Trace Anomalies and their applications

I)General Properties of Conformal Field Theories
II)Trace Anomalies. Classification.
III)Constraints on RG flows: a-theorem
IV)The Geometry of Moduli Space

I) General Properties of CFT

A CFT has a conserved and traceless energy momentum tensor at the quantum level:

$$\partial_\mu T^{\mu\nu} = 0 \qquad\qquad T^\mu_\mu = 0$$

As a consequence one has conserved currents:

$$J_\mu \equiv \zeta^\nu T_{\mu\nu}$$

provided the variations ζ_μ satisfy:

$$\partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu = 2\frac{\eta_{\mu\nu}}{d}\partial_\alpha \zeta^\alpha$$

The allowed variations contain Translations,Lorentz,Dilations and Special Conformal Transformations forming the $SO(2,d)$ group.

In order to construct conformal invariant actions and to control the conformal Ward identities it is convenient to couple to an external , c-number metric $\gamma_{\mu\nu}$ the action $S(\gamma_{\mu\nu},\phi)$ depending on it and on the true quantum fields ϕ

One requires the invariance of the action under :

1) diffeo transformations:

$$\delta_\zeta \gamma_{\mu\nu}(x) = \nabla_\mu \zeta_\nu(x) + \nabla_\nu \zeta_\mu(x) \qquad \delta_\zeta \phi = \zeta^\mu \partial_\mu \phi$$

2)Weyl transformations :

$$\delta_\sigma \gamma_{\mu\nu}(x) = 2\sigma(x)\gamma_{\mu\nu}(x) \qquad \delta_\sigma \phi = -\Delta\sigma\phi$$

Then the generating functional W defined as:

$$e^{iW(\gamma_{\mu\nu})} \equiv \int \mathfrak{D}\phi e^{iS(\gamma_{\mu\nu},\phi)}$$

generates the correlators of energy momentum tensors with the required symmetry properties. Define the energy momentum tensor

$$T_{\mu\nu} \equiv \frac{\delta W}{\delta \gamma^{\mu\nu}}$$

1)From diffeo invariance we obtain :

$$\nabla^{\mu}\zeta^{\nu}\frac{\delta W}{\delta \gamma^{\mu\nu}} = 0$$ i.e. the conservation $$\partial_{\mu}T^{\mu\nu} = 0$$

2)From Weyl invariance we obtain:

$$-2\sigma\gamma^{\mu\nu}\frac{\delta W}{\delta\gamma^{\mu\nu}} = 0$$ i.e. the tracelessness $$T^{\mu}_{\mu} = 0$$

Example: free massless scalar in d-dimensions.
The diffeo and Weyl invariant action is:

$$S = \int d^d x\sqrt{\gamma}[\gamma^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{d-2}{4(d-1)}R\phi^2]$$

which gives a conserved and traceless("improved") energy-momentum tensor:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi - \frac{d-2}{4(d-1)}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2$$

The multiple correlators of energy momentum tensors are obtained expanding in powers of $h_{\mu\nu}$, the perturbation around flat metric:

$$\gamma_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Formally they obey the Ward identities following from conservation and tracelessness.

II)Trace Anomalies . Classification.

In even dimensions the generating functional has necessarily contributions which cannot preserve both symmetries . If one decides to preserve diffeo invariance then the Weyl variation is not vanishing:

$$\delta_\sigma W = \int d^d x \sqrt{\gamma} \sigma(x) \mathcal{A}$$

The anomaly $\mathcal{A}$ fulfils the conditions :

1)It is a polynomial in curvatures and covariant derivatives("local")
2)Its additional variation obeys the condition following from the Abelian algebra of the Weyl transformations $[\delta_{\sigma_1}, \delta_{\sigma_2}] = 0$
3)It cannot be obtained from the variation of a local action

This “cohomological” problem reflects general analytic properties of energy momentum tensor correlators: the correlators are nonlocal but the possible violations occur in local(polynomial in momenta) terms . Since one can always add to an amplitude polynomials in momenta(“subtractions”) there is an anomaly only if it cannot be removed by such a local term .

Example: in d=4 the analysis following rules 1)-3) gives two solutions:

$$\delta_\sigma W = c \int d^4x \sqrt{\gamma}\sigma C_{\mu\nu\tau\rho}C^{\mu\nu\tau\rho} - a \int d^4x \sqrt{\gamma}\sigma E_4$$

where $$C_{\mu\nu\tau\rho} \equiv R_{\mu\nu\tau\rho} + \dots$$

is the Weyl tensor transforming homogenously under Weyl transformations:

$$\delta_\sigma C_{\mu\nu\tau\rho} = 2\sigma C_{\mu\nu\tau\rho}$$

and E_4 is the Euler density in d=4:

$$E_4 \equiv R_{\mu\nu\tau\rho}R^{\mu\nu\tau\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

$\sqrt{\gamma}E_4$ is a total derivative and on a compact manifold

its integral has a topological meaning.

The two anomalous structures in d=4show that there are two types of Trace Anomalies with different mechanisms.The distinction is based on the behaviour of the anomaly when the Weyl parameter is a constant:

1)Type B(the c-anomaly): if the anomaly does not vanish for $\sigma = ct$ it signals that there is a violation of the dilation charge. Therefore there is a logarithmically divergent counterterm consistent with conformal invariance. The normalization of the anomaly is related by a Ward identity to the normalization of this counterterm .

