A story of chaos and purity

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QMUL & WITS

Zakopane, May 29 2016

Outline

Chaos

- Diagnosing chaos with out-of-time-order (OTO) correlators.
- Chaos at large central charge.
- OTO in $SU(N)_k$ WZW models.
- OTO for rational CFT, and fractional statistics.

Purity

- Purity for excited states.
- Purity scrambling for large-c $SU(N)_k$.

Based on: 1602.06542 with P. Caputa and T. Numasawa

Diagnosing chaos with OTO

An indicator of the buterfly effect

$$\left\langle [\mathcal{O}_j(0), \mathcal{O}_i(t)]^2 \right\rangle_\beta \stackrel{?}{=} 0$$

Larkin, Ovchinnikov '69, Kitaev '14
Maldacena, Shenker, Roberts, Stanford '15

Consider late times behaviour

$$-\left\langle [\mathcal{O}_j, \mathcal{O}_i(t)]^2 \right\rangle_\beta = \langle \mathcal{O}_j \mathcal{O}_i(t) \mathcal{O}_i(t) \mathcal{O}_j \rangle_\beta + \langle \mathcal{O}_i(t) \mathcal{O}_j \mathcal{O}_j \mathcal{O}_i(t) \rangle_\beta$$
$$- \langle \mathcal{O}_j \mathcal{O}_i(t) \mathcal{O}_j \mathcal{O}_i(t) \rangle_\beta - \langle \mathcal{O}_i(t) \mathcal{O}_j \mathcal{O}_i(t) \mathcal{O}_j \rangle_\beta$$

In general, first line $\sim \langle \mathcal{O}_j \mathcal{O}_j \rangle_\beta \langle \mathcal{O}_i(t) \mathcal{O}_i(t) \rangle_\beta$

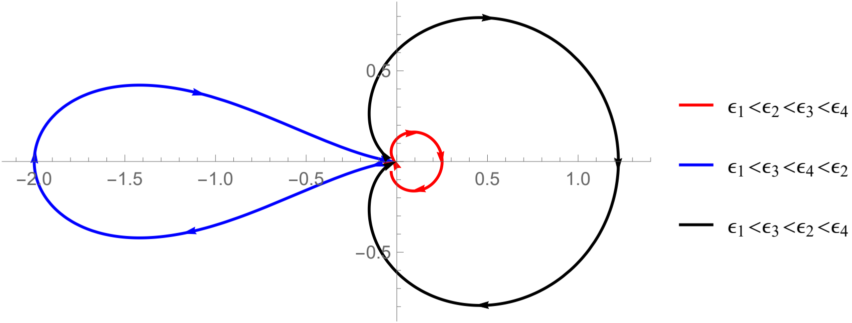
If the second line vanishes the system is chaotic

The correlators in the second line are out-of-time-order (OTO)

OTO in CFT_2

Chaos indicator $C^{\beta}_{ij}(t) \equiv \dfrac{\langle \mathcal{O}_i(t)\mathcal{O}_j\mathcal{O}_i(t)\mathcal{O}_j\rangle_\beta}{\langle \mathcal{O}_i\mathcal{O}_i\rangle_\beta \langle \mathcal{O}_j\mathcal{O}_j\rangle_\beta} = f(z,\bar{z})$

Cross-ratio z



OTOs $\langle \mathcal{O}_i(t)\mathcal{O}_j\mathcal{O}_j\mathcal{O}_i(t)\rangle_\beta$ go through the branch cut

Large-c and chaos

We can write (holography)

$$f(z,\bar{z}) \to \mathcal{F}(z)\bar{\mathcal{F}}(\bar{z})$$

For example, in the (heavy-heavy $\to$ light-light) regime

$$\mathcal{F}(z) \sim \left(\frac{z(1-z)^{-6h_w/c}}{1-(1-z)^{1-12h_w/c}} \right)^{2h_v}$$

Fitzpatrick, Kaplan, Walters '14

At late times the chaos indicator will vanish

$$C_{ij}^{\beta}(t) \sim \left(1 + \frac{2\pi h_w}{\epsilon_{12}^{*}\epsilon_{34}} \exp\left(\frac{2\pi}{\beta}(t-t_*) \right) \right)^{-2h_v}$$

