Thermalization of Wightman Two-Point Functions in AdS/CFT

[V. Keränen, PK: 1412.2806, 1511.08187]

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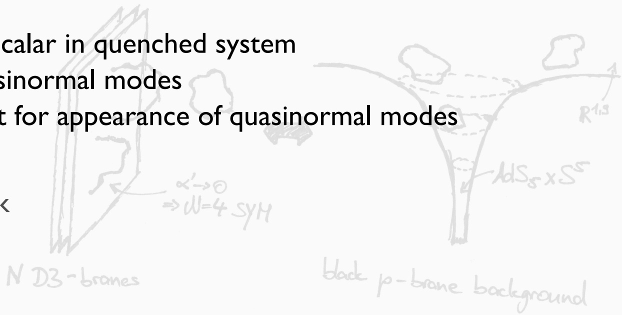
Outline

Motivation

- Why study out of equilibrium dynamics using holography?
- Why the Wightman function?

Holographic model of a quantum quench

- Global quench
- Thermalization of scalar in quenched system
- Appearance of quasinormal modes
- Analytical argument for appearance of quasinormal modes

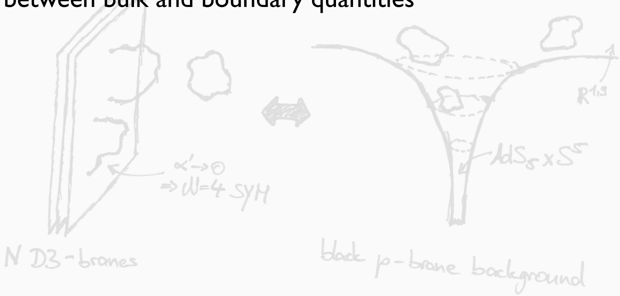


Conclusions & Outlook

Motivation

Why study far from equilibrium dynamics using holography?

- Study the holographic duality in extreme environments:
  - Well understood in and close to equilibrium
  - Going away from equilibrium is like testing the duality for finite N or λ
  - Test if the duality gives sensible results
  - Explore dictionary between bulk and boundary quantities



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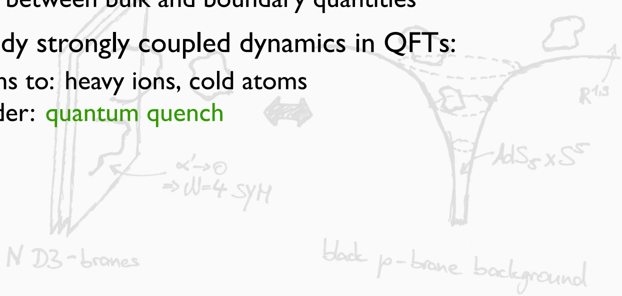
  - Going away from equilibrium is like testing the duality for finite N or λ
  - Test if the duality gives sensible results
  - Explore dictionary between bulk and boundary quantities
- Obtain a tool to study strongly coupled dynamics in QFTs:
  - Possible applications to: heavy ions, cold atoms
  - Example we consider: quantum quench

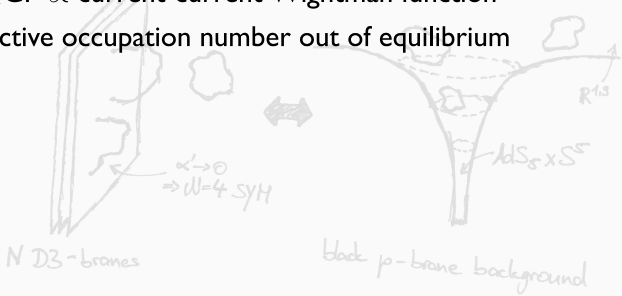


Motivation

Why do we use the Wightman function to study thermalization?

Questions:

- One-point function: $t_{eq}T \approx 1 \leftrightarrow$ non-local probes: $t_{eq} \propto l$
- Two-point functions only computed in geodesic approximation.
- Wightman functions quantify particle production rates, e.g. photon production in the QGP $\propto$ current-current Wightman function
- Definition of an effective occupation number out of equilibrium



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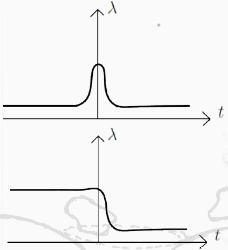
What we will see: Wightman functions in Fourier space thermalize on the same time scale as one-point functions set by the lowest quasinormal mode $\rightarrow$ unified picture of thermalization times?

