Anomalous transport: Theory and Applications - I

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Outline

- Anomalous Triangles
- Weyl vs. Dirac
- Covariant vs. Consistent Anomalies
- Landau Levels and spectral flow
- Landau Levels and Transport

Anomalous Triangles

Weyl = Dirac/2 $$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\phi}^{\dot{\alpha}} \end{pmatrix}$$

$$\mathcal{P}_{L,R} = \frac{1}{2}\left(1 \pm \gamma_5\right) \qquad \gamma_5 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}$$

massless case:

$$\psi_\alpha \to e^{i\phi}\psi_\alpha \quad , \quad \bar{\phi}^{\dot{\alpha}} \to e^{i\chi}\bar{\phi}^{\dot{\alpha}}$$

classically conserved currents:

$$J^\mu_{L,R} = \bar{\Psi}\gamma^\mu \mathcal{P}_{L,R}\Psi \quad , \quad \partial_\mu J^\mu_{L,R} = 0$$

Anomalous Triangles

Weyl equation:

When the restmass m=0 Weyl realized that the Dirac equations decomposes into two independent simpler equations

$$(i\vec{\sigma}\vec{p})\Psi_L = E\Psi_L$$

$$(-i\vec{\sigma}\vec{p})\Psi_R = E\Psi_R$$

- σ_i are 2x2 Pauli matrices
- $\psi_{L,R}$ are a 2 component spinors
- symmetries: $\psi_{L,R} \rightarrow e^{i\alpha_{L,R}} \psi_{L,R}$
- $n_{L,R}$ separately conserved
- Spin and momentum either aligned or anti-aligned: CHIRAL
- E=|p|

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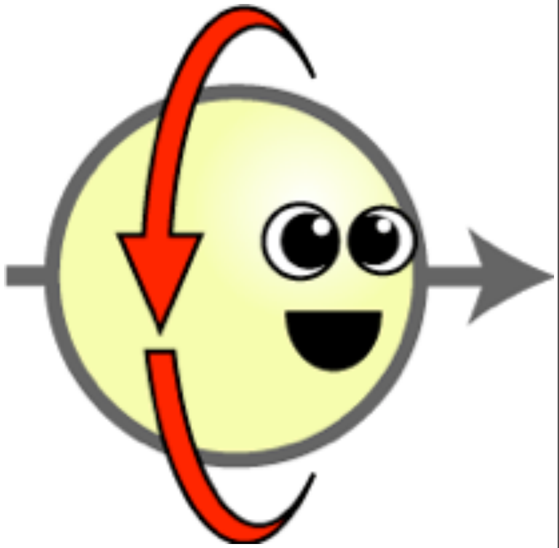

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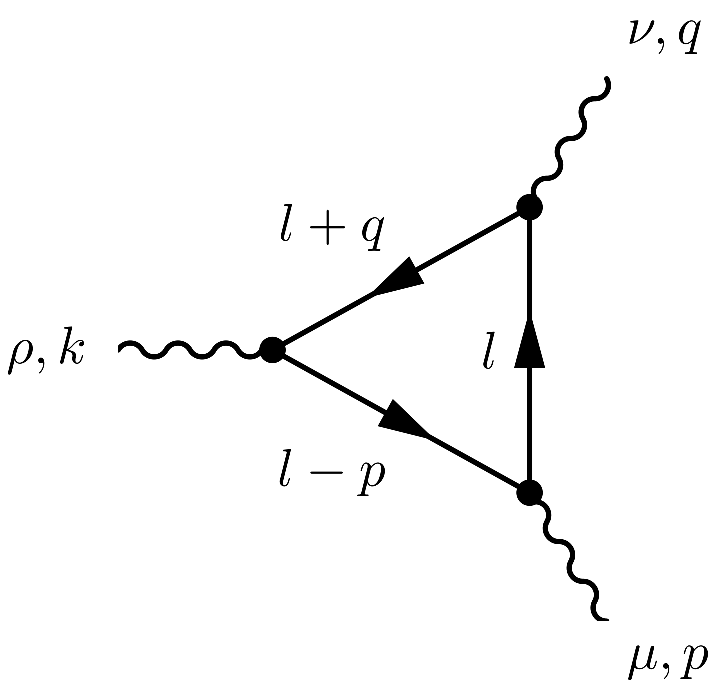
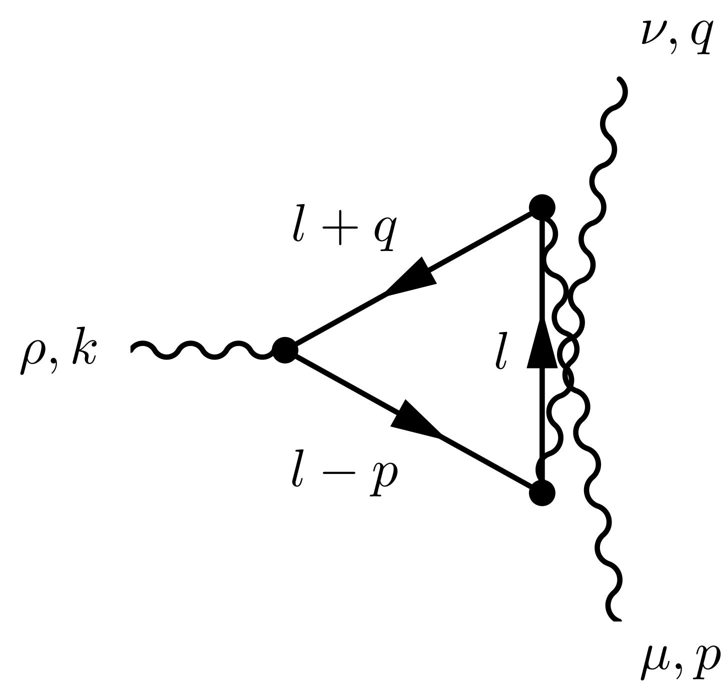
Anomalous Triangles





$$\langle J_L^\rho(k) J_L^\mu(p) J_L^\nu(q)\rangle = i \int \frac{d^4 l}{(2\pi)^4} \frac{N^{\mu\nu\rho}}{(l-p)^2 l^2 (l+q)^2}$$

naively symmetry implies:

$$k_\rho \langle J_L^\rho(k) J_L^\mu(p) J_L^\nu(q)\rangle = p_\mu \langle J_L^\rho(k) J_L^\mu(p) J_L^\nu(q)\rangle = q_\nu \rho \langle J_L^\rho(k) J_L^\mu(p) J_L^\nu(q)\rangle = 0$$

