# Out-of-Equilibrium Energy Flow and Entanglement Entropy in AdS/CFT

**Eugenio Megías**[1]

**In collaboration with: Johanna Erdmenger, Daniel Fernández, Mario Flory, Ann-Kathrin Straub, Piotr Witkowski**

**Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), Munich, Germany.**

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E.M. PoS EPS-HEP2015 (2015) 366, J. Erdmenger et al, in progress (2016).

# Issues

1. Motivation
2. Energy flow in strongly correlated systems
   - A universal regime of thermal transport
   - Holographic Model
   - Linearized solution
3. Information Flow
   - Entanglement Entropy
   - Mutual Information
   - Results in $d = 2$

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# Far from equilibrium dynamics

- Compute real time dynamics.
- Transition to hydrodynamics at late times.
- In strongly correlated systems:
  - Organizing principles out of equilibrium?
  - Universality classes?

**Examples:**

- Thermalization of quark-gluon plasma.
- Quenches in condensed matter systems.
- Fluctuations in the fractional Hall effect.

- **Much simpler class of problems** $\rightarrow$ (non)equilibrium steady state.
- **Concrete setup:** Two heat reservoirs at different temperatures are put in contact at time $t = 0$.

$$\text{Energy density} \equiv \varepsilon(x, t=0) = \frac{c\pi}{12}\left[T_L^2\Theta(-x) + T_R^2\Theta(x)\right]$$

# Issues

# A universal regime of thermal transport

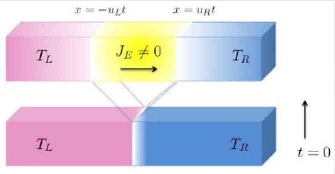
**[Bernard, Doyon '12], [Bhaseen, Doyon, Lucas, Schalm '13]**

- Two systems initially at equilibrium.
- Independently thermalized.
- Conservation equations and tracelessness:

$$\partial_x \langle T^{xx} \rangle = -\partial_t \langle T^{tx} \rangle = 0 \,, \qquad \langle T^{xx} \rangle = \langle T^{tt} \rangle$$

→ *"shockwaves"* emanating from interface: $\begin{cases} \langle T^{tx} \rangle = F(x-t) - F(x+t) \\ \langle T^{tt} \rangle = F(x-t) + F(x+t) \end{cases}$

- In $d+1$ dim: $\langle T^{\mu\nu} \rangle = a_d T^{d+1} \left( \eta^{\mu\nu} + (d+1) u^\mu u^\nu \right)$

# Hydrodynamics

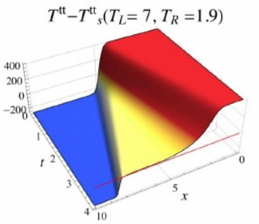
**[Bernard, Doyon '12], [Bhaseen, Doyon, Lucas, Schalm '13]**

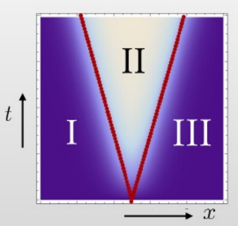
Prediction for the steady-state:

- Lorentz-boosted energy density with $u^\mu = (\cosh\theta, \sinh\theta, \vec{0})$ and temperature

$$T_L = T \cdot e^{\theta} , \qquad T_R = T \cdot e^{-\theta} \qquad \equiv \text{Doppler shift}$$

- Heat current: $J_E \equiv \langle T^{tx} \rangle_{st} = a_d \frac{T_L^{d+1} - T_R^{d+1}}{u_L + u_R}$ with $a_d \sim \frac{L^d}{G_N}$.
- Temperature: $T_{st} = \sqrt{T_L T_R}$
- Shockwave velocities: $c_s^2 = \frac{1}{d} = u_L u_R$.