Holographic model of a quantum quench

Global quench – field theory

Quantum quench: Prepare quantum system in vacuum, excite by injecting energy at $t = 0$.

Simple example: homogeneous, isotropic (global) quench



$$H(t) = H_0 + \lambda(t) \int d^3x \, \mathcal{O}(\boldsymbol{x})$$

Corresponds to an injection of energy equally distributed over space.

Interesting because

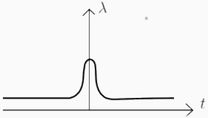
- Quenches can be studied experimentally in condensed matter and cold atom systems
- In these experiments, correlation functions can be measured
- Simplest far-from-equilibrium model

Holographic model of a quantum quench

Global quench – holographic model

A holographic model of a global quench is given by Vaidya spacetime:

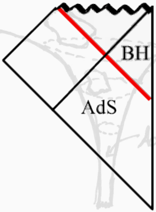
- Describes a collapsing shell of null matter forming a black hole



- Metric

$$ds^2 = \frac{1}{z^2}\left[-(1-\theta(v)z^2)dv^2 - 2dvdz + dx^2\right]$$

For $v < 0$ this is pure AdS_3 and for $v > 0$ the BTZ Black hole



- Does not solve the vacuum Einstein equations, but needs a source. It is however a good approximation of real scalar collapse

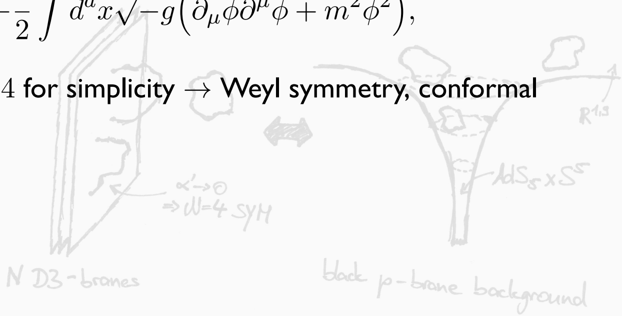
Holographic model of a quantum quench

Thermalization of scalar in quenched system

- We want to see how scalar correlators evolve in the quenched background,

$$S = -\frac{1}{2}\int d^d x \sqrt{-g}\Big(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2\Big),$$

and take $m^2 = -3/4$ for simplicity $\rightarrow$ Weyl symmetry, conformal coupling



Holographic model of a quantum quench

Thermalization of scalar in quenched system

- Consider the Wightman two-point function

$$G_+(x_2, x_1) = \langle \phi(x_2)\phi(x_1) \rangle$$

- We choose the vacuum state in AdS as initial state, so for $v_1, v_2 < 0$, G_+ is the AdS scalar vacuum correlator

$$G_+^{\text{AdS}}(v_2, x_2, z_2; v_1, x_1, z_1) = \frac{\sqrt{z_1 z_2}}{4\pi}\left(\frac{1}{\sqrt{-\Delta v^2 - 2\Delta v \Delta z + \Delta x^2 + i\theta(v2-v1)\epsilon}} - \frac{1}{\sqrt{-\Delta v^2 - 2\Delta v \Delta z + 4 z_1 z_2 + \Delta x^2 + i\theta(v2-v1)\epsilon}}\right)$$

where $\Delta v = v_2 - v_1$, $\Delta x = x_2 - x_1$, $\Delta z = z_2 - z_1$.