Anomalous Triangles

$$\mathrm{tr}[\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu \gamma_5] = 4i\epsilon^{\alpha\mu\beta\nu}$$

$$k_\rho A^{\rho\mu\nu}(k,p,q) = \epsilon^{\alpha\nu\beta\mu} \int \frac{d^4 l}{(2\pi)^2} \left(\frac{l_\alpha (l-p)_\beta}{(l-p)^2 l^2} - \frac{l_\alpha (l+q)_\beta}{(l+q)^2 l^2} \right) + (\mu \leftrightarrow \nu) = 0$$

$$p_\alpha p_\beta \qquad q_\alpha q_\beta$$

So far so good, but:

$$l \to l - q$$

$$p_\rho A^{\rho\mu\nu}(k,p,q) = \epsilon^{\alpha\nu\beta\mu} \int \frac{d^4 l}{(2\pi)^2} \left(\frac{l_\alpha q_\beta}{(l-p)^2 l^2} - \frac{(l-p)_\alpha (p+q)_\beta}{(l-p)^2 (l+q)^2} \right) + (\mu \leftrightarrow \nu)$$

Crux: linearly divergent integral

$$\int_{-\infty}^{\infty} dx f(x+a) = \int_{-\infty}^{\infty} f(x) + a f'(x) \Big|_{x=-\infty}^{x=+\infty}$$

Anomalous Triangles

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Anomalous Triangles

- Diagram is sensitive to momentum routing
- Most general routing

$$l^\mu \to l^\mu + c(p-q)^\mu + d(p+q)^\mu$$

$$p_\mu A^{\rho\mu\nu} = \frac{-i}{8\pi^2}(1-c)\epsilon^{\rho\nu\alpha\beta}q_\alpha p_\beta$$

$$q_\nu A^{\rho\mu\nu} = \frac{-i}{8\pi^2}(1-c)\epsilon^{\rho\mu\alpha\beta}q_\alpha p_\beta$$

$$q_\nu A^{\rho\mu\nu} = \frac{-i}{8\pi^2}2c\,\epsilon^{\rho\mu\alpha\beta}p_\alpha q_\beta$$

"When radiative corrections are finite but undetermined" [Jackiw, hep-th/9903044]

- What is the the right value for "c" ?

Anomalous Triangles

- Beyond Math: Physics constraints
- Single Weyl fermion

$$J^\mu = \frac{\delta\Gamma}{\delta A_\mu}$$

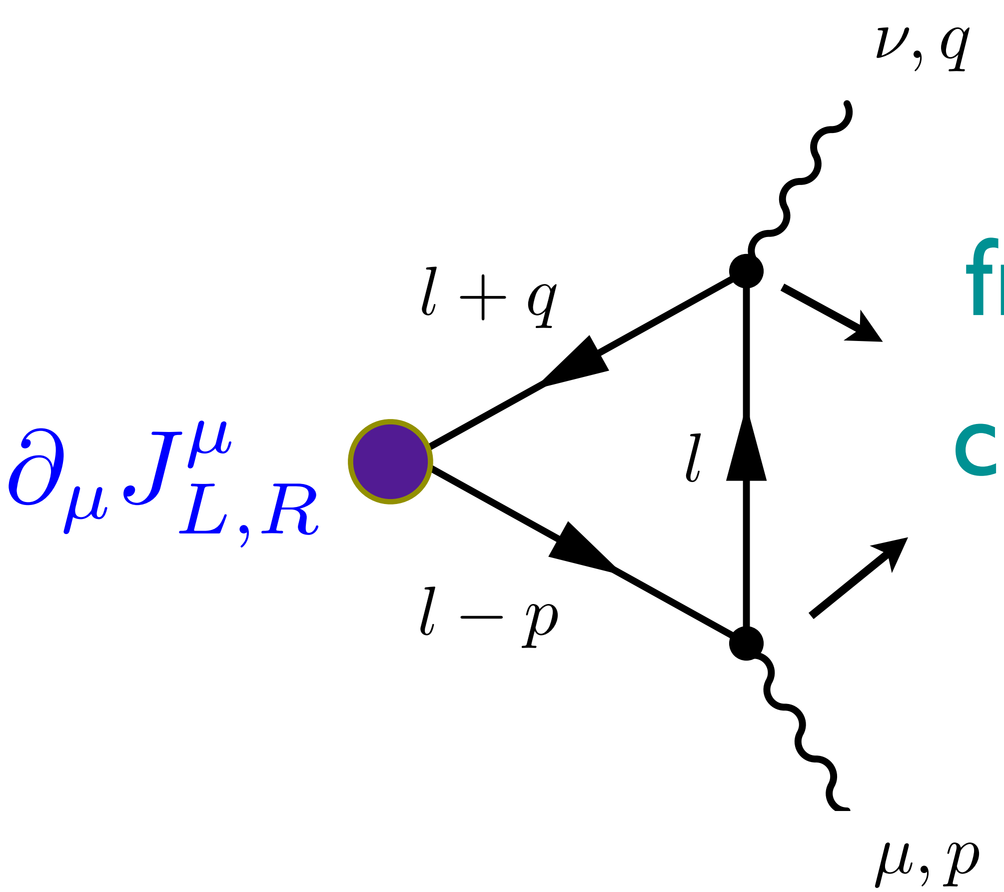
$$A^{\rho\mu\nu} = \frac{\delta^3\Gamma}{\delta A_\rho \delta A_\mu \delta A_\nu}$$

- Fixes c=1/3
- Coupling to external fields

$$\partial_\mu J^\mu_{L,R} = \pm \frac{1}{96\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$$

"consistent anomaly"

Anomalous Triangles



free of anomaly, c=1

current couples ***covariantly*** to external fields

$$\partial_\mu J^\mu_{L,R} = \pm \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$$

"covariant anomaly"

Bose symmetry is broken: $J^\mu_{\rm cov} \neq \frac{\delta\Gamma}{\delta A_\mu}$

Dirac Anomaly

$$\Psi_D = \Psi_L \oplus \Psi_R \qquad \text{(classically)}$$

Just add and subtract currents:

$$J^\mu = J^\mu_L + J^\mu_R \qquad J^\mu_5 = J^\mu_L - J^\mu_R$$

$$A^\mu = \frac{1}{2}(A^\mu_L + A^\mu_R) \qquad A^\mu_5 = \frac{1}{2}(A^\mu_L - A^\mu_R)$$

Trouble ahead:

$$\partial_\mu J^\mu = 4C\epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F^5_{\rho\lambda} \qquad C_{\text{cov}} = \frac{1}{32\pi^2}$$

$$\partial_\mu J^\mu_5 = 2C\epsilon^{\mu\nu\rho\lambda} \left(F_{\mu\nu}F_{\rho\lambda} + F^5_{\mu\nu}F^5_{\rho\lambda}\right) \qquad C_{\text{cons}} = \frac{1}{96\pi^2}$$

