Applications of On-Shell Physics

Jacob L. Bourjaily

Cracow School of Theoretical Physics
LVI Course, 2016
A Panorama of Holography

The Niels Bohr International Academy

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Organization and Outline

Spiritus Movens: One-Loop Generalized Unitarity

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The *Scalar* Box Decomposition

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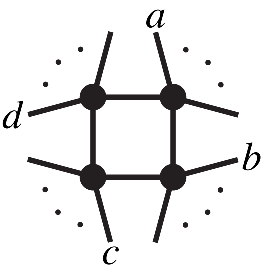

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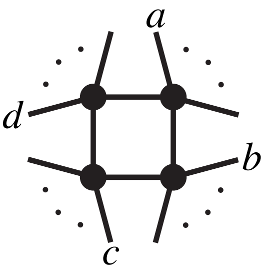

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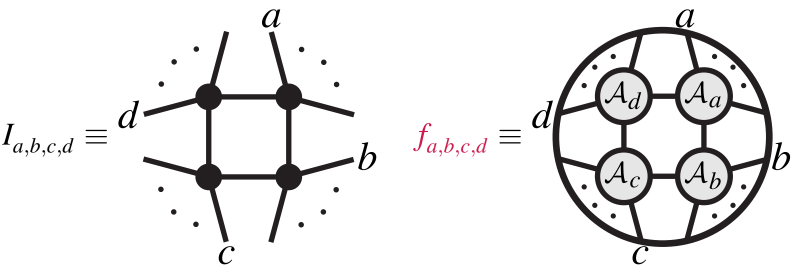

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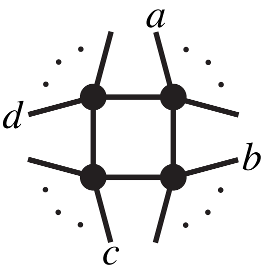

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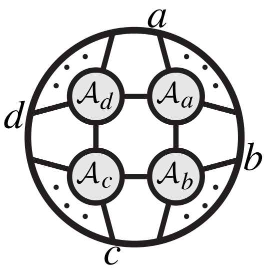

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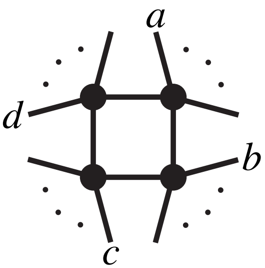

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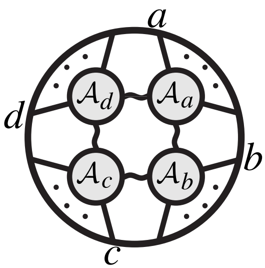

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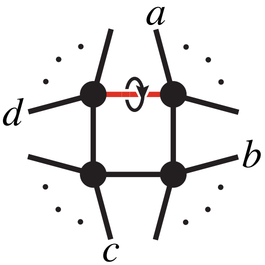

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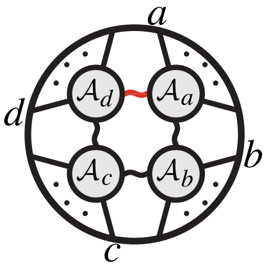

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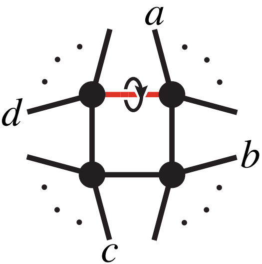

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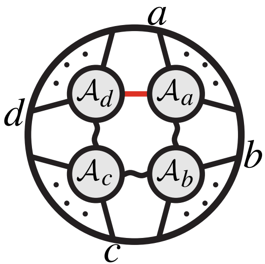

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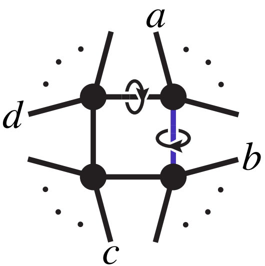

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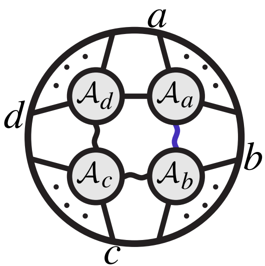

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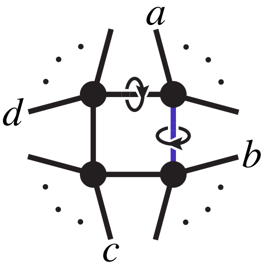
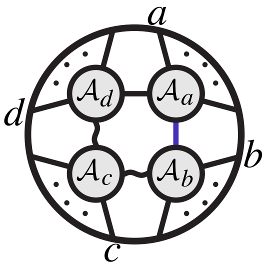
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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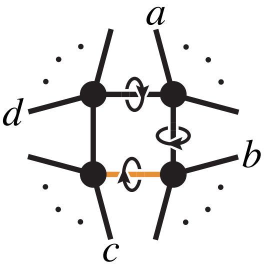

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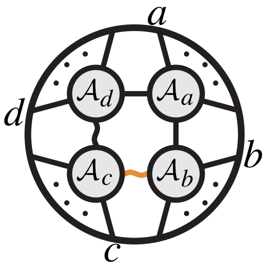

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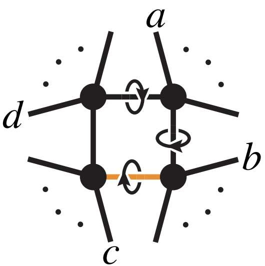

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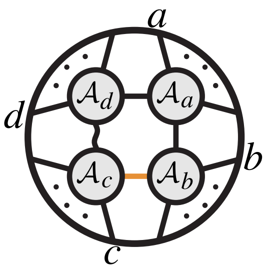

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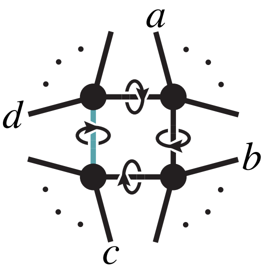

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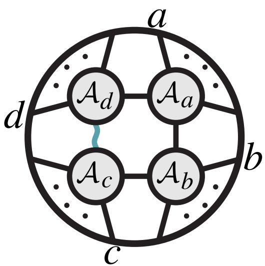

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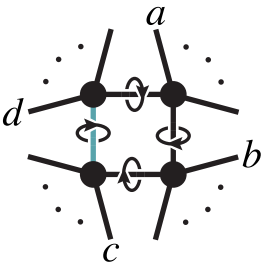

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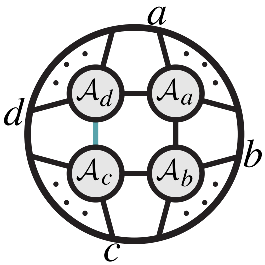

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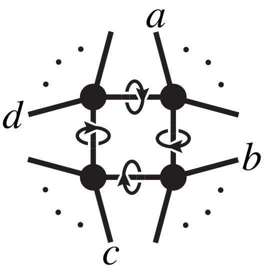

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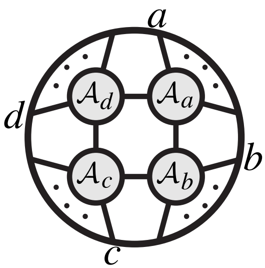

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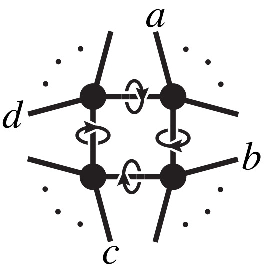

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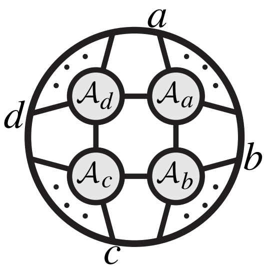

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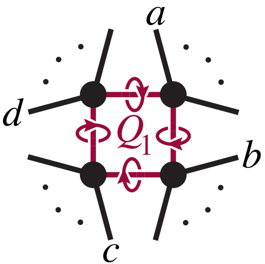
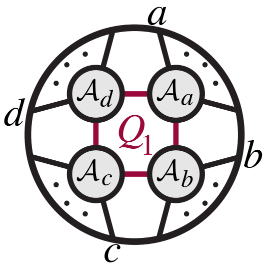
$I_{a,b,c,d} \equiv$ 

$f^1_{a,b,c,d} \equiv$ 

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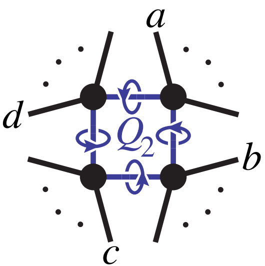

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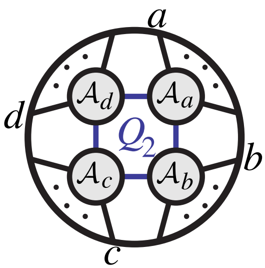

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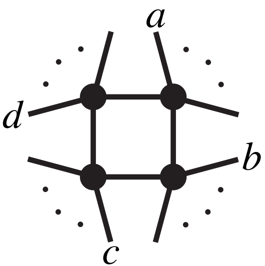
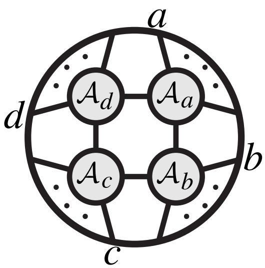
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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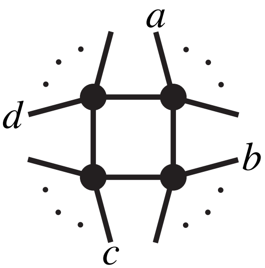

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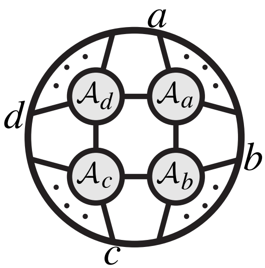

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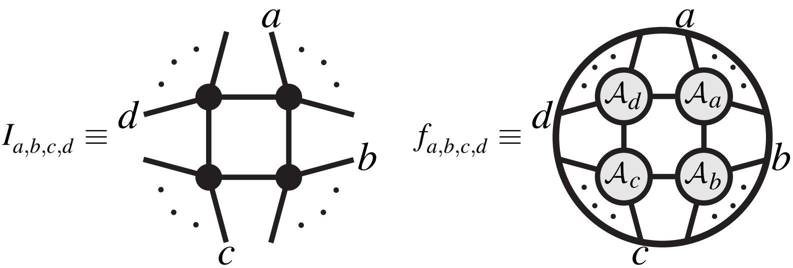

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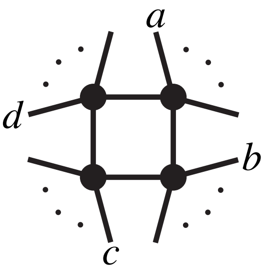
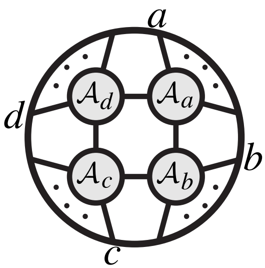
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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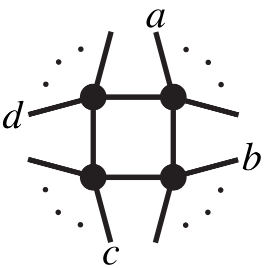

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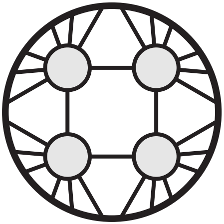

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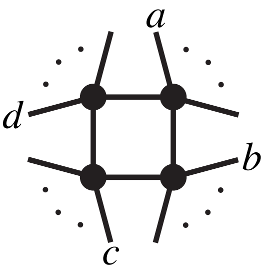

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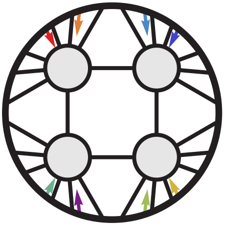

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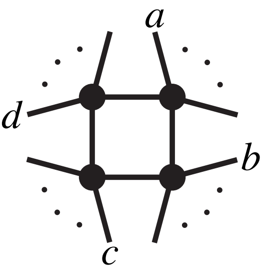
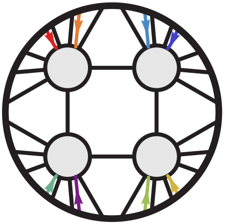
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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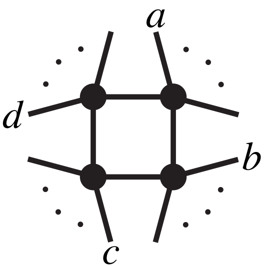

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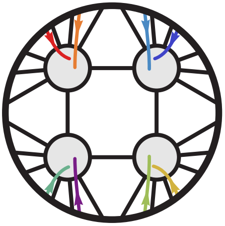

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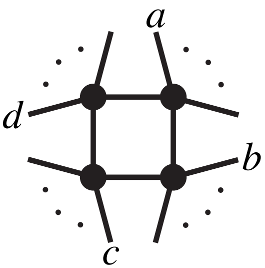
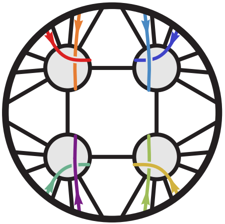
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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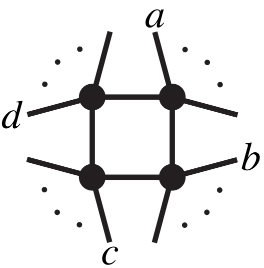
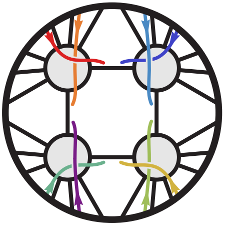
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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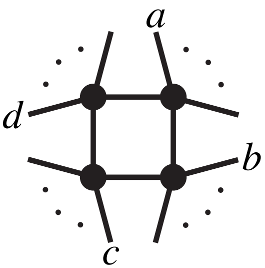
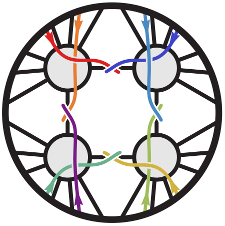
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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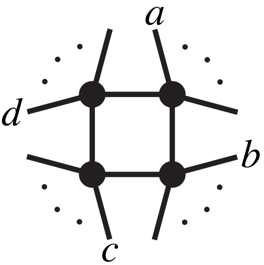

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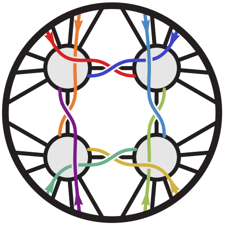

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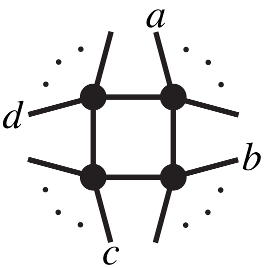

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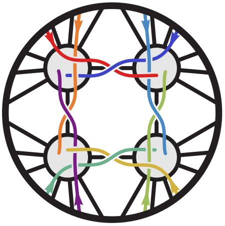

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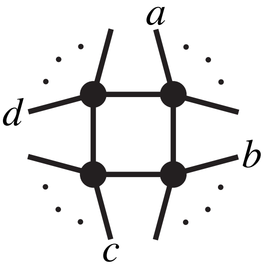

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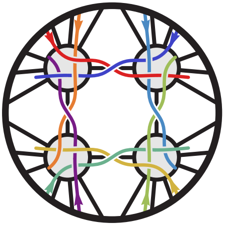

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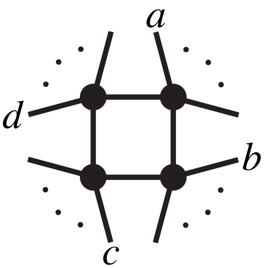

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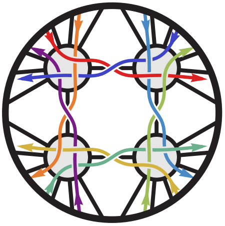

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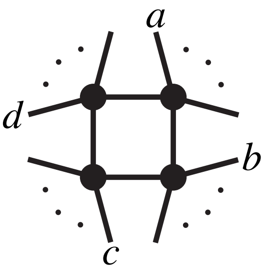

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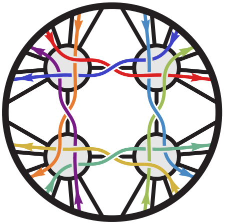