Schematically in dimensional regularization:

$$W = c[\frac{1}{\epsilon}\int d^d x\sqrt{\gamma}C\frac{1}{\Box^{\epsilon}}C - \frac{1}{\epsilon}\int d^d x\sqrt{\gamma}C\frac{1}{\mu^{2\epsilon}}C]$$

The first term is Weyl invariant while the second counterterm breaks it explicitly .Therefore the Weyl variation gives:

$$\delta_\sigma W = -c\frac{1}{\epsilon}\int d^d x(-\epsilon\sigma)\sqrt{\gamma}C\frac{1}{\mu^{2\epsilon}}C \to c\int d^4 x\sqrt{\gamma}\sigma C^2$$

Generically every Weyl invariant made of Weyl tensors could lead to a type B anomaly.

2)Type A(the a-anomaly):for $\sigma = ct$ the integrated anomaly vanishes showing that dilation is not violated i.e. there is no true UV divergence . The mechanism for avoiding the UV divergence can be seen in x or p-space:

a)In every even dimension there is one contraction of Weyl tensors of the right engineering dimension which vanishes due to the Schouten identity:

$$A^{\mu_1\nu_1}_{[\mu_1\nu_1} \ldots A^{\mu_n\nu_n}_{\mu_n\nu_n]} \equiv 0 \,, \qquad m < n \qquad d = 2m$$

Such a contraction does not need a counterterm.The limit in dimensional regularization respecting diffeo-invariance leads to a violation of Weyl invariance since the number of independent invariant amplitudes changed .

b)Analysing the Ward identities for a correlator of $m+1$ energy-momentum tensors in a kinematical configuration where only one invariant q^2 is nonvanishing leads to the relations between two invariant amplitudes $A(q^2)$ (of dimension 0) and $B(q^2)$ (of dimension -2):

$$A(q^2) = 0 \qquad\qquad A(q^2) - q^2 B(q^2) = 0$$

which "clash" in complete analogy to the situation for chiral anomalies. For the Imaginary part one has the nontrivial solution:

$$Im A(q^2) = 0 \qquad\qquad Im B(q^2) = a\delta(q^2)$$

Following from analyticity the amplitudes are :

$$A(q^2) = a \qquad\qquad B = \frac{a}{q^2}$$

which are anomalous.

Example: d=2(Type A) “Polyakov”anomaly.

-in momentum space:

$$\langle T_{\mu\nu}(q) T_{\rho\sigma}(-q)\rangle = c(d)\Gamma(2-d/2)(q^2)^{-1+\epsilon}\,\epsilon^{-1}[\tfrac{d-1}{2}(P_{\mu\rho}P_{\nu\sigma}+P_{\mu\sigma}P_{\rho\nu})-P_{\mu\nu}P_{\rho\sigma}]\,,$$

$$P_{\mu\nu} \equiv \eta_{\mu\nu}q^2 - q_\mu q_\nu$$

Using the kinematical identity valid in d=2 :

$$P_{\mu\nu} = \tilde{q}_\mu \tilde{q}_\nu\,,\ \tilde{q}_\mu \equiv \epsilon_{\mu\nu} q^\nu$$

and then taking the limit to d=2 one obtains:

$$\langle T_{\mu\nu}\, T_{\rho\sigma}\rangle_{d=2} = c(2) q^{-2}\, P_{\mu\nu}\, P_{\rho\sigma}$$

-in configuration space:

In dimensional regularization the Weyl invariant generating functional is:

$$W(\gamma_{\mu\nu}) = c(d)\,\Gamma(1-\epsilon)\,1/\epsilon \int d^dx\,\sqrt{-g}\left[C^{\mu\nu}_{[\rho\sigma}\,\Box^{\epsilon-1}\,C^{\rho\sigma}_{\mu\nu]} - R^{\mu\nu}_{[\rho\sigma}\Box^{\epsilon-1}R^{\rho\sigma}_{\mu\nu]}\right]$$
$$= c(d)\,\Gamma(1-\epsilon)\,1/\epsilon \int d^dx\,\sqrt{-g}\left[(d-1)R_{\alpha\beta}\,\Box^{\epsilon-1}R^{\alpha\beta} - \frac{d}{4}\,R\,\Box^{\epsilon-1}R\right]$$

There is no need for a counterterm since the numerator vanishes; in order to take the limit to d=2 one uses the curvature relations and one obtains:

$$W_{d=2} = c(2)/2 \int d^2x\,\sqrt{-g}\,R\,\Box^{-1}\,R$$

In any even dimension there is one Type A anomaly given by the Euler density and beginning in d=4 an increasing number of Type B anomalies.

III)Constraints on RG flows:a-theorem.

A generic massive QFT can be understood as a “flow” between two CFT:the UV one and the IR which is the theory of the massless degrees of freedom . The two limiting CFT are characterized by their trace anomalies . Zamolodchikov proved that in d=2 for a unitary flow the d=2 anomaly decreases along the flow .
In d=4 Cardy conjectured that for unitary flows again the Type A anomalies obey an inequality i.e. :

$$a_{IR} < a_{UV}$$

The intuitive reason behind these inequalities is that the RG leads to a “thining” of the degrees of freedom and a measures it though the relation is not straightforward:
the value is 1 for a scalar ,31 for each helicity for spin 1,etc.
The inequality implies that unitary flows are irreversible.

