Lecture 2
Scattering amplitudes and hidden symmetries in maximally supersymmetric gauge theory

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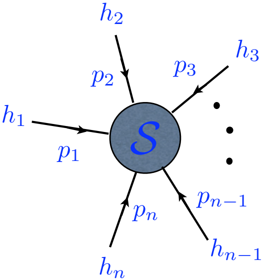
Scattering amplitudes in $\mathcal{N} = 4$ SYM

- Consider n-particle scattering amplitude



- Planar amplitudes most conveniently expressed in color ordered formalism:

$$A_n(\{p_i, h_i, a_i\}) = \delta^{(4)}(\sum_{i=1}^{n} p_i) \sum_{\sigma \in S_n/Z_n} g^{n-2} \, \mathrm{tr}[t^{a_{\sigma_1}} \ldots t^{a_{\sigma_n}}] \times \mathcal{A}_n(\{p_{\sigma_1}, h_{\sigma_1}\}, \ldots, \{p_{\sigma_1}, h_{\sigma_1}\}; \lambda = g^2 N)$$

$\mathcal{A}_n$: Color ordered amplitude. Color structure is stripped off.

Helicity of ith particle: $h_i = 0$ scalar, $h_i = \pm 1$ gluon, $h_i = \pm\frac{1}{2}$ gluino

Spinor helicity formalism

- Express momentum for massless particles via commuting spinors λ^α, $\tilde{\lambda}^{\dot{\alpha}}$:

$$p^{\alpha\dot{\alpha}} = (\sigma^\mu)^{\alpha\dot{\alpha}}\, p_\mu = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} = |\tilde{\lambda}^{\dot{\alpha}}]\,\langle\lambda^\alpha|$$

$$\Leftrightarrow \quad p_\mu\, p^\mu = \det p^{\alpha\dot{\alpha}} = 0$$

- Choice of helicity determines polarization vector ε^μ of external gluon

$$h = -1 \qquad \varepsilon_-^{\alpha\dot{\alpha}} = \frac{\lambda^\alpha \tilde{\mu}^{\dot{\alpha}}}{[\tilde{\lambda}\,\tilde{\mu}]} \qquad [\tilde{\lambda}\,\tilde{\mu}] := \epsilon^{\dot{\alpha}\dot{\beta}}\,\tilde{\lambda}_{\dot{\alpha}}\tilde{\mu}_{\dot{\beta}}$$

$$h = +1 \qquad \varepsilon_+^{\alpha\dot{\alpha}} = \frac{\mu^\alpha \tilde{\lambda}^{\dot{\alpha}}}{\langle\lambda\,\mu\rangle} \qquad \langle\lambda\,\mu\rangle := \epsilon_{\alpha\beta}\,\lambda^\alpha \mu^\beta$$

$\mu, \bar{\mu}$ arbitrary reference spinors.

- E.g. scalar products: $2\,p_1 \cdot p_2 = \langle\lambda_1\,\lambda_2\rangle\,[\tilde{\lambda}_2\,\tilde{\lambda}_1] = \langle 12\rangle\,[21]$
- Helicity assignments:

$$h(\lambda^\alpha) = -1/2 \qquad h(\tilde{\lambda}^{\dot{\alpha}}) = +1/2$$

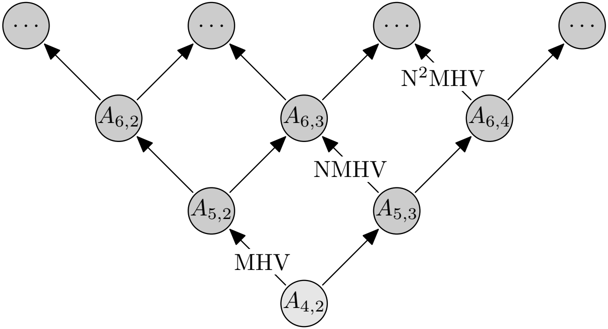
Gluon Amplitudes and Helicity Classification

Classify gluon amplitudes by # of helicity flips

- By SUSY Ward identities: $\mathcal{A}_n(1^+, 2^+, \ldots, n^+) = 0 = \mathcal{A}_n(1^-, 2^+, \ldots, n^+)$ true to all loops
- Maximally helicity violating (MHV) amplitudes

$$\mathcal{A}_n(1^+, \ldots, i^-, \ldots, j^-, \ldots n^+) = \delta^{(4)}\Big(\sum_i p_i\Big) \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$ [Parke, Taylor]