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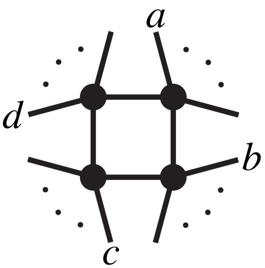
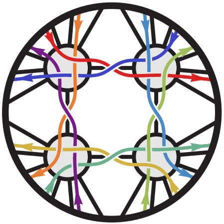
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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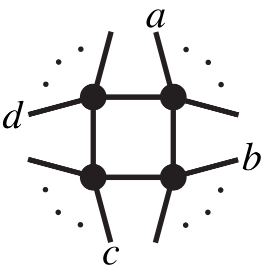

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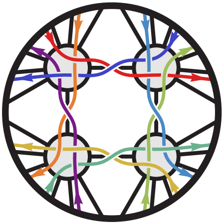

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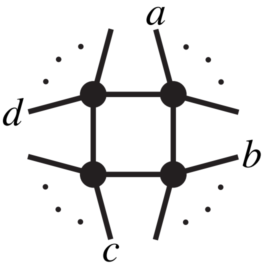
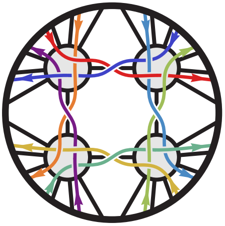
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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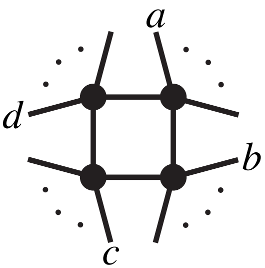

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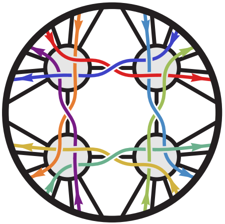

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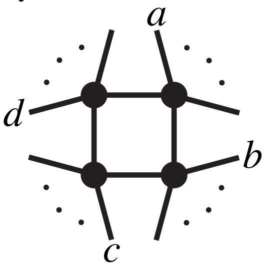
$I_{a,b,c,d} \equiv$ 

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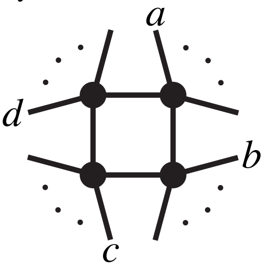

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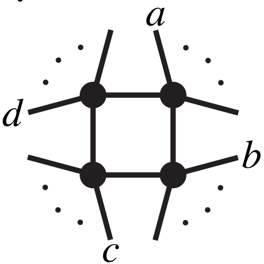

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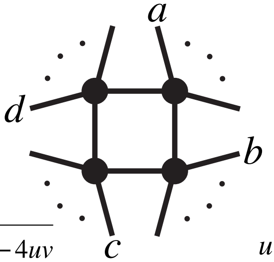

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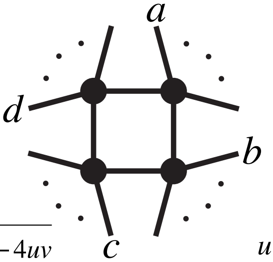

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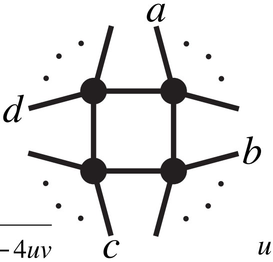

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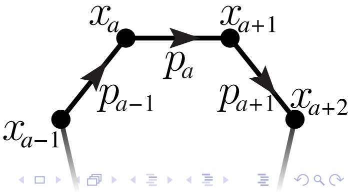

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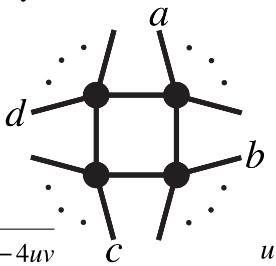

$$\equiv -\int_{\ell\in\mathbb{R}^{3,1}} d^4\ell\, \frac{(a,c)(b,d)\Delta}{(\ell,a)(\ell,b)(\ell,c)(\ell,d)}, \qquad \mathcal{I}_{a,b,c,d} \equiv -\int_{\ell\in\mathbb{R}^{3,1}} d^4\ell\, \frac{(a,c)(b,d)\Delta}{(\ell,a)(\ell,b)(\ell,c)(\ell,d)}$$

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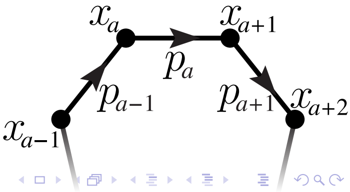

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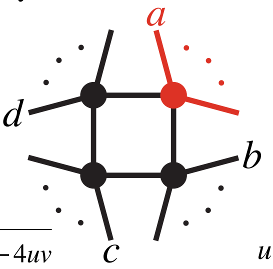

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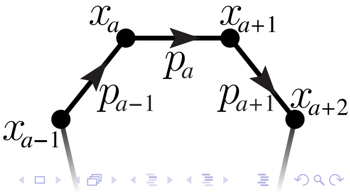

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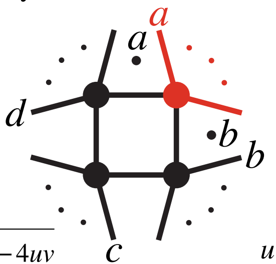

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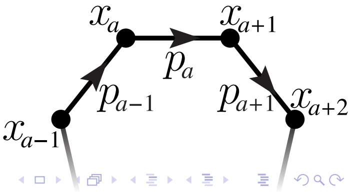

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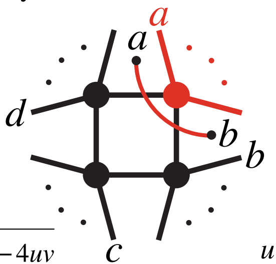

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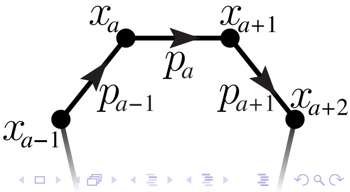

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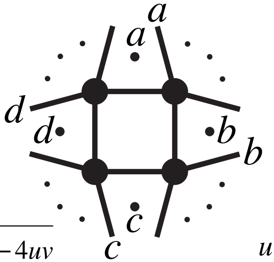

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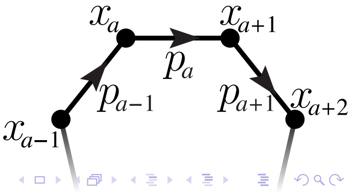

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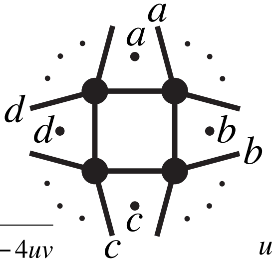

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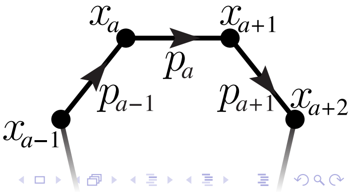

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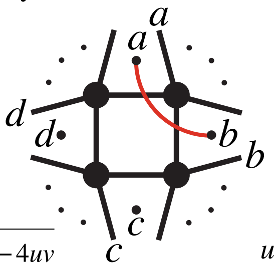

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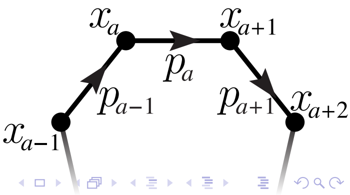

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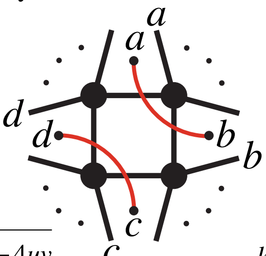

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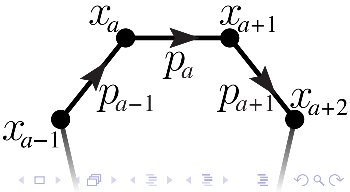

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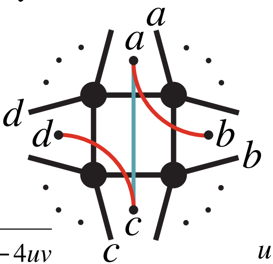

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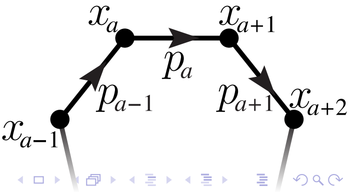

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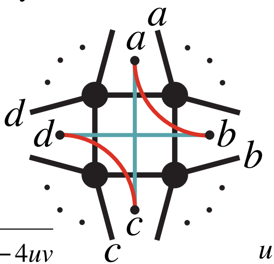

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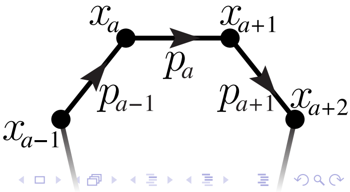

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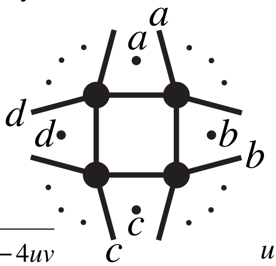

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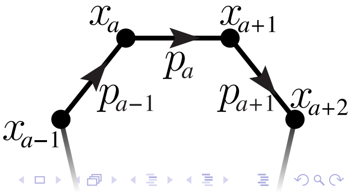

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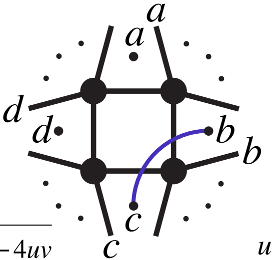

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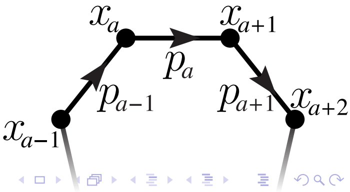

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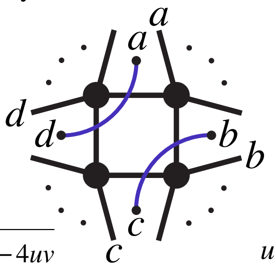

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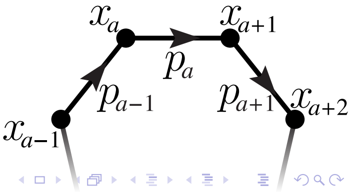

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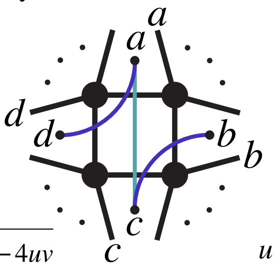

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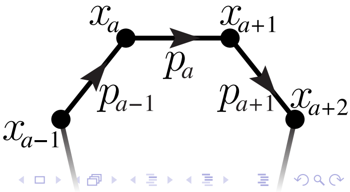

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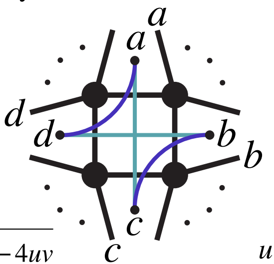

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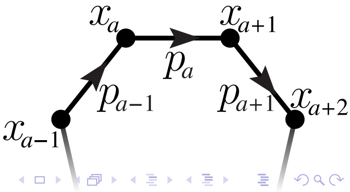

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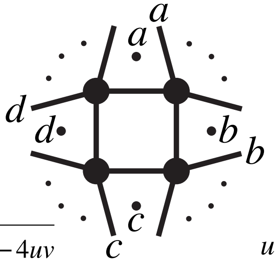

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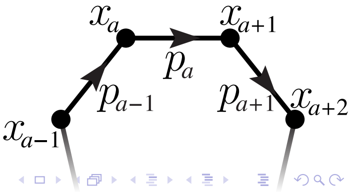

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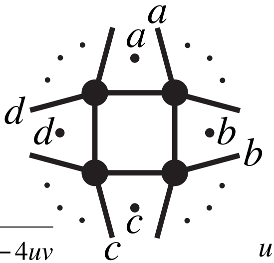

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$$\Delta \equiv \sqrt{(1-u-v)^2-4uv} \qquad u \equiv \frac{(a,b)(c,d)}{(a,c)(b,d)}, \quad v \equiv \frac{(b,c)(a,d)}{(a,c)(b,d)}$$

with $p_a \equiv x_{a+1} - x_a$, and where $1/(\ell,a)$ denotes the standard propagator:

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The Scalar "Four-Mass Box" Integral

The four-mass box integral is a manifestly finite, symmetric function of two dual-conformally invariant cross ratios, denoted u and v.

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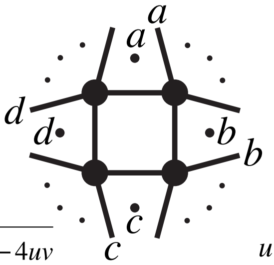

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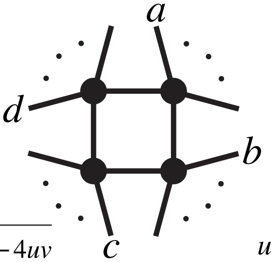

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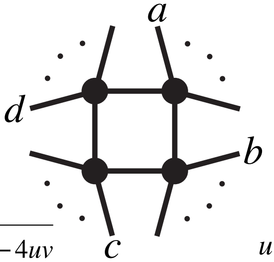

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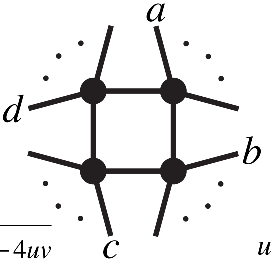

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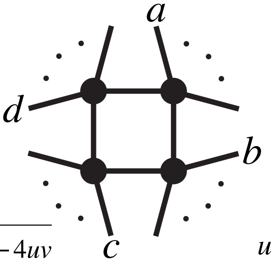

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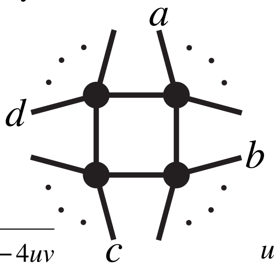

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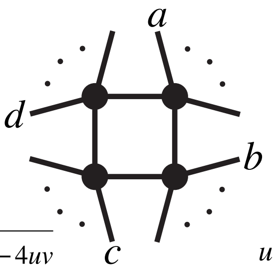

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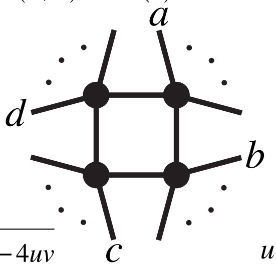

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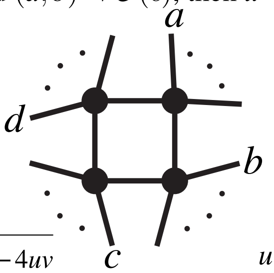

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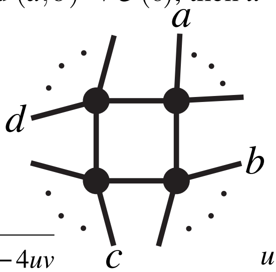

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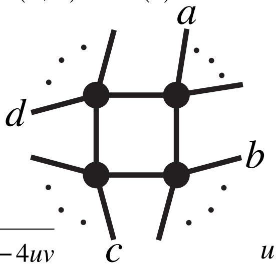
$I_{a,b,c,d}\equiv$  $\equiv -\displaystyle\int\limits_{\ell\in\mathbb{R}^{3,1}} d^4\ell\,\frac{(a,c)(b,d)\Delta}{(\ell,a)(\ell,b)(\ell,c)(\ell,d)},$

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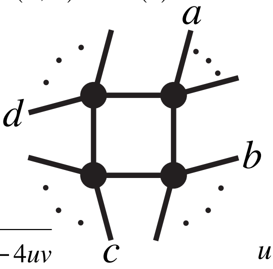

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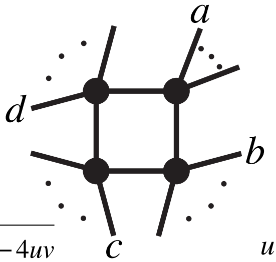

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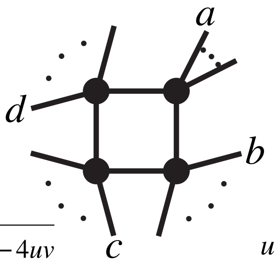

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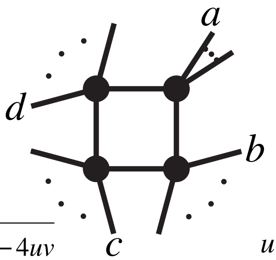

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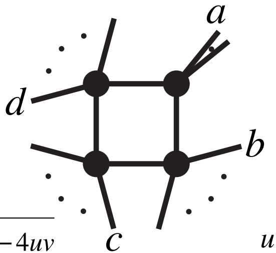
$I_{a,b,c,d} \equiv$ 