There are two basic mechanisms for producing a “flow” between the UV and IR:

1)Spontaneous breaking of Conformal Invariance

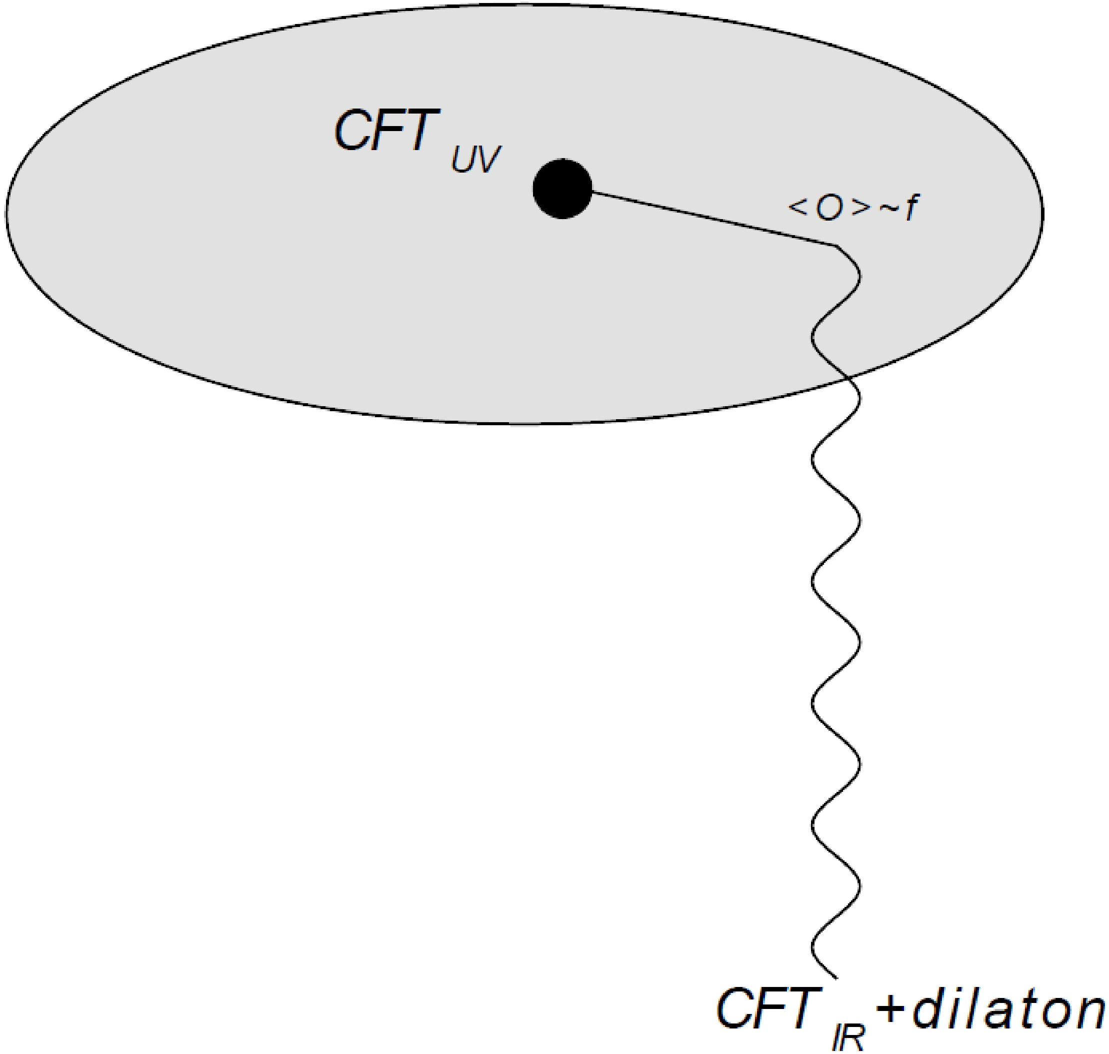
2)Perturbation of the UV CFT by a relevant or marginally relevant operator

An argument for the validity of the a-theorem is given for 1) and then a formal proof valid for both cases is sketched .

1)Spontaneous Breaking of Conformal Invariance .

In order for the Conformal Invariance to be spontaneously broken without breaking Poincare invariance a scalar operator $\mathcal{O}$ of dimension Δ should get a vacuum expectation value :

$$\langle 0|\mathcal{O}|0\rangle = v^{\Delta}$$



The breaking introduces a mass scale.The essential feature is that due to the Goldstone Theorem a massless dilaton with linear coupling to the e-m tensor joins the IR theory of the degrees of freedom which remained massless.

This general pattern is realized e.g. on the Coulomb Branch of N=4 SUSY gauge theory controllable in PT.

a) Goldstone Theorem for the Dilaton

When a symmetry is spontaneously broken the operatorial relations continue to be valid but being evaluated on a broken vacuum lead to the specific consequences of the spontaneous breaking .In particular consider the operatorial relation:

$$\int d^4x \partial^\mu T(J_\mu(x)\mathcal{O}(0)) = \Delta\mathcal{O}(0)$$

where $J_\mu \equiv x^\nu T_{\mu\nu}$ is the dilation current.