Roberts, Stanford '15

Fast scrambling time $t_* = \frac{\beta}{2\pi}\log(c)$

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Fitzpatrick, Kaplan, Walters '14

Holographic theories display the butterfly effect

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Roberts, Stanford '15

Matches bulk analysis (Shenker, Stanford '14)

WZW theories

CFT's that besides Virasoro they have an affine Lie algebra $\hat{\mathfrak{g}}_k$

$$\left[j^a_m, j^b_n\right] = i \sum_c f^{abc} j^c_{m+n} + k\, m\, \delta^{ab}\, \delta_{m+n}$$

Virasoro and $\hat{\mathfrak{g}}_k$ must be compatible

$$c = \frac{k \dim\, \mathfrak{g}}{k + C_{\mathfrak{g}}} \qquad \text{dual Coexeter number } C_{\mathfrak{g}}$$

We consider $SU(N)_k$ theories

$$\dim(\mathfrak{g}) = N^2 - 1 \qquad C_{\mathfrak{g}} = N \qquad c = \frac{k\,(N^2 - 1)}{k + N}$$

't Hooft large-N

$$N, k \to \infty \qquad \lambda_s = \frac{N}{k} \qquad \text{fixed}$$

Kiritsis, Niarchos '13

OTO in $SU(N)_k$ WZW

Affine primaries ($N-1$ rows, k columns)

Ex: $SU(3)_4$  $\dots$

Late time OTO fundamental ($\square$)

$$C^{\beta}_{gg}(t) \to q^{\frac{1}{N}+\frac{1}{2}} \frac{\left(q^{-\frac{N+2}{2}} + [N-1]\right)}{[N]}$$

Introducing the q-numbers

$$[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \qquad q = e^{-\frac{2\pi i}{N+k}}$$

Key point

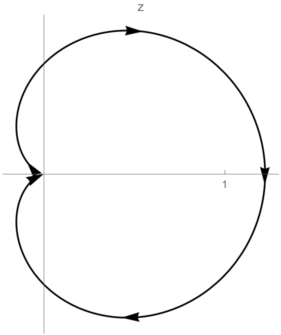
$$\text{At large-}c \qquad C^{\beta}_{gg}(t) = 1$$

Caputa, Numasawa, AVO '16

Late time OTO in RCFT

In a rational conformal field theory (RCFT)

$$f(z,\bar{z}) = \sum_p \mathcal{F}^{ii}_{jj}(p|z)\bar{\mathcal{F}}^{ii}_{jj}(p|\bar{z}).$$



We pick the monodromy

$$\mathcal{F}^{ii}_{jj}(p|z) \to \sum_q \mathcal{M}_{pq}\mathcal{F}^{ii}_{jj}(q|z)$$

Late time OTO

$$\boxed{C^{\beta}_{ij}(t) \to \frac{1}{d_i d_j}\frac{S^*_{ij}}{S_{00}}}$$

Quantum dimension (d_i)
Modular S-matrix (S_{ij})

Caputa, Numasawa, AVO '16, Gu, Qi '16

What is the quantum dimension?

Modular S-matrix

$$\chi_i(\tau) = \mathrm{Tr}_{\mathcal{H}_i}\left(e^{i\pi\tau(L_0 - c/24)}\right) \qquad \chi_i(-1/\tau) = \sum_j S_{ij}\chi_j(\tau)$$

Quantum dimension of $\mathcal{O}_i$

$$d_i = \lim_{\tau\to 0}\frac{\chi_i(\tau)}{\chi_0(\tau)} \qquad d_i = \frac{S_{0i}}{S_{00}}$$