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- G_+ obeys equations of motion:

$$(\Box_1 - m^2)G_+(x_2, x_1) = 0 = (\Box_2 - m^2)G_+(x_2, x_1)$$

which are a set of six-dimensional PDEs

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which are a set of six-dimensional PDEs

- Translation invariance in spatial direction $\rightarrow$ Fourier transform $\rightarrow$ four-dimensional PDE

Holographic model of a quantum quench

Thermalization of scalar in quenched system

We use a trick to solve this initial value problem:

- Method of Green's functions for linear differential equation

$$\phi(v_2, z_2, k) = i \int_{v_1=\text{const}} dz_1\, \phi(v_1, z_1, k)\, \overleftrightarrow{D}^{v_1} G_R(v_2, z_2; v_1, z_1; k)$$

where $D^v = \sqrt{-g} g^{v\nu} \partial_\nu$

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- Choose $v_1 = 0$, then $G_R = G_R^{BTZ}$

Holographic model of a quantum quench

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$$\phi^b(v_2,k) = i\int_0^{z_a^*} dz_1\,\phi(v_1,z_1,k)\,\overleftrightarrow{D}^{v_1} G_R^{bb}(v_2;v_1,z_1;k)$$

  where $D^v = \sqrt{-g}g^{v\nu}\partial_\nu$ and $\phi^b(v,k) = \sqrt{2\pi}\lim_{z\to 0} z^{-\Delta}\phi(v,z,k)$
- Choose $v_1 = 0$, then $G_R = G_R^{BTZ}$
- Take the boundary limit $z_2 \to 0$:

$$G_R^{bb}(v;0,z_1;k) = \sqrt{2\pi}\lim_{z_2\to 0} z_2^{-\Delta} G_R^{BTZ}(v,z_2;0,z_1;k)$$

$$= i\sqrt{2\pi}\Big[\tfrac{(1+4k^2)z_1^{\frac{3}{2}}}{8}\theta\left(z_a^*-z_1\right)\,{}_2F_1\Big(\tfrac{3}{4}-\tfrac{ik}{2},\tfrac{3}{4}+\tfrac{ik}{2},2,1-(\cosh v - z_1\sinh v)^2\Big)$$

$$-\tfrac{\sqrt{z_a^*}}{1+\cosh v}\delta\left(z_a^*-z_1\right)\Big]$$

where $z_a^* = \tanh\frac{v}{2}$

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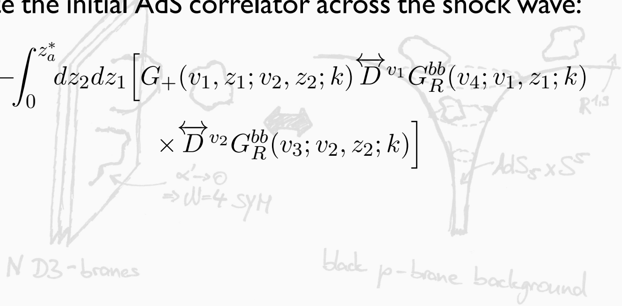
where $D^v = \sqrt{-g}g^{v\nu}\partial_\nu$

- Use this to propagate the initial AdS correlator across the shock wave:

$$G_+^{CFT}(v_4,v_3,k) = -\int_0^{z_a^*} dz_2 dz_1 \Big[G_+(v_1,z_1;v_2,z_2;k)\overleftrightarrow{D}^{v_1} G_R^{bb}(v_4;v_1,z_1;k) \times \overleftrightarrow{D}^{v_2} G_R^{bb}(v_3;v_2,z_2;k)\Big]$$



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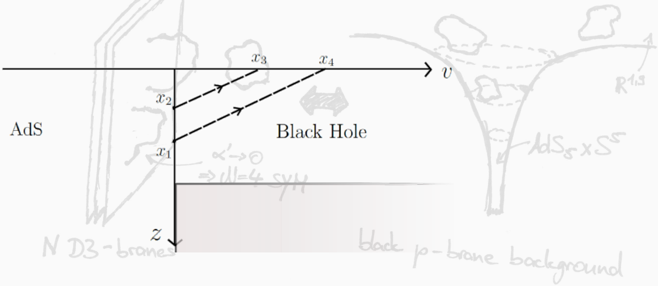

Holographic model of a quantum quench

Thermalization of scalar in quenched system

Problem: $G_+(x_1, x_2)$ diverges for lightlike separations, i.e.

$$G_+^{AdS}(x_1, x_2)\Big|_{v_1, v_2 = 0} = \infty$$

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Solution: Consider $\delta G_+(x_1, x_2) = G_+(x_1, x_2) - G_+^{\text{thermal}}(x_1, x_2)$