# Issues

# Gravitational Dual

- Energy transport: Lorentz-boosted thermal distribution
  ➞ Gravity dual: Boosted black brane

$$S = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-g}\left\{R+2\Lambda\right\} .$$

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= \frac{L^2}{z^2}\left[\frac{dz^2}{f(z)} - f(z)\left(\cosh\theta dt - \sinh\theta dx\right)^2 + \left(\cosh\theta dx - \sinh\theta dt\right)^2 + dx_\perp^2\right] \end{aligned}$$

where $f(z) = 1-\left(\frac{z}{z_h}\right)^d$ and $z_h = \frac{d}{4\pi T}$.

- $g_{\mu\nu}$ is a solution of e.o.m. as long as $z_h \neq z_h(t,x)$ and $\theta \neq \theta(t,x)$.
- $(t,x)$-dependent solutions ➞ linearization **[E.Megías '15].**

$$z_h(t,x) = z_h^{(0)} + \epsilon z_h^{(1)}(t,x) + \cdots , \qquad \theta(t,x) = \epsilon\theta^{(1)}(t,x) + \cdots .$$

# Issues

# Linearization

$$z_h(t,x) = z_h^{(0)} + \epsilon z_h^{(1)}(t,x) + \cdots, \qquad \theta(t,x) = \epsilon\theta^{(1)}(t,x) + \cdots$$

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \epsilon \begin{pmatrix} \delta g_{tt}(t,x,z) & \delta g_{tx}(t,x,z) & 0 & 0 \\ \delta g_{tx}(t,x,z) & \delta g_{xx}(t,x,z) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta g_{zz}(t,x,z) \end{pmatrix}$$

e.o.m. $\rightarrow$ $\begin{cases} 0 = 2\partial_t^2 z_h^{(1)}(t,x) - \partial_x^2 z_h^{(1)}(t,x) \\ 0 = -z_h^{(0)}\partial_x\theta^{(1)}(t,x) + 2\partial_t z_h^{(1)}(t,x) \end{cases}$

**Solution**: Assuming some initial profile $T_{ini}(x) \equiv T(t=0,x)$ $\rightarrow$

$$T(t,x) = \frac{3}{4\pi z_h(t,x)} = \frac{1}{2}\left[T_{ini}\left(x+\frac{t}{\sqrt{2}}\right) + T_{ini}\left(x-\frac{t}{\sqrt{2}}\right)\right]$$

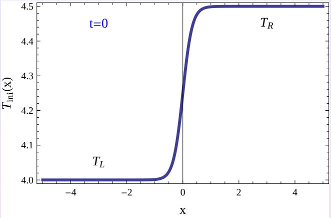
$$\theta(t,x) = -\epsilon\frac{2\sqrt{2}}{3}\pi z_h^{(0)}\left[T_{ini}\left(x+\frac{t}{\sqrt{2}}\right) - T_{ini}\left(x-\frac{t}{\sqrt{2}}\right)\right]$$

with $z_h^{(0)} = \frac{3}{2\pi(T_L+T_R)}$. Note $c_s = \frac{1}{\sqrt{2}}$.

# Linearization

- Let us consider the initial profile:

$$T_{\text{ini}}(x) = \frac{T_R + T_L}{2} + \frac{T_R - T_L}{2} \tanh(\alpha x)$$




- Then one gets the *t*-dependent solution:

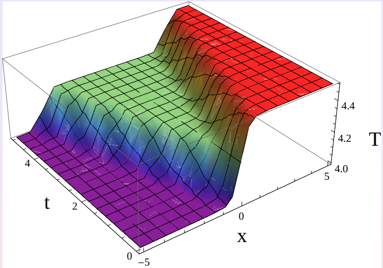
$$\langle T^{tt}(t,x)\rangle = \frac{(d-1)}{8G}\frac{1}{z_{h(0)}^{d-1}}\left[-\frac{(d-1)}{2\pi z_{h(0)}} + \epsilon\big(T_{\text{ini}}(x+c_s t) + T_{\text{ini}}(x-c_s t)\big)\right],$$

