Twistor Methods for AdS$_5$

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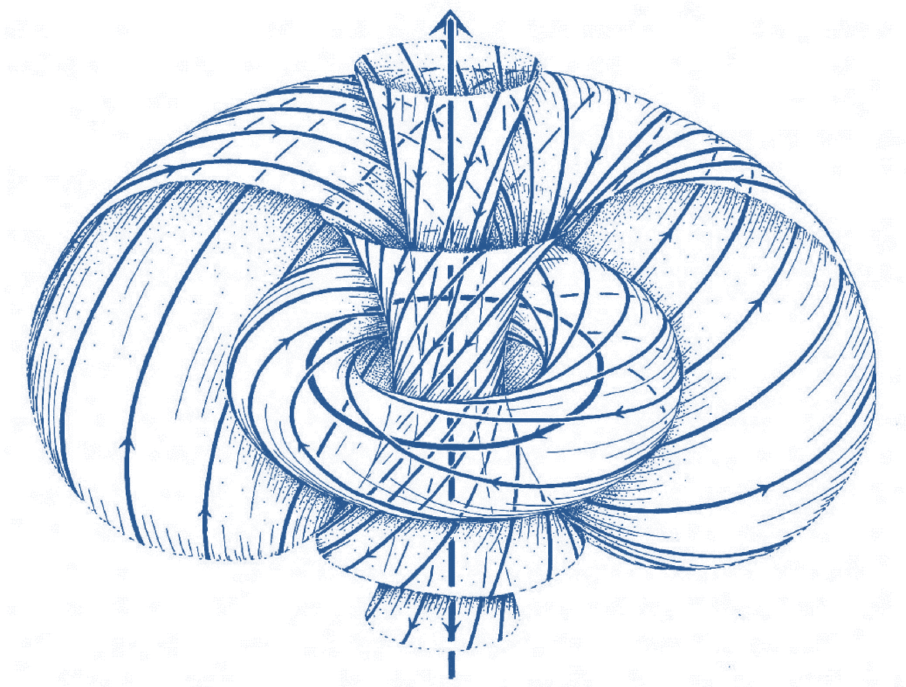
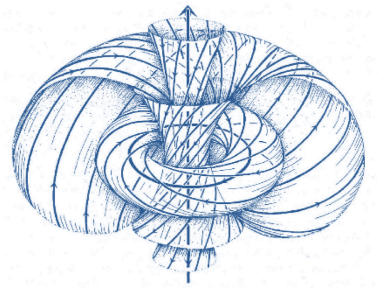
with David Skinner and Tim Adamo