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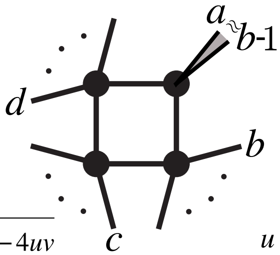

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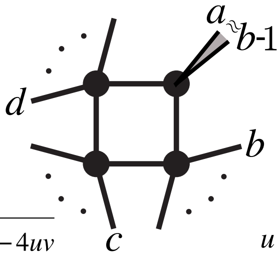

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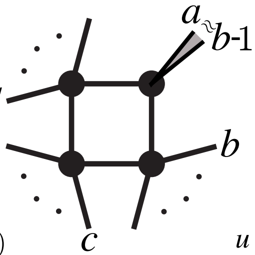
$I_{a,b,c,d}\equiv$ 

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$$-I_{a,b,c,d}(u,v)=\mathrm{Li}_2(\alpha)+\mathrm{Li}_2(\beta)-\mathrm{Li}_2(1)+\frac{1}{2}\log(u)\log(v)-\log(\alpha)\log(\beta)$$

$$\alpha\equiv\frac{1}{2}(1+u-v+\Delta)\quad\text{and}\quad\beta\equiv\frac{1}{2}(1-u+v+\Delta).$$

The Scalar "Four-Mass Box" Integral *and its Divergences*

When any corner becomes massless, the integral becomes **infrared divergent**

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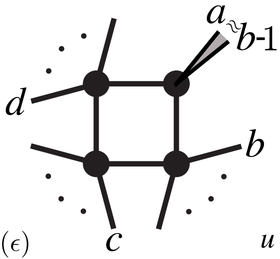

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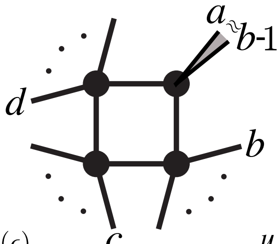

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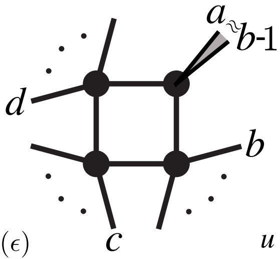

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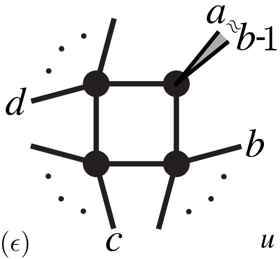

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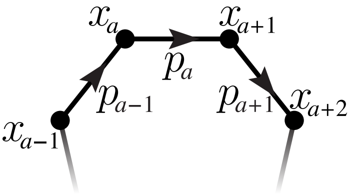

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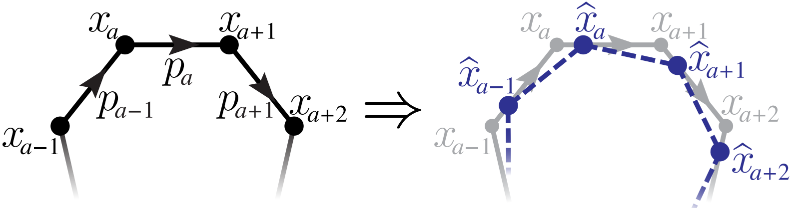

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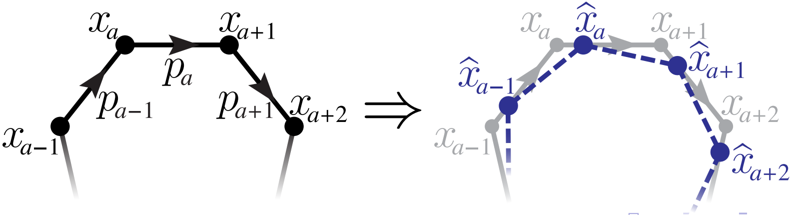

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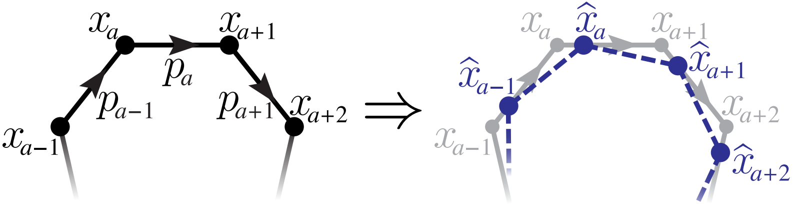

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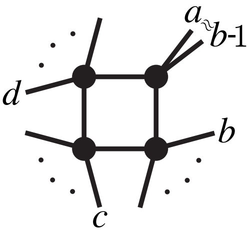

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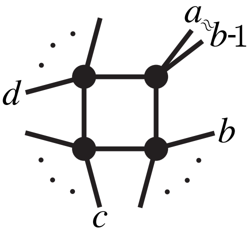

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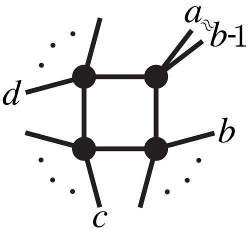

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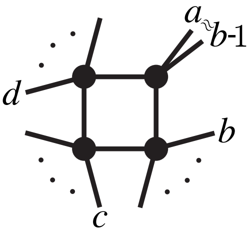

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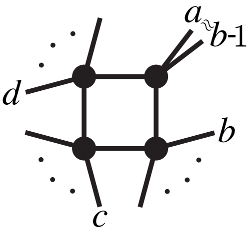

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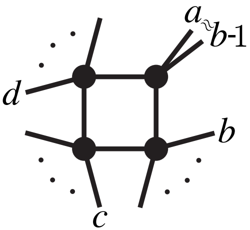

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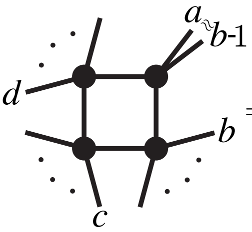

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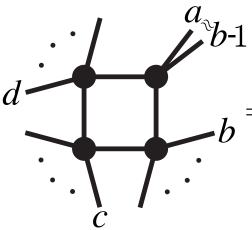

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$$\int d^4\ell\,\mathcal{A}_n^{(k),1} = \sum_{a,b,c,d} I_{a,b,c,d}\left(f^1_{a,b,c,d} + f^2_{a,b,c,d}\right)$$

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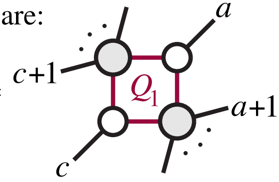

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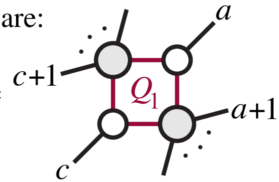

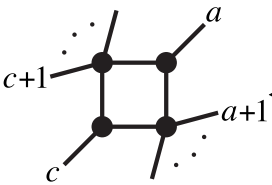

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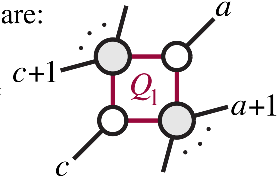

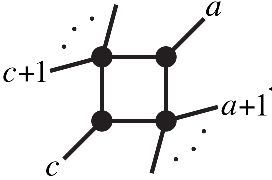

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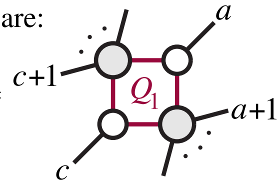

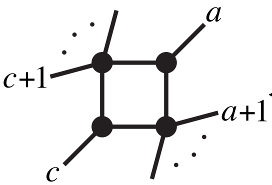

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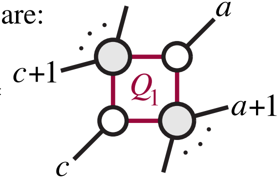
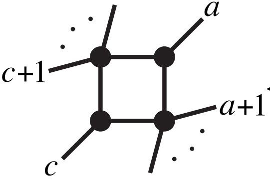
Consider for example the 'MHV' amplitude $(k=2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

$$f^1_{a,a+1,c,c+1} =$$





$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\, \mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the 'MHV' amplitude $(k{=}2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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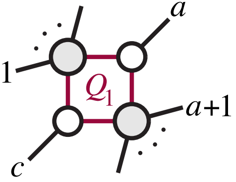

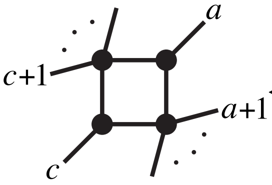

$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a{+}1)-(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\, \mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

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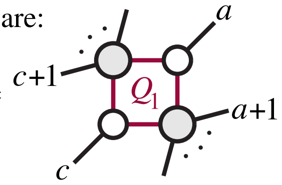

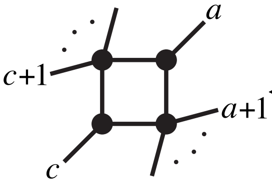

$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\,\mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

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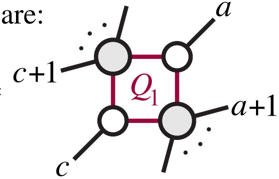

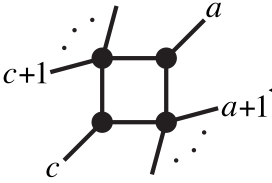

$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

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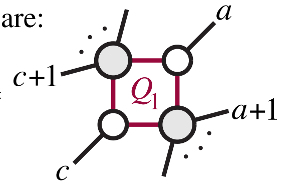

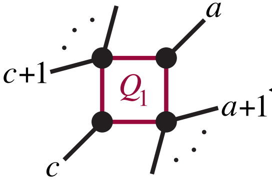

$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}\,\frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

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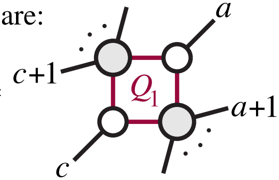

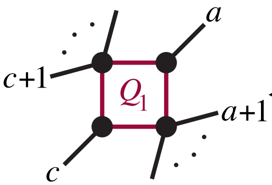

$$\Leftrightarrow \underbrace{\int d^4\ell\, \frac{(a,c)(a,a{+}1)-(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}\,\frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(2),1} \overset{¿?}{=} \sum_{a,c} \mathcal{I}^1_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the 'MHV' amplitude $(k=2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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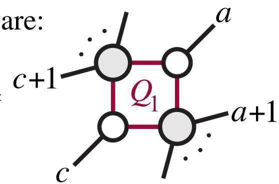

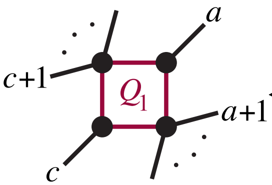

$$\Leftrightarrow \underbrace{\int d^4\ell\, \frac{(a,c)(a,a{+}1)-(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}\, \frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d}\right)$$

Consider for example the ‘MHV’ amplitude $(k=2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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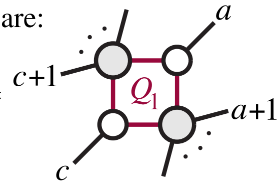

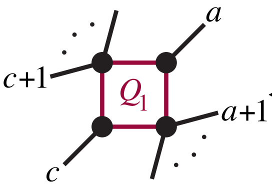

$$\Leftrightarrow \underbrace{\int d^4\ell \, \frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)} \frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d}\left(\mathcal{I}^1_{a,b,c,d}f^1_{a,b,c,d}+\mathcal{I}^2_{a,b,c,d}f^2_{a,b,c,d}\right)$$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

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This ansatz matches the correct integrand on **all** co-dimension four residues

A 'Box'-Expansion for One-Loop *Integrands*

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This ansatz matches the correct integrand on **all** co-dimension four residues *involving four distinct propagators*.

A 'Box'-Expansion for One-Loop *Integrands*

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$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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There are **also** co-dimension four residues involving only three propagators:

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d}\right)$$

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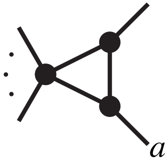

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d}\right)$$

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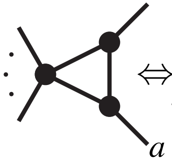

$$\Leftrightarrow \int d^4\ell \, \frac{(a-1, a+1)(a, X)}{(\ell, a-1)(\ell, a)(\ell, a+1)(\ell, X)}$$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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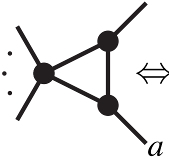
$$\mathcal{I}^a_{\text{div}} \equiv$$ 

$$\Leftrightarrow \int d^4\ell \, \frac{(a-1, a+1)(a, X)}{(\ell, a-1)(\ell, a)(\ell, a+1)(\ell, X)}$$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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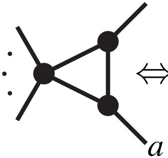
$$\mathcal{I}^a_{\text{div}} \equiv$$ 

$$\Leftrightarrow \int d^4\ell \, \frac{(a-1, a+1)(a, X)}{(\ell, a-1)(\ell, a)(\ell, a+1)(\ell, X)}$$

and the residue about the point $\ell \to x_a$ must be the tree amplitude: $\mathcal{A}_n^{(k),0}$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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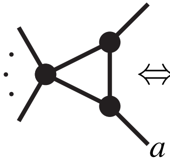
$\mathcal{I}^a_{\text{div}} \equiv$ 

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A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right) + \mathcal{A}_n^{(k),0} \sum_a \mathcal{I}^a_{\text{div}}$$

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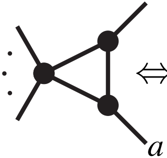
$\mathcal{I}^a_{\text{div}} \equiv$  $\Leftrightarrow \displaystyle\int d^4\ell \, \frac{(a-1, a+1)(a, X)}{(\ell, a-1)(\ell, a)(\ell, a+1)(\ell, X)}$

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A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d}\right) + \mathcal{A}_n^{(k),0} \sum_a \mathcal{I}^a_{\text{div}}$$

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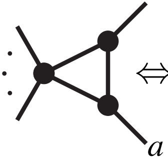
$\mathcal{I}^a_{\text{div}} \equiv$  $\Leftrightarrow \int d^4\ell \, \frac{(a-1, a+1)(a, X)}{(\ell, a-1)(\ell, a)(\ell, a+1)(\ell, X)}$

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A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} = \sum_{a,b,c,d}\left(\mathcal{I}^1_{a,b,c,d}f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d}f^2_{a,b,c,d}\right) + \mathcal{A}_n^{(k),0}\sum_a \mathcal{I}^a_{\text{div}}$$

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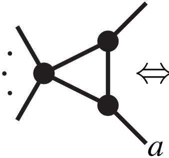
$\mathcal{I}^a_{\text{div}} \equiv$ 

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A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} = \sum_{a,b,c,d}\left(\mathcal{I}^1_{a,b,c,d}f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d}f^2_{a,b,c,d}\right) + \mathcal{A}_n^{(k),0}\sum_a \mathcal{I}^a_{\text{div}}$$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} = \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right) + \mathcal{A}_n^{(k),0} \sum_a \mathcal{I}^a_{\text{div}}$$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} = \underbrace{\sum_{a,b,c,d}\left(\mathcal{I}^1_{a,b,c,d}f^1_{a,b,c,d}+\mathcal{I}^2_{a,b,c,d}f^2_{a,b,c,d}\right)}_{\mathcal{A}_{n,\text{fin}}^{(k),1}} + \mathcal{A}_n^{(k),0}\sum_a \mathcal{I}^a_{\text{div}}$$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

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A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

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Constructing Local Integrands for Two-Loop Amplitudes

The Two-Loop *Chiral* Expansion

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"*Merging*" One-Loop, Chiral (X-dependent) **Integrands**

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$$|||$$

$$\frac{(b-1, b+1)(b, X)}{(\ell_1, b-1)(\ell_1, b)(\ell_1, b+1)(\ell_1, X)} \otimes \frac{(X, a)(a-1, a+1)}{(X, \ell_2)(\ell_2, a-1)(\ell_2, a)(\ell_2, a+1)}$$

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$$|||$$

$$\frac{(b-1, b+1)(b, a)(a-1, a+1)}{(\ell_1, b-1)(\ell_1, b)(\ell_1, b+1)(\ell_1, \ell_2)(\ell_2, a-1)(\ell_2, a)(\ell_2, a+1)}$$

Constructing Local Integrands for Two-Loop Amplitudes

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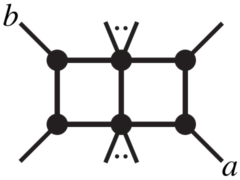

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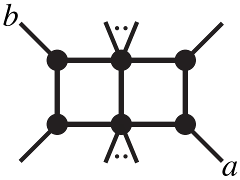

