The Cusp of Anomalous Dimension in GPPZ

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The Wilson Loop at Weak Coupling

Wilson loops a measure of the potential between particles in a given theory.

$$W = \frac{1}{N} \langle 0 | \mathrm{tr}\{ P \mathrm{exp} \left(i \oint_C dx \cdot A(x) \right) \} | 0 \rangle \tag{1}$$

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Here C specifies some a path with vectors v_1 and v_2 kept at some cusp angle θ, which we connect together at infinity.

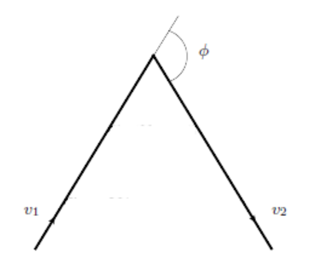
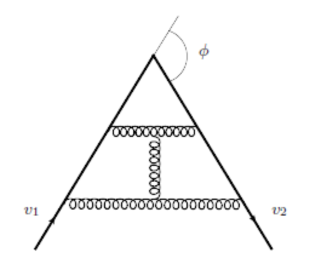


Figure: Path of the one cusp contour C.

The Wilson Loop at Weak Coupling

- A perturbative calculation of the Cusp of Anomalous Dimension for QCD is given to 3 loops in a 2014 paper by A. Grozin et al.

A. Grozin, J. Henn, G. Korchemsky and P. Marquard (arXiv:1409.0023 [hep-ph])



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The Wilson Loop at Strong Coupling

From the Gauge/Gravity correspondence we say that the Wilson loop is:

$$W \sim e^{-S} \tag{2}$$

where S is the action of a string with end points placed on the same contour C,

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and $h_{\alpha\beta}$ is the worldsheet metric. **The Wilson loop is related to the area the string traces out in spacetime.** The Cusp of Anomalous Dimension is a measure of how much this integral changes when we increase the cut-off Λ

$$\Gamma_{\mathsf{cusp}} = \frac{dS}{d\log\Lambda} = \Lambda\frac{dS}{d\Lambda}. \tag{4}$$

The Wilson Loop at Strong Coupling

For a general background that is Poincare invariant at the boundary:

$$ds^2 = e^{2\Phi(y)} \left(\eta_{\mu\nu} dx^\mu dx^\nu\right) + dy^2, \tag{5}$$

We orient the embedding co-ordinates as $x^\mu = (x^0, x^1, \mathbf{0}, y)$. Choosing the parameterization:

$$x^0 = s \sinh \eta \tag{6}$$

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$$\sqrt{-\mathrm{det}(h_{\alpha\beta})} = se^{\Phi(y)}\sqrt{e^{2\Phi(y)} - \left((\partial_s y)^2 - \frac{1}{s^2}(\partial_\eta y)^2\right)} \tag{8}$$

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This action does not depend explicitly on the rapidity η.

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Thus

$$y(s, \eta) = y(s), \tag{9}$$

and

$$S = \int ds\, s e^{\Phi(y)} \sqrt{e^{2\Phi(y)} - (\partial_s y)^2} \int d\eta. \tag{10}$$

The Wilson Loop at Strong Coupling

So we have that for **any** gauge with a gravitational dual, the Wilson loop at strong coupling with contour placed on the lightcone will have the form

$$W \sim e^{-\tilde{S}(\Lambda)(\eta_f - \eta_i)} \tag{11}$$

with

$$\tilde{S}(\Lambda) = \int ds\, s e^{\Phi(y)} \sqrt{e^{2\Phi(y)} - (\partial_s y)^2}. \tag{12}$$

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- With some effective theories (as we will see with GPPZ) we are required to include an effective "string tension" $T(y)$ in our Action to give:

$$\tilde{S}(\Lambda) = \int ds\, s T(y) e^{\Phi(y)} \sqrt{e^{2\Phi(y)} - (\partial_s y)^2}. \tag{13}$$

The AdS one Cusp Solution

For AdS space (dual to $N = 4$ SYM) we have $\Phi(y) = y$ so that

$$ds^2_{AdS} = e^{2y}\left(\eta_{\mu\nu}dx^\mu dx^\nu\right) + dy^2. \tag{14}$$

We choose $y = \log\frac{1}{\rho}$ to arrive at the more familiar form

$$ds^2_{AdS} = \frac{1}{\rho^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + d\rho^2\right). \tag{15}$$

with $\rho \in (0, \infty)$, with boundary at $\rho = 0$.