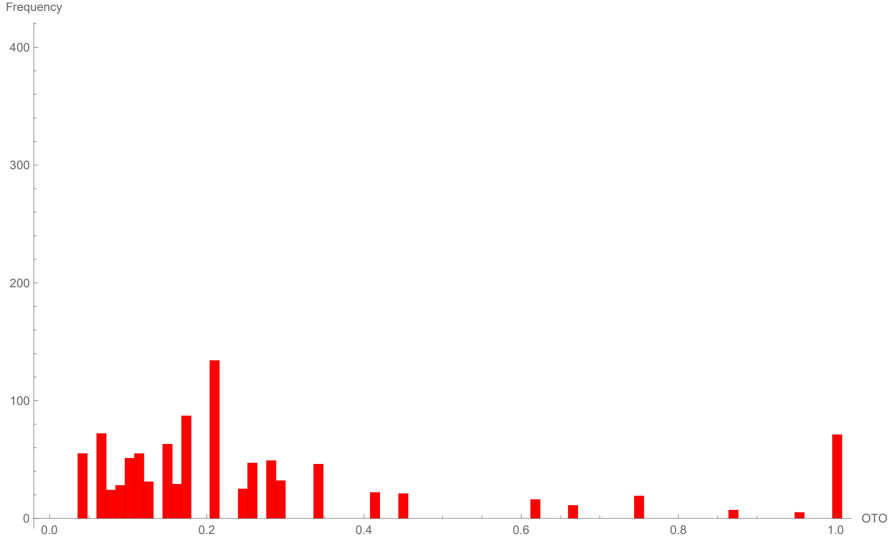
Dijkgraaf, Verlinde '88

Example: Ising model (1, σ, ϵ)

$$S_{ij} = \begin{array}{c} \begin{array}{ccc} 1 & \sigma & \epsilon \end{array} \\ \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ \sigma \\ \epsilon \end{array}$$

$$|C^{\beta}_{\sigma\sigma}| = 0$$
$$|C^{\beta}_{\epsilon\sigma}| = 1$$
$$|C^{\beta}_{\epsilon\epsilon}| = 1$$