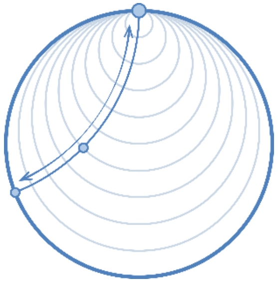


Outline

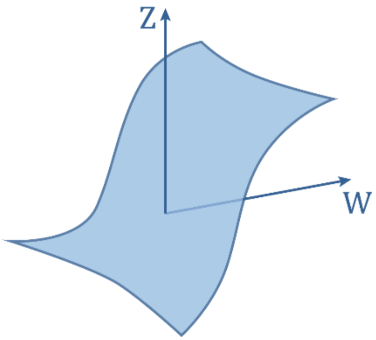
1. Twistor theory – motivation from 4D



2. Twistors into the bulk

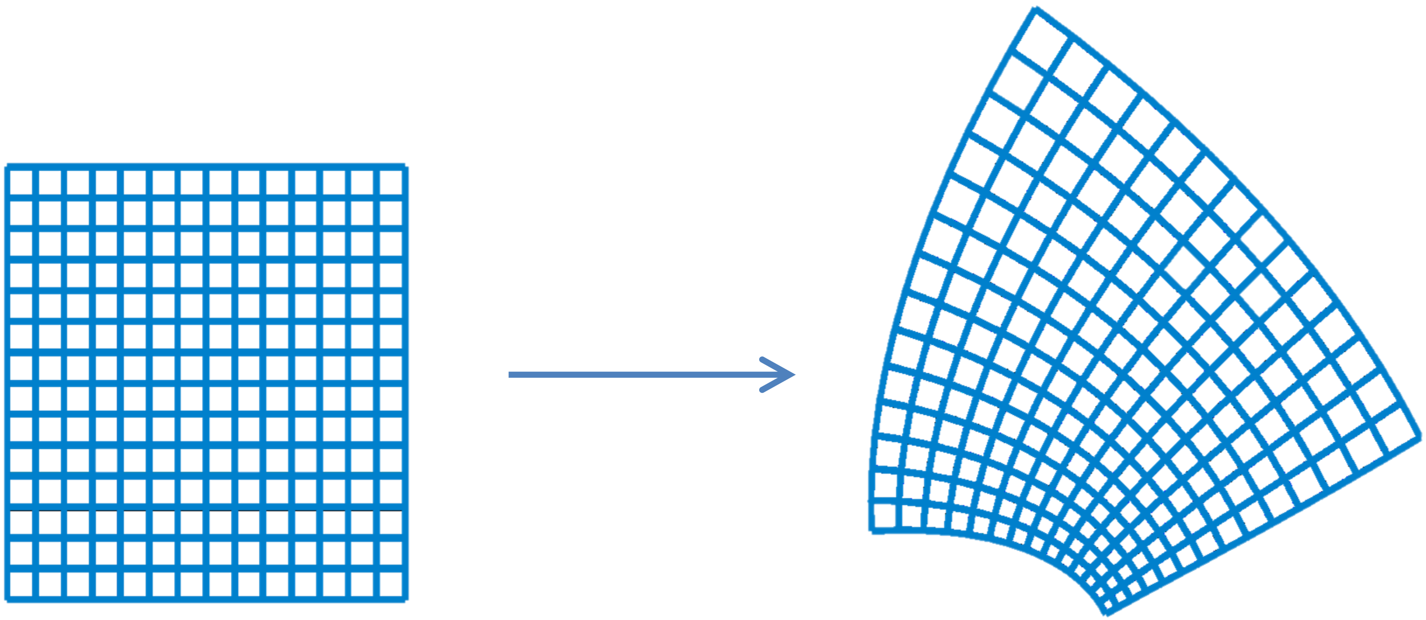


3. Twistors for holography



What is a twistor space?

Points in Minkowski space can be described by coordinates x^{μ}, but other choices are possible.



What is a twistor space?

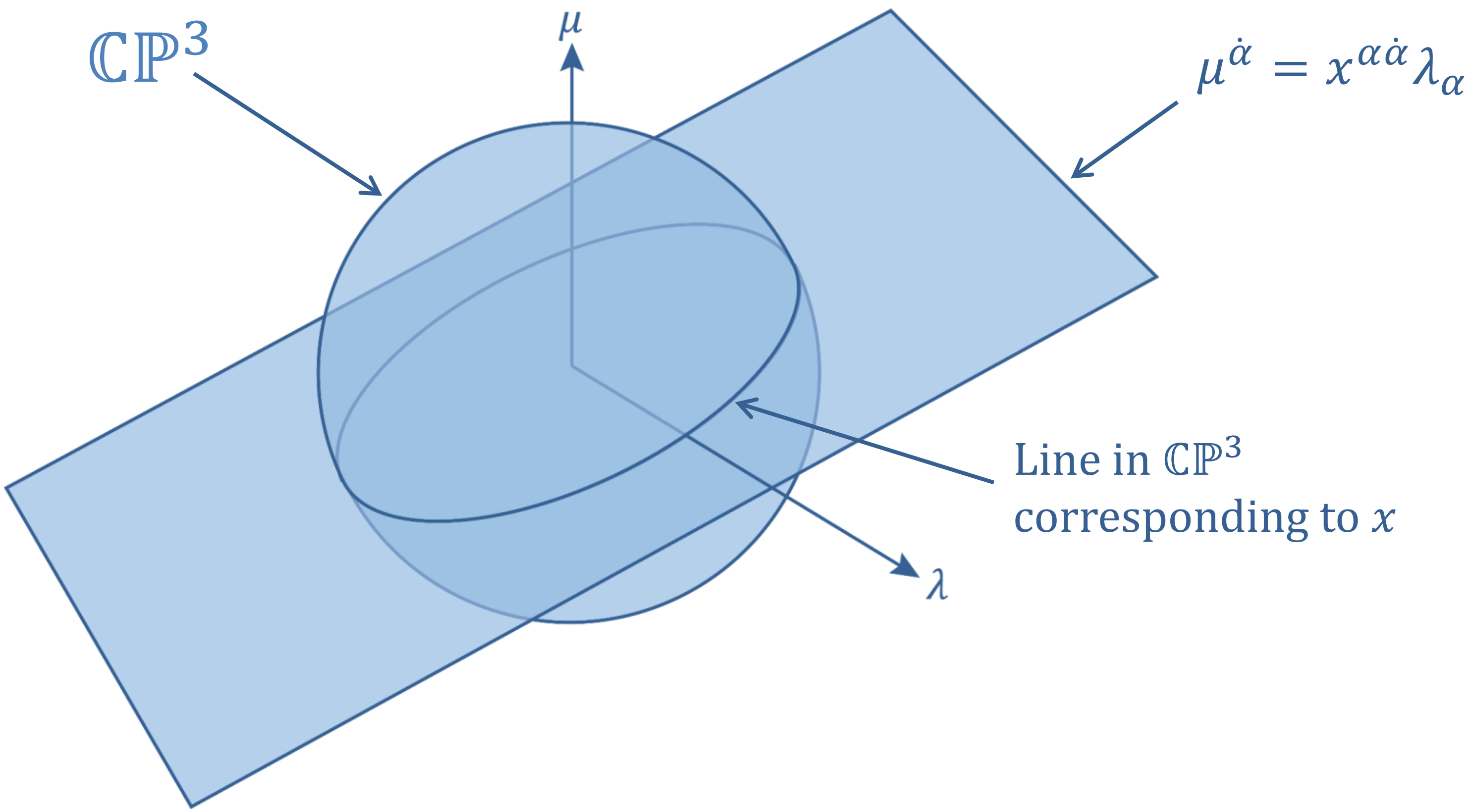
One choice is to introduce coordinates $x^{\alpha\dot{\alpha}}$, where $x^{\alpha\dot{\alpha}} = \sigma_{\mu}^{\alpha\dot{\alpha}} x^{\mu}$. (This makes use of the isomorphism $SO(4) \cong SU(2) \times SU(2)$.)

