The Vernacular of the S-Matrix

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LVI Course, 2016
A Panorama of Holography

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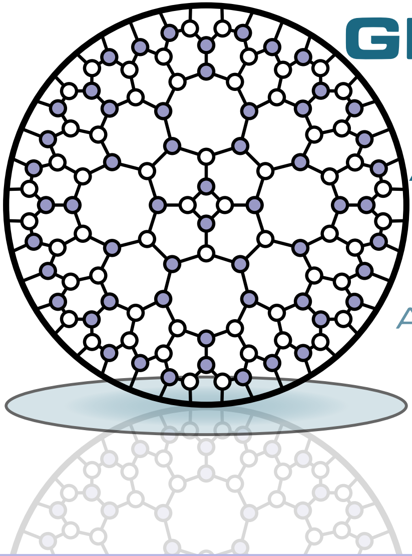

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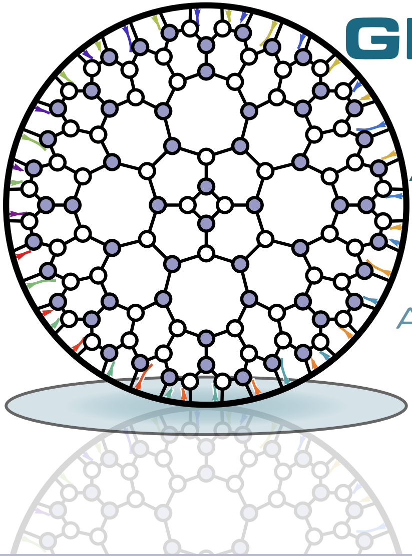

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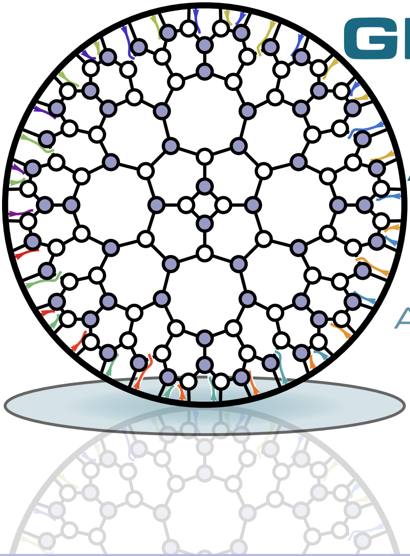

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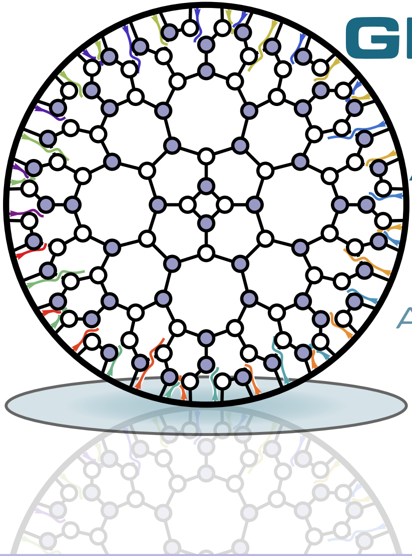

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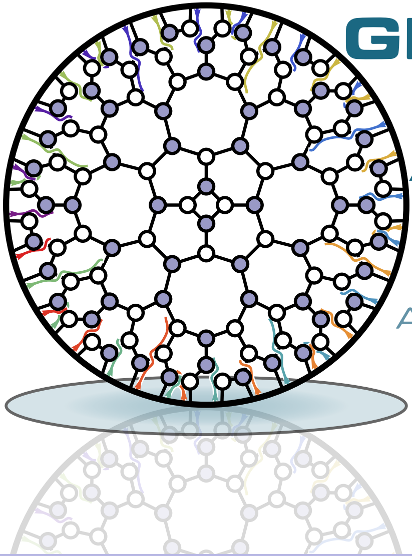