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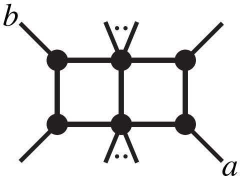

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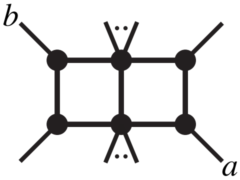

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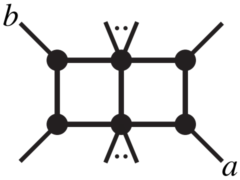

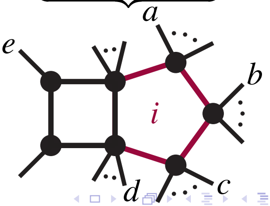

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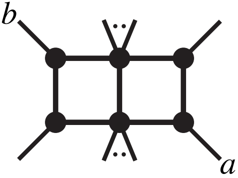

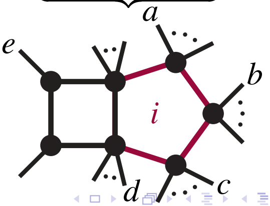

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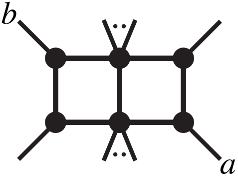

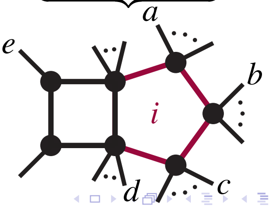

Finite Integrand Contributions to Two-Loop Amplitudes

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1. “Kissing” Boxes:

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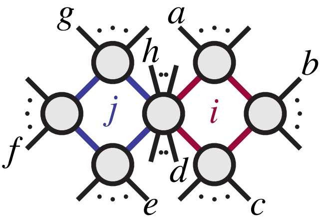

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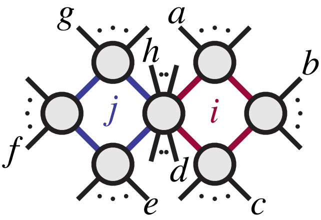

$$\times \left(\mathcal{I}^{j}_{e,f,g,h} \otimes \mathcal{I}^{i}_{a,b,c,d}\right)$$

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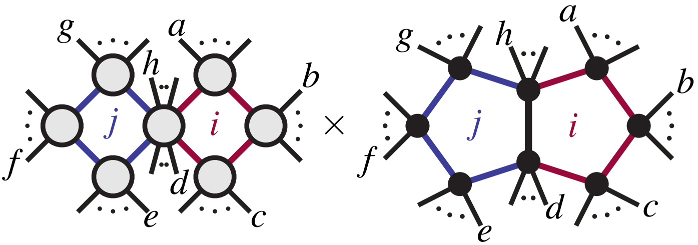

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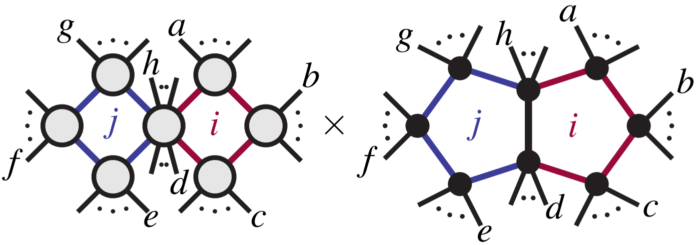

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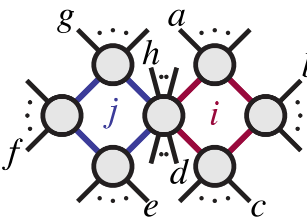
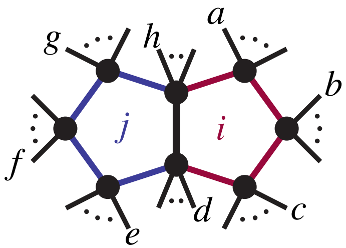


$\times$



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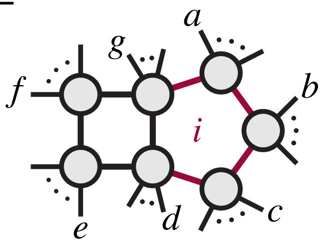

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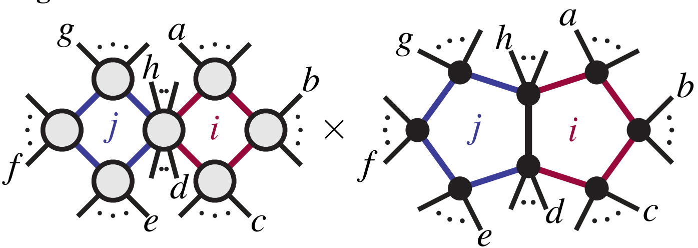

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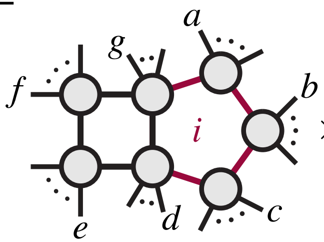

$\times \left(\mathcal{I}_{e,f,g,X} \otimes \mathcal{I}^{i}_{a,b,c,d}\right)$

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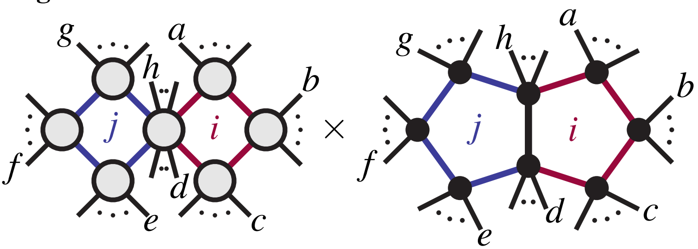

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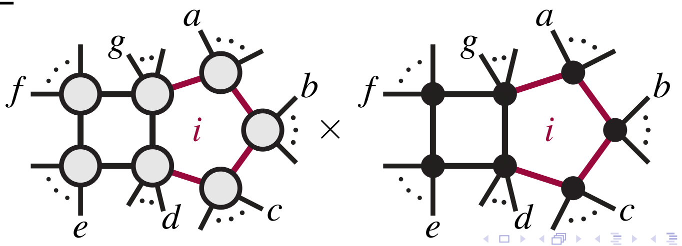

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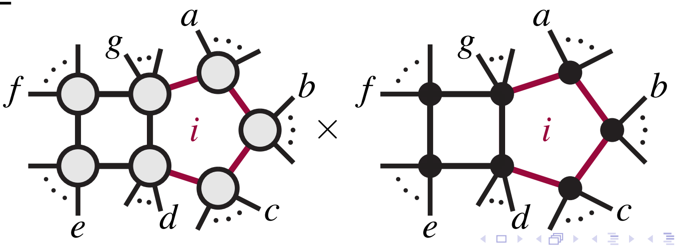

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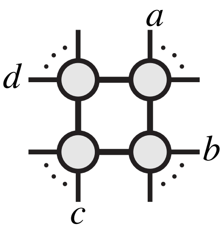

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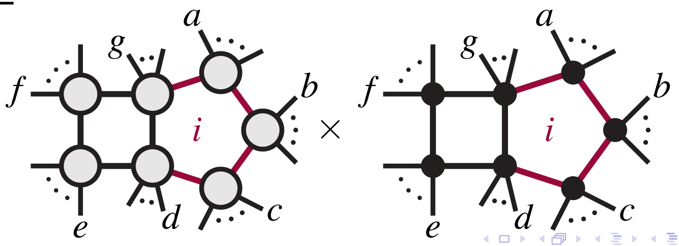

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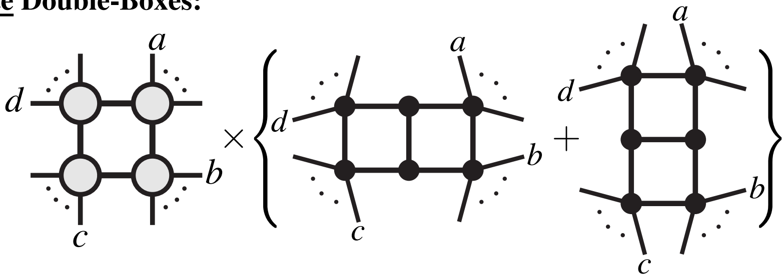

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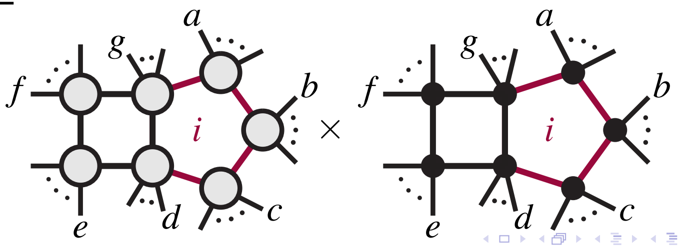

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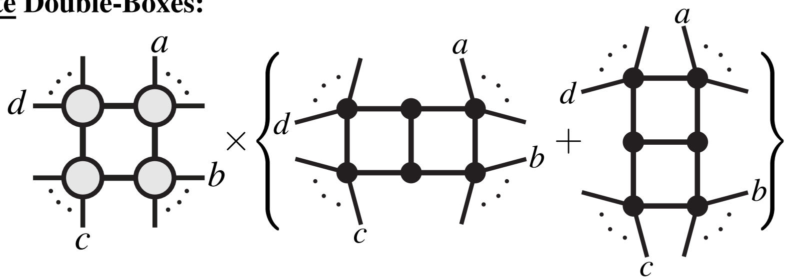

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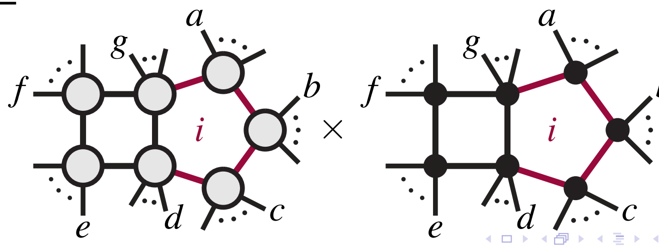

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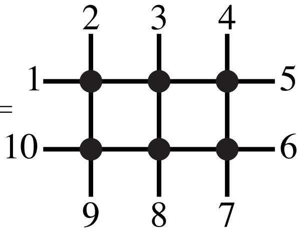

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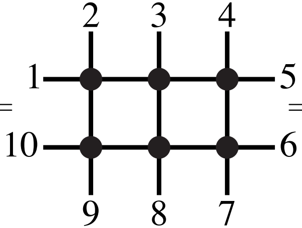
$=$ 

$$= \int \frac{d^4\ell_1 d^4\ell_2}{(\ell_1,9)(\ell_1,1)(\ell_1,3)(\ell_1,\ell_2)(\ell_2,4)(\ell_2,6)(\ell_2,8)}$$

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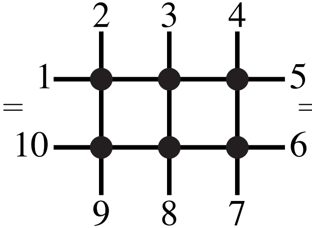

$$= \int \frac{d^4\ell_1 d^4\ell_2}{(\ell_1,9)(\ell_1,1)(\ell_1,3)(\ell_1,\ell_2)(\ell_2,4)(\ell_2,6)(\ell_2,8)}$$

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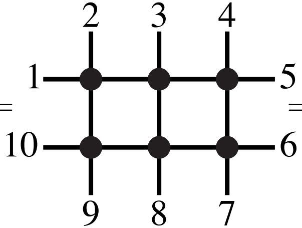

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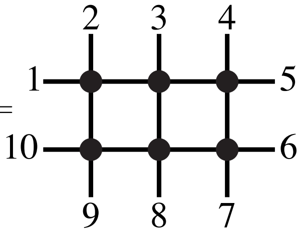
$=$  $\times$?

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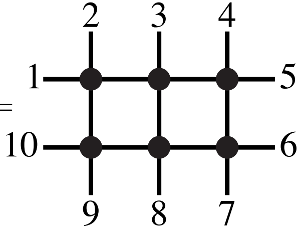
$=$ 

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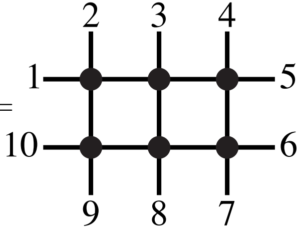
$=$ 

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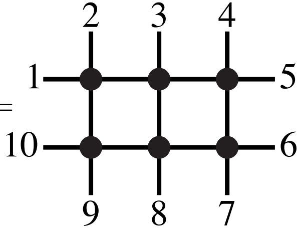
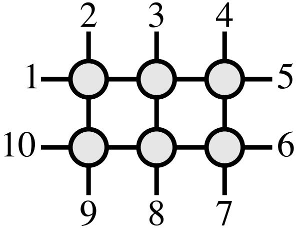
$=$  $\times$ 

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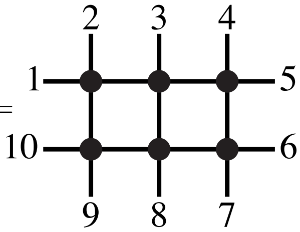
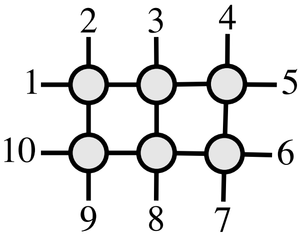
$=$ 

$\times$ 

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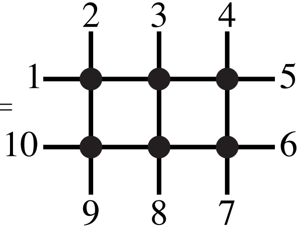
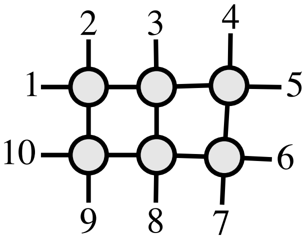
$=$ 

$\times$ 

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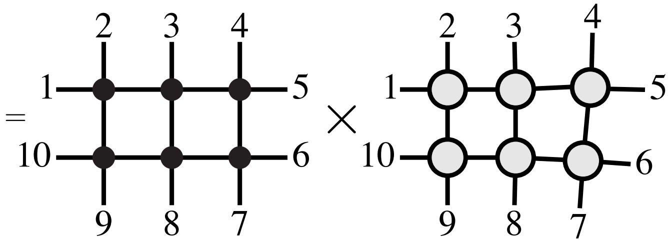

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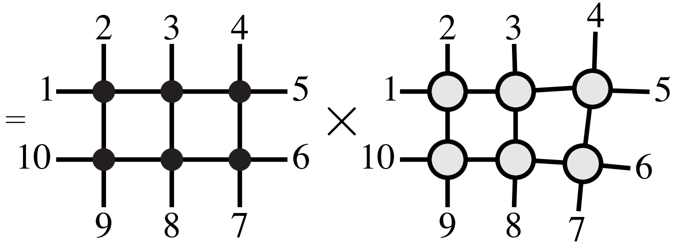

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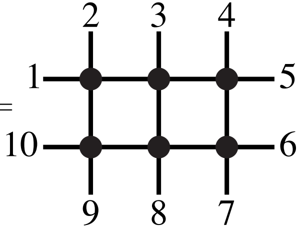

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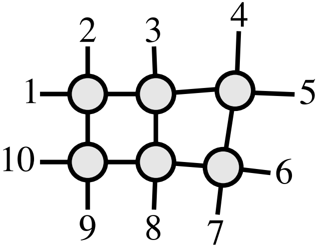

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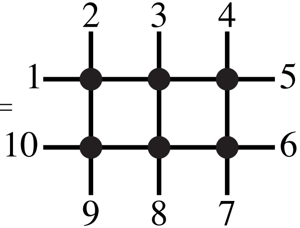
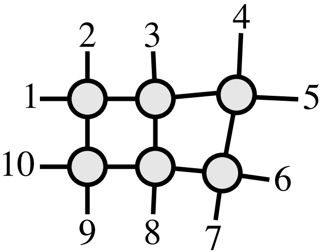
$=$ 

$\times$ 

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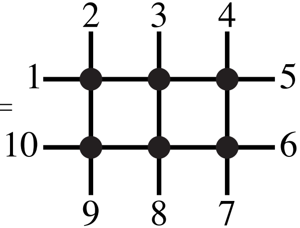
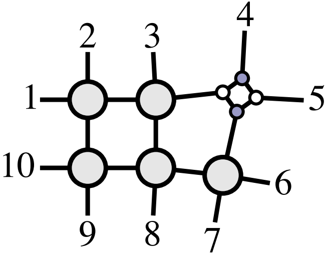
$=$  $\times$ 