- Next-to-maximally helicity amplitudes (N^{k}MHV) have more involved structure!



$$A_{n,m} : g_+^{n-m} g_-^m$$

[Picture from T. McLoughlin]

Obvious symmetries

- Translations:

$$p^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \lambda_i^{\alpha}\, \tilde{\lambda}_i^{\dot{\alpha}}\,, \qquad \text{with} \quad p^{\alpha\dot{\alpha}}\, \mathcal{A}_n(\lambda_i, \tilde{\lambda}_i) = 0$$

  True in the distributional sense $p\,\delta(p) = 0$, thanks to $\mathcal{A}_n(\lambda_i, \tilde{\lambda}_i) = \delta^{(4)}(\sum_i p_i)\, A_n(\lambda_i, \tilde{\lambda}_i)$.

- Lorentz generators in the helicity spinor basis come in two pairs of symmetric rank-two tensors $l_{\alpha\beta}$ and $\bar{l}_{\dot{\alpha}\dot{\beta}}$ originating from the projections $L^{\mu\nu}\,(\sigma_{\mu\nu})_{\alpha\beta} = l_{\alpha\beta}$ and $L^{\mu\nu}\,(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} = \bar{l}_{\dot{\alpha}\dot{\beta}}$,

$$l_{\alpha\beta} = \sum_{i=1}^{n} \lambda_{i\,(\alpha}\, \partial_{i\,\beta)}\,, \quad \bar{l}_{\dot{\alpha}\dot{\beta}} = \sum_{i=1}^{n} \tilde{\lambda}_{i\,(\dot{\alpha}}\, \partial_{i\,\dot{\beta})}\,,$$

  where $\partial_{i\alpha} := \frac{\partial}{\partial \lambda_i^{\alpha}}$, $\partial_{i\dot{\alpha}} := \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}}$ and $r_{(\alpha\beta)} := \frac{1}{2}\,(r_{\alpha\beta} + r_{\beta\alpha})$ denotes symmetrization with unit weight.

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- Invariance of $\mathcal{A}_n(\lambda_i, \tilde{\lambda}_i)$ under Lorentz-transformations

$$l_{\alpha\beta}\, \mathcal{A}_n(\lambda_i, \tilde{\lambda}_i) = 0 = \bar{l}_{\dot{\alpha}\dot{\beta}}\, \mathcal{A}_n(\lambda_i, \tilde{\lambda}_i)$$

is manifest, as

$$l_{\alpha\beta}\, \langle jk \rangle = \sum_{i=1}^{n} \lambda_{i\,(\alpha}\, \partial_{i\,\beta)}\, \lambda_j^{\gamma} \lambda_{k\,\gamma} = \lambda_{j\,\alpha}\lambda_{k\,\beta} - \lambda_{j\,\beta}\lambda_{k\,\alpha} + (\alpha \leftrightarrow \beta) = 0\,.$$

Less obvious symmetries: Dilatations

- Scale transformations are generated by the dilatation operator d,

$$d = \sum_{i=1}^{n} (\frac{1}{2}\lambda_i^{\alpha}\,\partial_{i\,\alpha} + \frac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}}\,\partial_{i\,\dot{\alpha}} + d_0)\,, \qquad d_0 \in \mathbb{R}\,,$$

reflecting the dilatation weight $\frac{1}{2}$ of the λ_i and $\tilde{\lambda}_i$, i.e. $[d, \lambda_i] = \frac{1}{2}\,\lambda_i$ and $[d, \tilde{\lambda}_i] = \frac{1}{2}\,\tilde{\lambda}_i$. The constant d_0 is undetermined at this point. It may be fixed by requiring invariance of the MHV amplitudes

$$\mathcal{A}_n^{\mathsf{MHV}} = \delta^{(4)}(\sum_i p_i)\,\frac{\langle\lambda_s\,\lambda_t\rangle^4}{\langle 12\rangle\ldots\langle n1\rangle}\,.$$

The dilatation operator d simply measures the weight in units of mass of the amplitude it acts on plus $n\,d_0$

$$d\,\mathcal{A}_n = ([\mathcal{A}_n] + n\,d_0)\,\mathcal{A}_n\,.$$

Note the weights $[\delta^{(4)}(p)] = -4$, $[\langle\lambda_s\,\lambda_t\rangle^4] = 4$ and $[\frac{1}{\langle 12\rangle\ldots\langle n1\rangle}] = -n$, hence

$$d\,\mathcal{A}_n^{\mathsf{MHV}} = (-4 + 4 - n + n\,d_0)\,\mathcal{A}_n^{\mathsf{MHV}}\,,$$

which vanishes for $d_0 = 1$.