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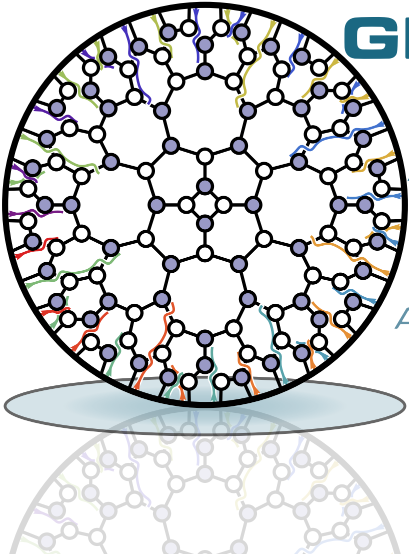

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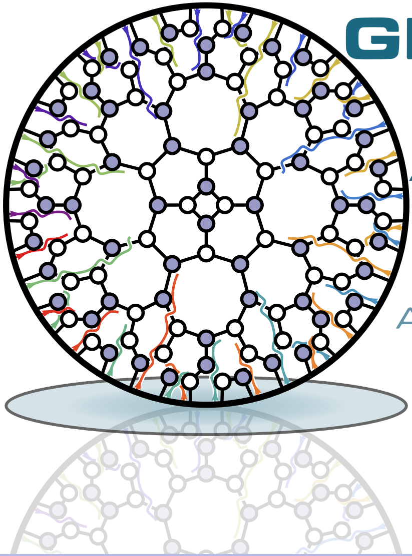

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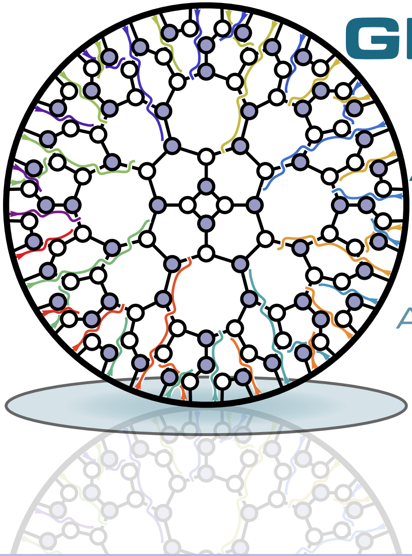

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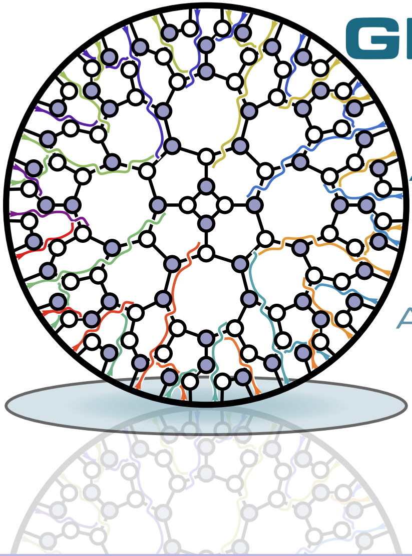

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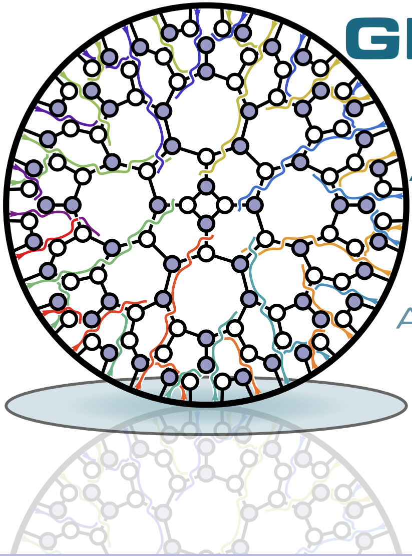

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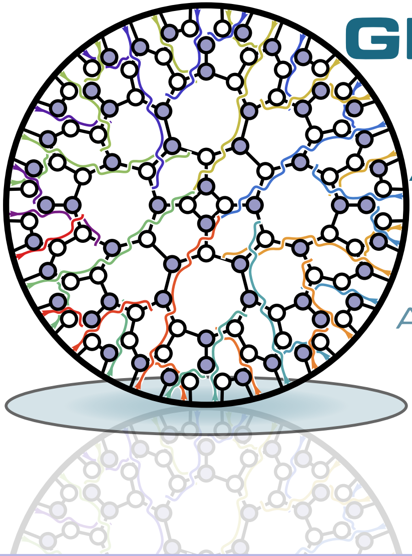