- $\delta G_+(x_1, x_2)$ obeys the same equations of motion
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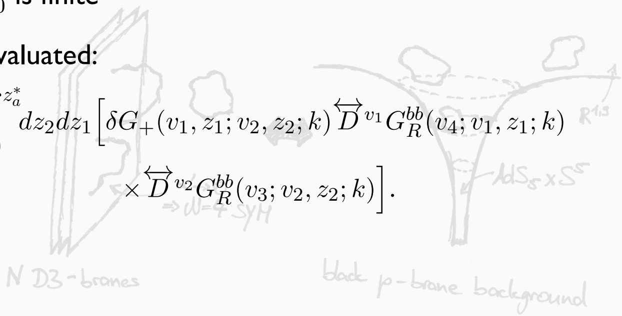
Simple integral to be evaluated:

$$\delta G_+^{CFT}(v_4, v_3, k) = -\int_0^{z_a^*} dz_2 dz_1 \Big[\delta G_+(v_1, z_1; v_2, z_2; k) \overleftrightarrow{D}^{v_1} G_R^{bb}(v_4; v_1, z_1; k) \times \overleftrightarrow{D}^{v_2} G_R^{bb}(v_3; v_2, z_2; k) \Big].$$



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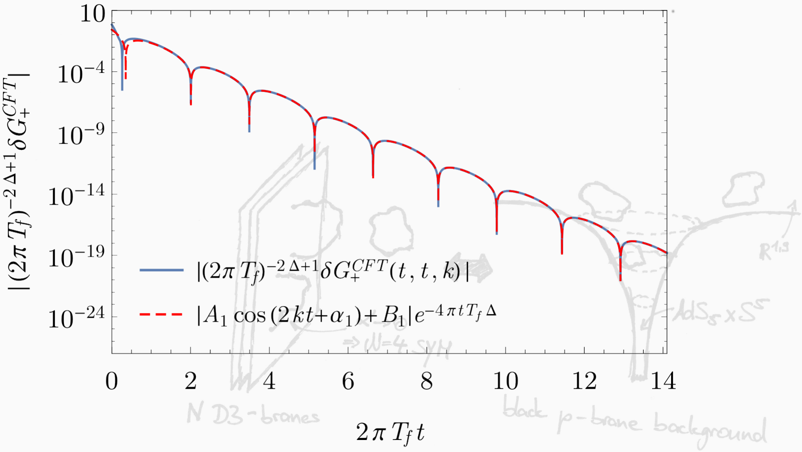
where

$$\delta\tilde{G}_+(z', z, k) = \left(\tfrac{1}{z^2 z'^2} - \tfrac{2}{z z'^2}\partial_z - \tfrac{2}{z^2 z'}\partial_{z'} + \tfrac{4}{z z'}\partial_z \partial_{z'}\right)\delta G_+(z', 0; z; 0, k).$$

Holographic model of a quantum quench

Appearance of quasinormal modes

The boundary correlator $G_+^{CFT}(t,t;k)$ thermalizes exponentially with a rate set by the lowest quasinormal mode, $\omega_0 = \pm k - i2\pi T_f \Delta$



Holographic model of a quantum quench

Analytical argument for appearance of quasinormal modes

QNMs are poles of the retarded correlator in Fourier space:

$$G_R^{bb}(v;v',z';k) \approx \sum_n c_n(z') e^{-i\omega_*^{(n)}(v-v')}, \qquad \omega_*^{(n)} = \pm k - i2\pi T_f(\Delta + 2n)$$

