# Anomalous transport: Theory and Applications - II

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$$\vec{J}_A = d_{ABC}\frac{\mu_B}{4\pi^2}\vec{B}_C + \left(d_{ABC}\frac{\mu_B\mu_C}{4\pi^2} + b_A\frac{T^2}{12}\right)\vec{\Omega}$$

$$\vec{J}_\epsilon = \left(d_{ABC}\frac{\mu_B\mu_C}{8\pi^2} + b_A\frac{T^2}{24}\right)\vec{B}_A + \left(d_{ABC}\frac{\mu_A\mu_B\mu_C}{6\pi^2} + b_A\frac{\mu_A T^2}{6}\right)\vec{\Omega}$$

# Outline

- Relativistic Hydrodynamics
- Kubo Formulas
- Anomalous Transport from Holography
- Renormalization
- Application: CME in QGP
- Application: NMR in WSM

# Relativistic Hydro

- Thermodynamics: maximal entropy = forget everything that can be forgotten (conserved charges)
  - Energy: Temperature $T$
  - Charge: Chemical potential $\mu$
  - Time: Frame $u^\mu \quad u^2 = -1$

$$Z = \mathrm{Tr} e^{-\beta(u^\mu P_\mu + \mu Q)}$$

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu}$$

$$J^\mu = \rho u^\mu$$

$$p = p(T,\mu) \quad , \quad \epsilon + p = sT + \rho\mu$$

# Relativistic Hydro

- Hydrodynamics: hypothesis of local thermal equilibrium
- Universal low energy, long wavelength behaviour
- Equations of motion: Conservation laws
- Ideal Hydrodyamics: $T(x), \mu(x), u^\mu(x)$

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\mu J^\mu = 0$$

$$s^\mu = s u^\mu$$

$$\partial_\mu s^\mu = 0$$

# Relativistic Hydro

- Viscous Hydrodynamics: add derivatives
- Effective Field Theory: all terms consistent with symmetries and 1 derivative

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu - pg^{\mu\nu} + \tau^{\mu\nu}$$
$$J^\mu = \rho u^\nu + \nu^\mu$$

- Ambiguities in local definitions of $T(x), \mu(x), u^\mu(x)$
- Frame choice: no corrections to $\epsilon, p$ and $u^\mu \tau^{\mu\nu} = 0$ (Landau frame)

$$\tau^{\mu\nu} = -\eta \mathcal{P}^{\mu\alpha}\mathcal{P}^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\partial_\lambda u^\lambda g_{\alpha\beta}\right) - \zeta \mathcal{P}^{\mu\nu}\partial_\lambda u^\lambda$$
$$\nu^\mu = \sigma\left[\mathcal{P}^{\mu\alpha}\partial_\alpha\left(\frac{\mu}{T}\right) - E^\mu\right]$$

# Relativistic Hydro

$$\tau^{\mu\nu} = -\eta \mathcal{P}^{\mu\alpha}\mathcal{P}^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\partial_\lambda u^\lambda g_{\alpha\beta}\right) - \zeta \mathcal{P}^{\mu\nu}\partial_\lambda u^\lambda$$

$$\nu^\mu = \sigma\left[\mathcal{P}^{\mu\alpha}\partial_\alpha\left(\frac{\mu}{T}\right) - E^\mu\right]$$

- Projector $\mathcal{P}^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$
- Local electric field $E^\mu = F^{\mu\nu} u_\mu$
- Transport coefficients: shear, bulk viscosities, conductivity
- Entropy production

$$s^\mu = s u^\mu - \frac{\mu}{T}\nu^\mu$$

$$\partial_\mu s^\mu \geq 0 \quad \Longrightarrow \quad \eta \geq 0 \;\; \zeta \geq 0 \;\; \sigma \geq 0$$