Non of the resulting currents is really conserved

$$\langle \partial_\mu J^\mu O_1 \cdots O_n \rangle = 0$$

Dirac Anomaly

Cure exists for consistent current: Bardeen counterterms

$$\Delta\Gamma = \int d^4x\, \epsilon^{\mu\nu\rho\lambda} A_\mu A^5_\nu \left(c_1 F_{\rho\lambda} + c_2 F^5_{\rho\lambda}\right)$$

$$J^\mu = \frac{\delta(\Gamma + \Delta\Gamma)}{\delta A_\mu} \qquad J^\mu_5 = \frac{\delta(\Gamma + \Delta\Gamma)}{\delta A^5_\mu}$$

starting from L,R symmetric treatment and chosing now $c_1 = -\frac{1}{12\pi^2} \quad , \quad c_2 = 0$

$$\partial_\mu J^\mu = 0$$

$$\partial_\mu J^\mu_5 = \epsilon^{\mu\nu\rho\lambda} \left(\frac{1}{16\pi^2} F_{\mu\nu} F_{\rho\lambda} + \frac{1}{48\pi^2} F^5_{\mu\nu} F^5_{\rho\lambda} \right)$$

WZ Consistency Condition

Anomaly: $\delta_\lambda \Gamma = \mathcal{A}_\lambda$

non-local local

$$[\delta_\lambda, \delta_\sigma]\Gamma = \delta_\lambda \mathcal{A}_\sigma - \delta_\sigma \mathcal{A}_\lambda = 0$$

BRST formalism: Grassmann valued gauge parameter "ghost"

$$\lambda \to c \quad , \quad \delta_c \to s \quad , \quad s^2 = 0$$

Cohomology problem

$$s\mathcal{A} = 0 \quad , \quad \mathcal{A} \neq s\mathcal{B}$$

consistent Anomaly = solution to consistency condition

consistent Anomaly = non-trivial element of the BRST cohomology on the space of local functionals

Covariant Anomaly

Consistent current is not gauge invariant

$$[\delta_\lambda, \frac{\delta}{\delta A_\mu}] = 0 \qquad\qquad \delta_\lambda J^\mu = \frac{\delta \mathcal{A}_\lambda}{\delta A_\mu} \neq 0$$

Define "covariant" current via

$$\delta_\lambda J^\mu = 0 \qquad\qquad \delta_\lambda J^{\mu,a} = i f^{abc} \lambda^b J^{\mu.c}$$ (non-abelian)

Solution: Bardeen-Zumino polynomial

$$J^{\mu,a}_{\mathrm{cov}} = J^{\mu,a}_{\mathrm{cons}} + Y^{\mu,a}[A]$$

chiral fermion: $$J^\mu_{L,R,\mathrm{cov}} = J^{\mu,a}_{R,L,\mathrm{cons}} \pm \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\lambda} A_\nu F_{\rho\lambda}$$

BZ polynomial is not variation of local functional

$$Y^{\mu,a}[A_\mu] \neq \frac{\delta X}{\delta A^a_\mu}$$

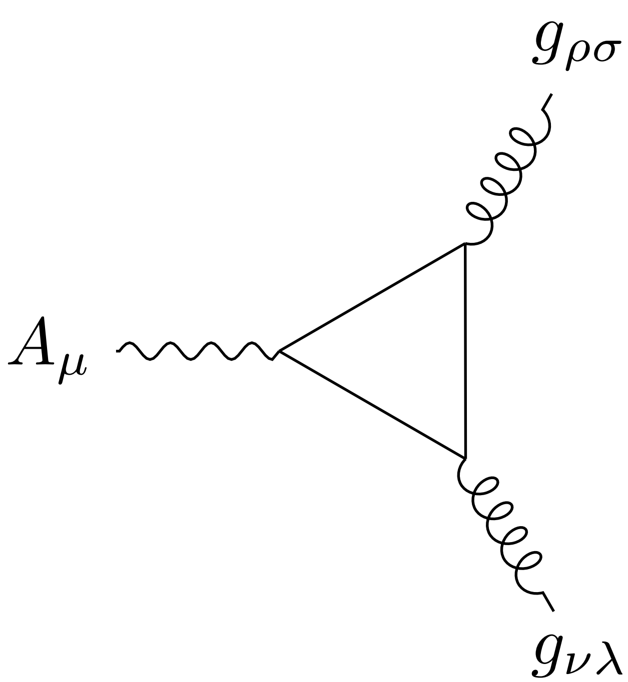
General Chiral Anomaly

$$(\nabla_\mu J^\mu)_A = \frac{d_{ABC}}{32\pi^2}\epsilon^{\mu\nu\rho\lambda}F^B_{\mu\nu}F^C_{\rho\lambda}$$

$$d_{ABC} = \mathrm{str}\left(T_A T_B T_C\right)_L - \mathrm{str}\left(T_A T_B T_C\right)_R$$

- Covariant Anomaly or totally symmetric consistent anomaly
- Consistent anomalies can be shifted between symmetries via Bardeen counterterms but not totally cancelled
- Covariant and Consistent currents are related by Bardeen-Zumino polynomials (Chern-Sions currents)

Gravitational chiral Anomaly



$$(D_\mu J^\mu_A) = \frac{b_A}{768\pi^2}\epsilon^{\mu\nu\rho\lambda}\mathcal{R}^\alpha{}_{\beta\mu\nu}\mathcal{R}^\beta{}_{\alpha\rho\lambda}$$

$$b_A = \mathrm{tr}(T_A)_L - \mathrm{tr}(T_A)_R = \sum_r q^r_A - \sum_l q^l_A$$

- No purely gravitational anomaly in 4D
- Mixed between (chiral) U(1) and diffeomorphisms
- Physical application: decay of π^0 into 2 gravitons
- Unmeasurable in HEP
- Contribution to transport (QGP, ? WSMs ?)

Anomalies and Landau levels

Chiral fermions in magnetic field:

Landau Levels: n>0

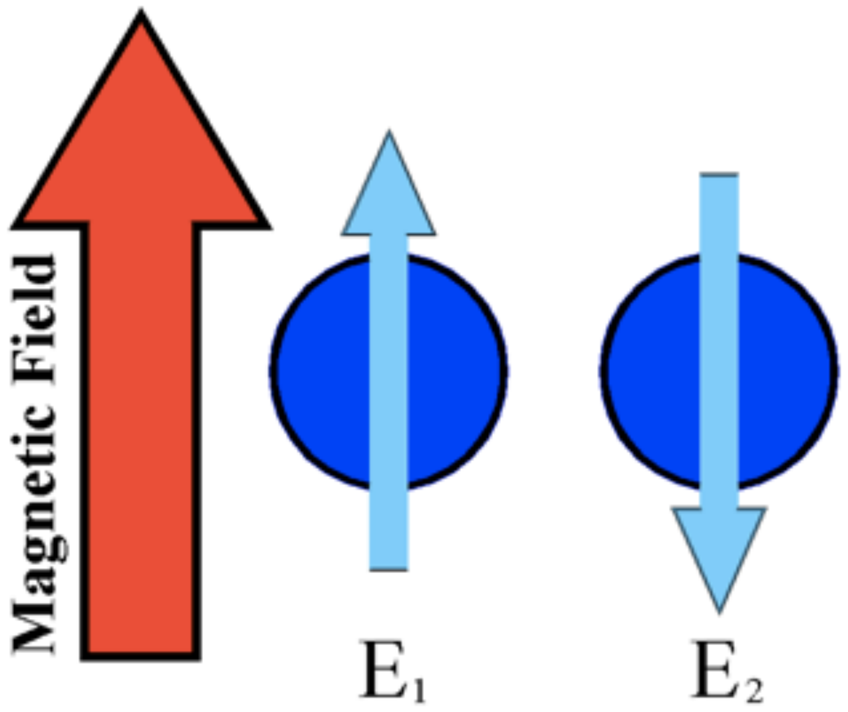
$$E_n = \pm\sqrt{p_z^2 + 2eB(n+1/2) + seB}$$

Lowest Landau Level : n=0

$$E_0 = \pm p_z$$

Sign is determined by chirality

Degeneracy of LL: $\frac{eB}{2\pi}$



Anomalies:

In additional electric field fermions feel Lorentz force

$$\frac{d}{dt}p_z = eE_z$$

The density of states changes as

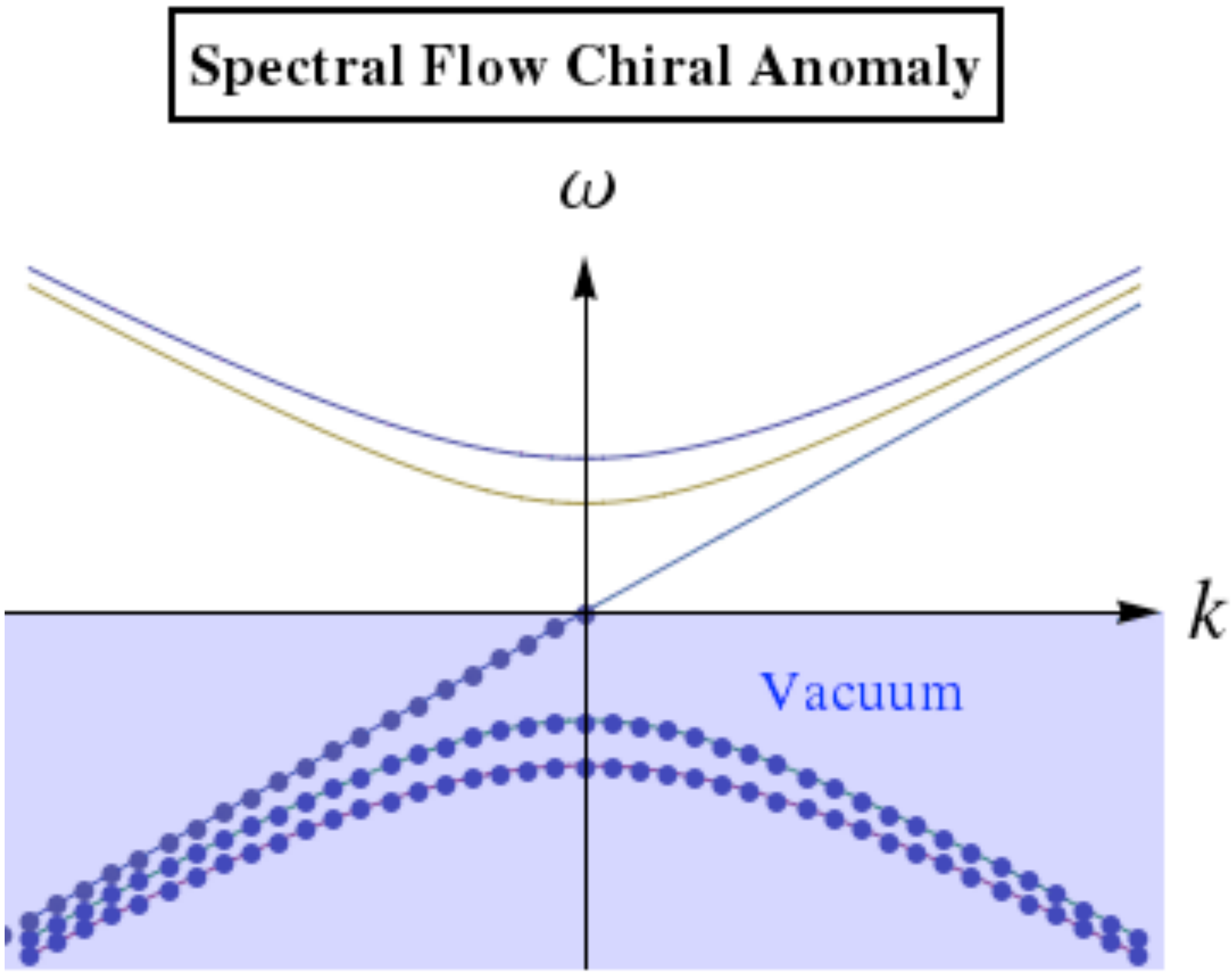
$$dn = \frac{dp}{2\pi}\frac{eB}{2\pi}$$

[Nielsen, Ninomiya], [Gribov]

Anomaly

$$\frac{dn}{dt} = \frac{e^2\vec{B}\vec{E}}{4\pi^2}$$

Anomalies:



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Anomaly →

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Anomalies:

$$n = n_R + n_L$$

$$n_5 = n_R - n_L$$

$$\frac{d(n_L + n_R)}{dt} = 0$$

$$\frac{d(n_L - n_R)}{dt} = \frac{1}{2\pi^2}(\vec{E}\vec{B})$$

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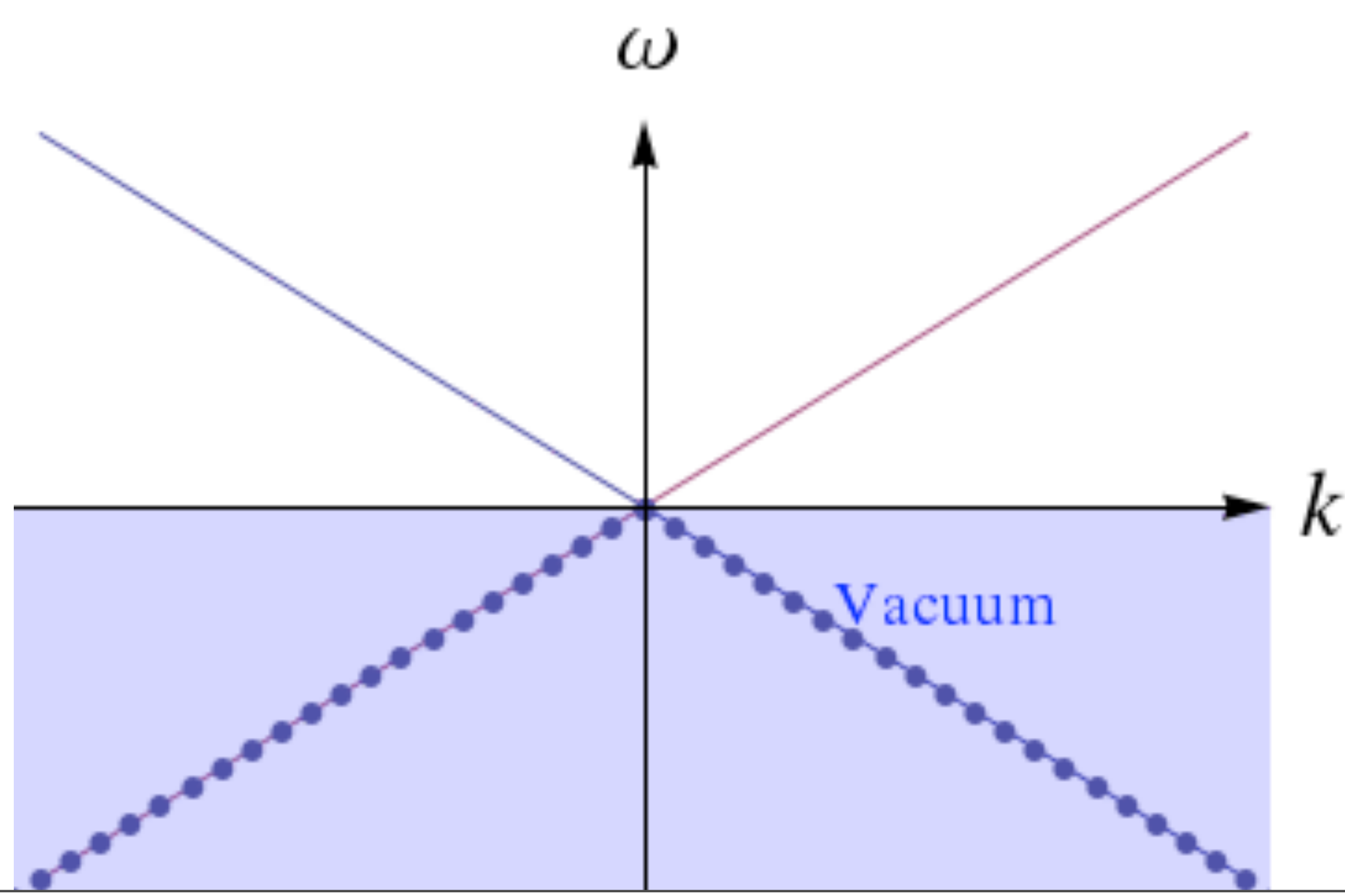
Spectral Flow Axial Anomaly



Anomalies:

sum of left and right ?

$$\frac{dn_{L,R}}{dt} = \pm\frac{1}{4\pi^2}\vec{E}\vec{B} = \pm\frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$$

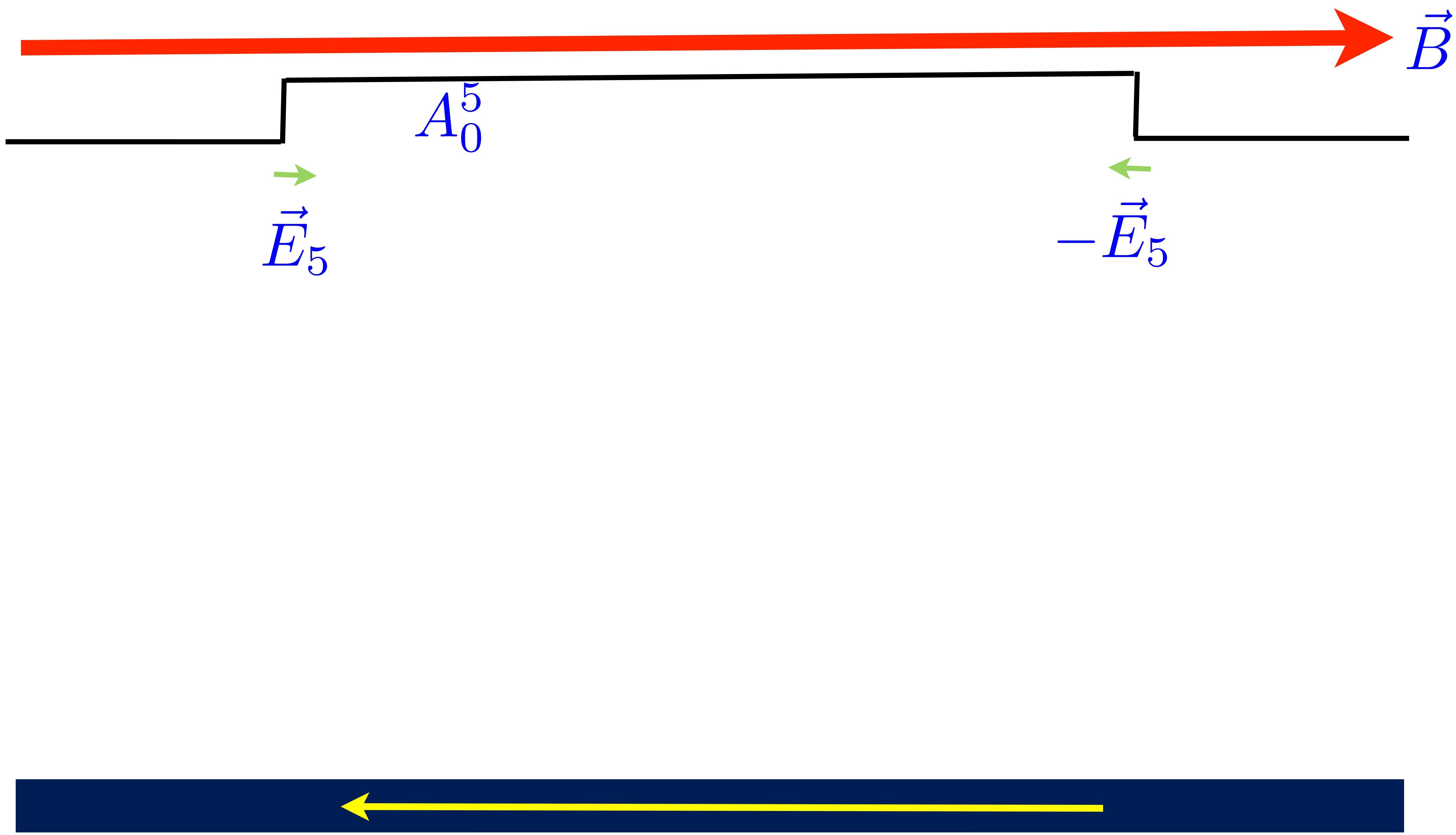
Assume independent gauge fields for left and right handed fermions

$$\frac{dn_L + n_R}{dt} = \pm\frac{1}{2\pi^2}(\vec{E}_5\vec{B} + \vec{E}\vec{B}_5)$$
$$\frac{dn_L - n_R}{dt} = \pm\frac{1}{2\pi^2}(\vec{E}\vec{B} + \vec{E}_5\vec{B}_5)$$

covariant anomaly

$$\partial\mu\partial_\nu F^{\mu\nu} = \partial_\mu J^\mu = 0$$!!!