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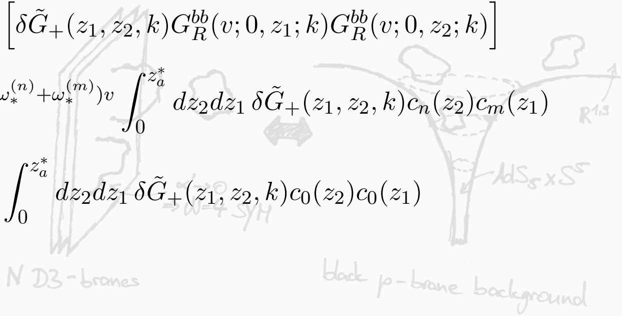
$$\begin{aligned}
\delta G_+^{CFT}(v, v, k) \\
&= -\int_0^{z_a^*} dz_2 dz_1 \left[\delta \tilde{G}_+(z_1, z_2, k) G_R^{bb}(v; 0, z_1; k) G_R^{bb}(v; 0, z_2; k)\right] \\
&= -\sum_{m,n} e^{-i(\omega_*^{(n)}+\omega_*^{(m)})v} \int_0^{z_a^*} dz_2 dz_1\, \delta \tilde{G}_+(z_1, z_2, k) c_n(z_2) c_m(z_1) \\
&\approx -e^{-2i\omega_*^{(0)}v} \int_0^{z_a^*} dz_2 dz_1\, \delta \tilde{G}_+(z_1, z_2, k) c_0(z_2) c_0(z_1)
\end{aligned}$$



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Only valid for $e^{-(v-v')} \ll \zeta$ where ζ = distance from lightcone

$$\begin{aligned}
&\delta G_+^{CFT}(v, v, k) \\
&= -\int_0^{z_a^*} dz_2 dz_1 \left[\delta\tilde{G}_+(z_1, z_2, k) G_R^{bb}(v; 0, z_1; k) G_R^{bb}(v; 0, z_2; k)\right] \\
&= -\sum_{m,n} e^{-i(\omega_*^{(n)} + \omega_*^{(m)})v} \int_0^{z_a^*} dz_2 dz_1\, \delta\tilde{G}_+(z_1, z_2, k) c_n(z_2) c_m(z_1) \\
&\approx -e^{-2i\omega_*^{(0)} v} \int_0^{z_a^*} dz_2 dz_1\, \delta\tilde{G}_+(z_1, z_2, k) c_0(z_2) c_0(z_1)
\end{aligned}$$

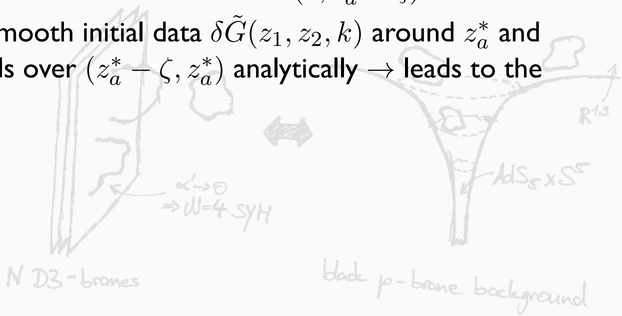
Wrong result!

Holographic model of a quantum quench

Analytical argument for appearance of quasinormal modes

Strategy:

- Split up the integrals into subintervals $(0, z_a^* - \zeta)$ and $(z_a^* - \zeta, z_a^*)$
- Use the QNM approximation in the interval $(0, z_a^* - \zeta)$
- Taylor expand the smooth initial data $\delta\tilde{G}(z_1, z_2, k)$ around z_a^* and perform the integrals over $(z_a^* - \zeta, z_a^*)$ analytically $\rightarrow$ leads to the right fall-off