# Hydro with anomalies

Conservation laws: effective action with sources $\Gamma[A_\mu, g_{\mu\nu}]$

Gauge Trafo: $\delta A_\mu = \partial_\mu \lambda$ Anomaly

Diffeo Trafo: $\delta g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$ $\delta A_\mu = \nabla_\mu(\epsilon^\nu A_\nu) + \epsilon^\nu F_{\mu\nu}$

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu - A^\nu \partial_\mu J^\mu$$

Rewrite with covariant current

$$\boxed{\begin{aligned} \partial_\mu T^{\mu\nu} &= F^{\nu\mu} J_\mu \\ \partial_\mu J^\mu &= 8\, C_{\rm cov}\, E^\mu B_\mu \end{aligned}}$$

$$E^\mu = F^{\mu\nu} u_\mu \quad , \quad B^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} u_\nu F_{\rho\lambda}$$

# Hydro with anomalies

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu$$
$$\partial_\mu J^\mu = 8\, C_{\rm cov}\, E^\mu B_\mu$$

Anomaly breaks parity: additional terms are allowed

$$\delta\tau^{\mu\nu} = \sigma_\epsilon^B (u^\mu B^\nu + u^\nu B^\mu) + \sigma_\epsilon^\omega (u^\mu \omega^\nu + u^\nu \omega^\mu)$$
$$\delta\nu^\mu = \sigma_B B^\mu + \sigma_\omega \omega^\mu$$
$$\delta s^\mu = X B^\mu + Y \omega^\mu$$

Vorticity: $\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda} u_\nu \partial_\rho u_\lambda$

Second law almost fixes them: $\partial_\mu s^\mu \geq 0$

$$\boxed{\begin{aligned} \sigma^B &= 24 C_{\rm cov}\mu & \sigma_\epsilon^B &= 12 C_{\rm cov}\mu^2 + \gamma T^2 \\ \sigma^\omega &= 24 C_{\rm cov}\mu^2 + 2\gamma T^2 & \sigma_\epsilon^\omega &= 8 C_{\rm cov}\mu^3 + 2\gamma\mu T^2 \end{aligned}}$$

# Hydro with anomalies

- Anomaly fixes dependence on chemical potentials
- Hydro best expressed in covariant currents
- Temperature enters with an undetermined integration constant
- Gravitational anomaly contribution is 4th order in derivatives, naively: no hydro
- Other frame choices are possible (Landau, Eckart, etc...)
- **Dissipationless**

# Kubo formulas

gravitomagnetic field = metric component

$$ds^2 = -dt^2 + 2A_i^g dt dx^i + d\vec{x}^2$$

$$B_i^g = \epsilon_{ijk}\partial_j A_k^g$$

Relation to vorticity $u^\mu = (1,0,0,0) \quad , \quad u_\mu = (-1, A_i^g)$

$$2\vec{\omega} = \vec{B}_i^g$$

Chiral vortical effect is chiral gravitomagnetic effect
(frame dragging, Thirring-Lense effect)

# Kubo formulas

Conductivity: $\langle J_i \rangle = \sigma(-i\omega)A_i \quad \Longrightarrow \quad \sigma = \lim_{\omega\to 0} \frac{i}{\omega}\langle J_i J_i \rangle$

CME: $\langle J_i \rangle = \sigma_B \epsilon_{ijk}(ip_j)A_j \Longrightarrow \sigma_B = \lim_{p_z\to 0} \frac{-i}{p_z}\langle J_x J_y \rangle$

CVE: $\langle J_i \rangle = \frac{1}{2}\sigma_\omega \epsilon_{ijk}(ip_j)A^g_j \Longrightarrow \sigma_\omega = 2\lim_{p_z\to 0} \frac{-i}{p_z}\langle J_x T_{0y} \rangle$

CME in energy current: $\sigma^B_\epsilon = \lim_{p_z\to 0} \frac{-i}{p_z}\langle T_{0x} J_y \rangle$

$$2\sigma^B_\epsilon = \sigma_\omega$$

# Holography

- Hydro has undetermined integration constant
- Weak coupling suggests relation to gravitational anomaly
- Answer with help of holographic model

# AdS/CFT

Motto:

"... if the gravitational field didn't exist, one could invent it for the purposes of this paper..."