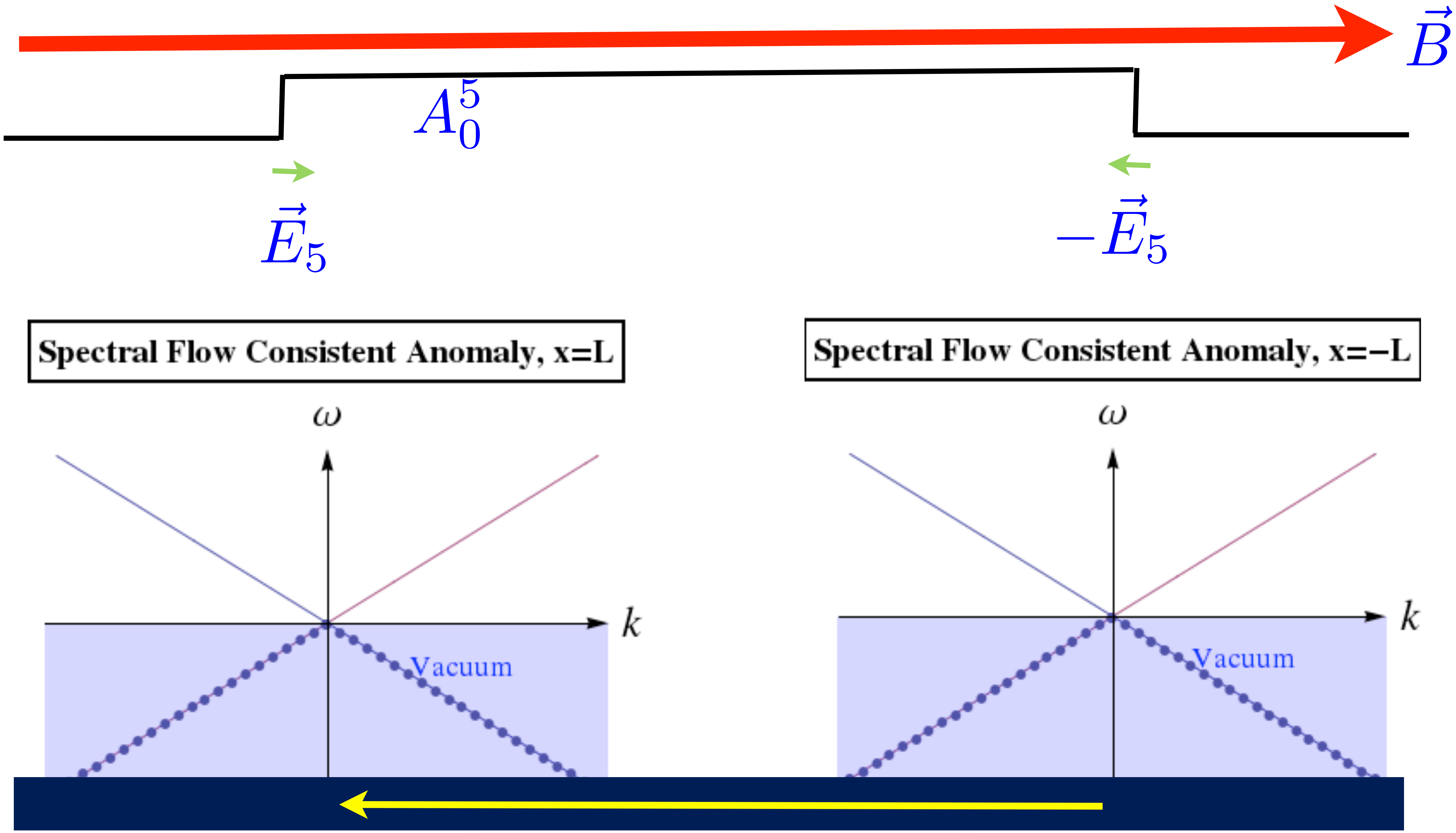
Anomalies: Covariant vs. Consistent:



$$\vec{J}_{C.S} = -\frac{1}{2\pi^2} A_0^5 \vec{B}$$

Chern-Simons current via boundary conditions at cutoff

Anomalies: Covariant vs. Consistent:



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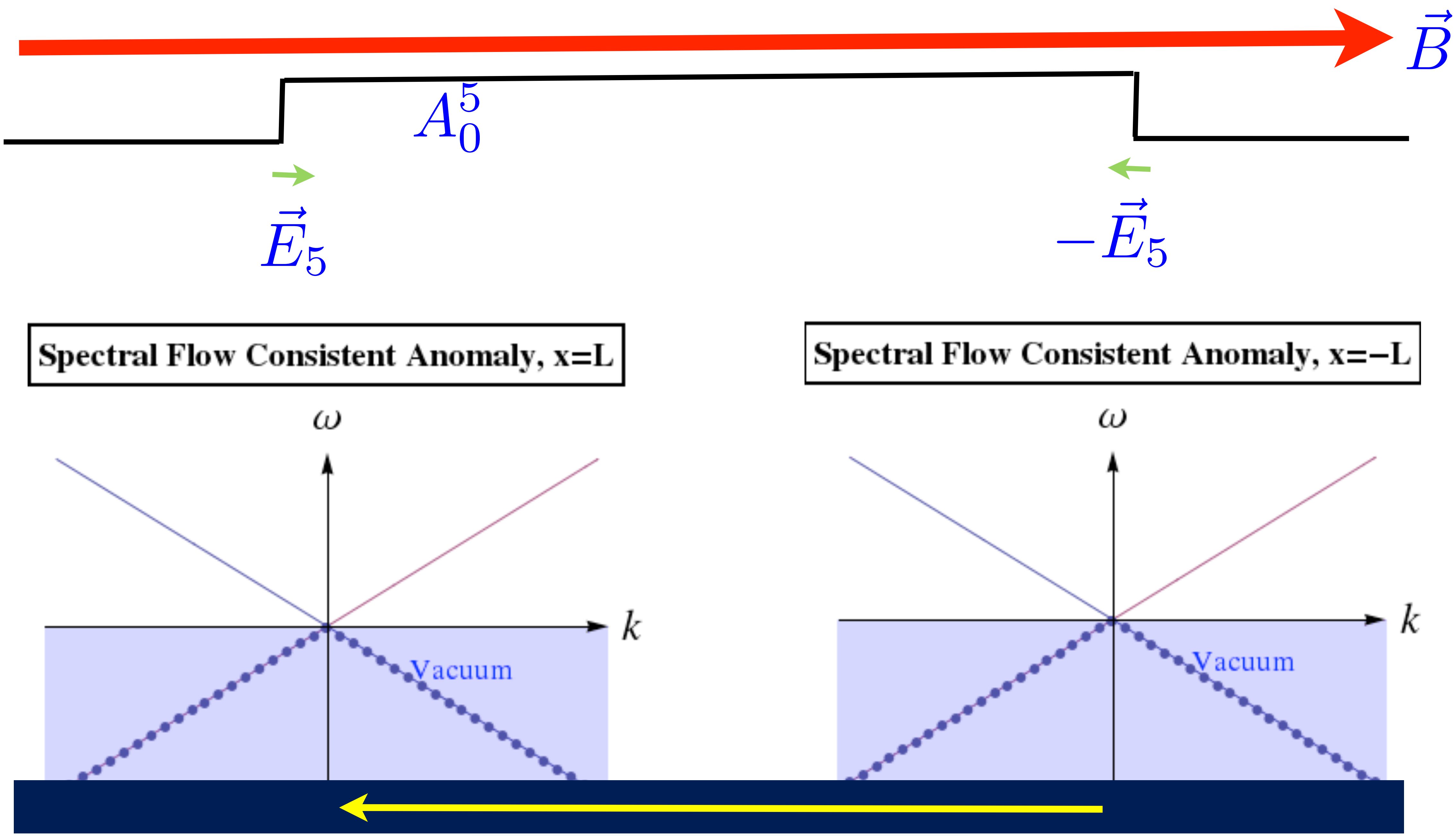

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Chern-Simons current via boundary conditions at cutoff

Anomalies: Covariant vs. Consistent:

$$J^\mu_{\rm cons} = J^\mu_{\rm cov} - \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\lambda} A^5_\nu F_{\rho\lambda}$$

$$\vec{J}_{CS} = -\frac{A^5_0}{2\pi^2}\vec{B} \qquad \vec{J}_{CS} = \frac{1}{2\pi^2}\vec{A}_5 \times \vec{E} \qquad J^0_{CS} = \frac{1}{2\pi^2}\vec{A}_5 . \vec{B}$$

- Covariant current is covariant under all "gauge' trafos (even anomalous ones)
- Consistent current is solution to Wess-Zumino consistency condition
- Consistent vector-current can be made exactly conserved: electric current
- Both are related by gauge invariant Chern-Simons current
- Consistent current automatic in gauge invariant regularization (DimReg, lattice)
- ∃ local counterterms to define consistent current (Bardeen)
- CS current = vacuum current (akin to quantum hall effect)
- Although this looks like the CME it is not (at least not all)

Landau Levels and Transport

Landau levels of chiral fermion in magnetic field

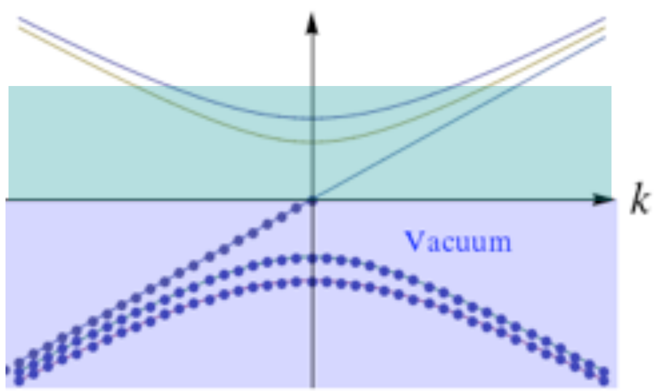
HLLs: $$E_n = \sqrt{p_z^2 + nB}$$

LLL: $$E_0 = \pm p_z$$

Current = charge.velocity

HLLs: $$\int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{\partial E_n}{\partial p_z} \left(\frac{1}{e^{\frac{E_n - \mu}{T}} + 1} - \frac{1}{e^{\frac{E_n + \mu}{T}} + 1} \right) = 0$$

LLL: $$\int_{-\infty}^{\infty} \frac{dp_z}{2\pi} (\pm 1) \left(\frac{1}{e^{\frac{E_n - \mu}{T}} + 1} - \frac{1}{e^{\frac{E_n + \mu}{T}} + 1} \right) = \frac{\mu}{2\pi}$$



Taking degeneracy of LLL into account :
Chiral Magnetic Effect (CME)