Consider an LSZ type decomposition:

$$< 0 \mid T(T_{\mu\nu}(x)\mathcal{O}(0)) \mid 0 >= \int dm^2 \rho(m^2) \int d^4k \frac{\eta_{\mu\nu}k^2 - k_\mu k_\nu}{k^2 - m^2 + i\epsilon} \exp ikx$$

Assuming the conservation for the e.m. tensor one obtains from the tracelessness:

$$\int d^4x \int dm^2 \rho(m^2) \frac{3k^2}{k^2 - m^2 + i\epsilon} \exp ikx = 0$$

which implies for the spectral density :

$$\rho(m^2) \sim \delta(m^2)$$

Therefore a 0-mass particle , the dilaton , should exist with a linear coupling to the e.m. tensor:

$$< 0 \mid T_{\mu\nu}(0) \mid G(k) >= f(-\eta_{\mu\nu} k^2 + k_\mu k_\nu)$$

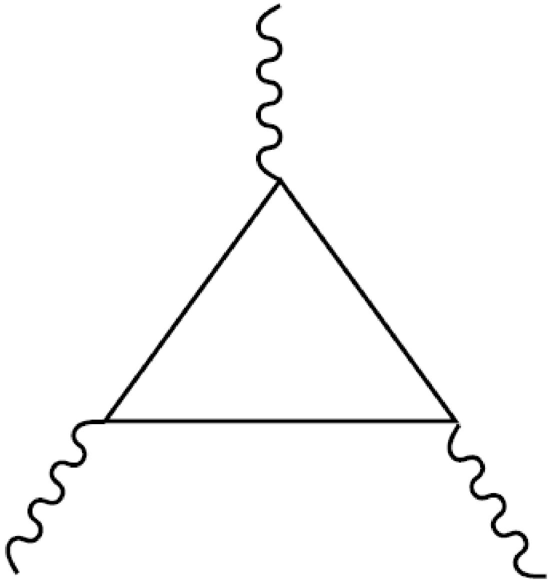
b) Trace Anomaly Matching in the Broken Phase .

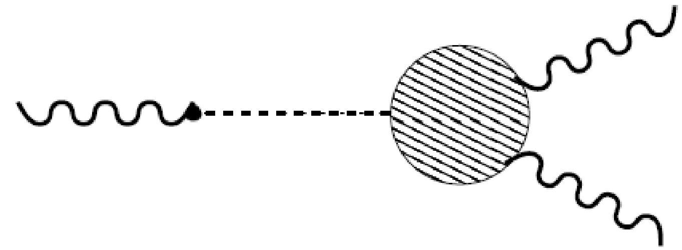
Since the trace anomaly relies on the operatorial Ward identity evaluated on a Poincare(only) invariant vacuum the unambiguous dimension -2 amplitude has an expression valid at any scale :

$$B(q^2) = \frac{a}{q^2}$$

For momenta much larger than the breaking scale the theory will be unbroken and therefore the normalization will be a_{UV}

On the other hand in the broken phase in order to produce singularities at vanishing momentum square one can have contributions from massless IR states of the theory or the pole contribution of the dilaton coupling, i.e the two type of diagrams:





Since one has to reproduce the same function it follows that the dilaton contributions of the anomaly (modulo one unit contribution where the dilaton behaves like a "normal" scalar) should be normalized to $(a_{UV} - a_{IR})$

We have to find therefore the expression of the dilaton-metric action whose Weyl variation reproduces the a-anomaly with this normalization .

The Weyl transformations of the fields are:

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu} e^{2\sigma} \qquad \text{and} \qquad \tau \to \tau + \frac{\sigma}{f}$$

The action whose Weyl variation is the normalized anomaly is:

$$-(a_{UV} - a_{IR}) \int d^4x \sqrt{\gamma} \left(\tau E_4 + 4\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) \partial_\mu \tau \partial_\nu \tau - 4(\partial\tau)^2 \Box \tau + 2(\partial\tau)^4 \right)$$

This action is universal :it follows unambiguously from anomaly matching and the Weyl and diffeo Ward identities. The terms which survive in flat background represent universal dilaton self interactions ("Wess-Zumino terms") normalized to $(a_{UV} - a_{IR})$

c)The Low Energy Theorem for Dilaton Scattering.

The universal Wess-Zumino term can be related to a physical process . We should first consider the additional , Weyl invariant terms in the dilaton effective action . These could be obtained from diffeoinvariant expressions constructed from the Weyl invariant metric :

$$\hat{\gamma}_{\mu\nu} \equiv \gamma_{\mu\nu} e^{-2f\tau}$$

In particular a term linear in $\hat{R}$ gives the Weyl invariant dilaton kinetic term:

$$f^2 \int d^4x \sqrt{\gamma} e^{-2\tau} [\gamma^{\mu\nu} \partial_\mu \tau \partial_\nu \tau + \frac{1}{6} R]$$

Quadratic in $\hat{R}$ terms could in principle modify the Wess-Zumino terms . Making however the allowed field redefinition

$$\phi \equiv f(1 - e^{-\tau})$$

we obtain the exact action up to four derivatives :

$$S = \int d^4x \sqrt{\gamma} \gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi +$$

$$+ (a_{UV} - a_{IR}) \int d^4x \sqrt{\gamma} [\frac{2}{f^2} \frac{(\partial\phi)^4}{(f - \phi)^2} + \Box\phi.....]$$

Therefore on mass shell we have an exact low energy theorem for the elastic dilaton scattering :

$$\mathcal{A}(s,t) = 2\frac{a_{UV} - a_{IR}}{f^2}(s^2 + t^2 + u^2) + \ldots$$

Considering the dispersion relation for the forward dilaton scattering amplitude :

$$\mathcal{A}(s,t=0) = \frac{1}{\pi}\int ds' \frac{Im\mathcal{A}(s',t=0)}{s'-s}$$

the low energy theorem translates into an exact sum rule :

$$a_{UV} - a_{IR} = \frac{f^4}{4\pi}\int ds' \frac{Im\mathcal{A}(s',t=0)}{(s')^3}$$

Using the positivity of the Imaginary part following from Unitarity this proves the a-Theorem:

$$a_{IR} < a_{UV}$$

Due to the linear coupling of the dilaton to the trace of the energy-momentum tensor the dilaton argument suggests that the general argument should involve a relation of four traces in the special kinematics . The general argument should be valid for both type of flows .