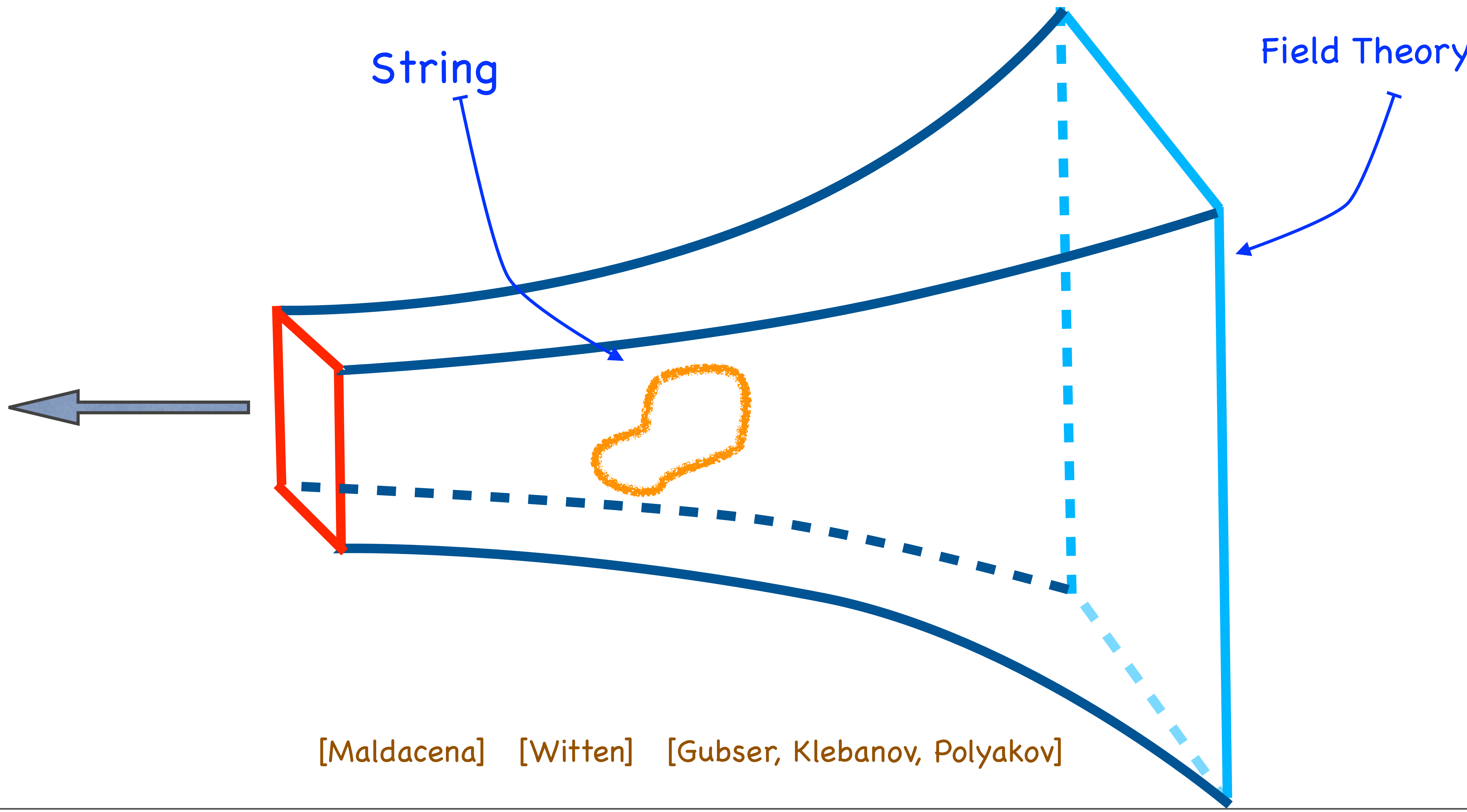
"Theory of Thermal Transport Coefficients"
Luttinger Phys. Rev. 135, A1505, (1964)

# AdS/CFT

"... if string theory didn't exist, one could invent it for the purposes of computing transport coefficients in strongly coupled theories..."