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with $\rho \in (0, \infty)$, with boundary at $\rho = 0$. Then we find

$$\tilde{S} = i \int ds\, s \frac{1}{\rho^2} \sqrt{\dot{\rho}^2 - 1}. \tag{16}$$

A solution of the Equations of Motion satisfying $\rho(s = 0) = 0$ is

$$\rho = \sqrt{2} s. \tag{17}$$

The GPPZ one Cusp Solution

$$ds^2_{GPPZ} = \left(e^{2y} - 1\right)\left(\eta_{\mu\nu} dx^\mu dx^\nu\right) + dy^2. \tag{18}$$

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- The theory requires an effective Tension term in our Action

$$S = \int d^2\sigma\, T(y)\sqrt{-\det(h_{\alpha\beta})} \qquad (19)$$

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In $y = \log \frac{1}{\rho}$ co-ordinates:

$$ds^2_{GPPZ} = \frac{1}{\rho^2}\left((1-\rho^2)\eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2\right). \tag{20}$$

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- We choose an Ansatz $\rho = \sin u$ to find:

$$ds^2_{GPPZ} = \cot^2 u\left(\eta_{\mu\nu}dx^\mu dx^\nu + du^2\right). \tag{22}$$

The GPPZ one Cusp Solution

In general we can take

$$\tilde{S}(\Lambda) = \int ds\, s T(y) e^{\Phi(y)} \sqrt{e^{2\Phi(y)} - (\partial_s y)^2}$$
$$= \int ds\, s \underbrace{T(y) e^{2\Phi(y)}}_{f(u)} \sqrt{1 - \underbrace{(\partial_s y)^2 e^{-2\Phi(y)}}_{(\partial_s u)^2}}$$

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This is equivalent to choosing the falling frame and as such this can be done generically.

$$ds^2 = e^{2u(y(u))}(\eta_{\mu\nu}dx^\mu dx^\nu + du^2). \qquad (25)$$

The GPPZ one Cusp Solution

For an Action of the form:

$$\tilde{S} = i \int ds\, s f(u) \sqrt{\dot{u}^2 - 1} \tag{26}$$

We will have an Equations of Motion:

$$s\ddot{u} = (1 - \dot{u}^2)(\dot{u} - s g(u)) \tag{27}$$

where $g(u) = \frac{\partial}{\partial u} \log f(u)$.

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- For the case of GPPZ $f(u) = 2\sqrt{2(1 + 3\sin^2 u)}\csc^2 u$ but we can make progress without knowing the form of this function.

We define

$$\dot{u} = \coth v \implies \frac{\ddot{u}}{1 - \dot{u}^2} = \dot{v}. \tag{28}$$

The GPPZ one Cusp Solution

Giving us the system of Equations

$$s\dot{v} = \dot{u} - sg(u), \tag{29}$$

$$\dot{u} = \coth v. \tag{30}$$

If $\dot{u}(s = s_i) = \coth v_i$ and $u(s = s_i)$ are real and $g(u)$ is a real-valued function then v must remain real for all s. Therefore $\dot{u}$ will be constrained to the region $\dot{u} \in (1, \infty)$.

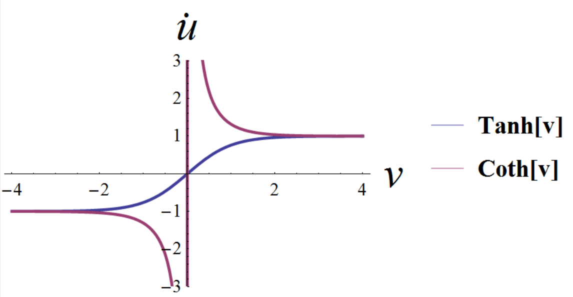

Figure: Path constraining $\dot{u} = \coth v$ for all real v.