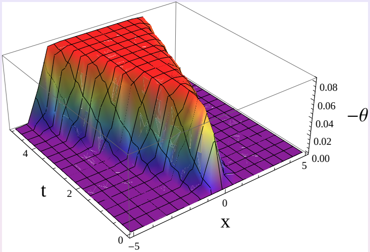
$$\langle T^{tx}(t,x)\rangle = -\epsilon\frac{1}{c_s}\frac{1}{8G}\frac{1}{z_{h(0)}^{d-1}}\left[T_{\text{ini}}(x+c_s t) - T_{\text{ini}}(x-c_s t)\right],$$

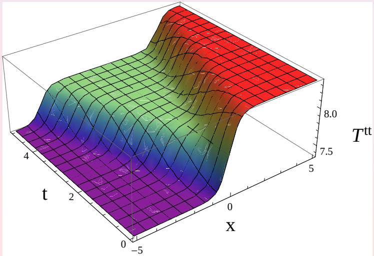
$\rightarrow$ $\mathcal{O}(\epsilon)$ are corrections in $\frac{|T_R - T_L|}{T_R + T_L} \ll 1$ $\rightarrow$ Small gradients in $T(t,x)$.

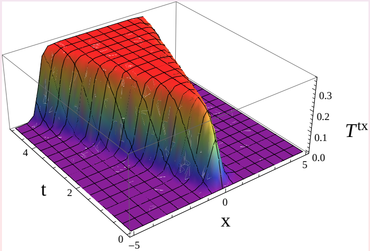
Example: $T_{\text{steady state}} \equiv T(t \to +\infty, x) = \sqrt{T_L T_R} = \frac{T_L + T_R}{2} + \mathcal{O}(\epsilon^2)$

# Linearization



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# Entanglement Entropy

How does information get exchanged between the systems which are isolated at $t = 0$?

- Entanglement entropy: Holographic measure $\Longrightarrow$ Generalization of Bekenstein-Hawking entropy formula

  **[Ryu, Takayanagi '06], [Hubeny, Rangamani, Takayanagi '07]**

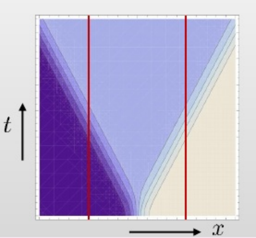
$$\mathcal{S}_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N^{(d+2)}}; \qquad \text{Area with boundary } \partial A$$

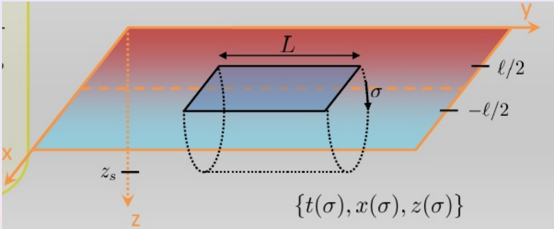
- Regularization: $\mathrm{Area}(\gamma_A^{div}) \sim \frac{1}{z_{uv}}$.
- Spatial induced metric on surface: $ds^2 = h_{\sigma\sigma} d\sigma^2 + g_{yy} dy^2$.
- Action: $\mathcal{S} = L \int ds \sqrt{\frac{\partial x^a}{\partial \sigma} \frac{\partial x^b}{\partial \sigma} \tilde{g}_{ab}}$
- Minimal surface $\longrightarrow$ Geodesic equations for metric $\tilde{g}_{ab} = g_{yy} g_{ab}$:

$$\frac{d^2 x^a}{ds^2} + \tilde{\Gamma}^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0 \, , \qquad a = t, x, z$$

$s \equiv$ affine parameter $\longrightarrow$ $\frac{\partial x^a}{\partial s} \frac{\partial x^b}{\partial s} \tilde{g}_{ab} = 1$ .