Less obvious symmetries: Special conformal transformation

- $k_{\alpha\dot\alpha} = \sum_{i=1}^{n} \partial_{i\,\alpha}\,\partial_{i\,\dot\alpha}$.
- Let's show invariance of the MHV amplitudes:

$$
\begin{aligned}
k_{\alpha\dot\alpha}\,\mathcal{A}_n^{\mathsf{MHV}} &= \sum_{i=1}^{n} \frac{\partial}{\partial\lambda_i^\alpha}\,\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}\,(\delta^{(4)}(p)\,A_n^{\mathsf{MHV}}) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial\lambda_i^\alpha}\left(\frac{\partial p^{\beta\dot\beta}}{\partial\tilde\lambda_i^{\dot\alpha}}\,\left(\frac{\partial}{\partial p^{\beta\dot\beta}}\,\delta^{(4)}(p)\right)\,A_n^{\mathsf{MHV}}\right) \\
&= \left[(n\,\frac{\partial}{\partial p^{\alpha\dot\alpha}} + p^{\beta\dot\beta}\,\frac{\partial}{\partial p^{\beta\dot\alpha}}\,\frac{\partial}{\partial p^{\alpha\dot\beta}})\,\delta^{(4)}(p)\right]\,A_n^{\mathsf{MHV}} \\
&\quad + (\frac{\partial\,\delta^{(4)}(p)}{\partial p^{\beta\dot\alpha}})\,\sum_{i=1}^{n}\lambda_i^\beta\,\frac{\partial}{\partial\lambda_i^\alpha}\,A_n^{\mathsf{MHV}}\,.
\end{aligned}
$$

For last term: Note the relation

$$
\sum_{i=1}^{n}\lambda_{i\,\alpha}\,\partial_{i\,\beta} = \sum_{i=1}^{n}\lambda_{i\,(\alpha}\,\partial_{i\,\beta)} + \frac{1}{2}\,\epsilon_{\alpha\beta}\,\sum_i \lambda_i^\gamma\,\partial_{i\,\gamma}\,,
$$

which follows from decomposing the l.h.s. in a symmetric and anti-symmetric piece. The blue term is the Lorentz generator $l_{\alpha\beta}$ which annihilates A_n^{MHV}.

The remaining operator just counts the # of λ's

$$\sum_{i=1}^{n} \lambda_i^{\beta} \, \frac{\partial}{\partial \lambda_i^{\alpha}} \, A_n^{\mathsf{MHV}} = \frac{1}{2} \, \delta_{\alpha}^{\beta} \, \sum_{i} \lambda_i^{\delta} \, \partial_{i\,\delta} \, A_n^{\mathsf{MHV}} = (4-n) \, A_n^{\mathsf{MHV}} \, .$$

Hence

$$k_{\alpha\dot\alpha} \, \mathcal{A}_n^{\mathsf{MHV}} = \Big[(4 \, \frac{\partial}{\partial p^{\alpha\dot\alpha}} + p^{\beta\dot\beta} \, \frac{\partial}{\partial p^{\beta\dot\alpha}} \, \frac{\partial}{\partial p^{\alpha\dot\beta}}) \, \delta^{(4)}(p) \Big] \, A_n^{\mathsf{MHV}} \, .$$

Indeed in a distributional sense we have

$$p^{\beta\dot\beta} \, \frac{\partial}{\partial p^{\beta\dot\alpha}} \, \frac{\partial}{\partial p^{\alpha\dot\beta}} \, \delta^{(4)}(p) = -4 \, \frac{\partial}{\partial p^{\alpha\dot\alpha}} \, \delta^{(4)}(p) \, ,$$

which one sees by integrating the second derivative expression against a test function $F(p)$,

$$\begin{aligned} \int d^4p \, F(p) \, p^{\beta\dot\beta} \, \frac{\partial}{\partial p^{\beta\dot\alpha}} \, \frac{\partial}{\partial p^{\alpha\dot\beta}} \, \delta^{(4)}(p) &= \int d^4p \Big(\Big[\frac{\partial}{\partial p^{\beta\dot\alpha}} \, F(p) \Big] 2 \, \delta_{\alpha}^{\beta} + \Big[\frac{\partial}{\partial p^{\alpha\dot\beta}} \, F(p) \Big] 2 \, \delta_{\dot\alpha}^{\dot\beta} \Big) \\ &= 4 \int d^4p \, [\frac{\partial}{\partial p^{\alpha\dot\alpha}} \, F(p) \,] \, \delta^{(4)}(p) \, . \end{aligned}$$

This proves the Poincare and conformal invariance of A_n^{MHV}.