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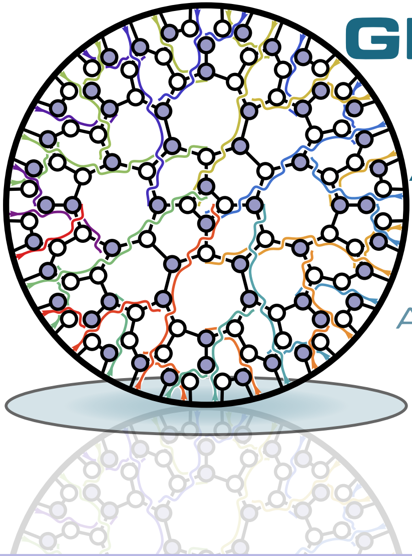

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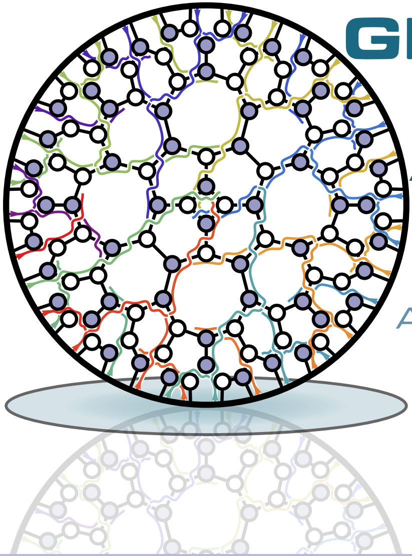

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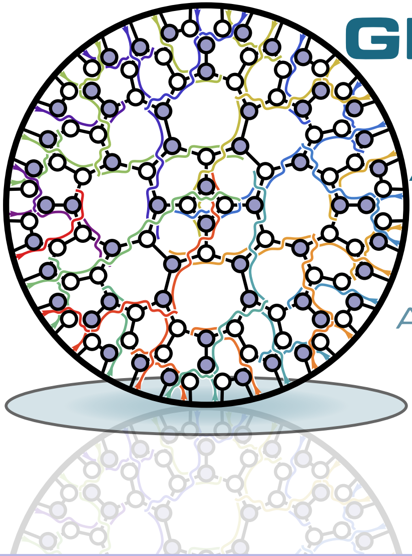

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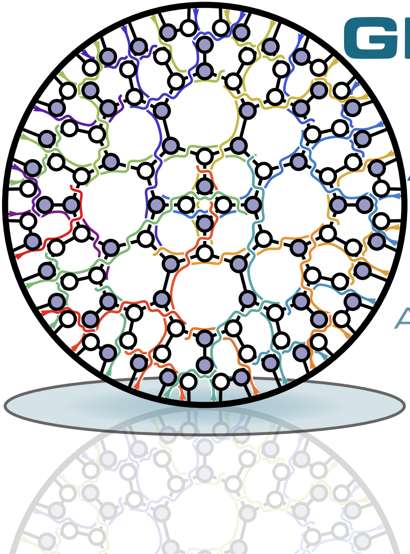

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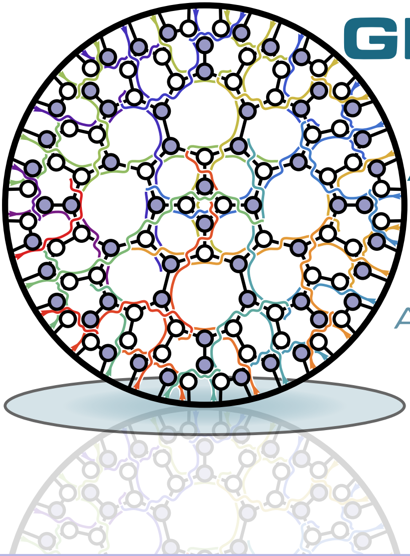