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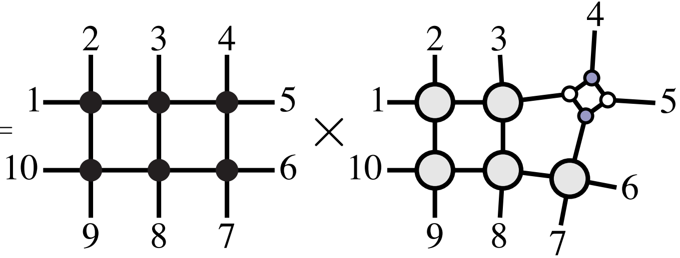

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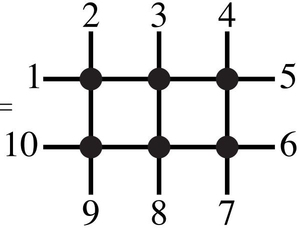
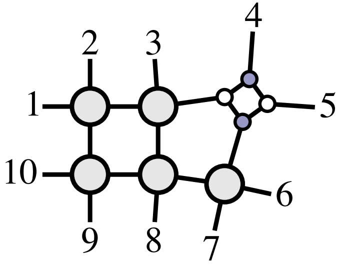
$=$  $\times$ 

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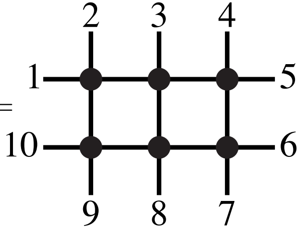
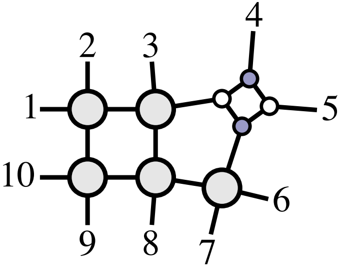
$=$ 

$\times$ 

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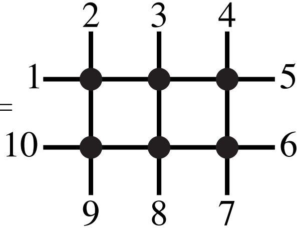
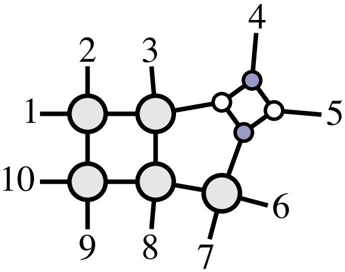
$=$  $\times$ 

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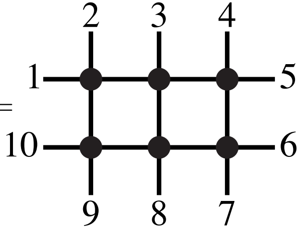
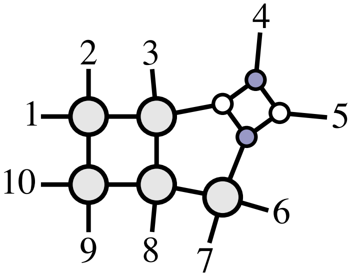
$=$  $\times$ 

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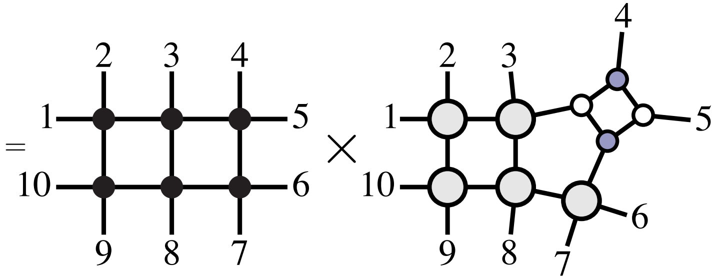

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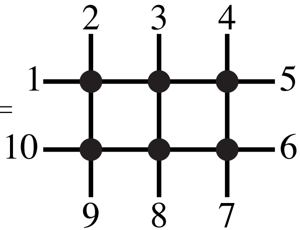
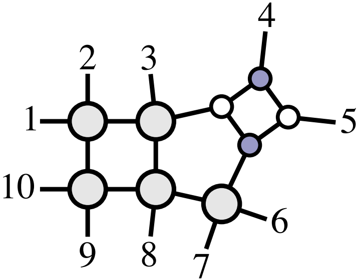
$=$ 

$\times$ 

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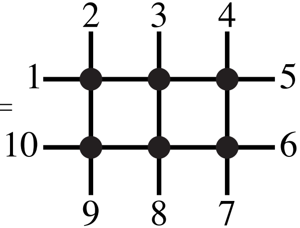
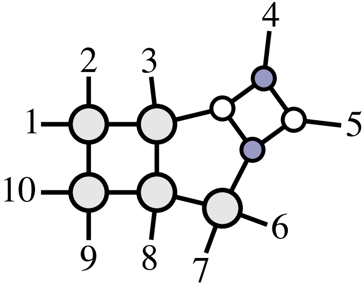
$=$ 

$\times$ 

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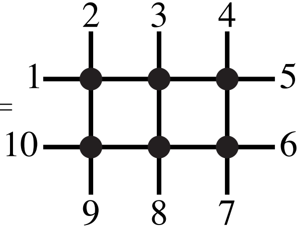
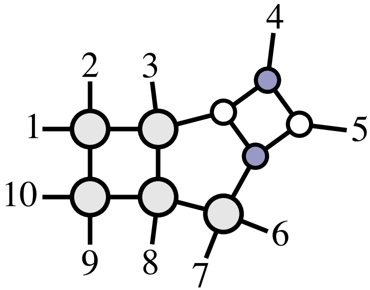
= 

× 

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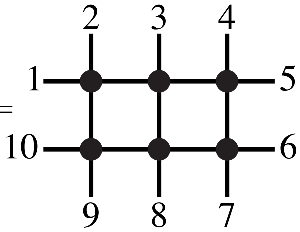
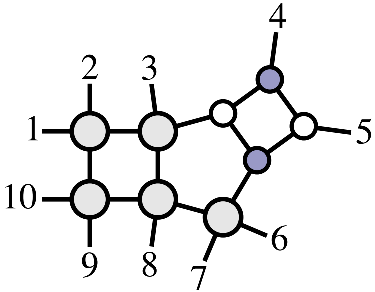
$=$ 

$\times$ 

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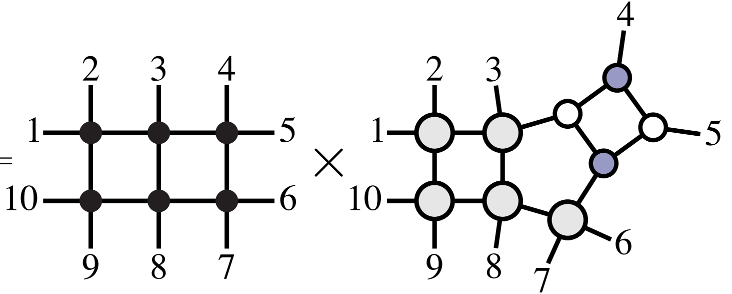
$=$ 

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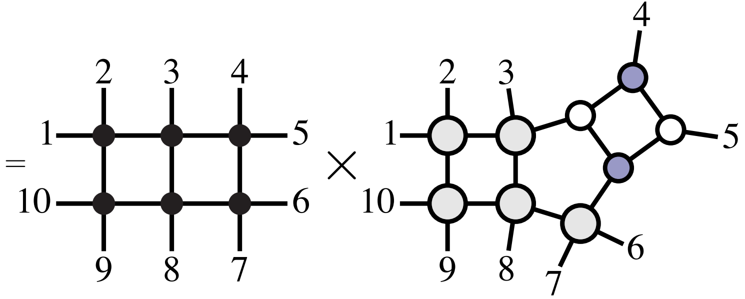

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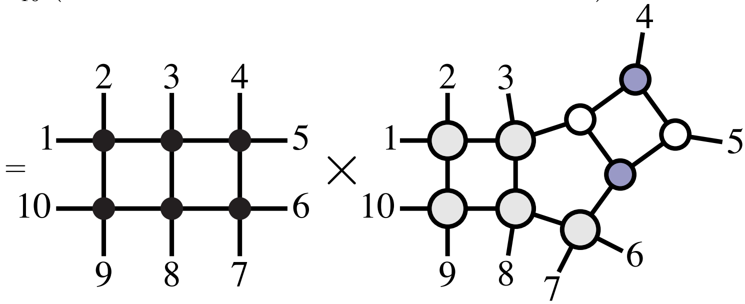

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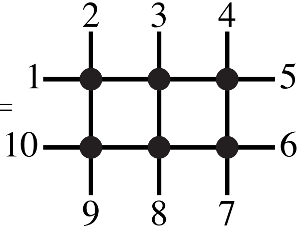
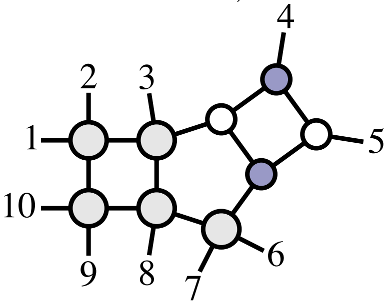
$=$ 

$\times$ 

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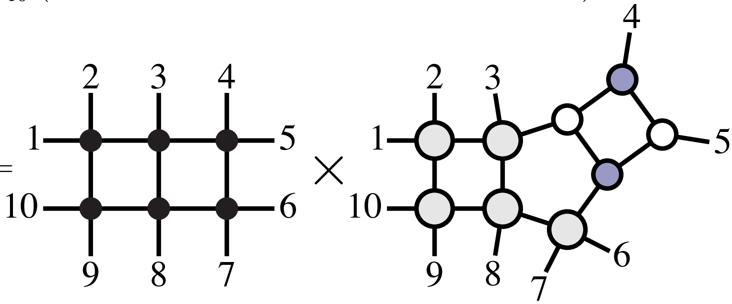
$=$ 

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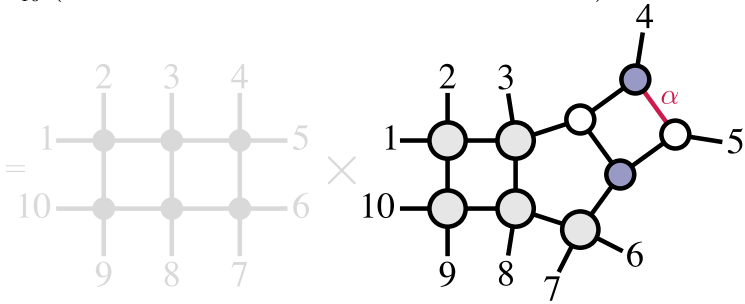

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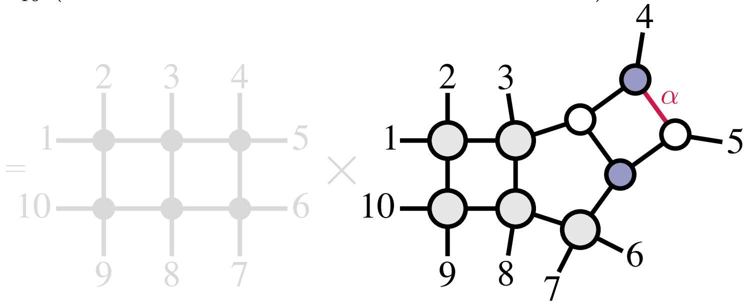

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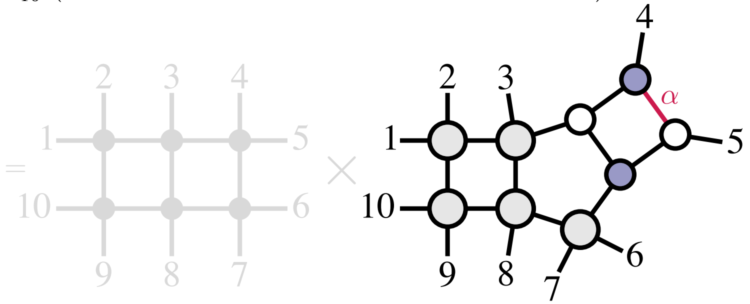

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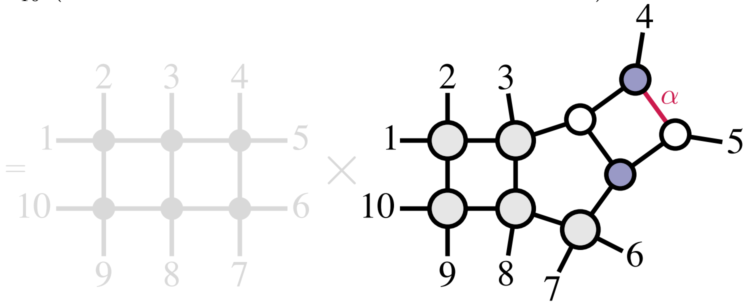

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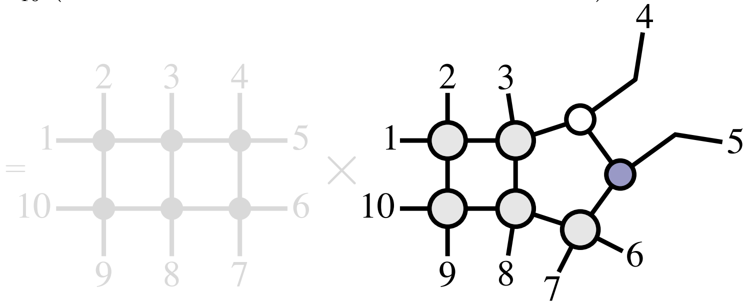

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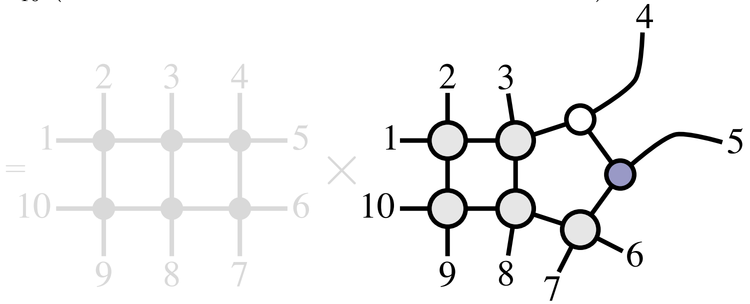

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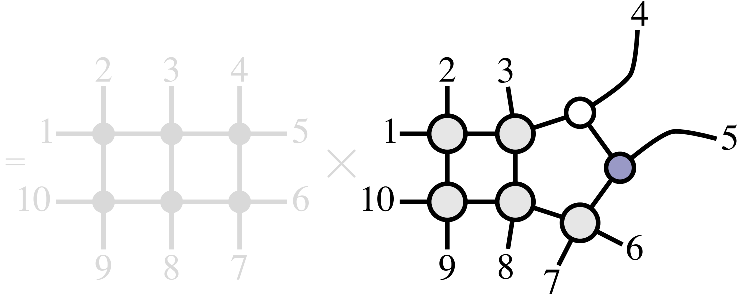

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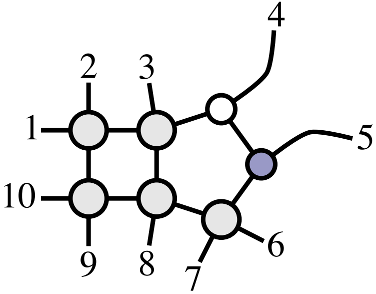
= × 

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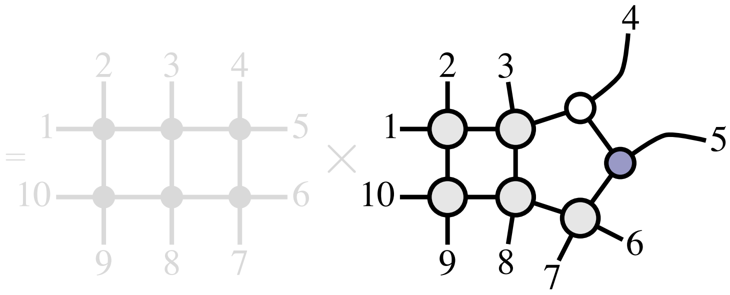

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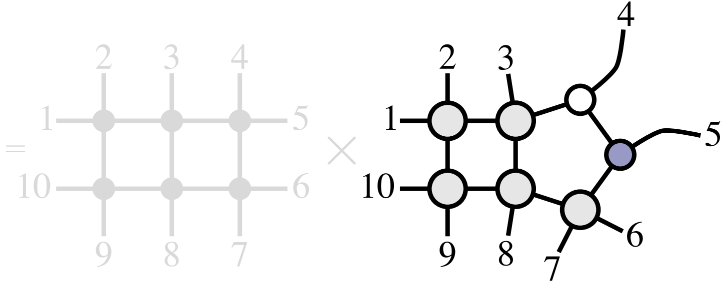