# AdS/CFT

$$ds^2 = \frac{r^2}{L^2}(dt^2 + d\vec{x}^2) + \frac{L^2 dr^2}{r^2}$$




[Maldacena] [Witten] [Gubser, Klebanov, Polyakov]

# AdS/CFT

$$\int_{\Phi|_\partial = \Phi_0} D\Phi e^{iS[\Phi]} = e^{iZ[\Phi_0]}$$

$$\frac{\delta^n Z[\Phi_0]}{\delta\Phi_0^1(x_1)\cdots\delta\Phi_0^n(x_n)} = \langle O_1(x_1)\ldots O_n(x_n)\rangle$$

Path integral (string theory) on AdS is hard. In practice resort to semi classical limit:

$$S_{grav}[\Phi_0] = Z[\Phi_0]$$

# AdS/CFT

- N=4 SYM best understood example: $\{\mathcal{A}_\mu, \Psi^a_\alpha, \phi^I\}$

$$g^2_{YM} N = \frac{R^4}{\alpha'^2} \qquad \frac{1}{N} \propto g_s$$

- *semiclassical gravity limit = large N, large coupling*

- *Interpretation of gravity solutions*

$$\Phi = \Phi_0 (r^{\Delta_-} + \cdots) + \langle \mathcal{O} \rangle (r^{\Delta_+} + \cdots)$$