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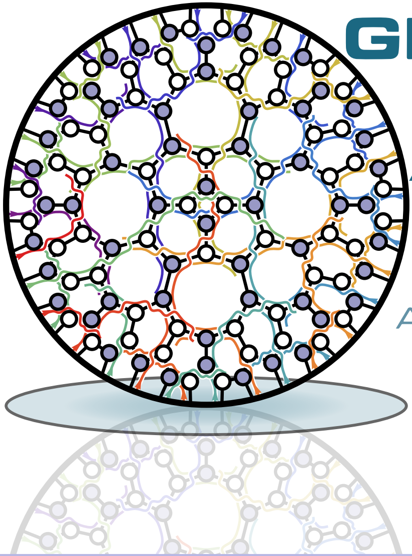

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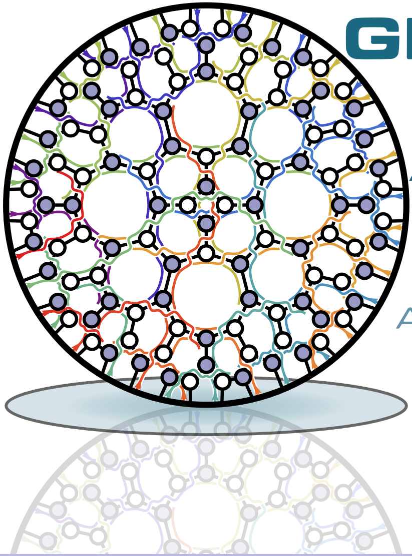

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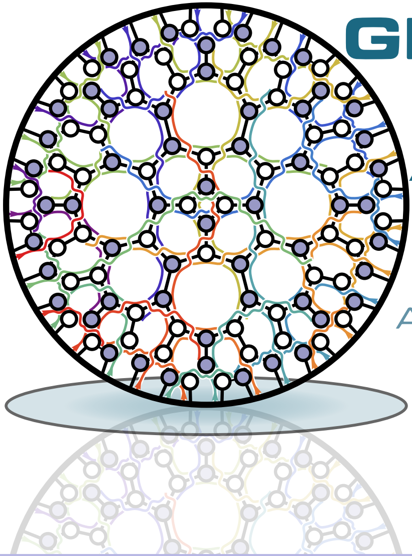

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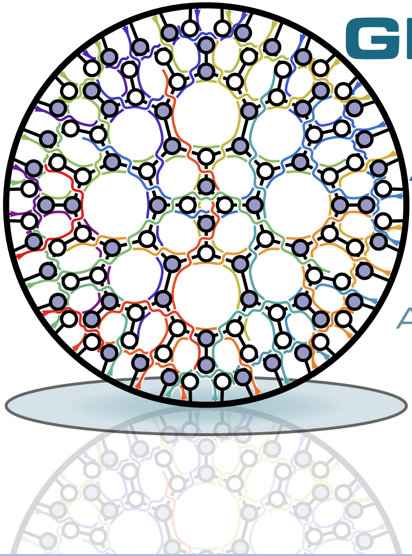

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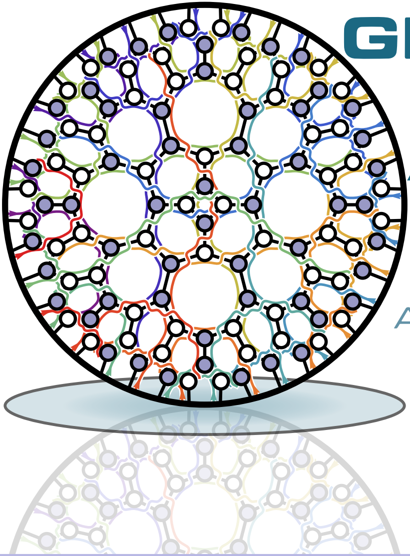

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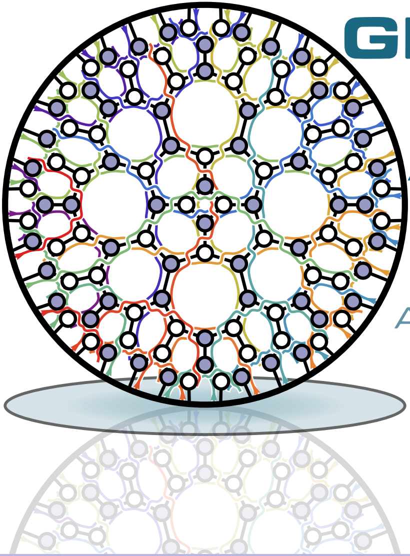