2)The Basic Sum Rule

Consider a massive unitary flow between two conformal theories :the UV and IR.

Then the correlator of four energy momentum tensors in the massive theory obeys:

$$\int_0^\infty ds \frac{ImS(s,M)}{s^3} = a_{UV} - a_{IR}$$

where

$$S(s,M) =< T[T_i^i(k_1)T_j^j(k_2)T_l^l(-k_1)T_m^m(-k_2)] >$$

and we are in the "forward kinematics " with

$s = (k_1 + k_2)^2$ and $k_1^2 = k_2^2 = 0$

$M =$ RG invariant mass scale

The a-Theorem follows immediately from the sum rule using the positivity of the integrand .

Analogous sum rules were used for chiral anomalies and by Cappelli et al for an alternative proof of Zamolodchikov's c-theorem in d=2.

The proof of the Sum Rule(Outline).

Consider the correlator of three energy momentum tensors in the massive theory. Decompose it in invariant amplitudes singling out the dimension -2 amplitude “responsible”

for the a-anomaly in the conformal theory:

$$\tilde{G}^{(3)}_{ij,kl,mn}(k_1,k_2,k_3) = B(s,M)A_{ij,kl}(k_1,k_2)F_{mn}(k_3) + \ldots$$

where $$F^{ij}(k) \equiv k^i k^j - k^2\eta^{ij}$$

and

$$
\begin{aligned}
A^{ij,kl}(k_1,k_2) &\equiv -\eta^{ij}\eta^{kl}(k_1\cdot k_2)^2 + k_1^k k_1^l k_2^i k_2^j \\
&- \frac{1}{2}(k_1\cdot k_2)\left(\eta^{ik}k_1^l k_2^j + \eta^{jk}k_1^l k_2^i + \eta^{il}k_1^k k_2^j + \eta^{jl}k_1^k k_2^i\right) \\
&- \frac{1}{2}\left(k_1^k k_1^i k_2^j k_2^l + k_1^k k_1^j k_2^i k_2^l + k_1^l k_1^i k_2^j k_2^k + k_1^l k_1^j k_2^i k_2^k\right) \\
&- \frac{1}{2}(k_1\cdot k_2)\left(\eta^{ik}k_1^j k_2^l + \eta^{jk}k_1^i k_2^l + \eta^{il}k_1^j k_2^k + \eta^{jl}k_1^i k_2^k\right) \\
&+ (k_1\cdot k_2)\left(\eta^{ij}k_2^k k_1^l + \eta^{ij}k_2^l k_1^k + \eta^{kl}k_1^i k_2^j + \eta^{kl}k_1^j k_2^i\right) \\
&+ k_1^i k_1^j k_2^k k_2^l + \frac{1}{2}(k_1\cdot k_2)^2\left(\eta^{ik}\eta^{jl} + \eta^{jk}\eta^{il}\right)
\end{aligned}
$$

for momenta which obey: $k_1^2 = k_2^2 = 0$

For $s \to \infty$ $B(s, M) \to -\frac{a_{UV}}{s}$

From the dispersion relation:

$$B(s, M) = \int_0^\infty ds' \frac{ImB(s', M)}{s' - s}$$

We obtain: $\int_0^\infty ds' ImB(s', M) = a_{UV}$

Since $ImB(s, M)$ contains $a_{IR}\delta(s)$

$$\int_0^\infty ds' \frac{Im[s'^3 B(s', M)]}{s'^3} = a_{UV} - a_{IR}$$

The four point function is related by diffeo invariance WI to the three point function:

$$\tilde{G}^{(4)}_{ij,kl,mn,pq}(k_1, k_2, -k_1, -k_2) =$$

$$= B(s, M)\Big(A_{ij,kl}(k_1, k_2) J_{mn,pq}(k_1, k_2) +$$

$$+ A_{mn,pq}(k_1, k_2) J_{ij,kl}(k_1, k_2)\Big) + ..$$

with :

$$J^{ij,kl}(k_1, k_2) \equiv \frac{1}{2}\left(\eta^{kl} k_2^i k_2^j + \eta^{ij} k_1^k k_1^l\right) - \frac{1}{4}\eta^{ij}\eta^{kl} k_1 \cdot k_2$$

$$- \frac{1}{4}\left(\eta^{jl} k_1^i k_2^k + \eta^{ik} k_1^j k_2^l + \eta^{jk} k_1^i k_2^l + \eta^{il} k_1^j k_2^k\right)$$

$$+ \frac{1}{4}\left(\eta^{ij} k_1^k k_2^l + \eta^{kl} k_1^i k_2^j + \eta^{ij} k_1^l k_2^k + \eta^{kl} k_1^j k_2^i\right)$$

$$- \frac{1}{4} \Big(\eta^{ik} k_2^j k_2^l + \eta^{jk} k_2^i k_2^l + \eta^{il} k_2^j k_2^k + \eta^{jl} k_2^i k_2^k +$$

$$+ \eta^{jl} k_1^i k_1^k + \eta^{il} k_1^j k_1^k + \eta^{jk} k_1^i k_1^l + \eta^{ik} k_1^j k_1^l \Big)$$

$$- \frac{1}{8} \Big(\eta^{ik} k_1^l k_2^j + \eta^{jk} k_1^l k_2^i + \eta^{il} k_1^k k_2^j + \eta^{jl} k_1^k k_2^i \Big)$$

$$+ \frac{3}{8} \left(\eta^{ik} \eta^{jl} + \eta^{il} \eta^{jk} \right) k_1 \cdot k_2$$