This means points are represented by 2×2 complex matrices

$$x^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix}.$$

What is a twistor space?

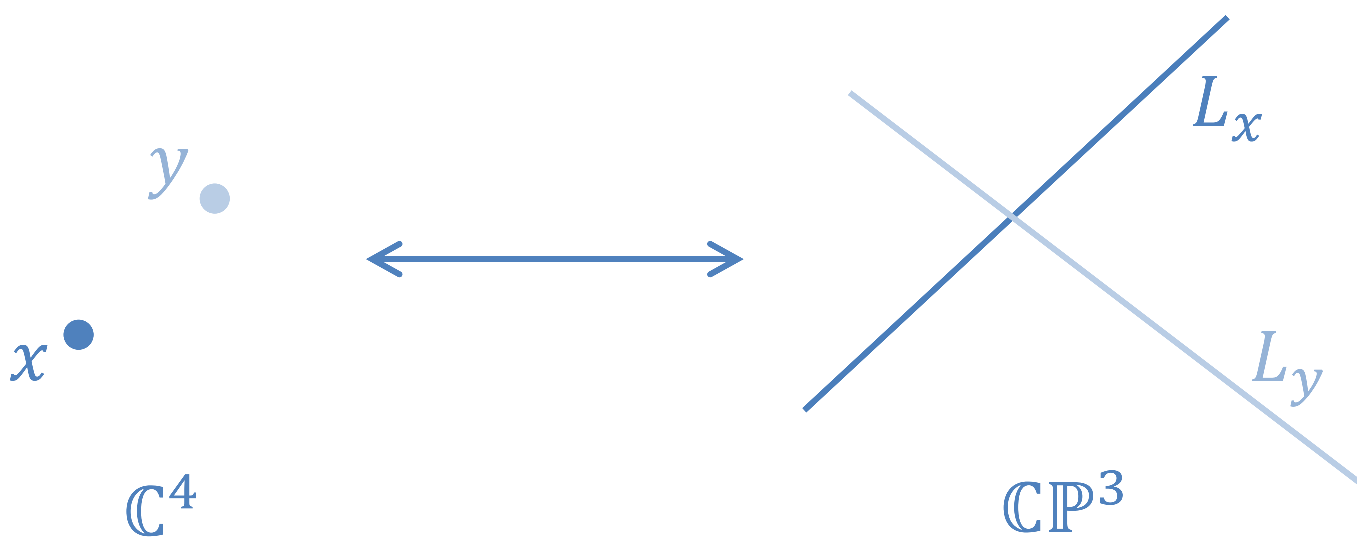
It is natural to let this matrix act on complex vectors as $\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha$.



What is a twistor space?

The coordinates $(\mu^{\dot{\alpha}}, \lambda_{\alpha})$ are homogeneous coordinates on *twistor space* $\mathbb{CP}^3$.

Points in spacetime correspond to lines in twistor space.



What is the point of this?

It helps us to solve equations.

Wave equation on spacetime

$$\Box\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$

In twistor space becomes

$$\Box\phi = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\partial_{\alpha\dot{\alpha}}\partial_{\beta\dot{\beta}}\phi = 0$$

What is the point of this?

Make a guess

$$\phi = \oint_{L_x} f(\mu, \lambda)$$

with $f(\mu, \lambda)$ *any* holomorphic function.

Then

$$\square\phi = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\partial_{\alpha\dot{\alpha}}\partial_{\beta\dot{\beta}}\phi$$

$$= \oint \varepsilon^{\alpha\beta}\lambda_\alpha\lambda_\beta\varepsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial^2}{\partial\mu^{\dot{\alpha}}\partial\mu^{\dot{\beta}}}f(x^{\alpha\dot{\alpha}}\lambda_\alpha, \lambda_\alpha)$$

$$= 0$$

What is the point of this?