Stability and Fixed Points

As $s \to 0$, our geometry becomes asymptotically AdS near the boundary.

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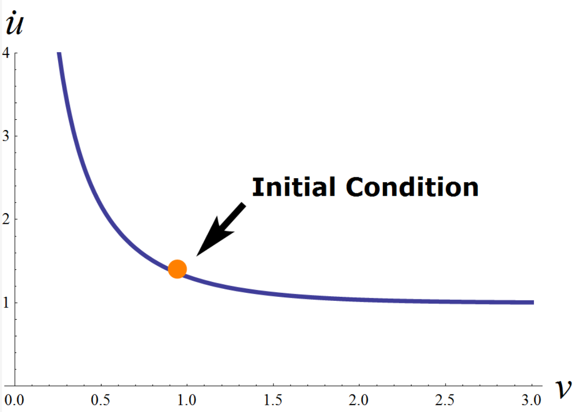
As $s \to 0$, our geometry becomes asymptotically AdS near the boundary. We know that $u = \sqrt{2}s$ is a solution to the AdS problem, so $\dot{u} = \sqrt{2}$ should be our second initial condition.



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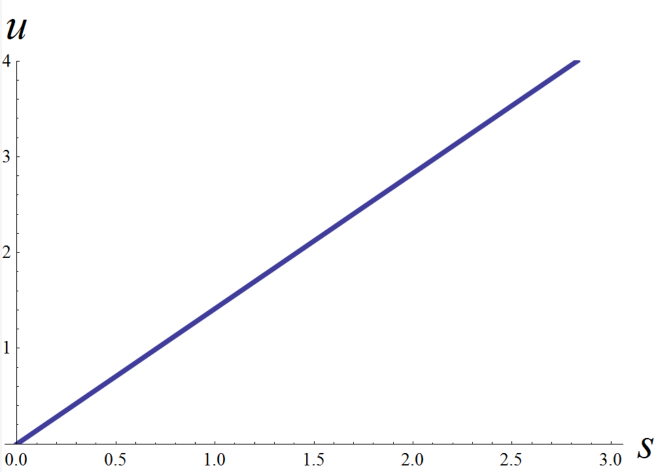

Figure: Solution $u = \sqrt{2}$ and numerical solutions for $g(u) < \frac{2}{u}$

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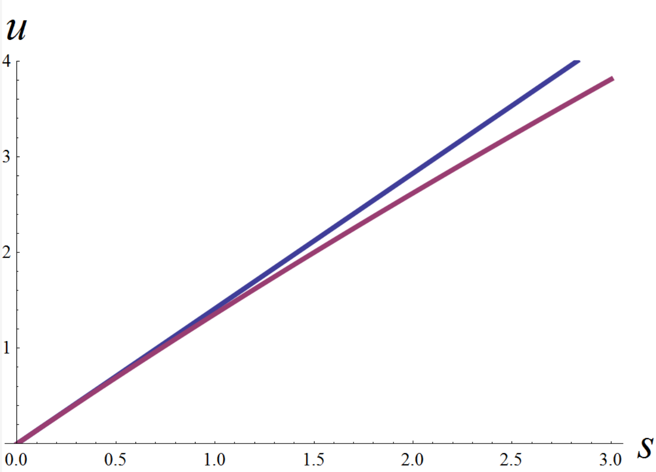