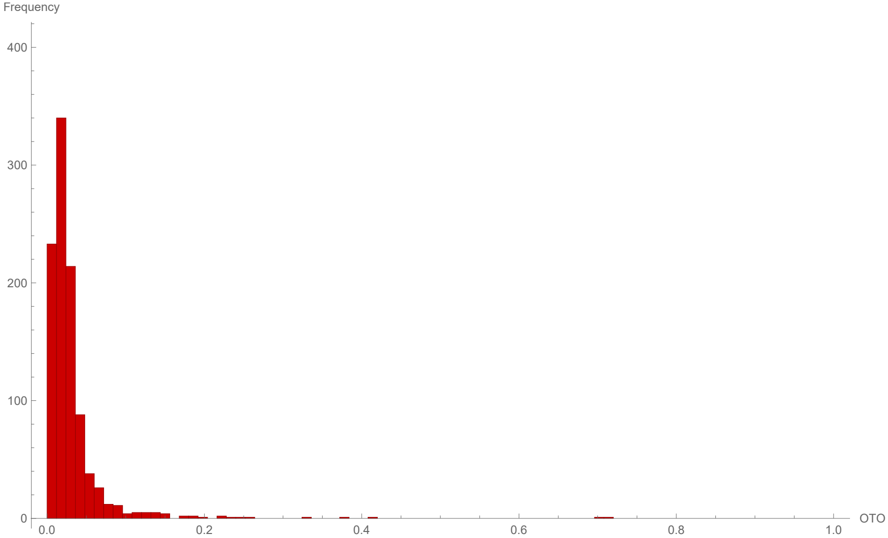
Random OTO in $SU(N)_2$, $N = 15$



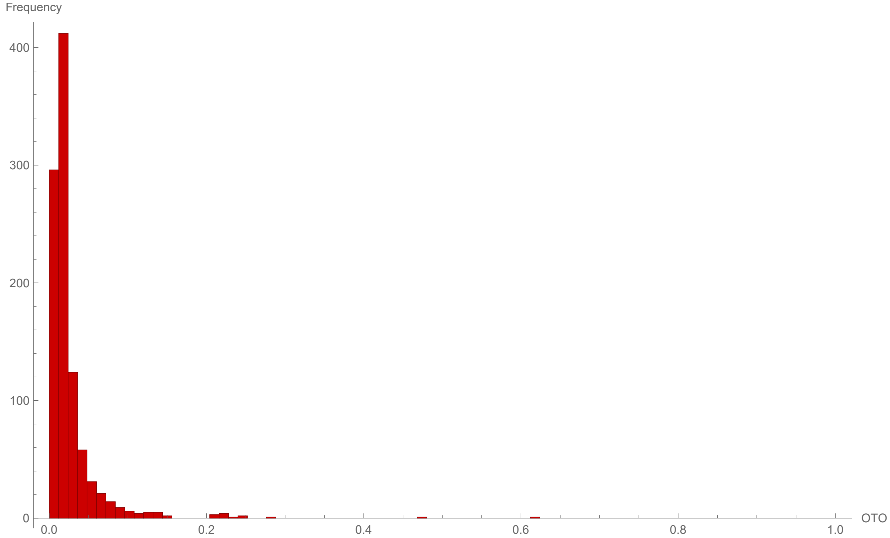
Random OTO in $SU(N)_2$, $N = 50$

Random OTO in $SU(N)_2$, $N = 100$

Random OTO in $SU(N)_2$, $N = 150$

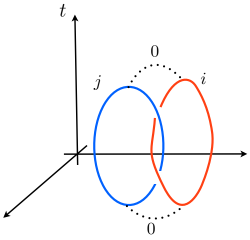


Random OTO in $SU(N)_2$, $N = 200$



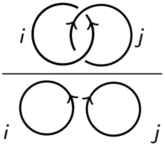
3d TQFT and anyon monodromy

Anyon worldlines



Hopf link

$C_{ij}^{\beta}(t) \rightarrow$



Caputa, Numasawa, AVO '16

Gu, Qi '16

This quantity can be measured in anyon interferometry experiments

Bonderson, Shtengel, Slingerland '06

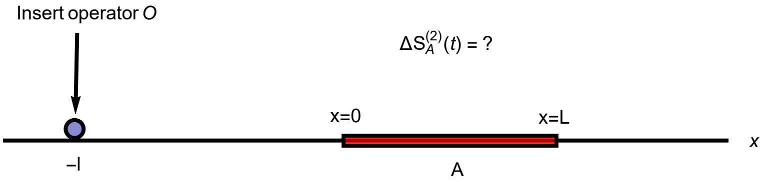
Connections between the buterfly effect and fractional statistics?

OTO for *Fractional Quantum Hall (FQH) fluids:*

Ising anyons, Fibonacci anyons, quantum magnets

Locally excited states

Physical setup:



- Start with the ground state of a CFT_2.
- At $t = 0$ insert an operator $\mathcal{O}(x, t)$ at $x = -l$.
- Smear the operator with ϵ.

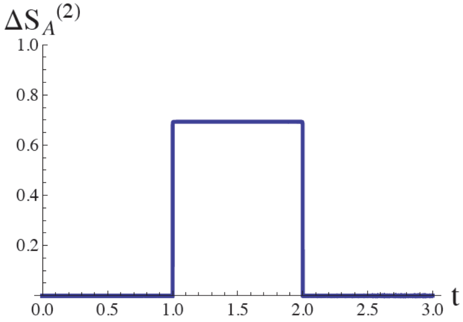
Purity $\Delta S_A^{(n)}(t)$ corresponds to a four-point function.

$$\text{Cross-ratios } z = z(l, L, t, \epsilon)$$

Purity and quantum dimension

For a Rational CFT

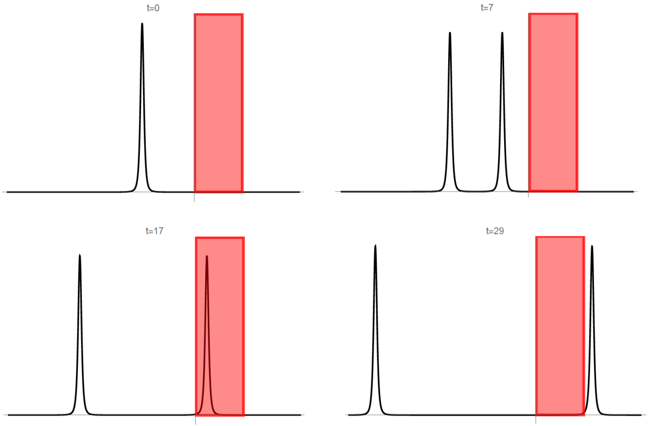
$$\Delta S_A^{(2)}(t) \simeq \begin{cases} 0 & t \notin [l, L+l] \\ \log(d_{\mathcal{O}}) & t \in [l, L+l] \end{cases}$$



He, Numasawa, Takayanagi, Watanabe '14; Caputa, AVO '15

Purity and quantum dimension

Quasiparticle interpretation

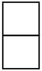


He, Numasawa, Takayanagi, Watanabe '14; Caputa, AVO '15

Purity in $SU(N)_k$ WZW

Some quantum dimensions

$$d_\mu = [N]$$



$$d_\mu = \frac{[N][N+1]}{[2]}$$

$$d_\mu = \frac{[N][N-1]}{[2]}$$

Removing the affine cut-off

$$\lim_{k\to\infty} [x] = x$$

Quantum dimension $\longrightarrow$ irrep dimension

The jump in purities $\Delta S_A^{(2)}$ will diverge in the large-c limit.