Any holomorphic function $f(\mu, \lambda)$ gives us a solution to the wave equation

$$\phi = \oint_{L_x} f(\mu, \lambda)$$

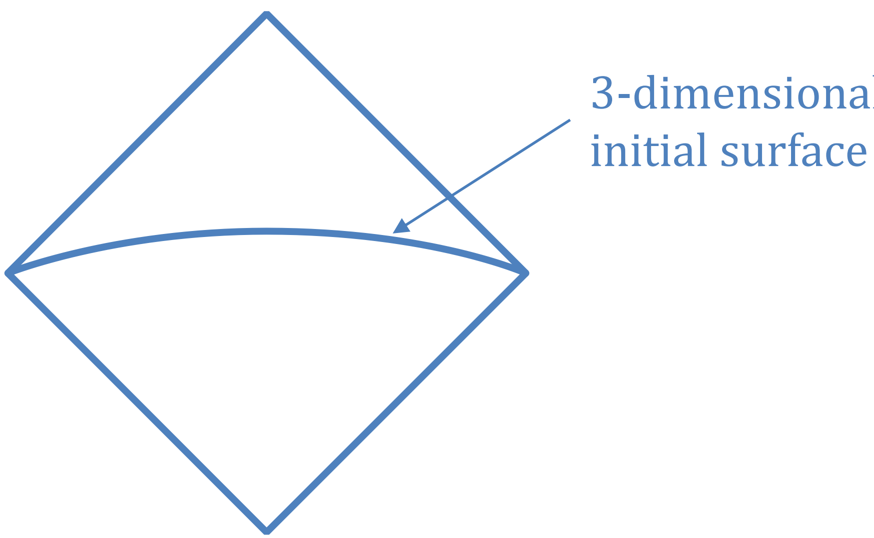
This is the *Penrose transform*.

The *Penrose transform* of an arbitrary function on twistor space gives us a solution to the spacetime wave equation.

Analogy with Green's functions

We expect a solution to $\Box\phi = 0$ to be determined by data on a 3-dimensional boundary.

The function $f: \mathbb{CP}^3 \longrightarrow \mathbb{C}$ is also a function of three variables.



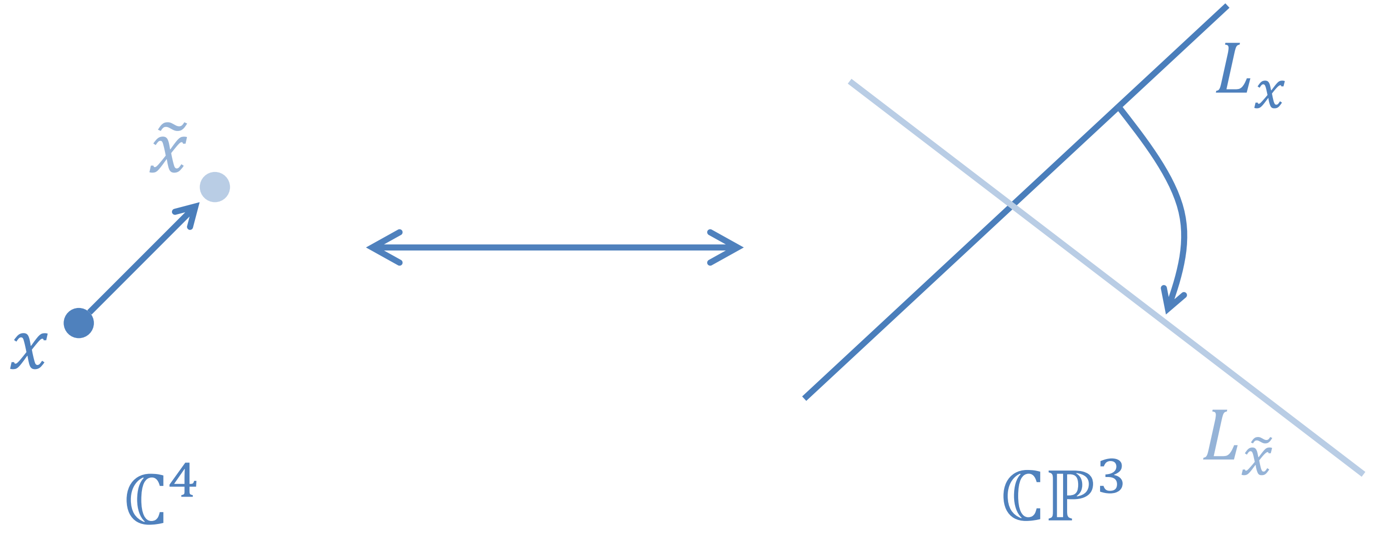
Other Penrose transforms

We can solve other equations too.

Wave equation	$\Box\phi = 0$	$\phi = \oint_{L_x} f(\mu, \lambda)$
Dirac equation	$\nabla^{\alpha\dot{\alpha}}\phi_\alpha = 0$	$\phi_\alpha = \oint_{L_x} \lambda_\alpha f(\mu, \lambda)$
$\vdots$	$\vdots$	$\vdots$
Higher spin	$\nabla^{\alpha\dot{\alpha}}\phi_{\alpha\dots\beta} = 0$	$\phi_{\alpha\dots\beta} = \oint_{L_x} \lambda_\alpha \cdots \lambda_\beta f(\mu, \lambda)$

Conformal invariance

In AdS/CFT, the boundary field theory is conformally invariant. Conformal invariance is naturally expressed in twistor variables.