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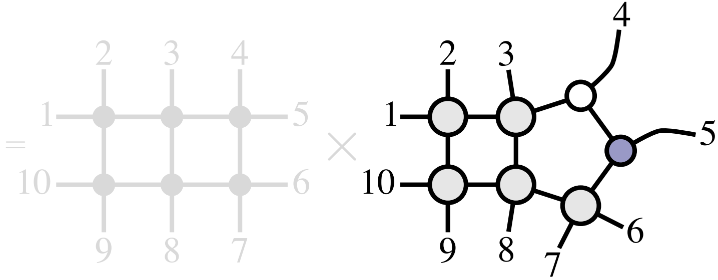

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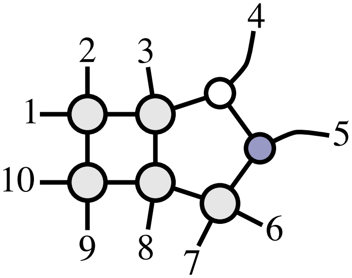
$\times$ 

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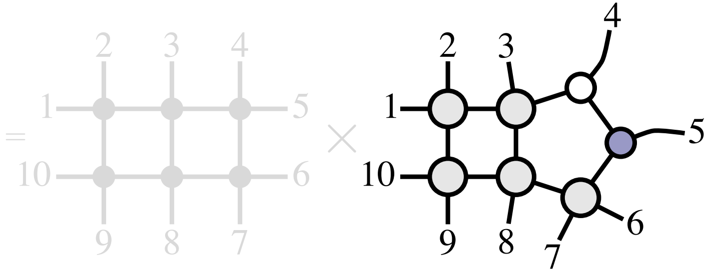

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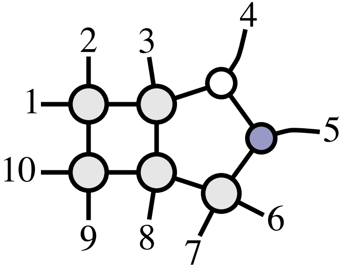
$=$ $\times$ 

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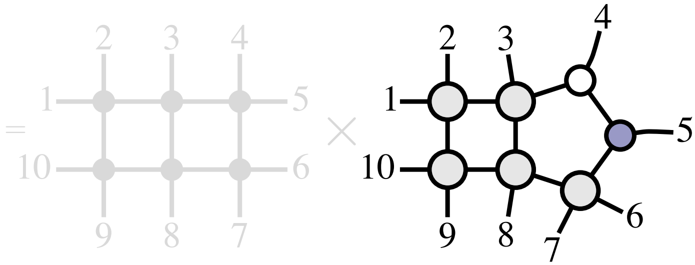

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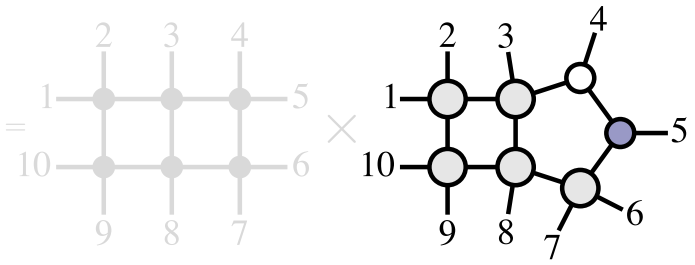

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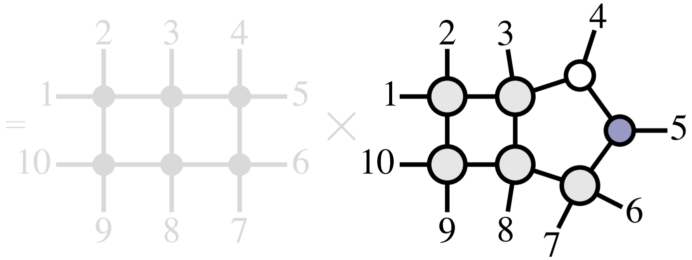

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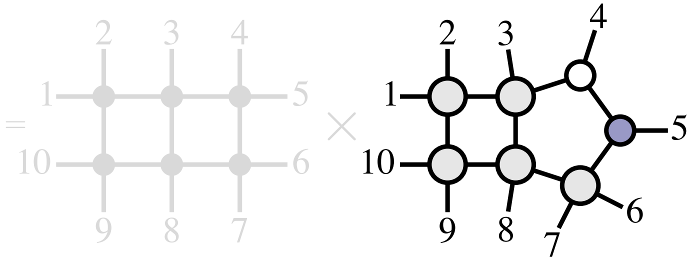

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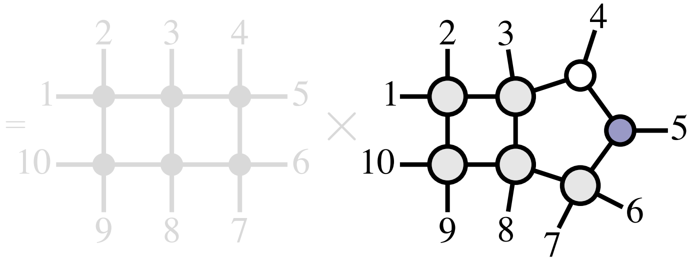

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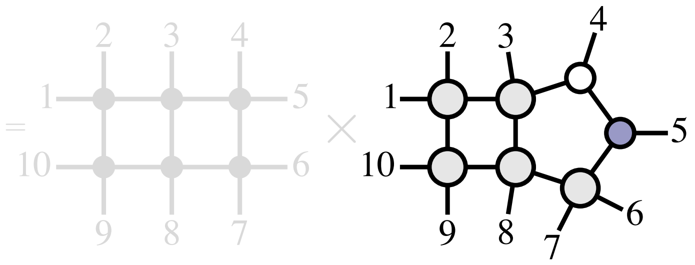

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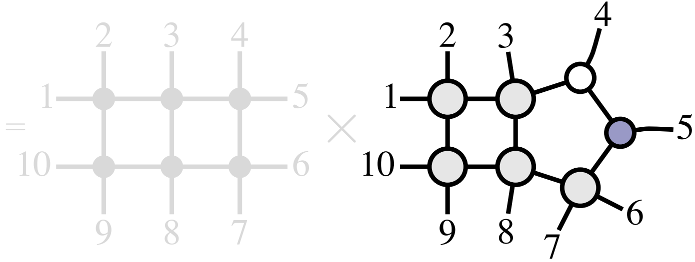

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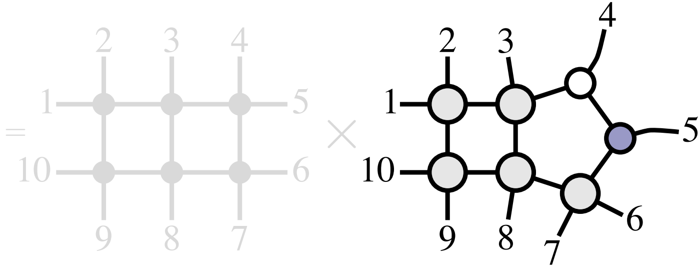

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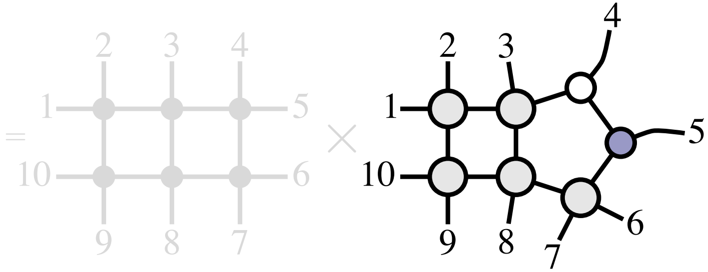

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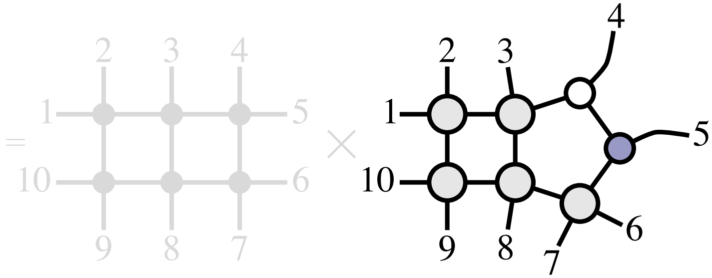

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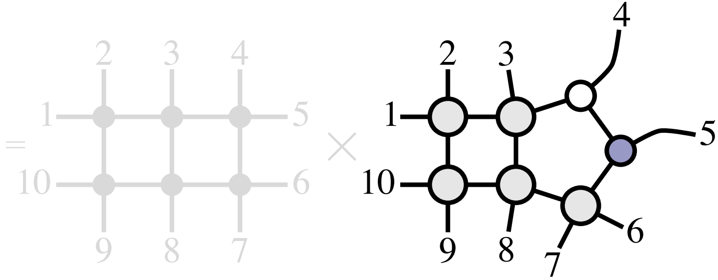

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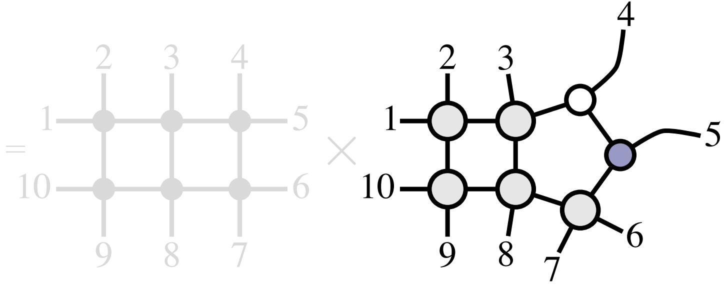

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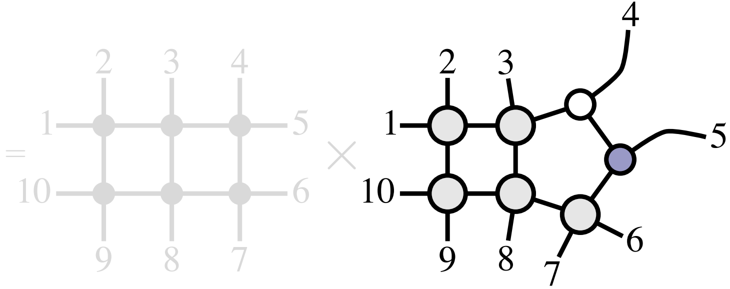

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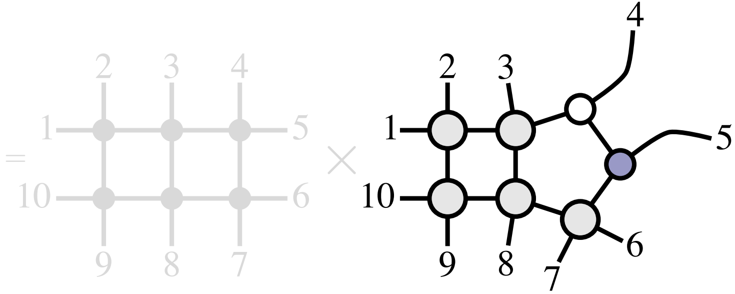

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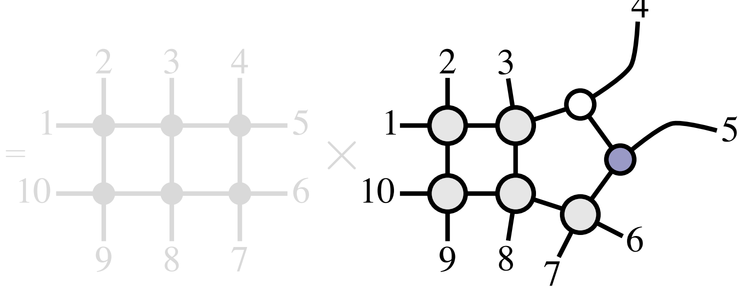

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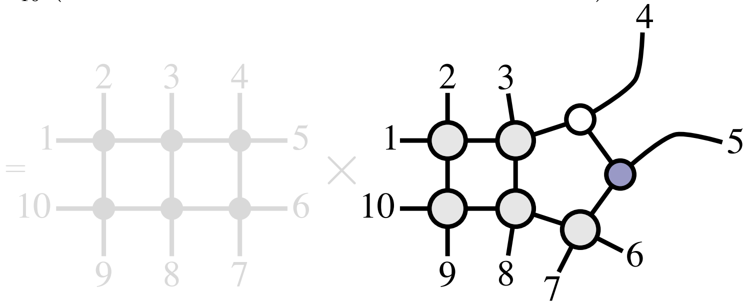

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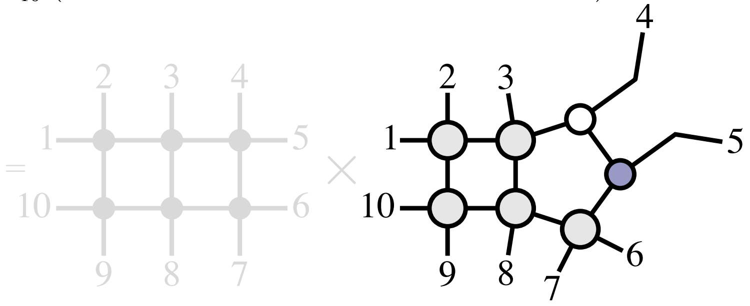

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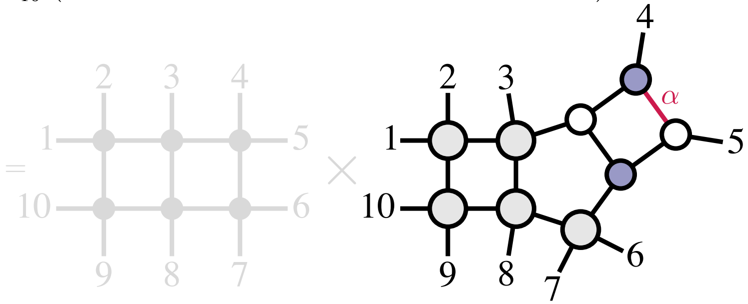
4. "Shifted" Double-Boxes:

It turns out that here are contributions to $\mathcal{A}_{n,\text{fin}}^{(k),2}$ which **cannot** be written as 'superfunction'×'integral'. To see this, consider the following 10-particle all-scalar, component amplitude:

$$\mathcal{A}_{10}^{(5)}(\varphi_{12}, \varphi_{12}, \varphi_{12}, \varphi_{23}, \varphi_{23}, \varphi_{34}, \varphi_{34}, \varphi_{34}, \varphi_{41}, \varphi_{41})$$



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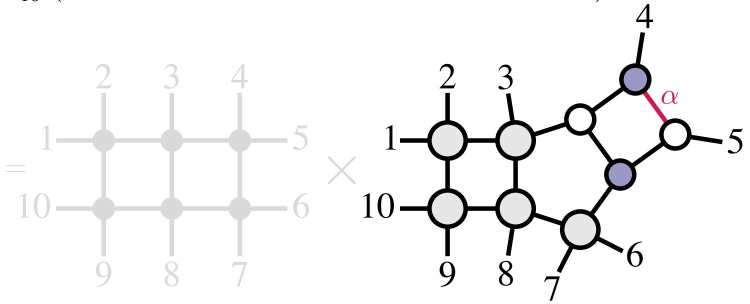

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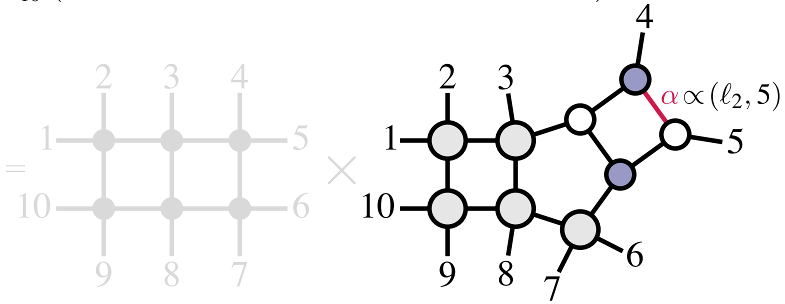

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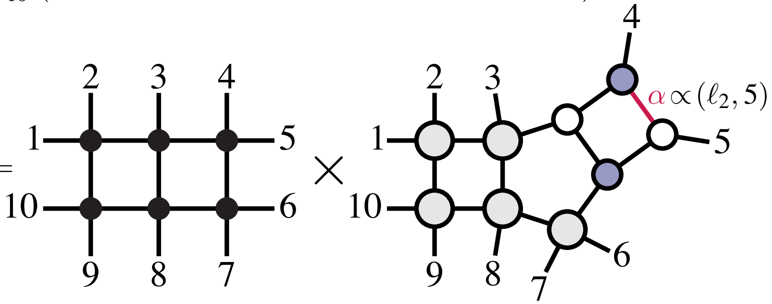
$=$ 