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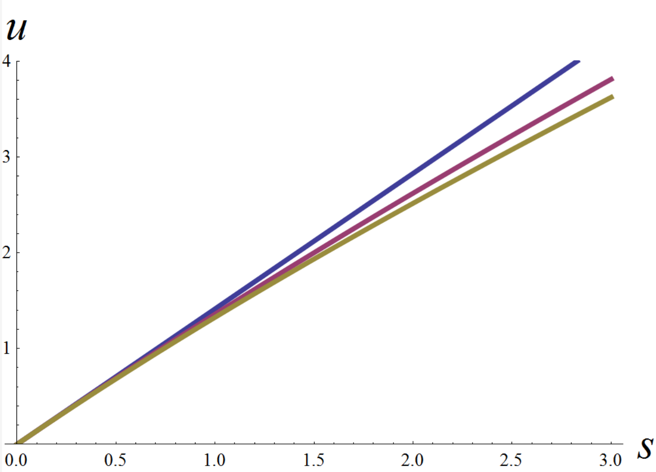

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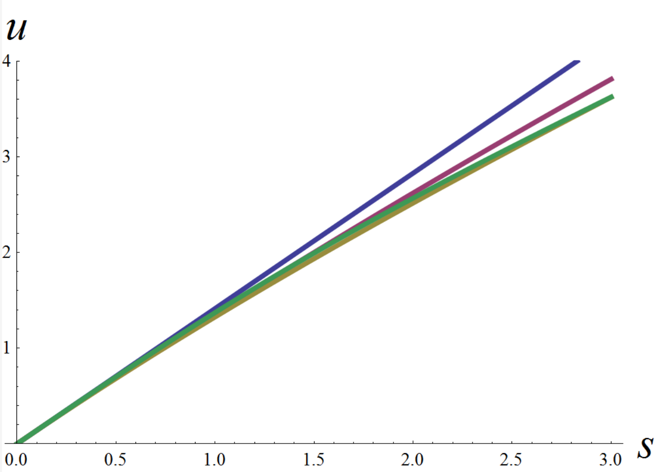

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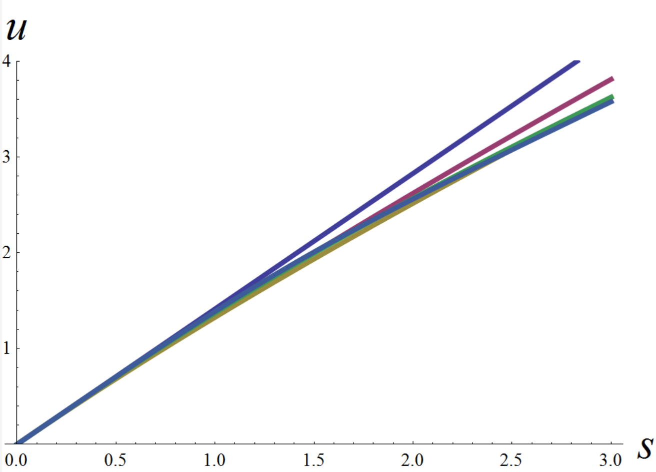

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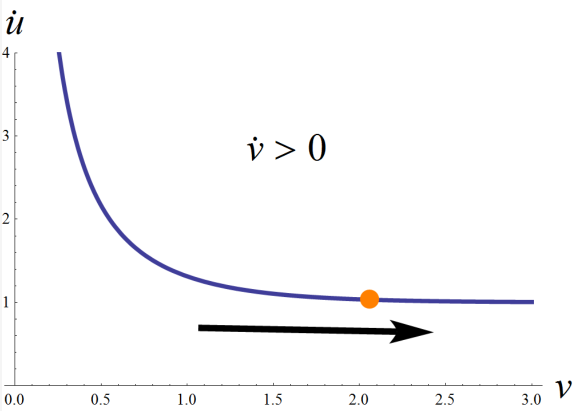
When $g(u)$ is bounded above by $\frac{2}{u}$ then the solution moves to the $\dot{u} \to 1$ fixed point.