- Non-normalizable
- Coupling in dual QFT

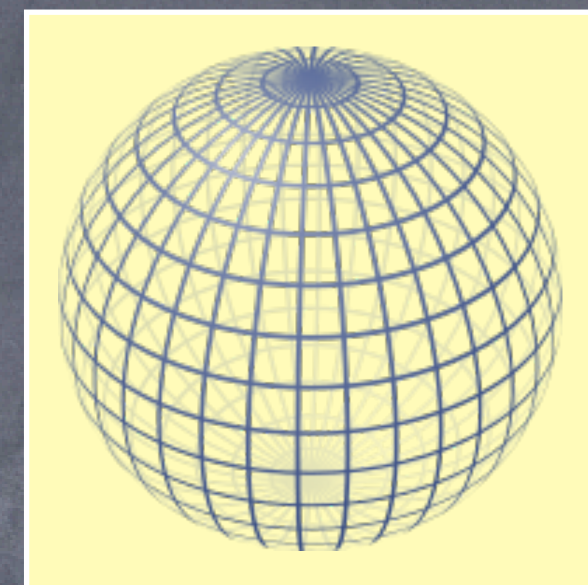
- Normalizable
- Vev in dual QFT

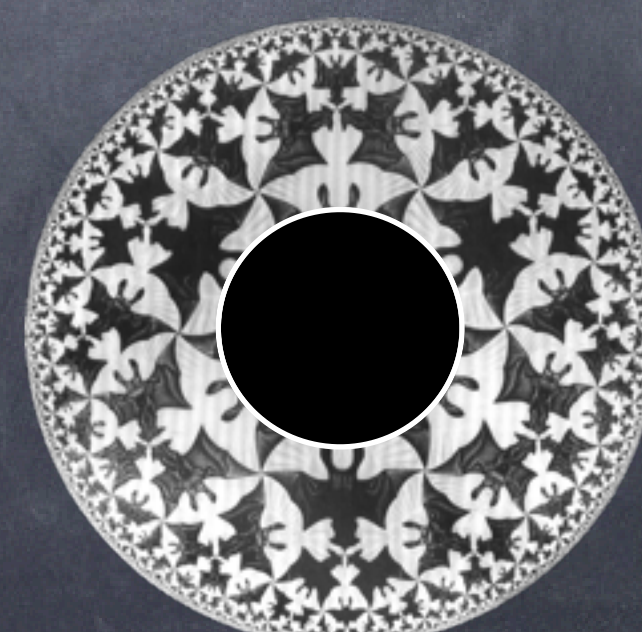
# AdS/CFT

## Dictionary

| AdS | Field Theory |
|---|---|
| five dimensional | four dimensional |
| r-direction | RG flow |
| strongly coupled | weakly coupled |
| gravity | no gravity |
| metric | energy momentum tensor |
| gauge field | current |
| scalar field | scalar operator |

# String Theory as spherical cow of sQGP





# AdS/CFT

- Holographic Model $S = S_{MEH} + S_{CS} + S_{GH} + S_{CSK}$

$$S_{MEH} = \int d^5x \sqrt{-g} \left[ \frac{1}{2\kappa^2} (\mathcal{R} + 2\Lambda) - \frac{1}{4} F^2 \right]$$

$$S_{CS} = \int d^5x \sqrt{-g} \epsilon^{MNPQR} A_M \left( \frac{\alpha}{3} F_{NP} F_{QR} + \lambda \mathcal{R}^A{}_{BNP} \mathcal{R}^B{}_{AQR} \right)$$

$$S_{GH} = \int_\partial d^4x \sqrt{-h} \frac{1}{\kappa^2} K$$

$$S_{CSK} = \int_\partial d^4x \sqrt{-h} \frac{4}{\kappa^2} \lambda n_M \epsilon^{MNPQR} A_N K_{PL} D_Q K^L_R$$

# AdS/CFT

- Anomaly: $\delta S = \int_\partial d^4x \sqrt{-h}\epsilon^{mnkl}\left(\frac{\alpha}{3}F_{mn}F_{kl} + \lambda \mathcal{R}^a{}_{bmn}\mathcal{R}^b{}_{akl}\right)$
- Background:

$$ds^2 = \frac{r^2}{L^2}\left(-f(r)dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2}dr^2 \qquad f = 1 - \frac{M}{r^4} + \frac{Q}{r^6}$$

$$A = (A_0 - \frac{\mu r_H^2}{r^2})dt$$

allow for general boundary value !

- Definition of Currents:

$$J^\mu = \sqrt{-h}\frac{1}{2\kappa^2}F^{\mu r} + \frac{4\alpha}{3}\epsilon^{mnkl}A_n F_{kl}$$

consistent current

covariant current

Bardeen-Zumino polynomial

# AdS/CFT

- Chiral Gravitomagnetic Effect via AdS boundary conditions

$$\delta h^t\, x\big|_{r=\infty} = B_g y \quad , \quad \delta\, A_x\big|_{r=\infty} = -\mu u y B_g$$

- Response parallel to gravitomagnetic field

$$\left[f a_z' - \mu h_z^t\right]' = -\alpha 4 \mu u B_g + \lambda 4 f' \left(2 u f'' + f'\right) B_g$$