# Entanglement Entropy




$$S_A^{reg} = \frac{1}{4G_N^{d+2}}\left(\mathrm{Area}(\gamma_A) - \mathrm{Area}(\gamma_A^{div})\right)$$

# Entanglement Entropy

$$\mathcal{S} = L \int_{-s_{uv}}^{s_{uv}} ds = 2L \cdot s_{uv}$$

$$\boxed{\mathcal{S}^{reg} = \mathcal{S} - \mathcal{S}^{div} = 2L \cdot \left( s_{uv} - \frac{1}{z(s_{uv})} \right)}$$

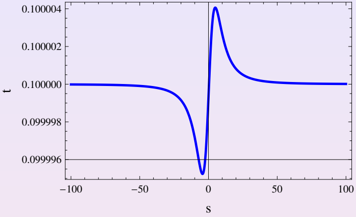
- Geodesic equations are 3 coupled equations of 2nd order $\Longrightarrow$ solutions:

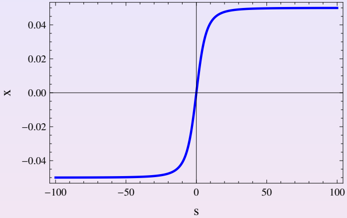
$$t = t(s)\,, \qquad x = x(s)\,, \qquad z = z(s)$$

→ 6 boundary conditions:

$$\begin{cases} t(s_{uv}) = t(-s_{uv}) = t_0 \\ x(s_{uv}) = -x(-s_{uv}) = \ell/2 \\ z(s_{uv}) = z(-s_{uv}) = z_{uv} \end{cases}$$

# Geodesics


0.100004
0.100002
0.100000
0.099998
0.099996
t
−100
−50
0
50
100
s


0.04
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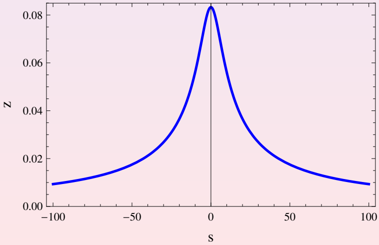
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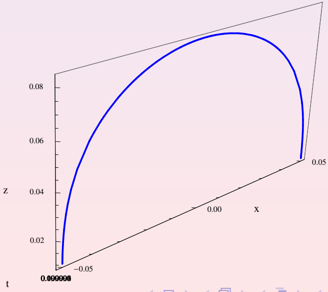
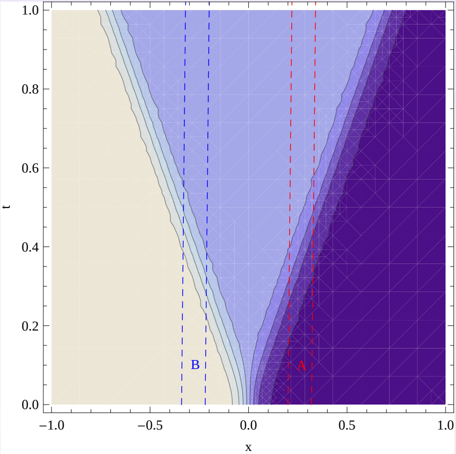
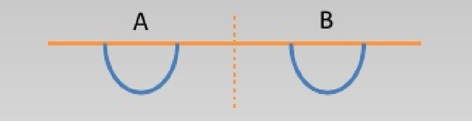

