Lecture 1

Yangian Symmetry and the Dilatation operator

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56 Cracow School for Theoretical Physics, Zakopane May 2016

Before we begin ...

Integrability in Gauge and String Theory 2016

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Campus Adlershof

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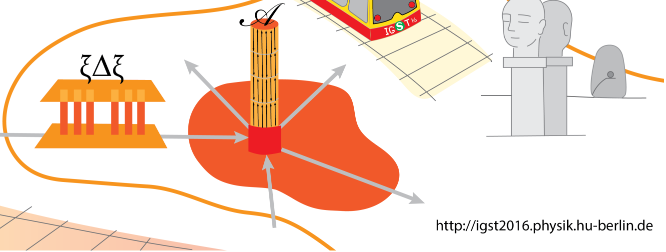

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http://igst2016.physik.hu-berlin.de

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Mathematica Summer School on Theoretical Physics

Random Matrix Theory

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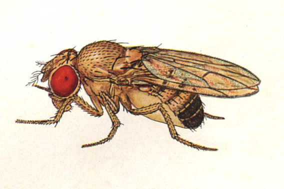
Recap

$\mathcal{N}=4$ super Yang-Mills



drosophila melanogaster

$\mathcal{N}=4$ SYM - The drosophila of quantum field theory

Field content of $\mathcal{N}=4$ SYM:

- One gauge field/gluon
  $A_\mu = A_\mu^n T^n$ $\quad$ T^n : $\mathrm{SU}(N)$ generators
- Six real scalars $\quad$ $\phi^i = \phi^{i,n} T^n$
- Four complex fermions $\quad$ $\psi^{a\alpha} = \psi^{a\alpha,n} T^n \,,\; \bar{\psi}_a^{\dot{\alpha}} = \bar{\psi}_a^{\dot{\alpha},n} T^n$

All fields transform in the adjoint rep of $\mathrm{SU}(N)$: $\quad U(x) = e^{i\xi^n(x)\, T^n}$

$$A_\mu \to U(x)\left(A_\mu + \partial_\mu\right) U^\dagger(x) \qquad \left(\phi^i, \psi^{a\alpha}, \bar{\psi}_a^{\dot{\alpha}}\right) \to U(x)\left(\phi^i, \psi^{a\alpha}, \bar{\psi}_a^{\dot{\alpha}}\right) U^\dagger(x)$$

Action:

$$S_{\mathcal{N}=4} = \frac{1}{g_{\mathsf{YM}}^2} \int \mathrm{d}^4 x \;\; \mathrm{Tr}\Big\{ -\tfrac{1}{2} F_{\mu\nu}^2 - \left(D_\mu \phi^i\right)^2 - \tfrac{1}{2}\left[\phi^i, \phi^j\right]\left[\phi_i, \phi_j\right] - 2\, \bar{\psi}_a^{\dot{\alpha}} \sigma_{\mu,\alpha\dot{\alpha}} D^\mu \psi^{a\alpha}$$
$$- \psi^{a\alpha} \Sigma^i_{ab} \varepsilon_{\alpha\beta}\left[\phi_i, \psi^{b\beta}\right] - \bar{\psi}_a^{\dot{\alpha}} \bar{\Sigma}^{i,ab} \varepsilon_{\dot{\alpha}\dot{\beta}}\left[\phi_i, \bar{\psi}_d^{\dot{\beta}}\right] \Big\}$$

$\mathcal{N}=4$ SYM has two freely tunable parameters $(N,\, g_{\mathsf{YM}})$ and we shall consider the theory in the planar limit $\left(N \to \infty\,,\; g_{\mathsf{YM}} \to 0 \text{ with } \lambda := g_{\mathsf{YM}}^2 N = \text{const.}\right)$.