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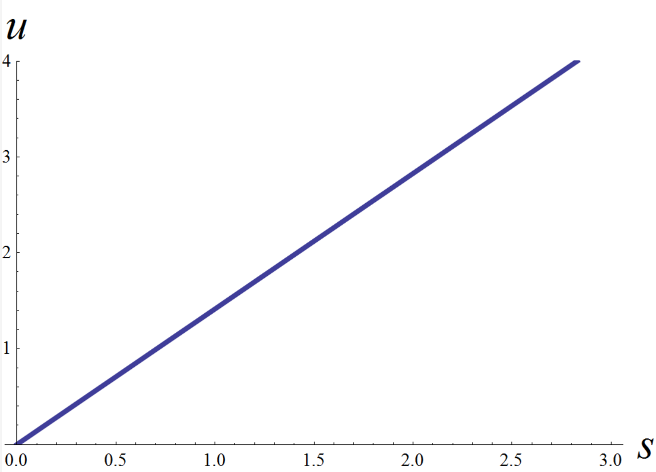

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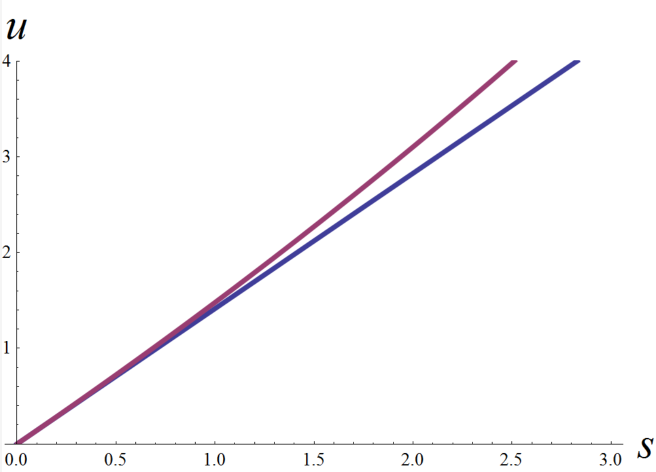

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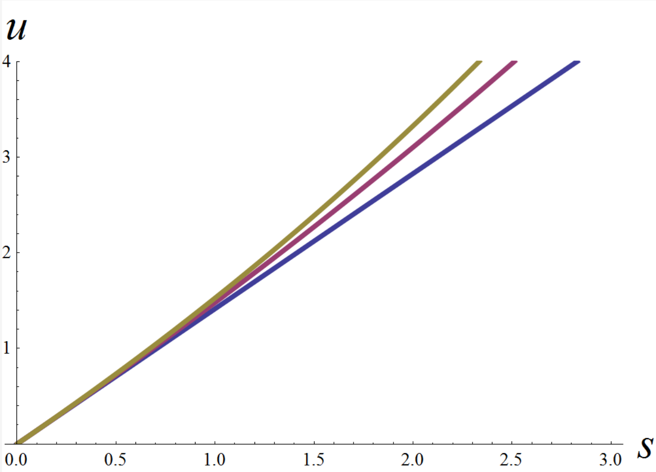

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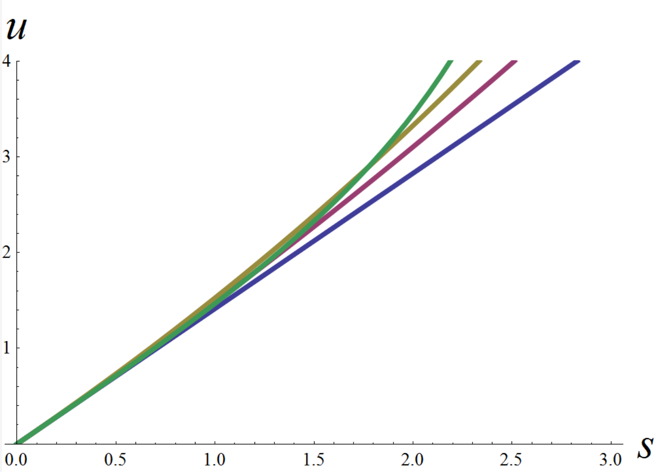