Conformal invariance

Lorentz	$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
Translations	$P_\mu = -i\partial_\mu$
Dilatations	$D = -ix^\mu \partial_\mu$
Special conformal	$K_\mu = i(x^2 \partial_\mu - x_\mu x^\nu \partial_\nu)$

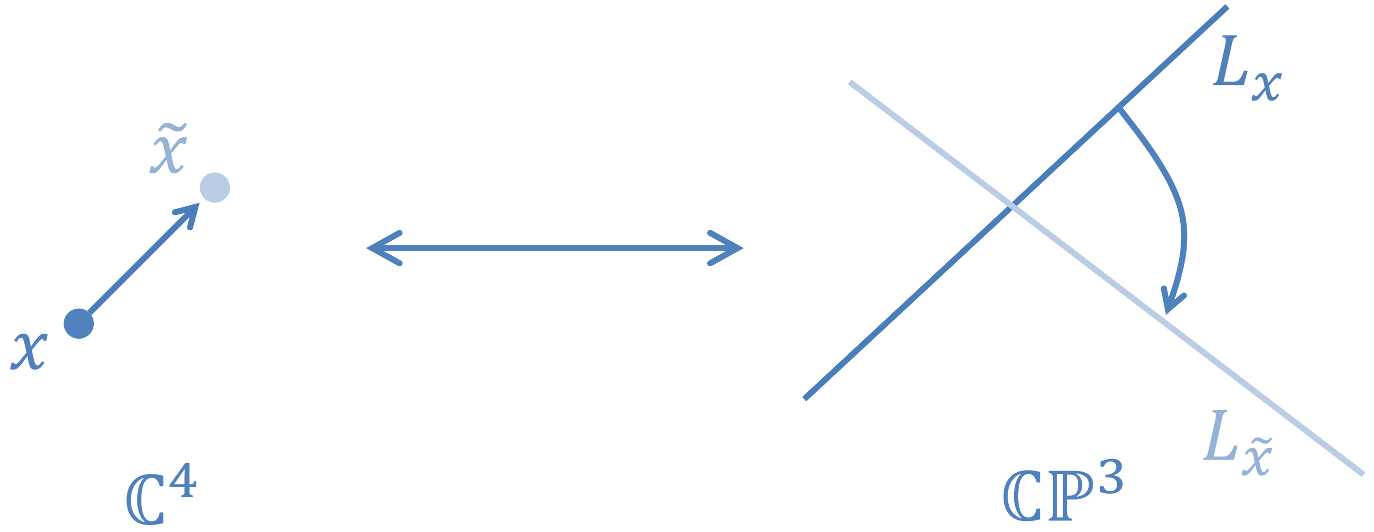
$$J^A{}_B = Z^A \frac{\partial}{\partial Z^B}$$

Conformal transformations act naturally on the twistor space $Z^A = \left(\mu^{\dot{\alpha}}, \lambda_\alpha\right)$.

Conformal invariance

The conformal transformations $J^A{}_B = Z^A \frac{\partial}{\partial Z^B}$ generate *linear* maps on twistor space $Z^A = \left(\mu^{\dot{\alpha}}, \lambda_\alpha\right)$.

So lines get mapped onto other lines.



Conformal invariance

The conformal group is $SL(4,\mathbb{C})$ (scales don't count in $\mathbb{CP}^3$).

Example: Dilatation

$$D = \mu^{\dot{\alpha}}\frac{\partial}{\partial\mu^{\dot{\alpha}}} - \lambda_\alpha\frac{\partial}{\partial\lambda_\alpha}$$

Scales $\mu^{\dot{\alpha}}$ and λ_α oppositely. So the line

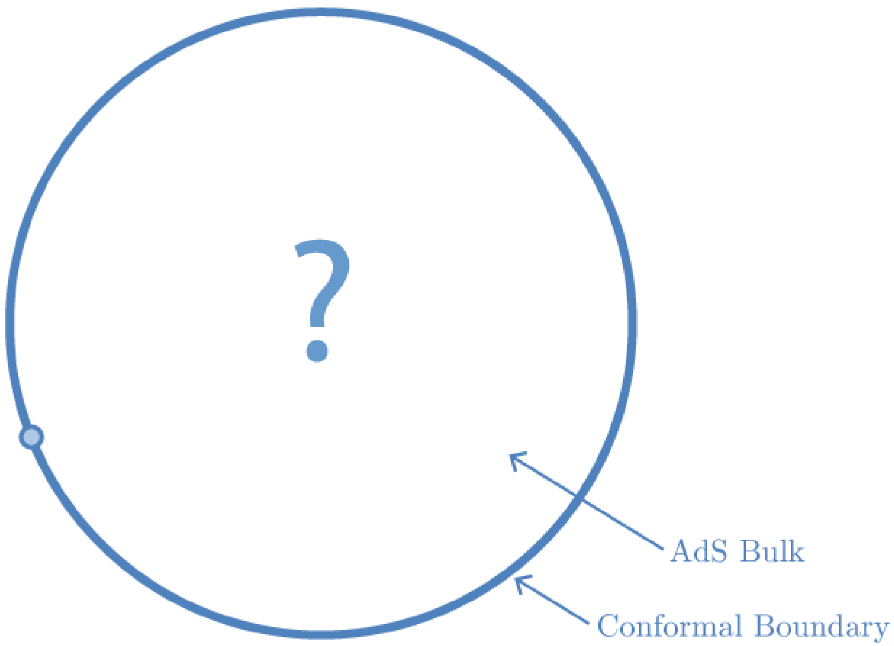
$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha$$