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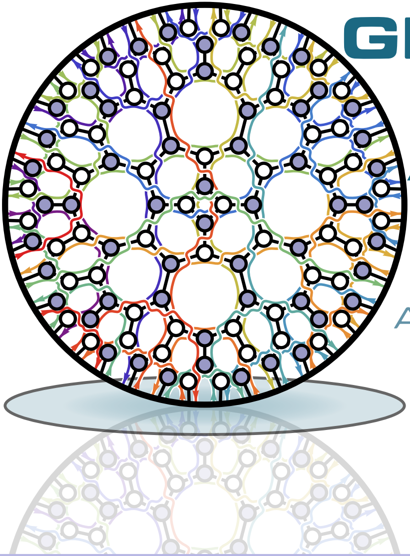

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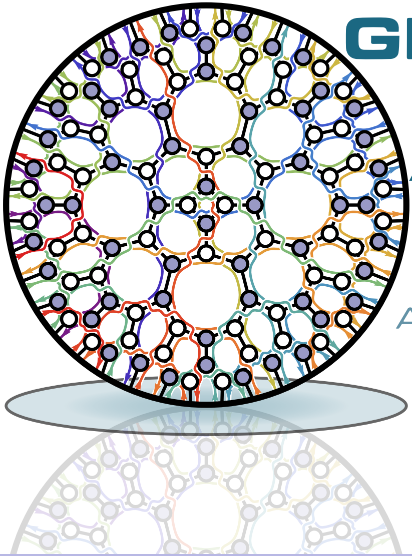

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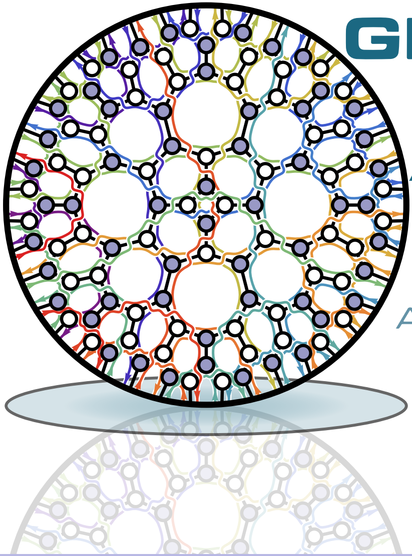

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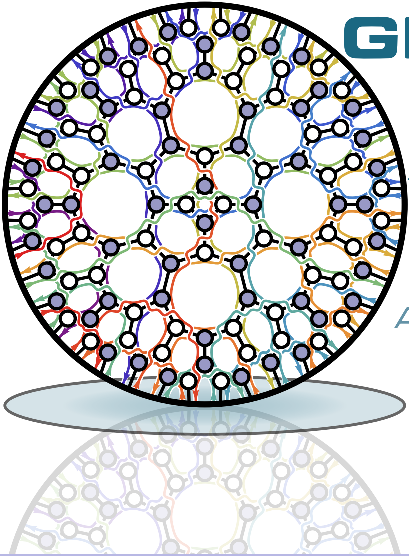

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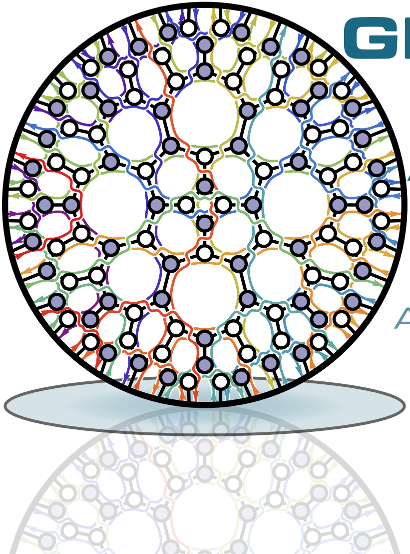