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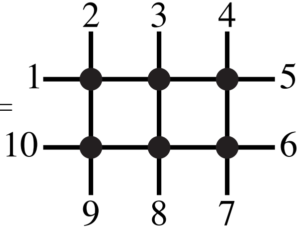
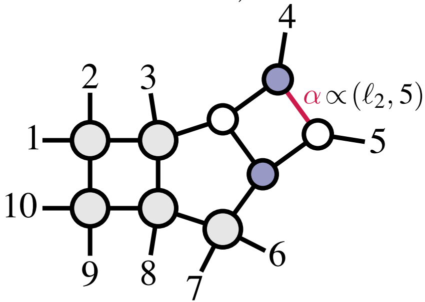
$=$ 

$\times$ 

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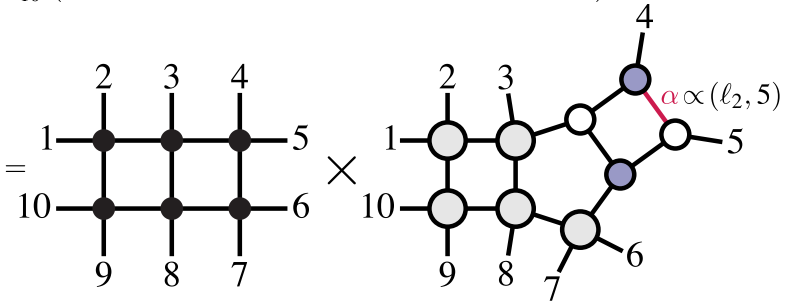

Spiritus Movens: Even More Shocking Simplicity is Found

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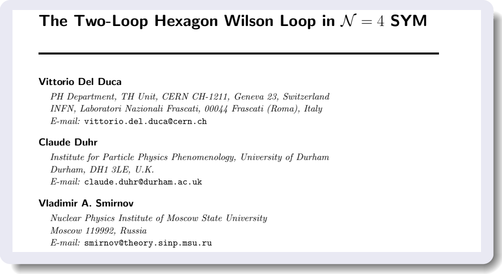

The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
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E-mail: smirnov@theory.sinp.msu.ru

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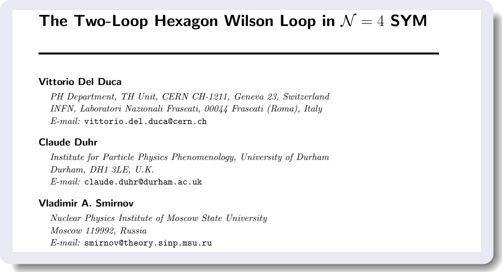

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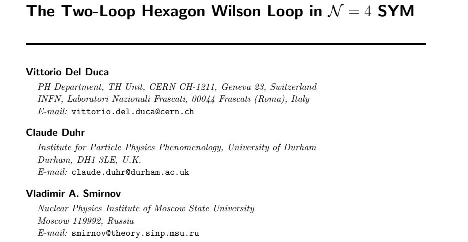

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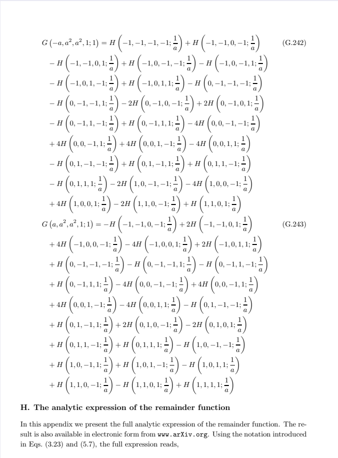

$$\begin{aligned}
G\left(-a,a^2,a^2,1;1\right) = {} & H\left(-1,-1,-1,-1;\frac{1}{a}\right) + H\left(-1,-1,0,-1;\frac{1}{a}\right) \\
& - H\left(-1,-1,0,1;\frac{1}{a}\right) + H\left(-1,0,-1,-1;\frac{1}{a}\right) - H\left(-1,0,-1,1;\frac{1}{a}\right) \\
& - H\left(-1,0,1,-1;\frac{1}{a}\right) + H\left(-1,0,1,1;\frac{1}{a}\right) - H\left(0,-1,-1,-1;\frac{1}{a}\right) \\
& - H\left(0,-1,-1,1;\frac{1}{a}\right) - 2H\left(0,-1,0,-1;\frac{1}{a}\right) + 2H\left(0,-1,0,1;\frac{1}{a}\right) \\
& - H\left(0,-1,1,-1;\frac{1}{a}\right) + H\left(0,-1,1,1;\frac{1}{a}\right) - 4H\left(0,0,-1,-1;\frac{1}{a}\right) \\
& + 4H\left(0,0,-1,1;\frac{1}{a}\right) + 4H\left(0,0,1,-1;\frac{1}{a}\right) - 4H\left(0,0,1,1;\frac{1}{a}\right) \\
& - H\left(0,1,-1,-1;\frac{1}{a}\right) + H\left(0,1,-1,1;\frac{1}{a}\right) + H\left(0,1,1,-1;\frac{1}{a}\right) \\
& - H\left(0,1,1,1;\frac{1}{a}\right) - 2H\left(1,0,-1,-1;\frac{1}{a}\right) - 4H\left(1,0,0,-1;\frac{1}{a}\right) \\
& + 4H\left(1,0,0,1;\frac{1}{a}\right) - 2H\left(1,1,0,-1;\frac{1}{a}\right) + H\left(1,1,0,1;\frac{1}{a}\right)
\end{aligned} \tag{G.242}$$

$$\begin{aligned}
G\left(a,a^2,a^2,1;1\right) = {} & -H\left(-1,-1,0,-1;\frac{1}{a}\right) + 2H\left(-1,-1,0,1;\frac{1}{a}\right) \\
& + 4H\left(-1,0,0,-1;\frac{1}{a}\right) - 4H\left(-1,0,0,1;\frac{1}{a}\right) + 2H\left(-1,0,1,1;\frac{1}{a}\right) \\
& + H\left(0,-1,-1,-1;\frac{1}{a}\right) - H\left(0,-1,-1,1;\frac{1}{a}\right) - H\left(0,-1,1,-1;\frac{1}{a}\right) \\
& + H\left(0,-1,1,1;\frac{1}{a}\right) - 4H\left(0,0,-1,-1;\frac{1}{a}\right) + 4H\left(0,0,-1,1;\frac{1}{a}\right) \\
& + 4H\left(0,0,1,-1;\frac{1}{a}\right) - 4H\left(0,0,1,1;\frac{1}{a}\right) - H\left(0,1,-1,-1;\frac{1}{a}\right) \\
& + H\left(0,1,-1,1;\frac{1}{a}\right) + 2H\left(0,1,0,-1;\frac{1}{a}\right) - 2H\left(0,1,0,1;\frac{1}{a}\right) \\
& + H\left(0,1,1,-1;\frac{1}{a}\right) + H\left(0,1,1,1;\frac{1}{a}\right) - H\left(1,0,-1,-1;\frac{1}{a}\right) \\
& + H\left(1,0,-1,1;\frac{1}{a}\right) + H\left(1,0,1,-1;\frac{1}{a}\right) - H\left(1,0,1,1;\frac{1}{a}\right) \\
& + H\left(1,1,0,-1;\frac{1}{a}\right) - H\left(1,1,0,1;\frac{1}{a}\right) + H\left(1,1,1,1;\frac{1}{a}\right)
\end{aligned} \tag{G.243}$$

H. The analytic expression of the remainder function

In this appendix we present the full analytic expression of the remainder function. The result is also available in electronic form from www.arXiv.org. Using the notation introduced in Eqs. (3.23) and (5.7), the full expression reads,

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$$
\begin{aligned}
&R_{6,WL}^{(2)}(u_1,u_2,u_3) = \qquad\qquad \text{(H.1)}\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\\
&\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\\
&\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+\\
&G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_1};1\right)+\\
&G\left(0,\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_3};1\right)+G\left(0,\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+\\
&\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)+\\
&\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_1};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)+\\
&G\left(0,\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\\
&\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)-\\
&\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-\\
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&\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3},0;1\right)+\frac{1}{2}G\left(\frac{1}{1-u_1},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+
\end{aligned}
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PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1,0;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},1;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_2};1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1},0;1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{1-u_2},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1,0;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},1;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_1};1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_2},0;1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{1-u_3},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1,0;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{79\pi^4}{360}+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)-G\left(\frac{1}{u_3},0,0,\frac{1}{u_1};1\right)-$$

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Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

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$$G\left(\frac{1}{u_3},0,0,\frac{1}{u_2};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_1+u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{132};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{312};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{123},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{132},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{132},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{213},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{213},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{312},\frac{1}{1-u_3};1\right)+\mathcal{G}\left(0,0,v_{321},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{132};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{231};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{312};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{321};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{u_1},0;1\right)-$$

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PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
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$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{1}{1-u_2},0;1\right)+\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{1}{1-u_2},1;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{u_3-1}{u_2+u_3-1},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{u_3-1}{u_2+u_3-1},\tfrac{1}{1-u_2};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{312},0,\tfrac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{1}{u_2},0;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{1}{1-u_3},0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{1}{1-u_3},1;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{1}{1-u_3},\tfrac{1}{1-u_3};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{u_1-1}{u_1+u_3-1},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{u_1-1}{u_1+u_3-1},\tfrac{1}{1-u_3};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{123},0,\tfrac{1}{1-u_1};1\right)-\tfrac{1}{2}\mathcal{G}\left(0,v_{123},1,\tfrac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{123},\tfrac{1}{1-u_1},0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(0,v_{123},\tfrac{1}{1-u_1},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{123},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{132},0,\tfrac{1}{1-u_1};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{132},\tfrac{1}{1-u_1},0;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{132},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{213},0,\tfrac{1}{1-u_2};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{213},\tfrac{1}{1-u_2},0;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{213},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2};1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{231},0,\tfrac{1}{1-u_2};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(0,v_{231},1,\tfrac{1}{1-u_2};1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{231},\tfrac{1}{1-u_2},0;1\right)-\tfrac{1}{2}\mathcal{G}\left(0,v_{231},\tfrac{1}{1-u_2},1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{231},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2};1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{312},0,\tfrac{1}{1-u_3};1\right)-\tfrac{1}{2}\mathcal{G}\left(0,v_{312},1,\tfrac{1}{1-u_3};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{312},\tfrac{1}{1-u_3},0;1\right)-\tfrac{1}{2}\mathcal{G}\left(0,v_{312},\tfrac{1}{1-u_3},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{312},\tfrac{1}{1-u_3},\tfrac{1}{1-u_3};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{321},0,\tfrac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{321},\tfrac{1}{1-u_3},0;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{321},\tfrac{1}{1-u_3},\tfrac{1}{1-u_3};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,0,v_{123};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,0,v_{132};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},0,\tfrac{1}{1-u_1},v_{123};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},0,\tfrac{1}{1-u_1},v_{132};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{123},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{123},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{132},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{132},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},0,v_{123};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},0,v_{132};1\right)-\tfrac{3}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{123};1\right)-\\
&\tfrac{3}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{132};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{123},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{123},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{132},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{132},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},0,1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},0,\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},1,0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{1}{1-u_1},0;1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{1}{1-u_1},1;1\right)+
\end{aligned}
$$

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Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
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$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{u_3},0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1,0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1,0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1,\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,0,v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,0,v_{231};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},0,\frac{1}{1-u_2},v_{213};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},0,\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},0,v_{213};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},0,v_{231};1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)-$$
$$\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0,1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},1,0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{u_1},0;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1},1;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1,0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1,\frac{1}{1-u_2};1\right)+$$

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Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}, 1; 1\right) +$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 0, 0; 1\right) -$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 0, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 0, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 1, 0; 1\right) -$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 1, \frac{1}{1-u_2}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, \frac{1}{1-u_2}, 0; 1\right) -$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, \frac{1}{1-u_2}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) -$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, 0, v_{312}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, 0, v_{321}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, 0, \frac{1}{1-u_3}, v_{312}; 1\right) -$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, 0, \frac{1}{1-u_3}, v_{321}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{312}, 1; 1\right) -$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{312}, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{321}, 1; 1\right) -$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{321}, \frac{1}{1-u_3}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, 0, v_{312}; 1\right) -$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, 0, v_{321}; 1\right) - \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{312}; 1\right) -$$
$$\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{321}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{312}, 1; 1\right) -$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{321}, 1; 1\right) -$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{321}, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, 0, 1; 1\right) +$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, 0, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, 1, 0; 1\right) +$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{1}{u_2}, 0; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{1}{1-u_3}, 0; 1\right) -$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{1}{1-u_3}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{1}{1-u_3}, \frac{1}{1-u_3}; 1\right) +$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{u_1-1}{u_1+u_3-1}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}; 1\right) +$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 0, 0; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 0, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 0, \frac{1}{1-u_3}; 1\right) -$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 1, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 1, \frac{1}{1-u_3}; 1\right) +$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}, 1; 1\right) +$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}, \frac{1}{1-u_3}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 0, 0; 1\right) -$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 0, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 0, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 1, 0; 1\right) -$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 1, \frac{1}{1-u_3}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, \frac{1}{1-u_3}, 0; 1\right) -$$

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Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
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$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},0,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},0,\frac{1}{1-u_1},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,0,\frac{1}{1-u_1};1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{123},1,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},0,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,1,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1,1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,1,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},1,0,\frac{1}{1-u_2};1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},1,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},0,1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,\frac{1}{1-u_3},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},1,0,\frac{1}{1-u_3};1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},1,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},0,1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,1,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1,1;1\right)-\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_1)-$$
$$\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_1)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_1)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_1)-\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_1)+$$
$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_1)-\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H(0;u_1)-$$
$$\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H(0;u_1)+\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_1)+$$
$$\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_1)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_1)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_1)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1};1\right)H(0;u_1)+$$

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Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
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$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1;1\right)H(0;u_1)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{2}G\left(\frac{1}{u_2},0,\frac{1}{u_1};1\right)H(0;u_1)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)H(0;u_1)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_1)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0;1\right)H(0;u_1)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_1)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_1};1\right)H(0;u_1)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H(0;u_1)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{u_1};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1};1\right)H(0;u_1)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H(0;u_1)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1;1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1;1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H(0;u_1)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H(0;u_1)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},1;1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{u_1};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0;1\right)H(0;u_1)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0;1\right)H(0;u_1)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H(0;u_1)-$$

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The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_1)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0;1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H(0;u_1)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_1)-$$
$$\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_1)-\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_1)+$$
$$\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_1)+\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_1)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_2)-\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)-$$
$$\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0;1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_2};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)-$$
$$\frac{3}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)H(0;u_2)-\frac{3}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)+$$
$$\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_2};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1;1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_2};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)+$$

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Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_1},v_{132};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_2},v_{213};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_2},v_{231};1\right)H(0;u_2)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_3},v_{312};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_3},v_{321};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{123},\tfrac{1}{1-u_1};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(0,u_{123},\tfrac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{1}{u_2};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{1}{1-u_3};1\right)H(0;u_2)+\\
&\tfrac{1}{2}\mathcal{G}\left(0,v_{123},\tfrac{1}{1-u_1};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(0,v_{213},\tfrac{1}{1-u_2};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{231},\tfrac{1}{1-u_2};1\right)H(0;u_2)-\tfrac{1}{2}\mathcal{G}\left(0,v_{312},\tfrac{1}{1-u_3};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{123};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{132};1\right)H(0;u_2)+\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{132};1\right)H(0;u_2)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},0;1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{1}{1-u_1};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},0;1\right)H(0;u_2)+\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},\tfrac{1}{1-u_1};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},0;1\right)H(0;u_2)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},\tfrac{1}{1-u_1};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{213};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{231};1\right)H(0;u_2)+\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{213};1\right)H(0;u_2)+\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{231};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},1;1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},\tfrac{1}{1-u_2};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},1;1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},\tfrac{1}{1-u_2};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,v_{312};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,v_{321};1\right)H(0;u_2)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{312};1\right)H(0;u_2)+\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{321};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},1;1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},\tfrac{1}{u_2};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},\tfrac{1}{1-u_3};1\right)H(0;u_2)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},0;1\right)H(0;u_2)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},\tfrac{1}{1-u_3};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},0;1\right)H(0;u_2)+\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},\tfrac{1}{1-u_3};1\right)H(0;u_2)+\\
&\tfrac{3}{4}\mathcal{G}\left(v_{123},1,\tfrac{1}{1-u_1};1\right)H(0;u_2)+\tfrac{3}{4}\mathcal{G}\left(v_{123},\tfrac{1}{1-u_1},1;1\right)H(0;u_2)-
\end{aligned}
$$