$\mathcal{N}=4$ SYM - Symmetries

Global symmetries of the classical theory:

Symmetry	Generators	Algebra	
Translations	P^{μ}		
Lorentz-trans.	$\mathrm{M}^{\mu\nu}$	$\mathfrak{so}(2,4) \simeq$	
conf. boosts	K^{μ}	$\mathfrak{su}(2,2)$	
Dilatations	D		$\mathfrak{psu}(2,2\vert 4)$
Supertranslations	$\mathrm{Q}_{a\alpha}, \bar{\mathrm{Q}}^{a}_{\dot{\alpha}}$		
Superboosts	$\mathrm{S}^{a\alpha}, \bar{\mathrm{S}}^{\dot{\alpha}}_{a}$		
R-Symmetry	$\mathrm{R}^{a}{}_{b}$	$\mathfrak{su}(4)$	

Symmetries of the quantized theory:

$\beta(g_{\mathsf{YM}}) = 0 \quad \rightarrow$ Symmetry algebra $\mathfrak{psu}(2,2|4)$ is unbroken

Yangian Symmetry: A hallmark of integrability

The Yangian algebra $Y[\mathfrak{g}]$ of a semi-simple Lie algebra $\mathfrak{g}$ is a quantum algebra spanned by [Drinfeld]

$$\text{Level-0: } \{\mathrm{J}^{\kappa}\} \qquad \text{Level-1: } \{\widehat{\mathrm{J}}^{\kappa}\},$$

satisfying the following axioms

(1) Lie Algebra $\quad [\mathrm{J}^{\kappa}, \mathrm{J}^{\rho}] = f^{\kappa\rho}{}_{\delta}\, \mathrm{J}^{\delta}$

(2) Level-one Relation $\quad [\mathrm{J}^{\kappa}, \widehat{\mathrm{J}}^{\rho}] = f^{\kappa\rho}{}_{\delta}\, \widehat{\mathrm{J}}^{\delta}$

(3) Serre Relation $\quad [\mathrm{J}^{\kappa}, [\widehat{\mathrm{J}}^{\rho}, \widehat{\mathrm{J}}^{\delta}]] + \text{cyclic} = f^{\kappa\mu}{}_{\sigma} f^{\rho\nu}{}_{\omega} f^{\delta\eta}{}_{\gamma} f_{\mu\nu\eta}\, \mathrm{J}^{(\sigma}\mathrm{J}^{\omega}\mathrm{J}^{\gamma)}$

$Y(\mathfrak{g})$ is an infinite-dimensional, graded algebra.

$$[\widehat{\mathrm{J}}^{\kappa}, \widehat{\mathrm{J}}^{\rho}] = f^{\kappa\rho}{}_{\delta}\, \widehat{\widehat{\mathrm{J}}}^{\delta} + X^{\kappa\rho}(\mathrm{J}, \widehat{\mathrm{J}})$$

Coproduct:

$$\Delta(\mathrm{J}^{\kappa}) = \mathrm{J}^{\kappa} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{J}^{\kappa} \qquad \Delta(\widehat{\mathrm{J}}^{\kappa}) = \widehat{\mathrm{J}}^{\kappa} \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{\mathrm{J}}^{\kappa} + \tfrac{1}{2} f^{\kappa}{}_{\rho\delta}\, \mathrm{J}^{\delta} \otimes \mathrm{J}^{\rho}$$

Tensor space representation(spin chains, scattering amplitudes, etc.):

$$\mathrm{J}^{\kappa} = \sum_{i} \mathrm{J}_i^{\kappa} \qquad \widehat{\mathrm{J}}^{\kappa} = \sum_{i} c_i\, \mathrm{J}_i^{\kappa} + f^{\kappa}{}_{\rho\delta} \sum_{i<j} \mathrm{J}_i^{\delta}\, \mathrm{J}_j^{\rho}$$

- Symmetry generators get deformed by radiative corrections but algebra is stable

$$J^a(\lambda) = J^a + \delta J^a(\lambda) \qquad \text{with} \qquad [J^a(\lambda), J^b(\lambda)] = f^{ab}_c \, J^c(\lambda)$$