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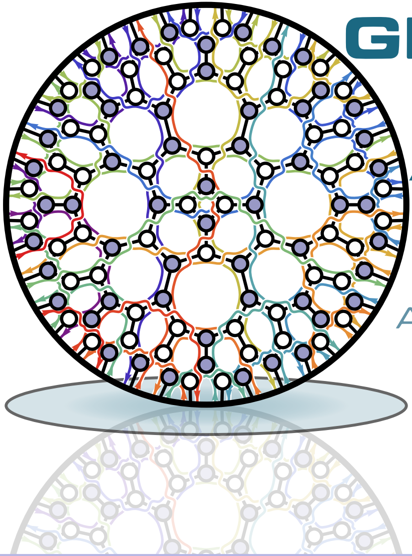

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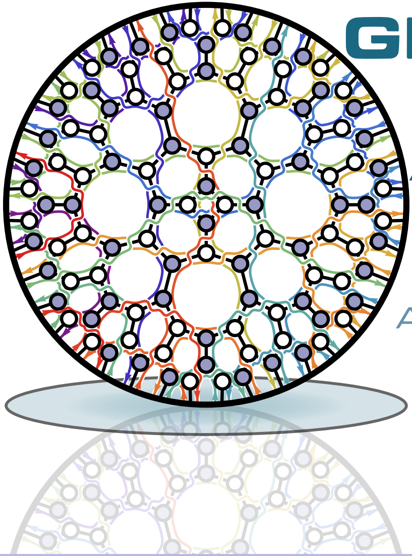

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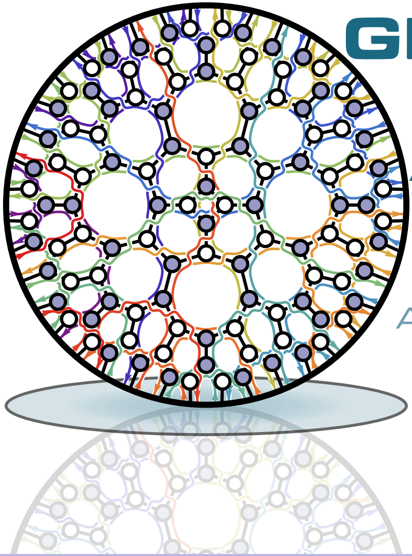

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Organization and Outline

1. *Spiritus Movens*: a moral parable
   - A Simple, Practical Problem in Quantum Chromodynamics
   - The *Shocking* Simplicity of Scattering Amplitudes
2. The Vernacular of the S-Matrix
   - Physically Observable Data Describing Asymptotic States
   - Massless Momenta and Spinor-Helicity Variables
   - (Grassmannian) Geometry of Momentum Conservation
3. The All-Orders S-Matrix for Three Massless Particles
   - Three Particle Kinematics and Helicity Amplitudes
   - Non-Dynamical Dependence: Coupling Constants & Spin/Statistics
4. Consequences of Quantum Mechanical Consistency Conditions
   - Factorization and Long-Range Physics: *Weinberg's Theorem*
   - Uniqueness of Yang-Mills Theory and the Equivalence Principle
   - The Simplest Quantum Field Theory: $\mathcal{N}=4$ super Yang-Mills

Supercomputer Computations in Quantum Chromodynamics

Consider the amplitude for two gluons to collide and produce four: $gg \to gggg$.

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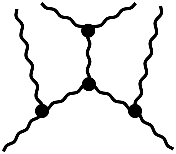

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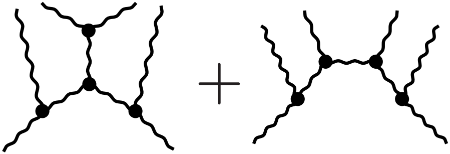

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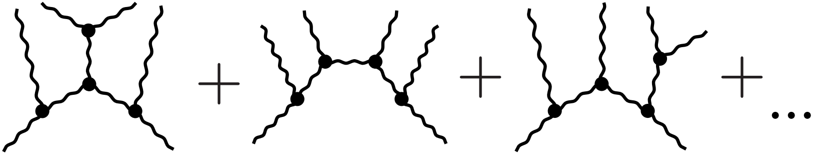