*4 derivatives !*

- Induced covariant current

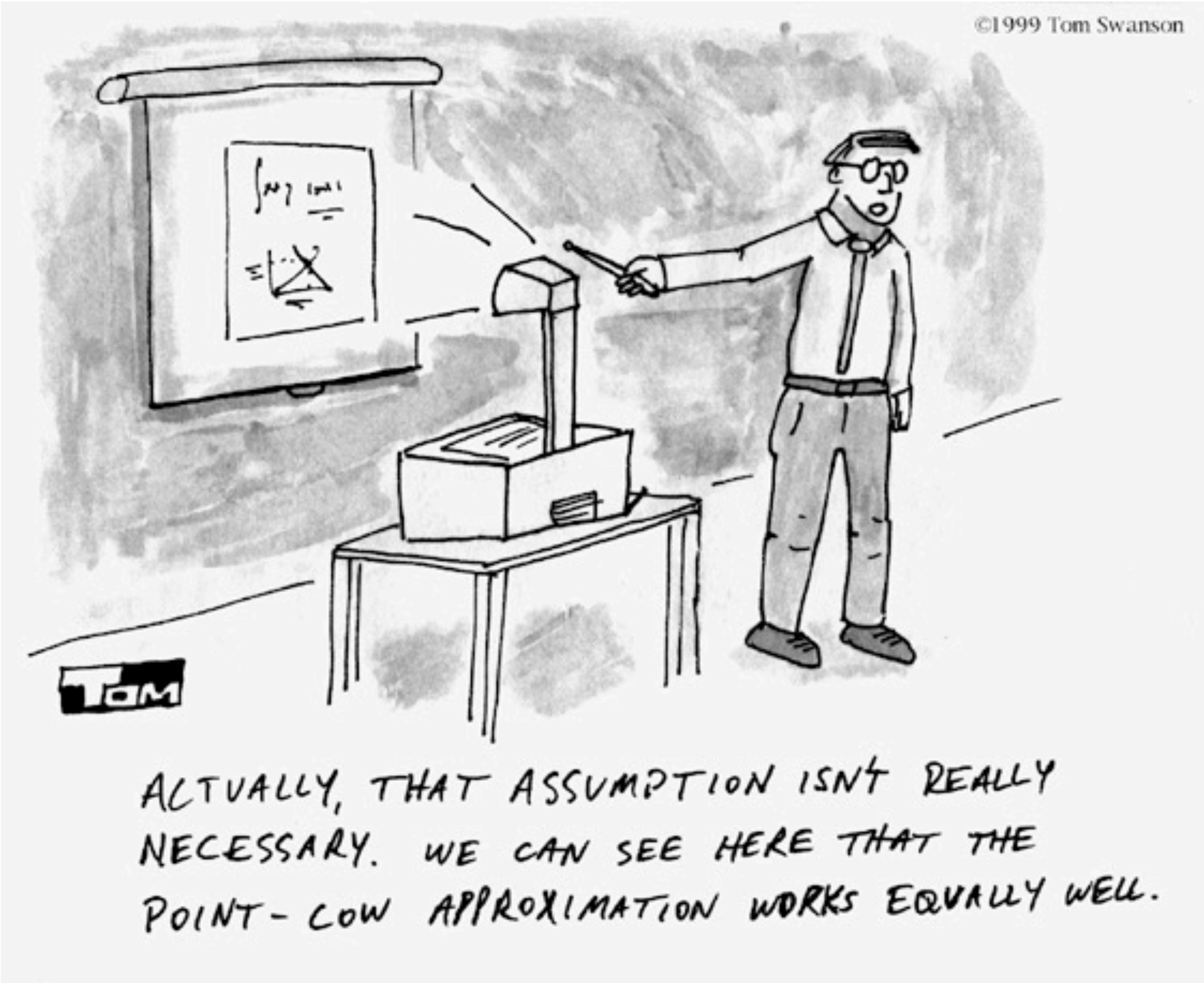
$$\vec{J}_{\rm cov} = (4\alpha\mu + 32\lambda T^2)\vec{B}_g$$