Holographic model of a quantum quench

Analytical argument for appearance of quasinormal modes

$$
\begin{aligned}
\delta G_+^{CFT}(v;v;k) &= -\int dz dz'\, G_R^{bb}(v;0,z';k) G_R^{bb}(v;0,z;k)\delta\tilde{G}_+(z',z,k) \\
&= -\int_0^{z_a^*-\zeta} dz dz'\, G_R^{bb}(v;0,z';k) G_R^{bb}(v;0,z;k)\delta\tilde{G}_+(z',z,k) \\
&\quad -2\left(\int_{z_a^*-\zeta}^{z_a^*} \frac{dz}{z^{\frac{3}{2}}} G_R^{bb}(v;0,z;k)\right)\int_0^{z_a^*-\zeta} dz'\, G_R^{bb}(v;0,z';k)\left({z_a^*}^{\frac{3}{2}}\delta\tilde{G}_+(z',z_a^*,k)+\mathcal{O}(\zeta)\right) \\
&\quad -\left(\int_{z_a^*-\zeta}^{z_a^*} \frac{dz}{z^{\frac{3}{2}}} G_R^{bb}(v;0,z;k)\right)^2\left({z_a^*}^3\delta\tilde{G}_+(z_a^*,z_a^*,k)+\mathcal{O}(\zeta)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
&G_R^{bb}(v;0,z_1;k) \\
&= i\sqrt{2\pi}\left[\frac{(1+4k^2)\, z_1^{\frac{3}{2}}}{8}\theta\left(z_a^*-z_1\right)\ {}_2F_1\!\left(\frac{3}{4}-\frac{ik}{2},\frac{3}{4}+\frac{ik}{2},2,1-(\cosh v-z_1\sinh v)^2\right)-\frac{\sqrt{z_a^*}}{1+\cosh v}\delta\left(z_a^*-z_1\right)\right]
\end{aligned}
$$

The near-lightcone integral is

$$
\int_{z_a^*-\zeta}^{z_a^*}\frac{dz}{z^{\frac{3}{2}}}G_R^{bb}(v;0,z;k)=\frac{i\cosh(\pi k)}{\sqrt{2}\pi\sinh v}F(1+\zeta\sinh v)=-i\left(\frac{2^{\frac{3}{2}}\zeta^{-\frac{1}{2}-ik}\Gamma(-ik)e^{-ikv}}{\Gamma\left(\frac{1}{2}-ik\right)}+c.c.\right)e^{-\frac{3}{2}v}+\mathcal{O}\left(\left(\zeta^{-1}e^{-v}\right)^{\frac{5}{2}}\right)
$$

Holographic model of a quantum quench

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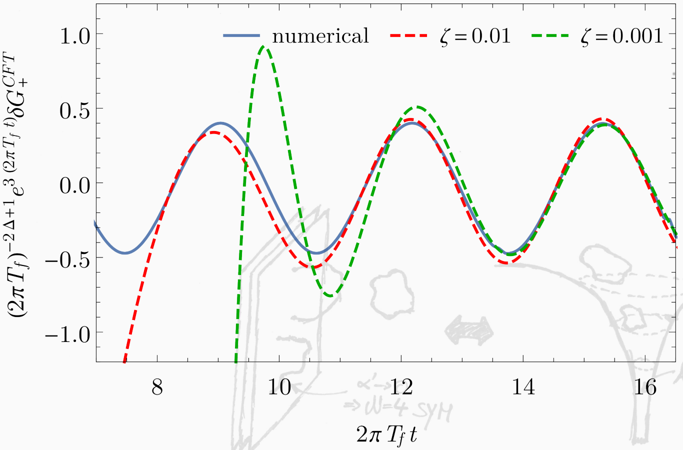
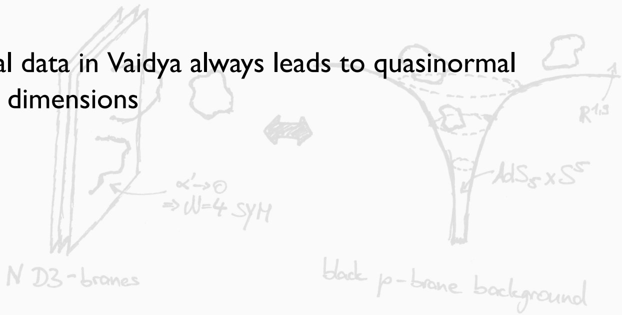


Conclusions & Outlook

We have seen:

- We applied the non-equilibrium dictionary for two-point functions to a quenched system and obtained reasonable results
- Far-from-equilibrium perturbations thermalize on the same time scale as infinitesimal perturbations: quasinormal modes
- Wightman functions in Fourier space thermalize on the same time scale as one-point functions $\rightarrow$ unified picture of thermalization times?
- Claim: Smooth initial data in Vaidya always leads to quasinormal decay, also in higher dimensions



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Open questions:

- Generalize the quench calculation to higher dimensions: retarded correlators in black hole backgrounds not known
- More realistic models of gravitational collapse