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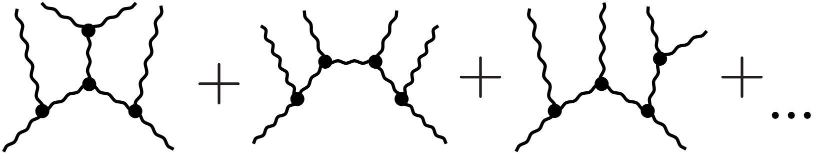

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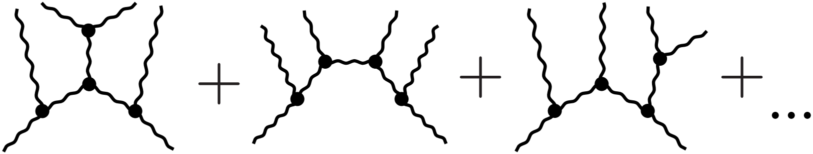

Supercollider physics

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Eichten *et al.* summarize the motivation for exploring the 1-TeV ($=10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.

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TeV. From Fig. 78 we find the corresponding two-jet cross section (at $p_\perp = 0.5$ TeV/c) to be about 7×10^{-2} nb/GeV, which is larger by an order of magnitude. Let us next consider the cross section in the neighborhood of the peak in Fig. 102. The integrated cross section in the bin $0.3 \lesssim \cos\theta \lesssim 0.4$ is approximately 0.1 nb/GeV, with transverse energy given roughly by $\langle E_T\rangle \approx (1\ \text{TeV}) \times \langle\cos\theta\rangle = 350$ GeV. The corresponding two-jet cross section, again from Fig. 78, is approximately 10 nb/GeV, which is larger by 2 orders of magnitude. In fact, we have certainly underestimated $\langle E_T\rangle$ and thus somewhat overestimated the two-jet/three-jet ratio in this second case.

We draw two conclusions from this very casual analysis:

At least at small-to-moderate values of E_T, two-jet events should account for most of the cross section.

The three-jet cross section is large enough that a detailed study of this topology should be possible.

It is apparent that these questions are amenable to detailed investigation with the aid of realistic Monte Carlo simulations. Given the elementary two→three cross sections and reasonable parametrizations of the fragmentation functions, this exercise can be carried out with some degree of confidence.

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

Another background source of four-jet events is double parton scattering, as shown in Fig. 103. If all the parton momentum fractions are small, the two interactions may be treated as uncorrelated. The resulting four-jet cross section with transverse energy E_T may then be approximated by

$$\sigma_4(E_T) \simeq \int_\varepsilon^{E_T-\varepsilon} dE_{T1} \int_\varepsilon^{E_T-\varepsilon} dE_{T2}\, \frac{\sigma_2(E_{T1})\sigma_2(E_{T2})\delta(E_{T1}+E_{T2}-E_T)}{\sigma_{\text{total}}}\ , \qquad (3.47)$$

where $\sigma_2(E_{T1})$ is the two-jet cross section and ε denotes the minimum E_T required for a discernable two-jet event. For a recent study of double parton scattering at $S\bar{p}pS$ and Tevatron energies, see Paver and Treleani (1983).

In view of the promise that multijet spectroscopy holds, improving our understanding of the QCD background is an urgent priority for further study.

D. Summary

We conclude this section with a brief summary of the ranges of jet energy which are accessible for various beam energies and luminosities. We find essentially no differences between pp and $\bar{p}p$ collisions, so only pp results will be given except at $\sqrt{s} = 2$ TeV where $\bar{p}p$ rates are quoted. Figure 104 shows the E_T range which can be explored at the level of at least one event per GeV of E_T per unit rapidity at 90° in the c.m. (compare Figs. 77—79 and 83). The results are presented in terms of the transverse energy per event E_T, which corresponds to twice the transverse momentum $p_\perp$ of a jet. In Fig. 105 we plot the values of E_T that distinguish the regimes in which the two-gluon, quark-gluon, and quark-quark final states are dominant. Comparing with Fig. 104, we find that while the accessible ranges of E_T are impressive, it seems extremely difficult to obtain a clean sample of quark jets. Useful for estimating trigger rates is the total cross section for two jets integrated over $E_T(=2p_\perp) > E_{T_0}$ for both jets in a rapidity interval of -2.5 to $+2.5$. This is shown for pp collisions in Fig. 106.