- Normalize to anomaly of 1 chiral fermion

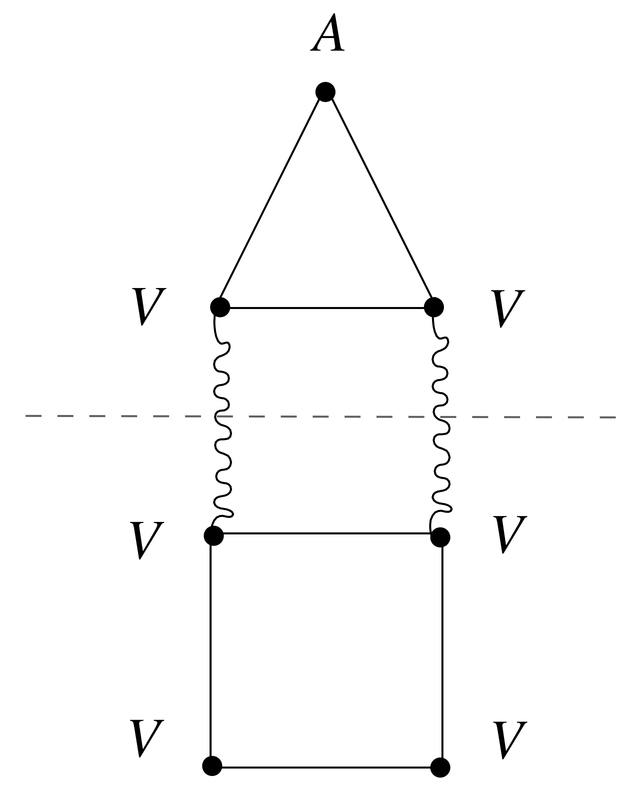
$$\alpha = \frac{1}{96\pi^2} \quad , \quad \lambda = \frac{1}{768\pi^2}$$

- Matches precisely the weak coupling result!!

# AdS/CFT


©1999 Tom Swanson
TOM
ACTUALLY, THAT ASSUMPTION ISN'T REALLY NECESSARY. WE CAN SEE HERE THAT THE POINT-COW APPROXIMATION WORKS EQUALLY WELL.

# Remarks




- Dynamical Gauge fields: radiative corrections due to rescattering
- Non-renormalization holds for 't Hooft anomalies (external gauge fields, anomalies with only global symmetries)
- Holographic model: Stückelberg axion
- Anomalous dimension of axial current = massterm for gauge field in AdS

$$\int d^5x\sqrt{-g}\,\left(-\frac{1}{4}F^2 + m^2(A_\mu + \partial_\mu\theta) + \frac{\alpha}{3}(A + d\theta)\wedge F\wedge F\right)$$

$$\partial_\mu J_5^\mu = \frac{\delta Z}{\delta\theta}$$

- Axial charge decay but basic transport phenomena persist

# Theory Summary

$$\vec{J} = \frac{\mu_5 - A_0^5}{2\pi^2}\vec{B}$$ Chiral Magnetic Effect (CME)