0.08
0.06
0.04
0.02
z
t
−0.05
0.00
0.05
x

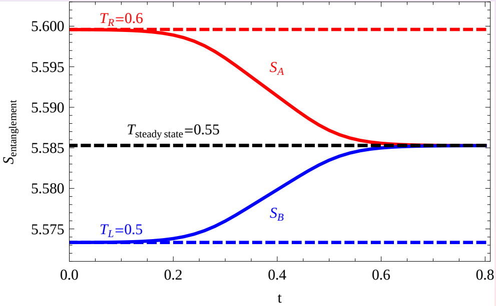
# Entanglement Entropy

**Contour plot of $T(t, x)$ with $T_L = 0.5$ and $T_R = 0.6$.**



# Entanglement Entropy



# Issues

# Mutual Information

## What do we learn about A by looking at B?

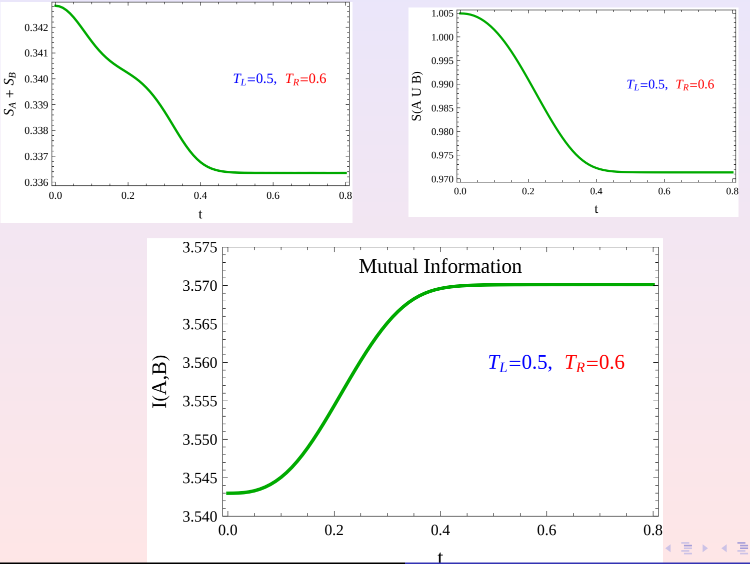
→ Mutual information: $I(A, B) = \mathcal{S}(A) + \mathcal{S}(B) - \mathcal{S}(A \, U \, B)$


**Basic features:**

- Shockwaves transport information.
- When does system A find out about the existence of system B?
- $I(A, B) =$ finite → no need to regularize!!!

We find: $\partial_t I(A, B) \geq 0$

# Mutual Information



# Issues

# Results in $d = 2$

- Exact analytical solution **[Bhaseen et al. '13]**

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + g_{\mu\nu} dx^\mu dx^\nu\right) ,$$

$$g_{tt} = -\left(1 - \frac{z^2}{L^2}\left(f_R(x-t) + f_L(x+t)\right)\right)^2 + \left(\frac{z^2}{L^2}\left(f_R(x-t) - f_L(x+t)\right)\right)^2$$

$$g_{tx} = -2\frac{z^2}{L^2}\left(f_R(x-t) - f_L(x+t)\right)$$

$$g_{xx} = \left(1 + \frac{z^2}{L^2}\left(f_R(x-t) + f_L(x+t)\right)\right)^2 - \left(\frac{z^2}{L^2}\left(f_R(x-t) - f_L(x+t)\right)\right)^2$$

- Energy-momentum tensor

$$\langle T^{tt}\rangle = \langle T^{xx}\rangle = \frac{c}{6\pi^2 L}\left(f_R(x-t) + f_L(x+t)\right)$$

$$\langle T^{tx}\rangle = \frac{c}{6\pi^2 L}\left(f_R(x-t) - f_L(x+t)\right)$$

# Results in $d = 2$

- Entanglement entropy in a boosted BH in $(\Delta x, \Delta t = 0)$ $\Longleftrightarrow$ Entanglement entropy in a non-boosted BH in $(\Delta\tilde{x}, \Delta\tilde{t})$

$$S_{\rm boost}(T_L, T_R, \ell) = 2\log\left(\frac{1}{\pi\sqrt{T_L T_R}}\sinh^{\frac{1}{2}}(\pi\ell T_L)\sinh^{\frac{1}{2}}(\pi\ell T_R)\right)$$

Valid in the steady-state regime $t > |x|$.