$$\vec{J}_{L,R} = \pm \frac{\mu}{4\pi^2} \vec{B}$$

Landau Levels and Transport

Energy transport = E . v = p = Momentum density

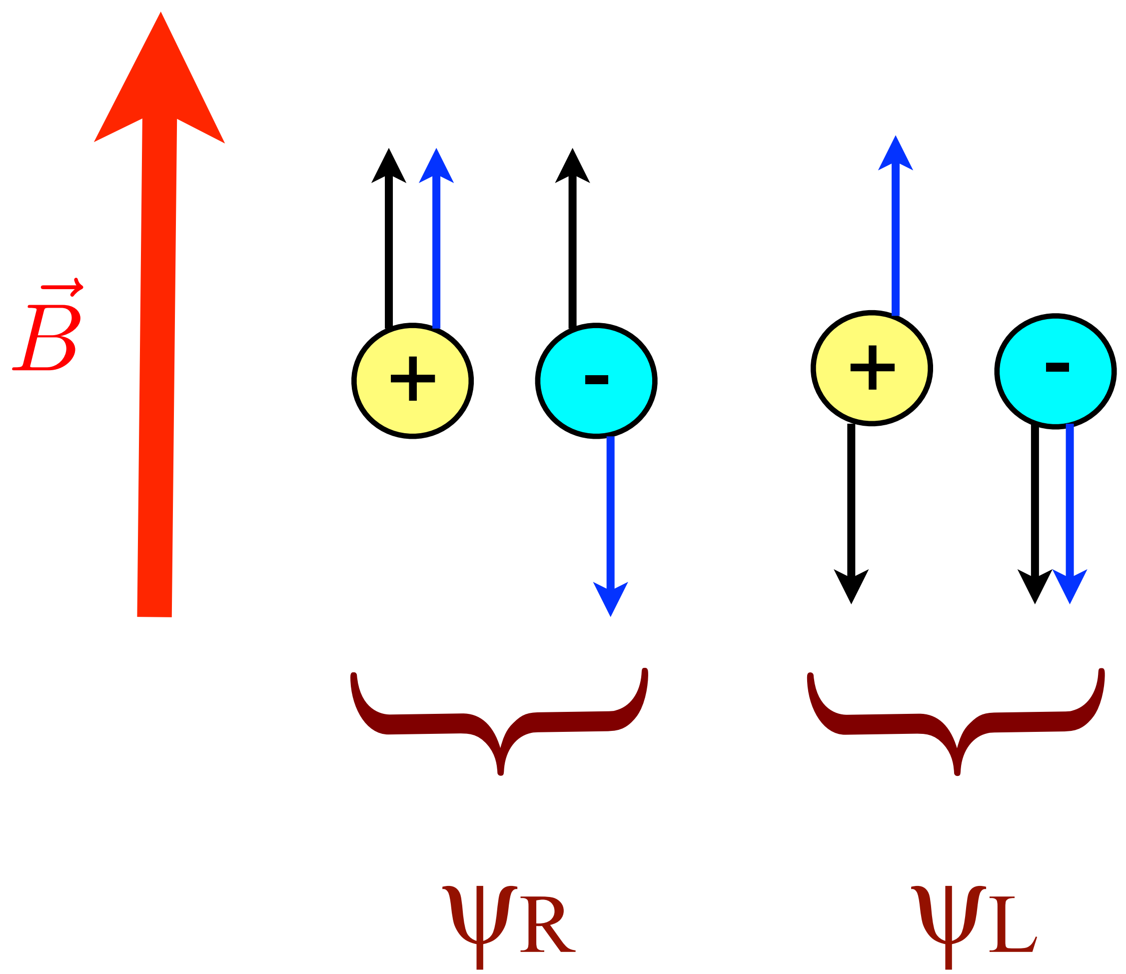
$$\vec{P}_{L,R} = \vec{J}_{\epsilon,L,R} = \pm \int_0^\infty \frac{dp_z}{2\pi} \left(\frac{p}{e^{\frac{p-\mu}{T}}+1} + \frac{p}{e^{\frac{p+\mu}{T}}+1} \right)$$
$$= \pm \frac{T^2}{2\pi} \left[Li_2(-e^{\mu/T}) + Li_2(-e^{-\mu/T}) \right]$$

Jonquiere inversion relations: Bernoulli polynomials

Chiral Magnetic Effect (CME) in energy current

$$\vec{J}_{\epsilon,L,R} = \pm \left(\frac{\mu_{L,R}^2}{8\pi^2} + \frac{T^2}{24} \right) \vec{B}$$

Chiral Magnetic Effect :



$$\vec{J}_{\mathrm{cov}} = \frac{\mu_5}{2\pi^2}\vec{B}$$

$$\mu_5 = \frac{1}{2}(\mu_R - \mu_L)$$

Axial chemical potential: counts occupied states above vacuum!

[Vilenkin], [Shaposhnikov, Giovannini], [Alekseev, Chaianov, Fröhlich] [Newman] [Kharzeev, Fukushima, Warringa],[Son,Surowka]

Rotation

Heuristics: similarity Coriolis force and Lorentz force

$$\vec{F} = 2m\vec{v} \times \vec{\Omega} \qquad \vec{F} = q\vec{v} \times \vec{B}$$

Charge is replaced by energy (E=m) $q\,\vec{B} \to E\,2\vec{\Omega}$

Chiral Vortical Effect (CVE)

$$\vec{J}_{\epsilon,L,R} = \pm \int_0^\infty \frac{dp_z}{2\pi} \left(\frac{p}{e^{\frac{p-\mu}{T}} + 1} + \frac{p}{e^{\frac{p+\mu}{T}} + 1} \right) \frac{\vec{\Omega}}{\pi} = \left(\frac{\mu_{L,R}^2}{4\pi^2} + \frac{T^2}{12} \right) \vec{\Omega}$$

$$\vec{J}_{\epsilon,L,R} = \pm \int_0^\infty \frac{dp_z}{2\pi} \left(\frac{p}{e^{\frac{p-\mu}{T}} + 1} + \frac{p}{e^{\frac{p+\mu}{T}} + 1} \right) \frac{\vec{\Omega}}{\pi} = \left(\frac{\mu_{L,R}^3}{6\pi^2} + \frac{\mu_{L,R} T^2}{6} \right) \vec{\Omega}$$

Relation to anomalies

Assume N_A different U(1) symmetries and N_f chiral fermion species with charges q^f_A

$$B \to q^f_A B_A \quad , \quad \mu \to q^f_A \mu_A \quad , \quad J \to J^f q^f_A = J_A$$

$$\vec{J}_A = d_{ABC} \frac{\mu_B}{4\pi^2} \vec{B}_C + \left(d_{ABC} \frac{\mu_B \mu_C}{4\pi^2} + b_A \frac{T^2}{12} \right) \vec{\Omega}$$

$$\vec{J}_\epsilon = \left(d_{ABC} \frac{\mu_B \mu_C}{8\pi^2} + b_A \frac{T^2}{24} \right) \vec{B}_A + \left(d_{ABC} \frac{\mu_A \mu_B \mu_C}{6\pi^2} + b_A \frac{\mu_A T^2}{6} \right) \vec{\Omega}$$

$$d_{ABC} = \sum_r q^r_A q^r_B q^r_C - \sum_l q^l_A q^l_B q^l_C \qquad\qquad b_A = \sum_r q^r_A - \sum_l q^l_A$$

Strong hint that currents are determined by anomalies

Examples

A-V theory: $d_{5VV} = d_{V5V} = d_{VV5} = 2$

Chiral Magnetic Effect (CME): $\vec{J} = \left(\frac{\mu_5}{2\pi^2} - \frac{A_0^5}{2\pi^2} \right) \vec{B}$

Chiral Separation Effect (CME): $\vec{J}_5 = \frac{\mu}{2\pi^2} \vec{B}$

Axial Magnetic Effect (AME): $\vec{J} = \frac{\mu}{2\pi^2} \vec{B}_5$

We will discuss applications of all three !

Summary:

- Triangle finite but regularization dependent ("finite but undetermined")
- Physical condition fixes consistent anomaly
- Covariant vs. Consistent current, Chern-Simons current
- Anomalies (covariant) and Landau Levels (spectral flow)
- Chiral Magnetic and Chiral Vortical effects from Lowest Landau Level
- Relation to Anomalies indicated by anomaly coefficients

Outlook:

- Relativistic Hydrodynamics
- CME and CVE from from 2nd law
- Application: Quark Gluon Plasma
- Application: Negative Magneto Resistivity (NMR)