mathbf{S}_j + \mathbf{S}_i \cdot \langle \mathbf{S}_j \rangle - \langle \mathbf{S_i} \rangle \cdot \langle \mathbf{S}_j \rangle).$$

For spin liquid states the order parameter $\langle S_i \rangle = 0$ - but then the mean field Hamiltonian is trivial. One has to perform mean field decoupling in more clever way!

Schwinger boson representation for $SU(2)$

One can introduce bosonic operators a_i, b_i for each lattice site:

$$[a_i, a_j^\dagger] = [b_i, b_j^\dagger] = \delta_{ij},$$

$$[a_i, a_j] = [b_i, b_j] = [a_i, b_j] = 0.$$

$$S_i^+ = a_i^\dagger b_i, \quad S_i^- = b_i^\dagger a_i, \quad S_i^z = \frac{1}{2}(a_i^\dagger a_i - b_i^\dagger b_i).$$

Moreover, one has constraint setting the representation of SU(2) (spin):

$$n_{ia} + n_{ib} = a_i^\dagger a_i + b_i^\dagger b_i = 2S = \kappa.$$

Bond operators

One can introduce the following, ***nonlocal*** bond operators:

$$F_{ij} = \frac{1}{2}(a_i^\dagger a_j + b_i^\dagger b_j),$$

$$A_{ij} = \frac{1}{2}(a_i b_j - a_j b_i).$$

Mean-Field Hamiltonian

$$H_{MF} = J\sum_{ij}(f_{ij}^\star F_{ij} + f_{ij} F_{ij}^\dagger - a_{ij}^\star A_{ij} - a_{ij} A_{ij}^\dagger) +$$

$$-J\sum_{<ij>}(f_{ij}^\star f_{ij} - a_{ij}^\star a_{ij}) + \mu\left(\sum_i (a_i^\dagger a_i + b_i^\dagger b_i) - \kappa\right).$$

Diagonalization of the mean-field Hamiltonian

Hamiltonian H_{MF} has only quadratic interactions

$$\Downarrow$$

Fourier and Bogoliubov transformation

$$\Downarrow$$

ground state $|\Psi_{MF}(f_{ij}, a_{ij}, \mu)\rangle$ + consistency relations:

$$\langle\Psi_{MF}| F_{ij} |\Psi_{MF}\rangle = f_{ij}, \quad \langle\Psi_{MF}| A_{ij} |\Psi_{MF}\rangle = a_{ij} \tag{1}$$

- Hilbert space of the mean field model is bigger then the physical Hilbert space - one has to project mean field ground state $|\Psi(f_{ij}, a_{ij})\rangle$ onto the subspace defined by the ***constraint*** $n_{ia} + n_{ib} = 2S$.

Gauge symmetry of the ansatz

The Hamiltonian is symmetric with respect to the following ***local*** gauge transformations:

$$\hat{a}_i \to G(\mathbf{r}_i)\hat{a}_i \equiv e^{i\phi(\mathbf{r}_i)}\hat{a}_i, \quad \hat{b}_i \to G(\mathbf{r}_i)\hat{b}_i \equiv e^{i\phi(\mathbf{r}_i)}\hat{b}_i$$

$$f_{ij} \to G^\dagger(\mathbf{r}_i)\, f_{ij}\, G(\mathbf{r}_j) \equiv e^{-i\phi(\mathbf{r}_i)}\, f_{ij}\, e^{i\phi(\mathbf{r}_j)}$$

$$a_{ij} \to G(\mathbf{r}_i)\, a_{ij}\, G(\mathbf{r}_j) \equiv e^{i\phi(\mathbf{r}_i)}\, a_{ij}\, e^{i\phi(\mathbf{r}_j)}$$

- Every gauge transformation G is associated with the function $\phi(\mathbf{r})$ defined on the lattice.
- Gauge transformations that do not change f_{ij}, a_{ij} (*Invariant gauge group*) form $\mathbb{Z}_2$.

In general gauge transformation changes the mean field ansaztz (a_{ij}, f_{ij}) But the ansätze related by the gauge transformation give the same ground state wave function after projection onto the subspace determined by the constraint:

$$\prod_i (1 - G(n_{ia} n_{ib})) \left| \Psi_{MF}(G(f_{ij}, a_{ij})) \right\rangle = \prod_i (1 - n_{ia} n_{ib}) \left| \Psi_{MF}(f_{ij}, a_{ij}) \right\rangle$$

One has many-to-one labeling between mean field ansätze and physical states!