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Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

$$\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_2)-$$
$$\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_2)-\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_1)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_1)H(0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)H(0;u_1)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H(0;u_1)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H(0;u_1)H(0;u_2)+\frac{5}{24}\pi^2H(0;u_1)H(0;u_2)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_3)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_3)+$$
$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_3)-\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)-$$
$$\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_3)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1;1\right)H(0;u_3)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_3};1\right)H(0;u_3)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0;1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_3)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_2},0,\frac{1}{u_3};1\right)H(0;u_3)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-$$
$$\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H(0;u_3)-\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H(0;u_3)+$$
$$\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_3)+$$
$$\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H(0;u_3)+$$

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The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_3},v_{312};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_3},v_{321};1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{123},\tfrac{1}{1-u_1};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(0,u_{123},\tfrac{1}{u_3};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{1}{1-u_2};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)-\\
&\tfrac{1}{2}\mathcal{G}\left(0,v_{123},\tfrac{1}{1-u_1};1\right)H(0;u_3)+\tfrac{1}{2}\mathcal{G}\left(0,v_{231},\tfrac{1}{1-u_2};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{312},\tfrac{1}{1-u_3};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(0,v_{321},\tfrac{1}{1-u_3};1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{123};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{132};1\right)H(0;u_3)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{123};1\right)H(0;u_3)+\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{132};1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},1;1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{1}{1-u_1};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{1}{u_3};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},0;1\right)H(0;u_3)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},\tfrac{1}{1-u_1};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},0;1\right)H(0;u_3)+\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},\tfrac{1}{1-u_1};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{213};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{231};1\right)H(0;u_3)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{213};1\right)H(0;u_3)+\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{231};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},0;1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{1}{1-u_2};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},0;1\right)H(0;u_3)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},\tfrac{1}{1-u_2};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},0;1\right)H(0;u_3)+\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},\tfrac{1}{1-u_2};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,v_{312};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,v_{321};1\right)H(0;u_3)+\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{312};1\right)H(0;u_3)+\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{321};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},1;1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},\tfrac{1}{1-u_3};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},1;1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},\tfrac{1}{1-u_3};1\right)H(0;u_3)-\\
&\tfrac{3}{4}\mathcal{G}\left(v_{123},1,\tfrac{1}{1-u_1};1\right)H(0;u_3)-\tfrac{3}{4}\mathcal{G}\left(v_{123},\tfrac{1}{1-u_1},1;1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(v_{132},1,\tfrac{1}{1-u_1};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(v_{132},\tfrac{1}{1-u_1},1;1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(v_{213},1,\tfrac{1}{1-u_2};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(v_{213},\tfrac{1}{1-u_2},1;1\right)H(0;u_3)+
\end{aligned}
$$

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Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

$$
\begin{aligned}
&\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_3)+\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_1)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_1)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_1)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H(0;u_1)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H(0;u_1)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H(0;u_1)H(0;u_3)+\\
&\frac{5}{24}\pi^2H(0;u_1)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H(0;u_2)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)H(0;u_3)+\frac{5}{24}\pi^2H(0;u_2)H(0;u_3)+\\
&3H(0;u_2)H(0,0;u_1)H(0;u_3)+3H(0;u_1)H(0,0;u_2)H(0;u_3)+\\
&\frac{1}{4}H(0;u_2)H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H(0;u_3)+\frac{1}{2}H(0;u_1)H(0,1;(u_1+u_3))H(0;u_3)+\\
&\frac{1}{4}H(0;u_1)H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H(0;u_3)+\frac{1}{2}H(0;u_2)H(0,1;(u_2+u_3))H(0;u_3)+\\
&\frac{3}{4}H(0;u_2)H(1,0;u_1)H(0;u_3)+\frac{3}{4}H(0;u_1)H(1,0;u_2)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H(0,0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H(0,0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H(0,0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H(0,0;u_1)-\frac{23}{24}\pi^2H(0,0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H(0,0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H(0,0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H(0,0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H(0,0;u_2)-\\
&\frac{25}{4}H(0,0;u_1)H(0,0;u_2)-\frac{23}{24}\pi^2H(0,0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H(0,0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H(0,0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H(0,0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H(0,0;u_3)+3H(0;u_1)H(0;u_2)H(0,0;u_3)-\\
&\frac{25}{4}H(0,0;u_1)H(0,0;u_3)-\frac{25}{4}H(0,0;u_2)H(0,0;u_3)-\frac{23}{24}\pi^2H(0,0;u_3)+\frac{1}{12}\pi^2H(0,1;u_1)+\\
&\frac{1}{12}\pi^2H(0,1;u_2)-\frac{1}{24}\pi^2H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H(0;u_1)H(0;u_2)H(0,1;(u_1+u_2))+
\end{aligned}
$$

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Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

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$$
\begin{aligned}
&\frac{1}{12}\pi^2 H(0,1;(u_1+u_2)) + \frac{1}{12}\pi^2 H(0,1;u_3) + \frac{1}{4}H(0;u_1)\,H(0;u_2)\,H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right) - \\
&\frac{1}{24}\pi^2 H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right) + \frac{1}{12}\pi^2 H(0,1;(u_1+u_3)) - \frac{1}{24}\pi^2 H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right) + \\
&\frac{1}{12}\pi^2 H(0,1;(u_2+u_3)) - \frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right) H(1,0;u_1) - \\
&\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right) H(1,0;u_1) + \frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right) H(1,0;u_1) + \\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right) H(1,0;u_1) + \frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right) H(1,0;u_1) + \\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right) H(1,0;u_1) + \frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right) H(1,0;u_1) - \\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right) H(1,0;u_1) - \frac{3}{4}H(0,0;u_2)\,H(1,0;u_1) - \frac{3}{4}H(0,0;u_3)\,H(1,0;u_1) + \\
&\frac{1}{4}H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right) H(1,0;u_1) - \frac{1}{3}\pi^2 H(1,0;u_1) - \frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right) H(1,0;u_2) - \\
&\frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right) H(1,0;u_2) + \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right) H(1,0;u_2) + \\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right) H(1,0;u_2) + \frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right) H(1,0;u_2) + \\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right) H(1,0;u_2) + \frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right) H(1,0;u_2) - \\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right) H(1,0;u_2) - \frac{3}{4}H(0,0;u_1)\,H(1,0;u_2) - \frac{3}{4}H(0,0;u_3)\,H(1,0;u_2) + \\
&\frac{1}{4}H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right) H(1,0;u_2) - \frac{1}{4}H(1,0;u_1)\,H(1,0;u_2) - \frac{1}{3}\pi^2 H(1,0;u_2) - \\
&\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right) H(1,0;u_3) - \frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right) H(1,0;u_3) + \\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right) H(1,0;u_3) + \frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right) H(1,0;u_3) + \\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right) H(1,0;u_3) - \frac{1}{3}\pi^2 H(1,0;u_3) + \frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right) H(1,0;u_3) + \\
&\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right) H(1,0;u_3) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right) H(1,0;u_3) + \\
&\frac{3}{4}H(0;u_1)\,H(0;u_2)\,H(1,0;u_3) - \frac{3}{4}H(0,0;u_1)\,H(1,0;u_3) - \frac{3}{4}H(0,0;u_2)\,H(1,0;u_3) + \\
&\frac{1}{4}H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right) H(1,0;u_3) - \frac{1}{4}H(1,0;u_1)\,H(1,0;u_3) - \frac{1}{4}H(1,0;u_2)\,H(1,0;u_3) + \\
&\frac{1}{24}\pi^2 H(1,1;u_1) + \frac{1}{24}\pi^2 H(1,1;u_2) + \frac{1}{24}\pi^2 H(1,1;u_3) + \frac{1}{2}H(0;u_2)\,H(0,0,0;u_1) + \\
&\frac{1}{2}H(0;u_3)\,H(0,0,0;u_2) + \frac{1}{2}H(0;u_1)\,H(0,0,0;u_3) - \frac{1}{2}H(0;u_2)\,H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right) - \\
&\frac{1}{2}H(0;u_3)\,H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right) - H(0;u_1)\,H(0,0,1;(u_1+u_2)) - \\
&H(0;u_2)\,H(0,0,1;(u_1+u_2)) - \frac{1}{2}H(0;u_1)\,H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right) -
\end{aligned}
$$

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The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

$$
\begin{aligned}
&\frac{1}{2}H(0;u_2)\,H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-H(0;u_1)\,H(0,0,1;(u_1+u_3))-\\
&H(0;u_3)\,H(0,0,1;(u_1+u_3))-\frac{1}{2}H(0;u_1)\,H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-\\
&\frac{1}{2}H(0;u_3)\,H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-H(0;u_2)\,H(0,0,1;(u_2+u_3))-\\
&H(0;u_3)\,H(0,0,1;(u_2+u_3))-\frac{1}{2}H(0;u_2)\,H(0,1,0;u_1)-\frac{1}{2}H(0;u_3)\,H(0,1,0;u_2)-\\
&\frac{1}{2}H(0;u_1)\,H(0,1,0;u_3)+\frac{1}{4}H(0;u_2)\,H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-\\
&\frac{1}{4}H(0;u_3)\,H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H(0;u_1)\,H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\\
&\frac{1}{4}H(0;u_2)\,H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)\,H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+\\
&\frac{1}{4}H(0;u_3)\,H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(0;u_2)\,H(1,0,0;u_1)-\frac{1}{2}H(0;u_3)\,H(1,0,0;u_1)-\\
&\frac{1}{2}H(0;u_1)\,H(1,0,0;u_2)+\frac{1}{2}H(0;u_3)\,H(1,0,0;u_2)+\frac{1}{2}H(0;u_1)\,H(1,0,0;u_3)-\\
&\frac{1}{2}H(0;u_2)\,H(1,0,0;u_3)-\frac{1}{4}H(0;u_3)\,H\left(1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-\\
&\frac{1}{4}H(0;u_2)\,H\left(1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)\,H\left(1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-\\
&7H(0,0,0,0;u_1)-7H(0,0,0,0;u_2)-7H(0,0,0,0;u_3)+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\\
&3H(0,0,0,1;(u_1+u_2))+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+3H(0,0,0,1;(u_1+u_3))+\\
&\frac{3}{2}H\left(0,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+3H(0,0,0,1;(u_2+u_3))+\frac{9}{4}H(0,0,1,0;u_1)+\\
&\frac{9}{4}H(0,0,1,0;u_2)+\frac{9}{4}H(0,0,1,0;u_3)-\frac{1}{2}H(0,1,0,0;u_1)-\frac{1}{2}H(0,1,0,0;u_2)-\\
&\frac{1}{2}H(0,1,0,0;u_3)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&\frac{1}{2}H\left(0,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+H(0,1,1,0;u_1)+H(0,1,1,0;u_2)+H(0,1,1,0;u_3)-\\
&\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\\
&\frac{1}{4}H\left(0,1,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+H\left(1,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+H\left(1,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&H\left(1,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+2H(1,0,1,0;u_1)+2H(1,0,1,0;u_2)+2H(1,0,1,0;u_3)+\\
&\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&\frac{1}{4}H\left(1,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(1,1,1,0;u_1)+\frac{1}{2}H(1,1,1,0;u_2)+\frac{1}{2}H(1,1,1,0;u_3)-\\
&\frac{1}{24}\pi^2H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{u_{123}}\right)-\frac{1}{24}\pi^2H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{u_{231}}\right)-\frac{1}{24}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{u_{312}}\right)+\\
&\frac{1}{8}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{123}}\right)-\frac{1}{8}\pi^2H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{24}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-
\end{aligned}
$$

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Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

$$\frac{1}{24}\pi^2 H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-\frac{1}{24}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{24}\pi^2 H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{8}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2 H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{v_{312}}\right)-$$
$$\frac{1}{8}\pi^2 H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{312}}\right)+\frac{1}{24}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-\frac{1}{24}\pi^2 H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{4}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)-\frac{1}{4}H(1,0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H(1,0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-$$
$$\frac{1}{4}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-\frac{1}{4}H(1,0;u_1)\,\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-$$
$$\frac{1}{4}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+$$
$$\frac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)-$$
$$\frac{1}{4}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+$$
$$\frac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)-$$
$$\frac{1}{4}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{2}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{1}{2}H(0,0;u_2)\,\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{2}H(0,0;u_3)\,\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)-\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{132}}\right)-$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{213}}\right)-\frac{1}{2}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{1}{2}H(0,0;u_1)\,\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{2}H(0,0;u_3)\,\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)-\frac{1}{2}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{2}H(0,0;u_1)\,\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{1}{2}H(0,0;u_2)\,\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)-$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{321}}\right)+\frac{1}{2}H(0;u_2)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+\frac{1}{2}H(0;u_3)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+$$
$$\frac{1}{2}H(0;u_1)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H(0;u_3)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H(0;u_1)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{2}H(0;u_2)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{u_{231}}\right)+$$

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INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
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Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
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E-mail: smirnov@theory.sinp.msu.ru

$$
\begin{aligned}
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-\\
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-\\
&\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+\\
&\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+\\
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{u_{231}}\right)+\\
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-\\
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-\\
&\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+\\
&\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+\\
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+H(0;u_2)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)-H(0;u_3)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)-\\
&H(0;u_1)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H(0;u_3)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H(0;u_1)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-\\
&H(0;u_2)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{123}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{231}}\right)-\\
&\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{312}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{132}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{213}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{321}}\right)-\\
&\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{312}}\right)-\\
&\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right)+\\
&\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{123}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{132}}\right)+\zeta_3H(0;u_1)+\zeta_3H(0;u_2)+\zeta_3H(0;u_3)+\\
&\frac{5}{2}\zeta_3H(1;u_1)+\frac{5}{2}\zeta_3H(1;u_2)+\frac{5}{2}\zeta_3H(1;u_3)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{123}}\right)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{231}}\right)+\\
&\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{312}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{312}}\right)+\\
&\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{132}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{312}}\right)+\\
&\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{213}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{312}}\right)+\\
&\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{123}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{213}}\right)+\\
&\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{312}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{123}}\right)+\\
&\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{213}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{312}}\right)+
\end{aligned}
$$

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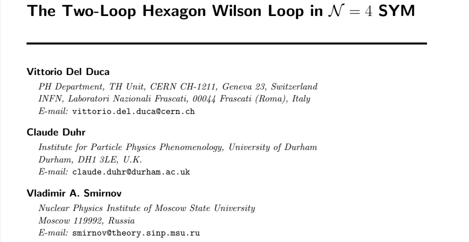

The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
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Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

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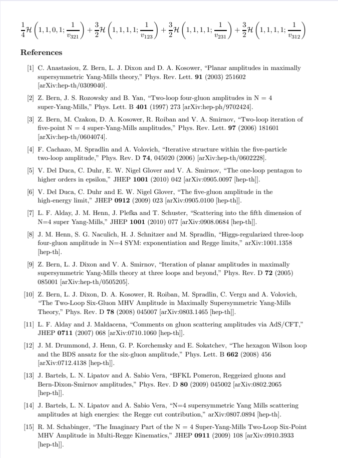

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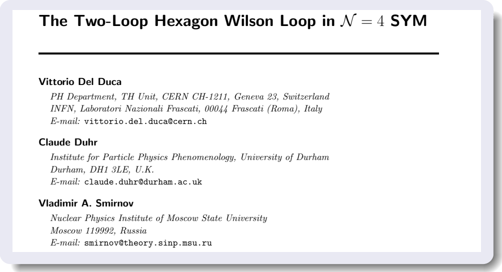

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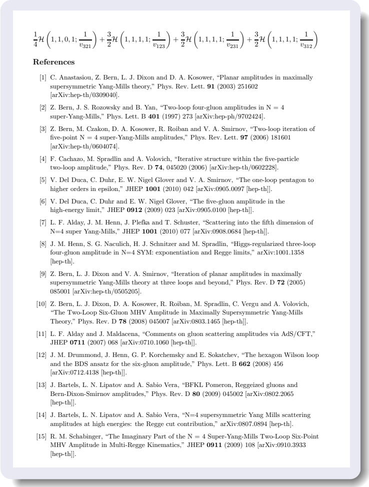

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Spiritus Movens: Even More Shocking Simplicity is Found

Upon consulting with Goncharov about his polylogarithms, these 18 pages were found to simplify, [arXiv:1006.5703]:

$$R(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{J^4}{24} + \chi\frac{\pi^2}{12}\left(J^2+\zeta(2)\right).$$

The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

$$\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{321}}\right) + \frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{123}}\right) + \frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{231}}\right) + \frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{312}}\right)$$

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