On-shell superspace

- Augment λ_i^α and $\tilde{\lambda}_i^{\dot{\alpha}}$ by Graßmann-odd variables $\eta_i^A \quad A = 1, 2, 3, 4$ [Nair]
- On-shell superspace $(\lambda_i^\alpha, \tilde{\lambda}^{\dot{\alpha}}, \eta_i^A)$ with on-shell superfield:

$$\varphi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^-(p)$$

- Superamplitudes: $\left\langle \varphi(\lambda_1, \tilde{\lambda}_1, \eta_1)\, \varphi(\lambda_2, \tilde{\lambda}_2, \eta_2) \ldots \varphi(\lambda_n, \tilde{\lambda}_n, \eta_n) \right\rangle$
  Packages all n-parton gluon$^\pm$-gluino$^{\pm 1/2}$-scalar amplitudes
- General form of **tree superamplitudes**:

$$\mathcal{A}_n = \frac{\delta^{(4)}(\sum_i \lambda_i \tilde{\lambda}_i)\, \delta^{(8)}(\sum_i \lambda_i \eta_i)}{\langle 12\rangle \langle 23\rangle \ldots \langle n1\rangle} \mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$$

Conservation of super-momentum: $\delta^{(8)}(\sum_i \lambda^\alpha \eta_i^A) = (\sum_i \lambda^\alpha \eta_i^A)^8$

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Superamplitudes and N^{k}MHV expansion

$$\mathcal{A}_n(\{\lambda_i,\tilde\lambda_i,\eta_i\}) = \frac{\delta^{(4)}(\sum_i \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha})\,\delta^{(8)}(\sum_i \lambda_i^\alpha \eta_i^A)}{\langle 1,2\rangle\,\langle 2,3\rangle \ldots \langle n,1\rangle}\,\mathcal{P}_n(\{\lambda_i,\tilde\lambda_i,\eta_i\})$$

- η-expansion of $\mathcal{P}_n$ yields N^{k}MHV-classification of superamps as helicity $h(\eta) = -1/2$

$$\boxed{\mathcal{P}_n = 1 + \eta^4\,\mathcal{P}_n^{\text{NMHV}}(\lambda,\tilde\lambda) + \eta^8\,\mathcal{P}_n^{\text{NNMHV}}(\lambda,\tilde\lambda) + \ldots + \eta^{4n-8}\,\mathcal{P}_n^{\overline{\text{MHV}}}(\lambda,\tilde\lambda)}$$

- Expansion in powers of $\eta^4 := \epsilon_{abcd}\eta^a\eta^b\eta^c\eta^d$ due to R-symmetry $R^a{}_b$ invariance

The full $\mathfrak{su}(2,2|4)$ symmetry

- Superamplitude: $(i = 1, \ldots, n)$

$$\mathcal{A}_n^{\text{tree}}(\{\lambda_i, \tilde\lambda_i, \eta_i\}) = \frac{\delta^{(4)}(\sum_i \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha})\, \delta^{(8)}(\sum_i \lambda_i^\alpha \eta_i^A)}{\langle 12\rangle \langle 23\rangle \ldots \langle n1\rangle}\, \mathcal{P}_n(\{\lambda_i, \tilde\lambda_i, \eta_i\})$$

- Representation of $\mathfrak{su}(2,2|4)$ generators in **on-shell superspace**, e.g. [Witten]

$$p^{\alpha\dot\alpha} = \sum_{i=1}^n \lambda_i^\alpha\, \tilde\lambda_i^{\dot\alpha} \qquad q^{\alpha\, A} = \sum_{i=1}^n \lambda_i^\alpha\, \eta_i^A \qquad \Rightarrow \text{ obvious symmetries}$$

$$k_{\alpha\dot\alpha} = \sum_{i=1}^n \frac{\partial}{\partial\lambda_i^\alpha}\, \frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}} \qquad s_{\alpha\, A} = \sum_{i=1}^n \frac{\partial}{\partial\lambda_i^\alpha}\, \frac{\partial}{\partial\eta_i^A} \qquad \Rightarrow \text{ less obvious sym}$$