gets mapped to a line with $x^{\alpha\dot{\alpha}}$ rescaled.

Twistor variables for AdS/CFT

Twistor variables describe the boundary theory nicely. But the bulk theory is supposed to be the same as the boundary.

Do these variables extend into the bulk?



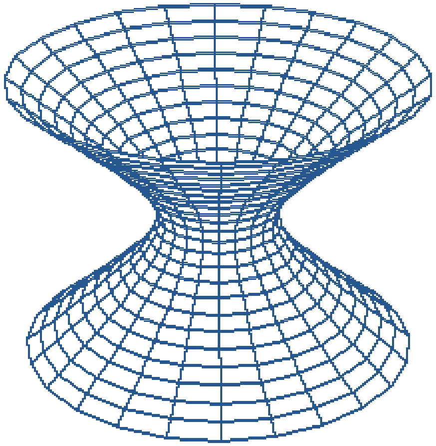
(Ambi-)twistor space of AdS_5

There are two common constructions of AdS_5.

Hyperboloid Model

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu$$

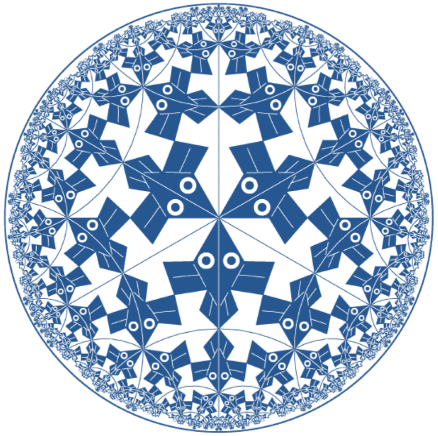
$$X^2 = 1$$



Poincaré coordinates

$$ds^2 = \frac{dr^2 + d\vec{z}^2}{r^2}$$

$$r > 0, \vec{z} \in \mathbb{C}^4$$



(Ambi-)twistor space of AdS_5

We can write the components of the flat 6d X^μ in a 4×4 antisymmetric matrix X^{AB} with the inner product

$$\delta_{\mu\nu} X^\mu Y^\nu = X \cdot Y = \varepsilon_{ABCD} X^{AB} Y^{CD}$$

Then AdS_5 is the subset $X^2 = 1$ with this inner product.

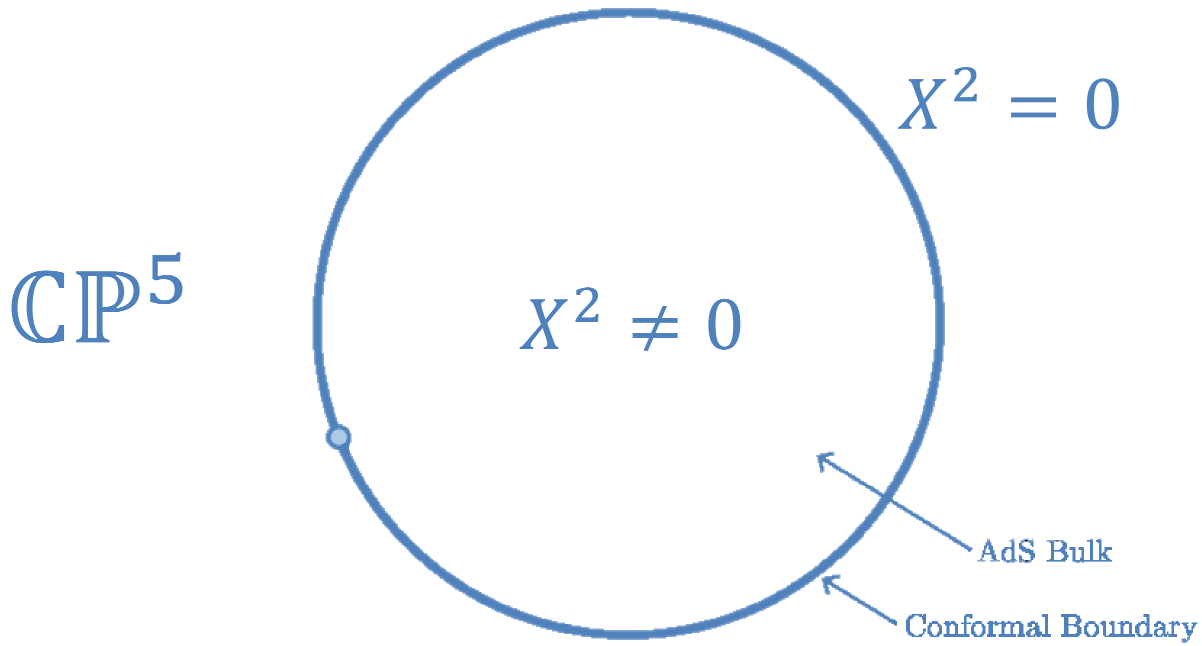
(Ambi-)twistor space of AdS_5

Alternatively, we could regard the coordinates X^{AB} as projective coordinates on