Taking the trace on all the indices we get:

$$\tilde{G}^{(4)}_{ij,kl,mn,pq} \eta^{ij} \eta^{kl} \eta^{mn} \eta^{pq} = s^3 B(s, M) + ...$$

One should consider the contributions to the sum rule of other invariant amplitudes .We distinguish between a)amplitudes present in the CFT and b)amplitudes present only in the massive theory ,being “killed” by the conformal Ward identities.

For a): besides the amplitude B related to the “a” anomaly there is an amplitude related to the “c” anomaly and one which is non-anomalous. The last two amplitudes vanish when more than one trace is taken and cannot contribute to the sum rule.

For b) we start with amplitudes $A(s, M)$ with positive dimensions $2r = 0, 2, 4$

Since the amplitudes cannot contribute in a CFT their imaginary parts should be subdominant in the UV and the IR:

$$ImA(s, M) \rightarrow s^{r-\epsilon} M^{2\epsilon} \quad \text{for} \quad s \rightarrow \infty$$

$$ImA(s, M) \rightarrow s^{r+\epsilon} M^{-2\epsilon} \quad \text{for} \quad s \rightarrow 0$$

It follows that we can write an unsubtracted dispersion relation for $\frac{\partial^r A}{\partial s^r}$

From the vanishing of the derivative in the IR :

$$\frac{\partial^r A}{\partial s^r} = 0 \qquad \text{at} \qquad s = 0$$

and the dispersion relation we obtain :

$$\int_0^\infty ds \frac{Im A(s, M)}{s^{r+1}} = 0$$

which is the condition that the amplitude does not contribute (through its descendent) to the sum rule .

For negative dimension $(= -2)$ amplitudes one repeats the argument used for the B-amplitude . Since there is no $1/s$ in the UV or IR one obtains the sum rule for B with 0 r.h.s. i.e. these amplitudes do not contribute either .

The suppressions of the Im parts do not need to be powerlike :a logarithmic one (like in QCD) allows the previous arguments to go through as well.

The possible additional contributions to the sum rule involve either four point functions which are descendants through the diffeo Ward identities from three point functions or contributions which “start” with four point functions .There are no trace anomalies which start with the four point functions . These type of contributions are either amplitudes which survive the conformal limit and therefore cannot contribute to traces or amplitudes which are suppressed in the conformal limit .For the second type an

analysis completely analogous to the one performed for the three point amplitudes shows that the suppression "kills" the contribution to the sum rule. In conclusion the sum rule gets its only contribution from the B amplitude .
The vanishing of the other contributions is after being integrated :locally the sum of all contributions is positive .
The convergence of the sum rule is guaranteed by the way the CFT in the UV and IR are approached from the massive theory:

In the UV:

for $s \to \infty$ $ImS(s, M) \to s^{2-\epsilon} M^{\epsilon}$

In the IR :

for $s \to 0$ $ImS(s, M) \to s^{2+\epsilon} M^{-\epsilon}$

The limits can be approached logarithmically e.g. in the first line one could have :

$$\frac{s^2}{(log(s/M^2))^{\epsilon}}$$

Further comments:

a)The sum rule is equivalent to the dilaton effective action argument since:

$$\frac{d^2 S(s=0,M)}{ds^2} = 2\int ds' \frac{ImS(s')}{s'^3}$$

b)For a spontaneously broken conformal theory the sum rule is valid for the couplings excluding the dilaton poles.

c)The general pattern in dimension d=2n is a sum rule for the amplitudes of 2n traces though the positivity is not obvious in d>4.

IV)The Geometry of the Moduli Space

(with J.Gomis,P-S.Nazgoul-Hsin,Z.Komargodski,N.Seiberg,S.Theisen)

1)Moduli

If the CFT has a set of scalar operators $\mathcal{O}_I$ of dimension d one can add them to the action :

$$\delta S = \sum_I \lambda_I \int d^d x \mathcal{O}_I$$

a)if their three point functions are vanishing no beta function will be produced and the theory stays conformal:truly marginal=Moduli
b)the 0-dimensional sources $\lambda_I(x)$ are reparametrization invariant in field space

2)The Geometry of Moduli Space . Classification of Moduli Anomalies .

Zamolodchikov proposed a metric $g_{IJ}(\lambda^K)$ on the moduli space:

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(y)\rangle_{\lambda^K} = \frac{g_{IJ}(\lambda^K)}{(x-y)^{2d}}$$

Kutasov clarified the role of the other geometric quantities i.e. connections,curvatures on moduli space. For d=2 if the CFT is the world sheet theory for a String Compactification the geometric quantities calculate terms in the target space effective action .

The essential feature of correlators of moduli in even dimensions is the presence of logarithms in momentum space . This leads to trace anomalies in joint correlators of moduli and energy momentum tensors i.e. in the generating functional depending on the metric and the moduli sources .