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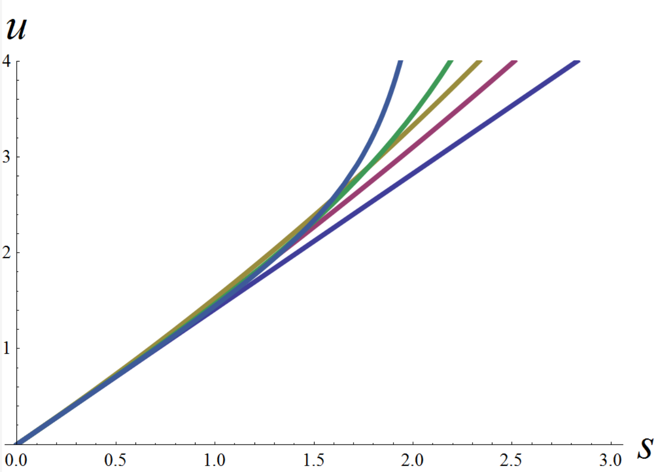

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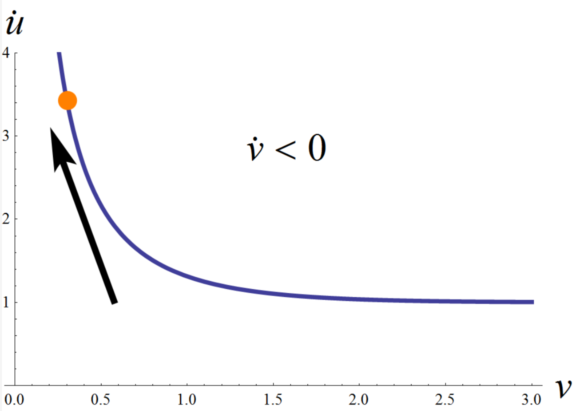
When $g(u)$ is bounded below by $\frac{2}{u}$ then the solution moves to the $\dot{u} \to \infty$ fixed point.



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GPPZ Results

For GPPZ $g(u) = 11\cot u - 3\cos 3u \csc z10 - 6\cos(2u)$ and is bounded above by $\frac{2}{u}$.