$$\mathbb{CP}^5 \backslash \{X^2 = 0\} \cong \{X^2 = 1\} = \mathrm{AdS}_5$$

$\mathbb{CP}^5$ is compact, but $\mathrm{AdS}_5 = \mathbb{CP}^5 \backslash \{X^2 = 0\}$ is not.

(Ambi-)twistor space of AdS_5



The conformal boundary of AdS_5 is the set $X^2 = 0$ in $\mathbb{CP}^5$.

(Ambi-)twistor space of AdS_5

To make a connection to Poincaré coordinates, we will need to choose one point on the boundary, the *infinity twistor* I^{AB}.

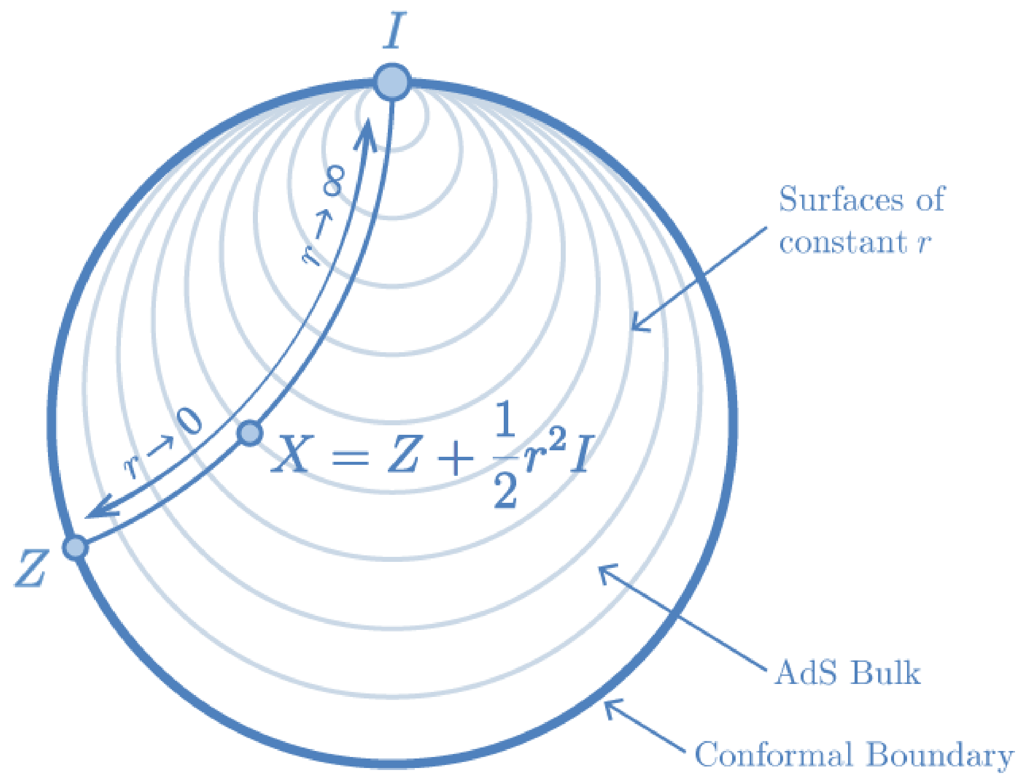
Then a general point in $\mathbb{CP}^5$ can be written as

$$X^{AB} = Z^{AB} + \frac{1}{2} r^2 I^{AB}$$

where Z^{AB} is a boundary point and r is a real parameter.