- Dilatation operator:
  Scaling dimensions $\Delta_{\mathcal{A}}$ are eigenvalues of the dilatation operator $D(\lambda) = D + \delta D(\lambda)$

$$D(\lambda)\,|\mathcal{O}\rangle = \Delta_{\mathcal{A}}|\mathcal{O}\rangle \qquad \text{with} \qquad |\mathcal{O}\rangle = \sum_{\{i_k\}} c^{i_1 \dots i_L}\,|X_{i_1} \dots X_{i_L}\rangle$$

- Captured by mixing matrix Z of renormalization of $\mathcal{O}_{\mathcal{A}}$

$$\delta D \sim Z^{-1}\,\mu\frac{dZ}{d\mu}$$

  Constructable from Feynman diagrammatics

- δD is invariant under $\mathfrak{psu}(2,2|4)$

$$[D, J^a] = \underbrace{\Delta^a}_{\text{number}} \; J^a \qquad \Rightarrow \qquad [D(\lambda), J^a(\lambda)] = \Delta^a \, J^a(\lambda)$$

$$\Rightarrow \qquad [D, \delta J^a] + [\delta D, J^a] = \Delta^a \, \delta J^a$$

  in perturbation thy operators only mix with other operators of the same classical dimension, hence $[D, \delta J^a] = \Delta^a \, \delta J^a$ thus

$$\boxed{[\delta D, J^a] = 0}$$

- One loop dilatation operator:
  Look at simple subsector closed under renormalization
  $X = \phi_1 + i\phi_2 \qquad Y = \phi_3 + i\phi_4 \qquad Z = \phi_5 + i\phi_6$
  - Groundstate: $|0_L\rangle \hat{=} \mathrm{Tr}(Z^L)$ indeed $\Delta_{0_L} = 0$ (1/2 BPS state)
  - Consider excitations (insert W's) $\Rightarrow$ SU(2) sector $|\uparrow\rangle \hat{=} Z$ $|\downarrow\rangle \hat{=} W$

$$\mathrm{Tr}(Z...ZWZ..W) \hat{=} |\uparrow\rangle \otimes ...|\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes ...|\uparrow\rangle$$

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$$\operatorname{Tr}(Z...ZWZ..W) \hat{=} |\uparrow\rangle \otimes ...|\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes ...|\uparrow\rangle$$

- Explixit computation in SU(2) sector yields

$$\boxed{\delta D_{SU(2)} = \frac{\lambda}{2} \sum_{k=1}^{L} (\mathbf{1}_k - \mathbb{P}_{k,k+1}) + \mathcal{O}(\lambda^2)} \qquad \mathbb{P}_{k,k+1} = \vec{\sigma}_k \cdot \vec{\sigma}_{k+1}$$

- Obviously $\delta D_{SU(2)}$ is invariant under su(2): For $J_a \in$ su(2)

$$[J_a, \delta D_{SU(2)}] = 0 \qquad \text{with} \qquad J_a = \frac{1}{2i} \sum_{k=1}^{L} \sigma_{a,k}$$

- Yangian symmetry $Y[\mathfrak{su}(2)]$
Level one generators (in trivial evaluation representation):

$$\hat{J}_a = \epsilon_{abc} \sum_{1 \le k < j \le L} J_{b,j}\, J_{c,k}$$

Note: Definition singles out site 1 and L! Breaks cyclicity.