Purity in $SU(N)_k$ WZW, large-c

The jump in purity comes from the divergence

$$\mathcal{G}(z,\bar{z}) \to d_{\mathcal{O}}^{-1}\left((1-z)\,\bar{z}\right)^{-2h} + \dots$$

as $(z,\bar{z}) \to (1,0)$.

If we take $c \to \infty$ we find

$$\Delta S_A^{(2)}(t) \simeq 2h \log\left(\frac{2t}{\epsilon}\right) - \log(2)$$

This matches holographic computations.

Punchline

No quasiparticles $\to$ Memory loss $\to$ Entanglement scrambling

Caputa, Numasawa, AVO '16

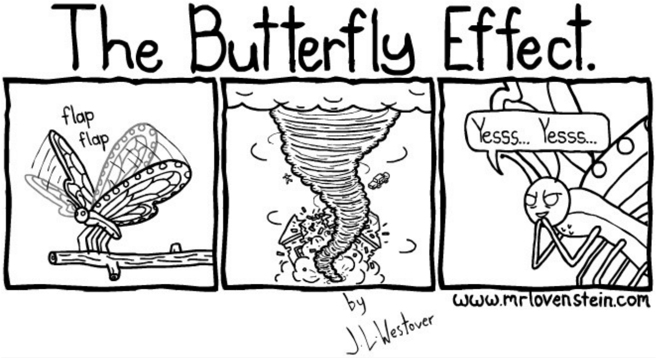
Conclusions

- OTO correlators serve to characterize the buterfly effect.
- Late OTO in RCFT are captured by the (measurable) monodromy scalar.
- There are non-chaotic channels even at large-c.
- The purity grows logarithmically (scrambles) at large-c.

Puzzles

- What is the precise relationship between chaos, holography and scrambling?
- Fractional statistics and the butterfly effect?
- Other settings: Higher-spin, coset models, GCFT, LCFT ...
- Could OTO and purity be explored experimentally?
- Interesting models with signatures of chaos: Spin chains, SYK, FKV, etc

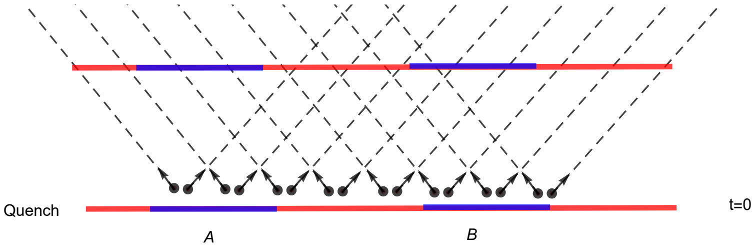
... and many other interesting questions?



Thank you for your attention!!

Entanglement scrambling

After introducing a quench



The mutual information

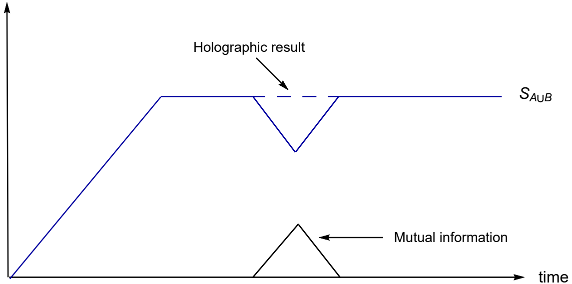
$$I(A, B) = S_A + S_B - S_{A \cup B}$$

Displays a long term memory effect

Calabrese, Cardy '05

Entanglement scrambling

Mutual information spike



Spike fades in holographic theories

Breakdown of quasiparticles $\longleftrightarrow$ Entanglement scrambling

Asplund, Bernamonti, Galli, Hartman '15