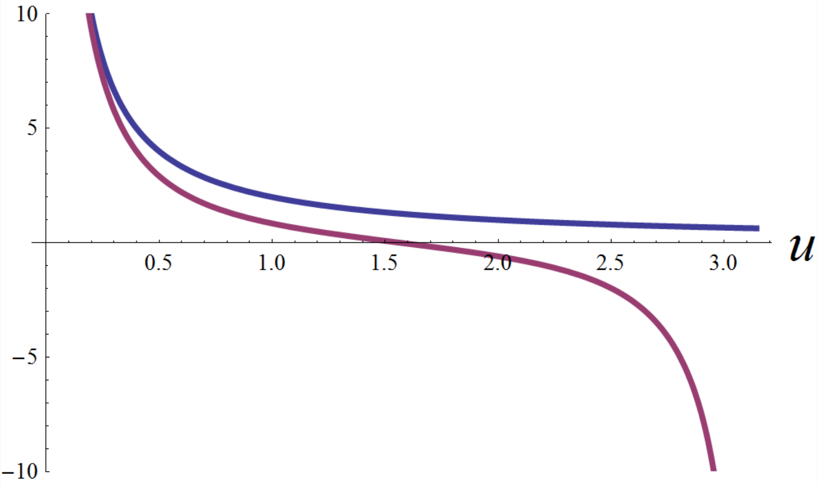

Figure: Comparason of $g(u)$ to $\frac{2}{u}$

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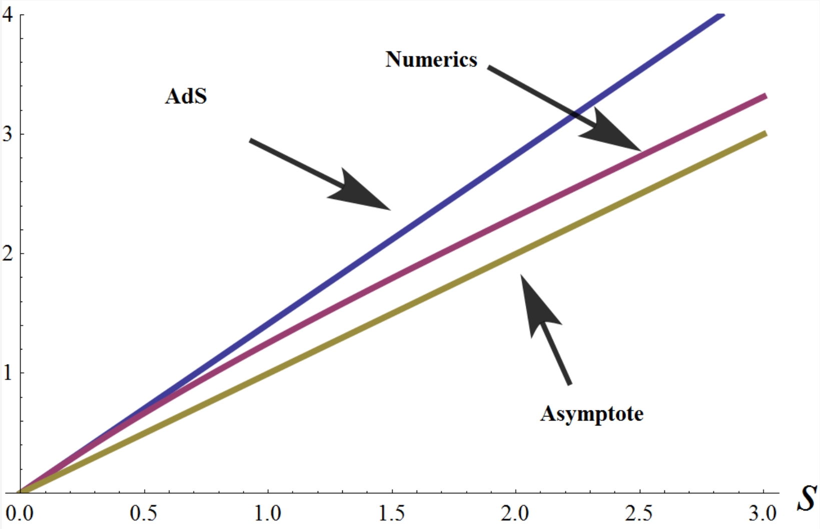

Figure: Numerical plot of $u(s)$, $\sqrt{2}s$ and s for comparason

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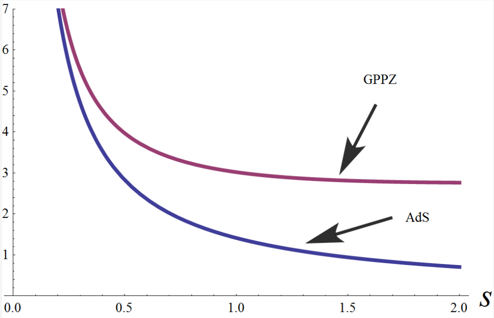

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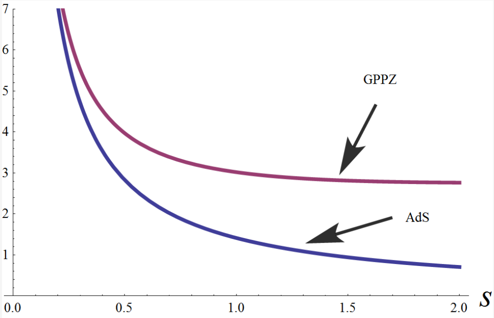

Figure: GPPZ and AdS Solutions substituted back into their respective Lagrangian Densities $\mathcal{L} = sf(u)\sqrt{\dot{u}^2 - 1}$.

We can note that although $\dot{u} \to 1$, then $f(u)$ term diverges causing a non-vanishing expression.

$$\mathcal{L} = s2\sqrt{2(1 + 3\sin^2 u)}\csc^2 u\sqrt{\dot{u}^2 - 1} \tag{31}$$

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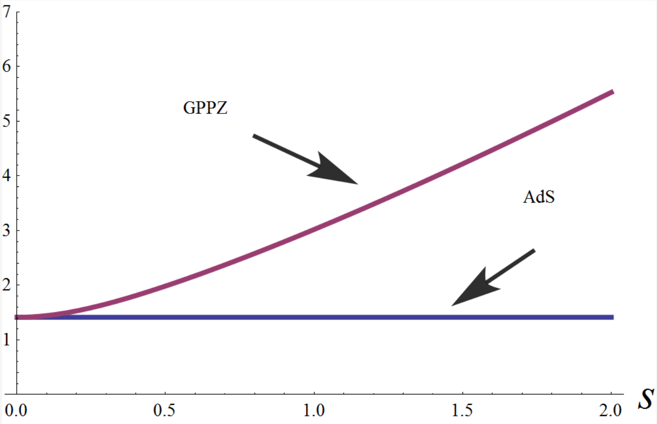

Figure: The cusp Γ_{cusp} for both GPPZ and AdS.

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- Computed the cusp of anomalous dimension associated with a light-line Wilson loop in a non-conformal theory.
- We could perform a similar analysis for different types of loops.

Conclusion

- We've analysed general behaviour regarding string solutions in multiple gravitational theories.
- Computed the cusp of anomalous dimension associated with a light-line Wilson loop in a non-conformal theory.
- We could perform a similar analysis for different types of loops.

Thank You.