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Level one generator in su(2) sector

$$\hat{J}_a = \epsilon_{abc} \sum_{1 \leq k < j \leq L} J_{b,j}\, J_{c,k}$$

- One shows: $\epsilon_{abc}\,[J_b \otimes J_c, \sigma_d \otimes \sigma_d] = J_a \otimes \mathbf{1} - \mathbf{1} \otimes J_a$
- Thus with $\delta D_{SU(2)} = \sum_{k=1}^{L} (\mathbf{1}_k - \mathbb{P}_{k,k+1})$

$$[\hat{J}_a, \delta D_{SU(2)}] = J_{a,1} - J_{a,L} \,\hat{=}\, J_a \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{1} \otimes \ldots \otimes J_a$$

- Hence $\delta D_{SU(2)}$ is invariant up to boundary terms!
- Consequence on non-cyclicity of $\hat{J}$.

Full $\mathfrak{psu}(2,2|4)$ spin chain

Message: Surprisingly little changes

- Let V_F be 1-particle states in free $\mathcal{N}=4$ SYM
- δD acts on 2-particle states in $V_F \otimes V_F$
- Decomposition of $V_F \otimes V_F$ into irreps of $\mathfrak{psu}(2,2|4)$ surprisingly simple

$$V_F \otimes V_F = \bigoplus_{j=0}^{\infty} V_j$$

  j characterizes the multiplet whose conformal primary is an R-singlet of angular momentum $(j-2)$.
- Quadratic Casimir operator of the 2-particle system acts as

$$J_{12}^2 \, V_j = (J_1^a + J_2^a)^2 \, V_j = j(j+1) \, V_j \qquad j = 0, 1, 2, \ldots$$

  Just as in SU(2) !
- Allows us to write an $\mathfrak{psu}(2,2|4)$ invariant ansatz for the one-loop dilatation operator δD:

$$\delta D = \sum_{j=0}^{\infty} h(j) \, P_{12,j} \qquad P_{12,j} : \text{Projector of } V_F \otimes V_F \text{ on } V_j$$

- Yangian symmetry $\boxed{[\hat{J}^a, \delta D] = J_1^a - J_L^a}$ demands $h(j) = \sum_{n=1}^{j} \frac{2}{n}$
- Proof:

Define $\hat{J}^a_{ij} := f^a_{bc}\, J^a_i\, J^b_j$. We note $\boxed{\hat{J}^a_{ij} = \frac{1}{4}\,[J^2_{ij}, q^a_{ij}]}$ with $q^a_{ij} := J^a_i - J^a_j$.

$$\Rightarrow \quad [\delta D_{ij}, \hat{J}^a_{ij}] = \frac{1}{4}[\delta D_{ij}, [\hat{J}^2_{ij}, q_{ij}]\,]$$

Important relation (w/o proof) $\quad q^a_{ij} : V_j \to V_{j-1} \oplus V_{j+1}$

$$q^a_{ij}|\lambda(j)\rangle = |\chi^a(j-1)\rangle + |\rho^a(j+1)\rangle$$

Let's us conclude

$$\Rightarrow [\delta D_{ij}, \hat{J}^a_{ij}]\,|\lambda(j)\rangle = \ldots = \frac{j}{2}\Big(h(j) - h(j-1)\Big)|\chi^a(j-1)\rangle + \frac{j+1}{2}\Big(h(j+1) - h(j)\Big)|\rho^a(j+1)\rangle$$

$$\stackrel{!}{=} q^a_{ij}|\lambda(j)\rangle = |\chi^a(j-1)\rangle + |\rho^a(j+1)\rangle$$

$$\Rightarrow \quad \boxed{h(j) - h(j-1) = \tfrac{2}{j}}.$$

- Thus Yangian symmetry, which here means that

$$\boxed{[\hat{J}^a, \delta D] = J_1^a - J_L^a}$$

completely determines the one loop anomalous dimensions in the theory!

- Result is consistent with explicit field theory compuations
- Note: Yangian only a symmetry of the "bulk" Hamiltonian.
- At higher loop orders the deformations of $J^a(\lambda)$ and $\hat{J}^a(\lambda)$ can be constructed based on a preserved algebra. Result now unique due to possible similarity transformations $J^a \to X\, J^a\, X^{-1}$.