Therefore classifying the Weyl anomalies of this generating functional will give the general geometric structure on the moduli space. In addition to the usual rules for Weyl anomalies we should require also reparametrization invariance in the space of the sources $\lambda_I(x)$

In d=4 we get the list:

a) $$\delta_\sigma \log Z = -\frac{1}{192\pi^2}\int d^4x\sqrt{\gamma}\,\sigma\left(g_{IJ}\widehat{\Box}\lambda^I\widehat{\Box}\lambda^J - 2\,g_{IJ}\partial_\mu\lambda^I(R^{\mu\nu} - \frac{1}{3}\gamma^{\mu\nu}R)\partial_\nu\lambda^J\right)$$

where $$\widehat{\Box}\lambda^I = \Box\lambda^I + \Gamma^I_{JK}\partial^\mu\lambda^J\partial_\mu\lambda^K$$

The integrability condition is fulfilled and the tensor and connection correspond to the Zamolodchikov metric: a Ward identity connects the anomaly to the Zamolodchikov correlator.

b) $$\delta_\sigma \log Z = \int d^4x\,\sigma\sqrt{\gamma}\,c_{IJKL}\partial_\mu\lambda^I\partial^\mu\lambda^J\partial_\nu\lambda^K\partial^\nu\lambda^L$$

where c_{IJKL} is a new, in principle independent tensor on moduli space though a higher symmetry may relate it to g_{IJ}

c) $$\delta_\sigma log Z = \int d^4x\sqrt{\gamma}\sigma e_I \hat{\Delta}_4 \lambda^I$$

where $$\Delta_4 = \Box^2 + \frac{1}{3}\nabla^\mu R \nabla_\mu + 2\, R^{\mu\nu}\nabla_\mu \nabla_\nu - \frac{2}{3} R\Box$$

is the appropriately covariantized in source space FTPR operator transforming homogenously under Weyl transformations .

The difference between the structures a) and c) generates a new type of anomaly but its normalization vanishes in a unitary CFT.

d) In addition there are unnatural parity anomalies.

e) There are many cohomologically trivial terms e.g.:

$$\delta_\sigma log Z = \int d^4x\sqrt{\gamma}\Box\sigma(x) h_{IJ}\partial_\mu \lambda^I \partial^\mu \lambda^J \quad \text{or} \quad \int d^4x\sqrt{\gamma}\Box\sigma R f(\lambda^I)$$

related to the variation of the local terms

$$\int d^4x\sqrt{\gamma} R h_{IJ}\partial_\mu \lambda^I \partial^\mu \lambda^J \quad \text{and} \quad \int d^4x\sqrt{\gamma} R^2 f(\lambda^I)$$

3) Exact Partition Functions

Generically partition functions of a QFT on a compact manifold(e.g. with S_4 topology) are not universal . Regularization dependence shows up through local counterterms with arbitrary coefficients consistent with the symmetries .

With SUSY (N=2 in d=4) the moduli dependent part of the partition function becomes universal : the anomalous (nonlocal) action requires a completion by local terms fixed by the anomaly i.e. cohomologically trivial terms become nontrivial . The nonlocal action in superspace leads to local terms in ordinary space.

In order to study Weyl anomalies for a N=2 superconformal theory one has to couple it to a full N=2 supergravity background .

The moduli are in chiral(anti-chiral) multiplets as well as their sources $\Phi^A, \overline{\Phi}^{\overline{A}}$ which have 0- Weyl weight .

The Weyl parameter is imbedded into chiral(anti-chiral) superfields Σ , $\overline{\Sigma}$.

In N=2 superspace the anomaly has a simple, unique form :

$$\delta_\sigma log Z = \int d^4x\, d^4\theta\, d^4\overline{\theta}\, E\,(\Sigma + \overline{\Sigma}) K(\Phi^A, \overline{\Phi^A})$$

where E is the superfield analogue of the metric determinant and $K(\Phi^A, \overline{\Phi^A})$ is the "Kahler potential" on moduli space defined through the anomaly. Using the Kahler potential the other geometric quantities are defined :

the Kahler metric : $g_{I\overline{J}} = \partial_I \partial_{\overline{J}} K$

connection : $\Gamma^I_{JK} = g^{I\overline{L}} \partial_J \partial_K \partial_{\overline{L}} K$

and curvature: $\mathcal{R}_{I\overline{J}K\overline{L}} = \partial_I \partial_{\overline{J}} g_{K\overline{L}} - g^{M\overline{N}}\, \partial_I g_{K\overline{L}}\, \partial_{\overline{J}} g_{M\overline{N}}$

The expansion of the anomaly in components , normalized to the Zamolodchikov metric gives:

$$
\begin{aligned}
\delta_\Sigma \log Z = +\frac{1}{192\pi^2}\int d^4x\sqrt{g}\Big\{ & (\sigma+\overline{\sigma})\mathcal{R}_{I\overline{K}J\overline{L}}\nabla^\mu\phi^I\,\nabla_\mu\phi^J\,\nabla^\nu\overline{\phi}^{\overline{K}}\,\nabla_\nu\overline{\phi}^{\overline{L}} \\
& + (\sigma+\overline{\sigma})g_{I\overline{J}}\left(\widehat{\Box}\phi^I\,\widehat{\Box}\overline{\phi}^{\overline{J}} - 2\left(R^{\mu\nu}-\frac{1}{3}R\,\gamma^{\mu\nu}\right)\nabla_\mu\phi^I\,\nabla_\nu\overline{\phi}^{\overline{J}}\right) \\
& + \frac{1}{2}K\,\Box^2(\sigma+\overline{\sigma}) + \frac{1}{6}K\,\nabla^\mu R\,\nabla_\mu(\sigma+\overline{\sigma}) + K\,\left(R^{\mu\nu}-\frac{1}{3}\gamma^{\mu\nu}R\right)\nabla_\mu\nabla_\nu(\sigma+\overline{\sigma}) \\
& - 2\,g_{I\overline{J}}\,\nabla^\mu\phi^I\,\nabla^\nu\overline{\phi}^{\overline{J}}\,\nabla_\mu\nabla_\nu(\sigma+\overline{\sigma}) + g_{I\overline{J}}\left(\widehat{\nabla}^\mu\widehat{\nabla}^\nu\phi^I\,\nabla_\nu\overline{\phi}^{\overline{J}} - \widehat{\nabla}^\mu\widehat{\nabla}^\nu\overline{\phi}^{\overline{J}}\,\nabla_\nu\phi^I\right)\nabla_\mu(\sigma-\overline{\sigma}) \\
& - \frac{1}{2}\left(\widehat{\nabla}_I\widehat{\nabla}_J K\,\nabla^\mu\phi^I\nabla_\mu\phi^J - \widehat{\nabla}_{\overline{I}}\widehat{\nabla}_{\overline{J}}K\,\nabla^\mu\overline{\phi}^{\overline{I}}\nabla_\mu\overline{\phi}^{\overline{J}} + \nabla_I K\,\widehat{\Box}\phi^I - \nabla_{\overline{I}}K\,\widehat{\Box}\overline{\phi}^{\overline{I}}\right)\Box(\sigma-\overline{\sigma}) \\
& + \left(R^{\mu\nu}-\frac{1}{3}R\,\gamma^{\mu\nu}\right)\left(\nabla_I K\,\nabla_\mu\phi^I - \nabla_{\overline{I}}K\,\nabla_\mu\overline{\phi}^{\overline{I}}\right)\nabla_\nu(\sigma-\overline{\sigma})\Big\}
\end{aligned}
$$

One recognizes in □ (blue box) the Zamolodchikov anomaly.

In □ (red box) the four-index anomaly appears but N=2 fixed the tensor to be the curvature of the Zamolodchikov metric.

In $\square$ and after , all the terms are cohomologically trivial i.e. are variations of local terms . In particular the terms multiplying the Kahler potential integrate to :

$$\frac{1}{2}\square^2\sigma + \frac{1}{6}\nabla^\mu R\nabla_\mu\sigma + \left(R^{\mu\nu} - \frac{1}{3}\gamma^{\mu\nu}R\right)\nabla_\mu\nabla_\nu\sigma = \frac{1}{2}\Delta_4\sigma$$
$$= \delta_\sigma\left(\frac{1}{8}E_4 - \frac{1}{12}\square R + c\,C^2\right)$$

where C is the Weyl tensor , E_4 the Euler density and c an arbitrary constant .

Therefore if the metric on the compact manifold is conformally flat one can calculate exactly the partition function by taking the moduli sources to constants $\lambda, \bar{\lambda}$, the values of the lowest components of the superfield .

$$Z[S^4] = \left(\frac{r}{r_0}\right)^{-4a} e^{K/12}$$

4)Global Properties of the Moduli Space

The “Kahler potential” is defined by the Anomaly .Since the Anomaly is ambiguous up to local terms in superspace the Kahler potential inherits this ambiguity .

The local term in superspace has the form :

$$\int d^4x\, d^4\theta\, \mathcal{E} F(\Phi)\left(\Xi - W^{\alpha\beta}W_{\alpha\beta}\right) + \text{c.c.}$$

whose Weyl variation in components reduces to:

$$\frac{1}{192\pi^2}\int d^4x\,\sqrt{\gamma}\,(F\Delta_4\overline{\sigma} + \text{c.c.})$$

corresponding to the anomaly for a holomorphic function. Therefore the Anomaly defines a full Kahler structure on the Moduli Space.

The Kahler ambiguity has an interesting mapping into a Weyl variation of the Anomaly polynomial . Making a second Weyl variation on the Euler part of the Anomaly one obtains:

$$\delta_{\widetilde{\Sigma}}\delta_{\Sigma} \log Z = -\frac{a}{8\pi^2}\int d^4x\, d^4\theta\, \mathcal{E}\, \delta\Sigma\, \overline{\Delta}\, \delta\overline{\widetilde{\Sigma}} + \text{c.c.}\,.$$

while the Kahler shift produces a change :

$$\delta_F\delta_{\Sigma} \log Z = \frac{1}{192\pi^2}\int d^4x\, d^4\theta\, \mathcal{E}\, \delta\Sigma\, \overline{\Delta}\, \overline{F} + \text{c.c.}$$

Therefore making a Weyl transformation with a parameter correlated with a Kahler shift

$$\delta\widetilde{\Sigma} = \frac{1}{24a}F$$

leaves the Anomaly polynomial invariant.

This implies that the Moduli space is Kahler- Hodge:its second cohomology class is integral being related to the Weyl transformation which contains U(1) gauge transformations acting on integral charge fermions (Bagger-Witten) .

The Kahler class of the Moduli space however is plausibly trivial anyhow: the transition functions in different patches are the result of a specific regularization. It can produce different local terms in different patches but not an ambiguity in a given patch(holonomy) which would be a sign of a nontrivial Kahler structure .