(Ambi-)twistor space of AdS_5

$$X^{AB} = Z^{AB} + \frac{1}{2} r^2 I^{AB}$$



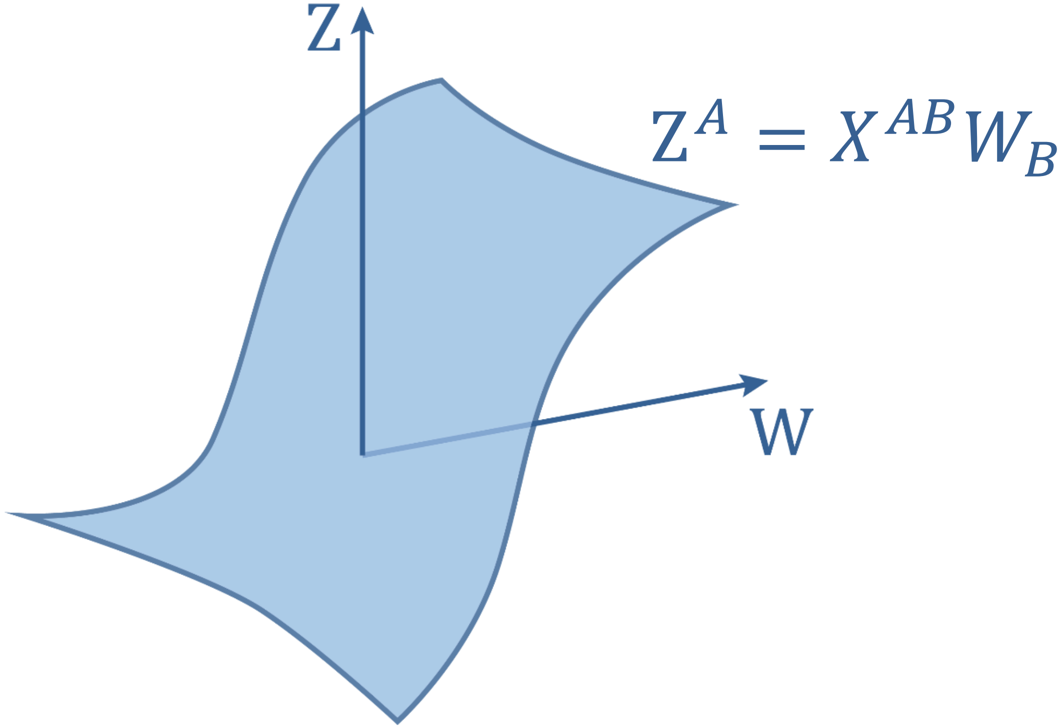
(Ambi-)twistor space of AdS$_5$

The matrices X^{AB} can act on complex 4d vectors too.

$$\mathrm{Z}^A = X^{AB} W_B$$

This describes a line in $\mathbb{CP}^3 \times \mathbb{CP}^3$. Because X^{AB} is antisymmetric, we have $Z \cdot W = 0$.

(Ambi-)twistor space of AdS_5



So each point in AdS_5 corresponds to a hyperplane in $Q = \{Z \cdot W = 0\} \subset \mathbb{CP}^3 \times \mathbb{CP}^3$, the *ambitwistor space* of AdS_5.

Twistor variables in AdS/CFT

Solutions to the wave equation in AdS_5 can also be obtained as the Penrose transform of holomorphic functions on ambitwistor space.

$$\phi = |X|^{\Delta} \int_{X} D^3W \, f(Z, W)$$

Twistor variables in AdS/CFT

Solutions to the wave equation in AdS_5 can also be obtained as the Penrose transform of holomorphic functions on ambitwistor space.

$$\phi = |X|^{\Delta} \int_X D^3W \, f(X \cdot W, W)$$

Twistor variables in AdS/CFT

Twistor variables are a natural description for both the conformal boundary and the bulk space.

Can we express the basic ingredients of AdS/CFT in a purely twistorial way?

One of the most important objects in holography is the *bulk-to-boundary propagator*.

Twistor variables in AdS/CFT

We can write the scalar bulk to boundary propagator in twistor notation.

The scalar field equation

$$\Box\phi = \Delta(\Delta - 4)\phi$$

has bulk-to-boundary propagator

$$\phi(X;Y) = \frac{|X|^{\Delta}}{(X \cdot Y)^{\Delta}}$$

Twistor variables in AdS/CFT

This can be written as the Penrose transform of a function on twistor space!

$$\phi(X;Y) = |X|^{\Delta} \int_X D^3W \, \frac{\bar{\delta}^3(W,A)}{(Z\cdot B)^{\Delta}}$$

This idea can also be extended to bulk-to-boundary propagators of higher spin fields.

Canonical Pairing in Twistor space

For any Penrose representative f, one can construct an *indirect Penrose representative* $\tilde{f}$.

$$\phi(X) = |X|^{-\Delta} \int_X D^3Z\, \tilde{f}(Z,W)$$

The two-point function in the CFT can be obtained from f and its indirect partner $\tilde{f}$:

$$\int_Q D^3Z\, D^3W\, \bar{\delta}(Z \cdot W)\, f(Z,W) \wedge \tilde{f}(Z,W)$$

Canonical Pairing in Twistor space

There is also a natural local action on twistor space for the scalar field.

$$S = \int_Q D^3Z\, D^3W\, \bar{\delta}(Z \cdot W)\, f(Z,W) \wedge \bar{\partial}\tilde{h}(Z,W)$$

The equation of motion is simply that f must be holomorphic: $\bar{\partial} f = 0$.

Conclusions and outlook

Conformal invariance of the boundary theory is most naturally expressed in twistor language.

Important ingredients on the bulk side also have a purely twistorial description.

Some bulk and boundary calculations can be performed twistorially.