- Entanglement entropy in a stepwise BH

$$ds^2 = ds_L^2\Theta(-x) + ds_R^2\Theta(x)$$

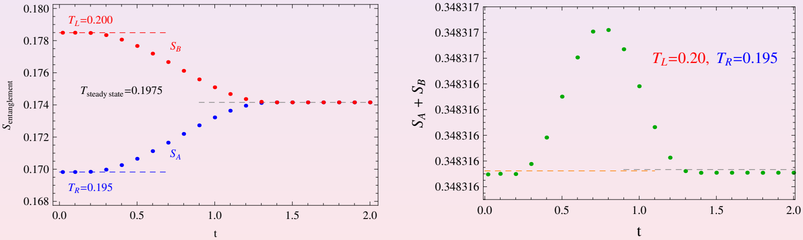
$$S(T_L, x_L; T_R, x_R) =$$
$$= 2\log\left[\frac{T_{\rm L}\cosh(\pi T_{\rm L}x_{\rm L})\sinh(\pi T_{\rm R}x_{\rm R}) - T_{\rm R}\sinh(\pi T_{\rm L}x_{\rm L})\cosh(\pi T_{\rm R}x_{\rm R})}{\pi T_{\rm L}T_{\rm R}}\right]$$

Valid in the regime $t = 0$.

# Results in $d = 2$

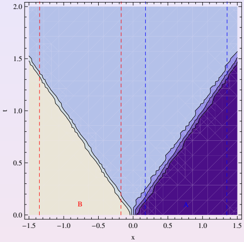
- *'Conservation'* of entanglement entropy ($t = 0 \leftrightarrow t = \infty$):

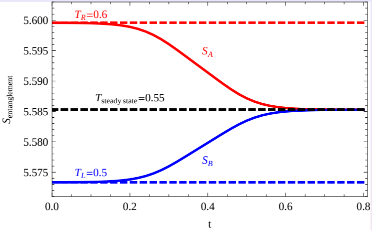
$$\boxed{S(T_L, \ell; t = 0) + S(T_R, \ell; t = 0) = 2S_{\text{boost}}(T_L, T_R, \ell; t = \infty)}$$

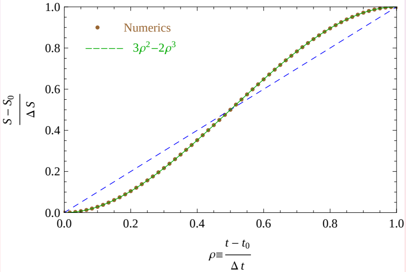



$\Longrightarrow S_A(t) + S_B(t)$ not exactly conserved at intermediate times.

# Universal t-evolution of entanglement entropy ($d = 2$)




$$\frac{S(t) - S(t=0)}{S(t=\infty) - S(t=0)} = f(\rho) = $$
$$= 3\rho^2 - 2\rho^3$$

with $\rho \equiv (t - t_0)/\Delta t = (t - x_L)/\ell$

Good for $|T_L - T_R| < T_L + T_R$

# Conclusions and future directions

- We have studied a holographic model for time-dep energy flow.
- Linearization for $d \geq 3$ ➞ analytical solution valid for $|T_L - T_R| \ll T_L + T_R$ *(linear response regime)*.
- Exact solution for $d = 2$ **[Bhaseen et al. '13].**
- We have computed the time evolution of Entanglement Entropies ➞ Mutual information grows with time.
- Universal time evolution of entanglement entropy in the linear response regime ($d = 2$).

What next?

- Complete analysis beyond linear response regime for $d \geq 3$: $0 < T_L/T_R < 1$ ➞ Full numerical solution of Einstein e.o.m. **[Chesler, Yaffe '09], [Amado,Yarom '15], [Ecker, Grumiller, Stricker '15], [Erdmenger, Fernandez, Flory, EM, Straub, Witkowski '16].**
- Other possible solutions? **[Chang, Karch, Yarom '14].**
- Higher dimensions: What changes?

# Thank You!