- Not all local $U(1)$ gauge transformations are allowed - they have to be consistent with lattice symmetries (SG).
- For triangular lattice there are 4 generators : T_1, T_2, R, σ corresponding to translations, rotation through $\frac{\pi}{3}$ and reflection through origin.
- They satisfy several relations, for example:

$$T_1^{-1} T_2 T_1 T_2^{-1} = \mathbb{I}. \tag{2}$$

- Every symmetry transformation can be written as a composition of generators

Symmetries of the mean-field ansatz

With every $U \in SG$ one can associate gauge transformation G_U :

$$a_{ij} = G_U \, a_{U(i)\,U(j)}$$

Generators T_1, T_2, R, σ correspond to *space - dependent* phases $\phi_{T_1}(\mathbf{r})$, $\phi_{T_2}(\mathbf{r})$, $\phi_R(\mathbf{r})$, $\phi_\sigma(\mathbf{r})$.

Projective Symmetry Group (PSG) consists of transformations of the form $G_U \, U$ - but relations between its elements have to be consistent with relations for Symmetry Group (SG)!

- Equations for generators of lattice symmetry group $\Leftrightarrow$ Equations for corresponding phases

Solution to Projective symmetry group equations on the triangular lattice (F. Wang at al.,2006)

$$\phi_{T_1}(r_1, r_2) = 0 \tag{3}$$

$$\phi_{T_2}(r_1, r_2) = p_1 \pi r_1 \tag{4}$$

$$\phi_{\sigma}(r_1, r_2) = p_2 \pi/2 + p_1 \pi r_1 r_2 \tag{5}$$

$$2\phi_R(r_1, r_2) = p_3 \pi + p_1 \pi r_2 (r_2 - 1 + 2r_1) \tag{6}$$

where $p_1, p_2, p_2 \in \{0, 1\}$.

Classification of spin liquid phases

Conjecture (X.G. Wen, Phys. Lett. A 300, 175(2002)) :

Different Projective symmetry groups represent different spin liquid phases (i.e. they determine the different universality classes) in the similar way as different Symmetry groups correspond to different phases in classical Landau theory.

Nearest-neighbor models

Assuming that nearest-neighbor bonds a_{ij} are non-zero, one is left with two-possible solutions:

Zero-flux ansatz

$$p_1 = 0,\ p_2 = 0,\ p_3 = 1$$

$$\phi_{T_1}(r_1, r_2) = 0$$

$$\phi_{T_2}(r_1, r_2) = 0$$

$$\phi_{\sigma}(r_1, r_2) = 0$$

$$\phi_R = \pi/2$$

π-flux ansatz

$$p_1 = 1,\ p_2 = 1,\ p_3 = 0$$

$$\phi_{T_1}(r_1, r_2) = 0$$

$$\phi_{T_2}(r_1, r_2) = \pi r_1$$

$$\phi_{\sigma}(r_1, r_2) = \pi/2 + \pi r_1 r_2$$

$$\phi_R = \pi r_1 r_2 + (\pi/2) r_2 (r_2 - 1)$$

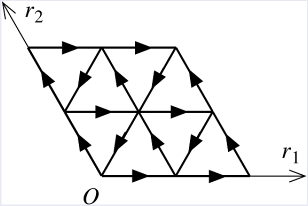
Z_2 gauge theory

Zero flux ansatz

$$a_{(r_1,r_2),(r_1+1,r_2)} = a_{(r_1,r_2),(r_1,r_2+1)} =$$

$$= -a_{(r_1,r_2),(r_1+1,r_2+1)}$$

$$f_{ij} = \text{const}$$

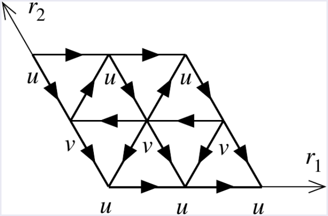


π-flux ansatz

$$a_{(r_1,r_2),(r_1+1,r_2)} = -a_{(r_1,r_2),(r_1+1,r_2+1)}$$

$$= (-1)^{r_2+1} a_{(r_1,r_2),(r_1,r_2+1)}$$

$$f_{ij} = \text{const} = 0$$



Aplications

- Starting from the values a_{01} on particular bond and using gauge transformations between different sites of the lattice one can generate a_{ij} for the whole lattice.
- Zero energy gap $\Rightarrow$ condensation of Schwinger bosons $\Rightarrow$ long range ordering of spins, equivalent to spin wave expansion
- Non - vanishing gap indicates that the ground state of a model does not break the symmetry and is probably a spin liquid state

- X.-G. Wen, Phys. Lett. A **300**, 175 (2002)
- X.-G. Wen, "Quantum Field Theory of Many-Body Systems", Oxford University Press 2004
- F. Wang and A. Vishwanath, Phys. Rev. B **74**, 174423 (2006)
- S. Bieri at al, *Projective symmetry group classification of chiral spin liquids*, Phys. Rev. B. **93**, 094437