- Invariance: $\{\, p, k, m, \bar m, d, r, q, \bar q, s, \bar s, c_i \,\} \circ \mathcal{A}_n^{\text{tree}}(\{\lambda_i^\alpha, \tilde\lambda_i^{\dot\alpha}, \eta_i^A\}) = 0$

- N.B.: **Local** invariance $\quad h_i\, \mathcal{A}_n = 1 \cdot \mathcal{A}_n$

Helicity operator: $\quad h_i = -\tfrac{1}{2}\, \lambda_i^\alpha\, \partial_{i\,\alpha} + \tfrac{1}{2}\, \tilde\lambda_i^{\dot\alpha}\, \partial_{i\,\dot\alpha} + \tfrac{1}{2}\, \eta_i^A\, \partial_{i\,A} = 1 - c_i$

$\mathfrak{su}(2,2|4)$ invariance

- The representation of the $\mathfrak{su}(2,2|4)$ generators in on-shell superspace $(\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)$:

$$p^{\dot{\alpha}\alpha} = \sum_i \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^\alpha \,, \qquad k_{\alpha\dot{\alpha}} = \sum_i \partial_{i\alpha}\partial_{i\dot{\alpha}} \,,$$

$$\bar{l}_{\dot{\alpha}\dot{\beta}} = \sum_i \tilde{\lambda}_{i(\dot{\alpha}}\partial_{i\dot{\beta})}, \qquad l_{\alpha\beta} = \sum_i \lambda_{i(\alpha}\partial_{i\beta)} \,,$$

$$d = \sum_i [\tfrac{1}{2}\lambda_i^\alpha \partial_{i\alpha} + \tfrac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}}\partial_{i\dot{\alpha}} + 1], \qquad r^A{}_B = \sum_i [-\eta_i^A \partial_{iB} + \tfrac{1}{4}\delta_B^A \eta_i^C \partial_{iC}] \,,$$

$$q^{\alpha A} = \sum_i \lambda_i^\alpha \eta_i^A \,, \qquad \bar{q}_A^{\dot{\alpha}} = \sum_i \tilde{\lambda}_i^{\dot{\alpha}} \partial_{iA} \,,$$

$$s_{\alpha A} = \sum_i \partial_{i\alpha}\partial_{iA}, \qquad \bar{s}_{\dot{\alpha}}^A = \sum_i \eta_i^A \partial_{i\dot{\alpha}} \,,$$

$$\partial_{i\alpha} := \frac{\partial}{\partial \lambda_i^\alpha} \qquad \partial_{iA} := \frac{\partial}{\partial \eta_i^A} \,.$$

Dual conformal and Yangian symmetries

- Superconformal + Dual superconformal algebra [Drummod,Henn,Korchemsky,Sokatchev]

  $= $ Yangian $Y[\mathfrak{psu}(2,2|4)]$ algebra [Drummond, Henn, JP]

$$[J_a^{(0)}, J_b^{(0)}\} = f_{ab}{}^c\, J_c^{(0)}$$ conventional superconformal symmetry

$$[J_a^{(0)}, J_b^{(1)}\} = f_{ab}{}^c\, J_c^{(1)}$$ from dual conformal symmetry

$$[J_a^{(1)}, J_b^{(1)}\} = f_{ab}{}^c\, J_c^{(2)} + g_{ab}(J^{(0)}, J^{(1)})$$

$$\vdots$$ and super Serre relations [Dolan,Nappi,Witten]

- Coproducts:

$$\Delta(J_a^{(0)}) = J_a^{(0)} \otimes 1 + 1 \otimes J_a^{(0)} \qquad \Delta(J_a^{(1)}) = J_a^{(1)} \otimes 1 + 1 \otimes J_a^{(1)} + f_a^{cb}\, J_b^{(0)} \otimes J_c^{(0)}$$

- Or explicitly

Local generators $$J_a^{(0)} = \sum_{i=1}^{n} J_{a,i}^{(0)}$$

Nonlocal generators $$J_a^{(1)} = \sum_{i=1}^{n} J_{a,i}^{(1)} + f^{cb}{}_a \sum_{1<j<i<n} J_{i,b}^{(0)}\, J_{j,c}^{(0)}$$

Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ SYM

- Evaluation representation $J^{(1)}_{a,i} = u_i\, J^{(0)}_{a,i}$

$$J^{(1)}_a = \sum_{i=1}^{n} u_i\, J^{(0)}_{a,i} + f^{cb}{}_a \sum_{1<j<i<n} J^{(0)}_{i,b}\, J^{(0)}_{j,c}$$

- For tree-level superamplitudes $u_i = 0$ (trivial evaluation representation)
- Explicit example: Bosonic invariance $\boxed{p^{(1)}_{\alpha\dot\alpha}\,\mathcal{A}_n = 0}$ with

$$\begin{aligned} p^{(1)}_{\alpha\dot\alpha} &= \tfrac{1}{2}(l + \bar{l} - d) \otimes p + \bar{q} \otimes q \\ &= \tfrac{1}{2} \sum_{i<j} (l_{i,\,\alpha}{}^{\gamma} \delta^{\dot\gamma}_{\dot\alpha} + \bar{l}_{i,\,\dot\alpha}{}^{\dot\gamma} \delta^{\gamma}_{\alpha} - d_i\, \delta^{\gamma}_{\alpha} \delta^{\dot\gamma}_{\dot\alpha})\, p_{j,\,\gamma\dot\gamma} + \bar{q}_{i,\,\dot\alpha C}\, q^{C}_{j,\alpha} - (i \leftrightarrow j) \end{aligned}$$

- In fact $J^{(0)}_a$ and $p^{(1)}$ generate all of $Y[\mathfrak{psu}(2,2|4)]$

Proof of Yangian invariance for MHV super-amplitudes

$$p^{(1)}_{\alpha\dot\alpha}\,\mathcal{A}_n^{MHV} = 0$$

$$p^{(1)}_{\alpha\dot\alpha} = \tfrac{1}{2}\sum_{i<j}(l_{i,\,\alpha}{}^{\gamma}\delta^{\dot\gamma}_{\dot\alpha} + \bar{l}_{i,\,\dot\alpha}{}^{\dot\gamma}\delta^{\gamma}_{\alpha} - d_i\,\delta^{\gamma}_{\alpha}\delta^{\dot\gamma}_{\dot\alpha})\,p_{j,\,\gamma\dot\gamma} + \bar{q}_{i,\,\dot\alpha C}\,q^{C}_{j,\alpha} - (i \leftrightarrow j)$$

This is a non-local single derivative operator.

- Action on delta-function cancels straightforwardly and one is left with

$$p^{(1)}_{\alpha\dot\alpha}\,\frac{1}{\langle 12\rangle\ldots\langle n1\rangle} = \sum_{j<k}(\lambda_{j\alpha}\,\tilde\lambda_{k\dot\alpha}\,\langle k\partial_j\rangle - \lambda_{k\dot\alpha}\,\tilde\lambda_{k\alpha}\,d_j)\frac{1}{\langle 12\rangle\ldots\langle n1\rangle} - (j \leftrightarrow k)$$

- Standard manipulations and shifts in summation variables yield

$$p^{(1)}_{\alpha\dot\alpha}\,\mathcal{A}_n^{MHV} = \frac{\lambda_{1\,\gamma}\lambda_{n\,\alpha} + \lambda_{1\,\alpha}\lambda_{n\,\gamma}}{\langle n1\rangle}\,P^{\gamma}{}_{\dot\alpha}\,\mathcal{A}_n^{MHV} = 0$$

Cyclicity?

- While amplitudes are cyclic the level one generators are not:

$$J_a^{(1)} = fbc_a \sum_{1<j<i<n} J_{i,b}^{(0)} J_{j,c}^{(0)} \qquad \tilde{J}_a^{(1)} = f_a^{bc} \sum_{2<j<i<n+1} J_{i,b}^{(0)} J_{j,c}^{(0)}$$

- Difference of the two

$$J_a^{(1)} - \tilde{J}_a^{(1)} = f_a^{bc} f_{bc}^{d} J_{1,d} - f_a^{bc} J_1^b J^c$$

  Acting on an amplitude the second term vanishes by superconformal invariance. First term vanishes for $\mathfrak{psu}(2,2|4)$ as its dual coxeter number is zero.

- This prooves the Yangian invariance of the tree-level MHV amplitudes
- General proof of Yangian invariance of superamplitudes works via super-BCFW recursion.
- The Yangian symetry may be extended to loop-order, and in fact constrain amplitudes as well as integrands of amplitudes. Again sees deformations of the generators. Yet, the status is not satisfactory. Problem: IR divergencies need a regulator which breaks conformal and Yangian symmetry.

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