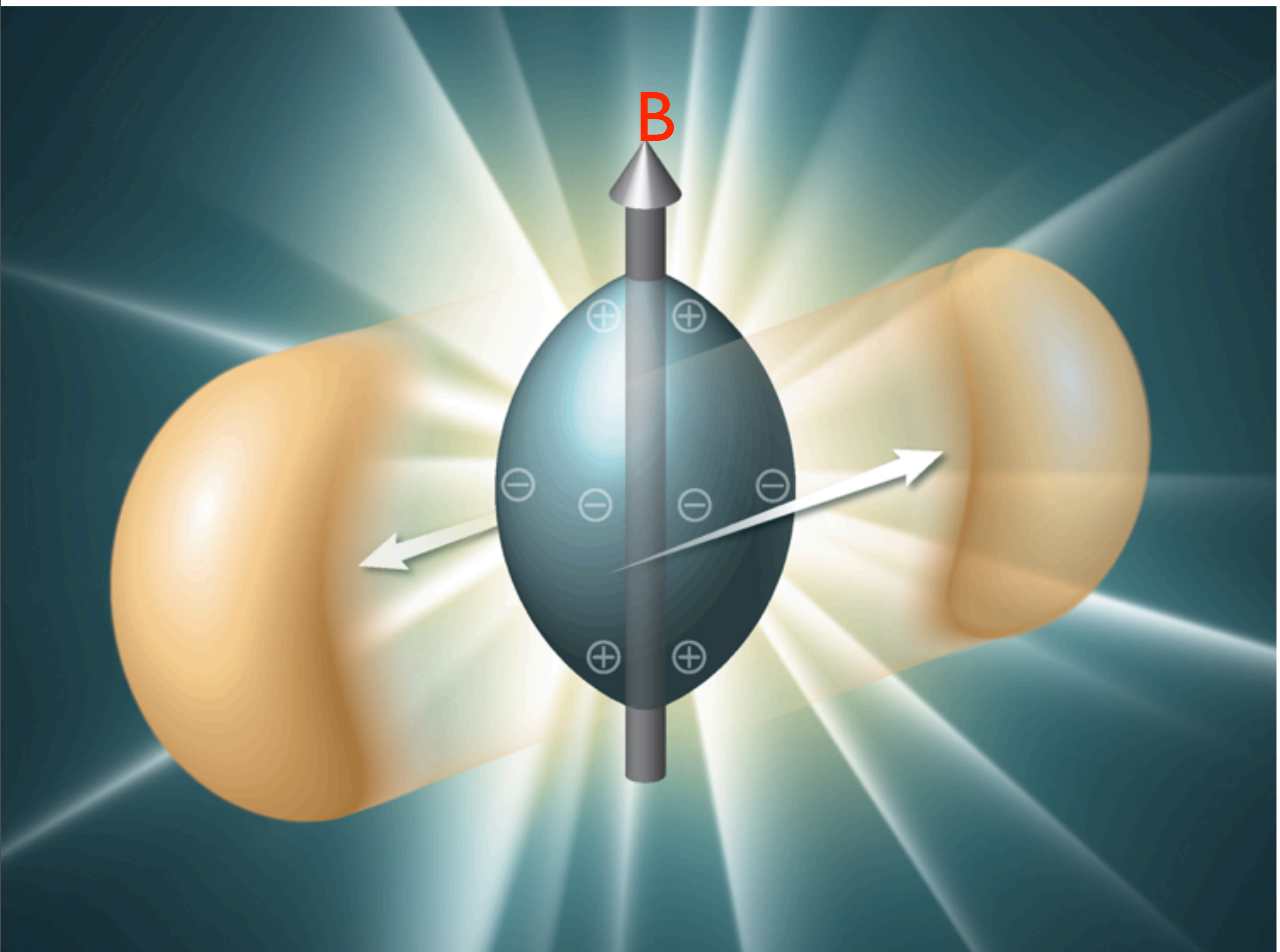
$$\vec{J}_5 = \frac{\mu}{2\pi^2}\vec{B}$$ Chiral Separation Effect (CSE)

$$\vec{J} = \frac{2\mu\mu_5}{\pi^2}\vec{\Omega}$$ Chiral Vortical Effects (CVE)

$$\vec{J}_5 = \left(\frac{\mu^2 + \mu_5^2}{\pi^2} + \frac{T^2}{12}\right)\vec{\Omega}$$

# Applicaton in QGP

## Induced Quadrupole moment in HICs




2 steps

$$\vec{J}_5 = \frac{\mu}{2\pi^2}\vec{B}$$

$$\vec{J} = \frac{\mu_5}{2\pi^2}\vec{B}$$

propagating wave of axial-electric charge conversion: Chiral Magnetic Wave

# Summary

- Hydrodynamics with anomalies
- Local form of 2nd law almost fixes CME, CVE
- Derivative mismatch for grav. anomaly in CVE avoided in Holography (additional direction)
- (Non)-Renormalization
- Applications: CMV

# Outlook

- Weyl Semi-metals and CME and NMR
- Edge physics (Fermi arcs) from CVE and CME
- Anomalous Hall Effect (AHE)
- Holographic model of WSM