IV. ELECTROWEAK PHENOMENA

In this section we discuss the supercollider processes associated with the standard model of the weak and electromagnetic interactions (Glashow, 1961; Weinberg, 1967; Salam, 1968). By "standard model" we understand the $SU(2)_L \otimes U(1)_Y$ theory applied to three quark and lepton doublets, and with the gauge symmetry broken by a single complex Higgs doublet. The particles associated with the electroweak interactions are therefore the (left-handed) charged intermediate bosons $W^\pm$, the neutral intermedi-

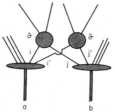

FIG. 103. Four-jet topology arising from two independent parton interactions.

Supercomputer Computations in Quantum Chromodynamics

Consider the amplitude for two gluons to collide and produce four: $gg \to gggg$.
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- 220 Feynman diagrams, thousands of terms

Supercollider physics

E. Eichten
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Eichten *et al.* summarize the motivation for exploring the 1-TeV ($=10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.

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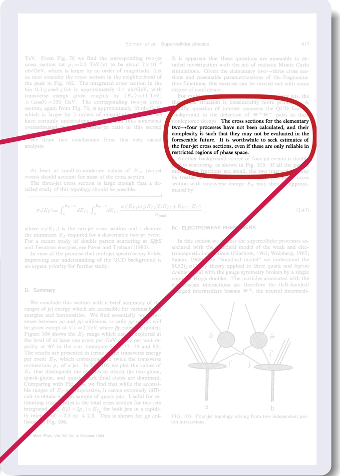

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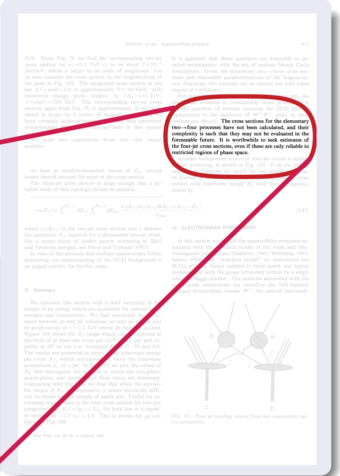
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gluons. The cross section for the scattering of two gluons with momenta p_1, p_2 into four gluons with momenta p_3, p_4, p_5, p_6 is obtained from eq. (5) by setting $I=2$ and replacing the momenta p_3, p_4, p_5, p_6 by $-p_3, -p_4, -p_5, -p_6$.

As the result of the computation of two hundred and forty Feynman diagrams, we obtain

$$A_{\binom{0}{2}}(p_1, p_2, p_3, p_4, p_5, p_6) = (\mathscr{D}^\dagger, \mathscr{D}^\dagger_\rho, \mathscr{D}^\dagger_\sigma, \mathscr{D}^\dagger_\tau)_{\binom{0}{2}} \cdot \begin{pmatrix} K & K_\rho & K_\sigma & K_\tau \\ K_\rho & K & K_\tau & K_\sigma \\ K_\sigma & K_\tau & K & K_\rho \\ K_\tau & K_\sigma & K_\rho & K \end{pmatrix} \cdot \begin{pmatrix} \mathscr{D} \\ \mathscr{D}_\rho \\ \mathscr{D}_\sigma \\ \mathscr{D}_\tau \end{pmatrix}_{\binom{0}{2}}, \quad (6)$$

where $\mathscr{D}$, $\mathscr{D}_\rho$, $\mathscr{D}_\sigma$ and $\mathscr{D}_\tau$ are 11-component complex vector functions of the momenta p_1, p_2, p_3, p_4, p_5 and p_6, and K, K_ρ, K_σ and K_τ are constant 11×11 symmetric matrices. The vectors $\mathscr{D}_\rho$, $\mathscr{D}_\sigma$ and $\mathscr{D}_\tau$ are obtained from the vector $\mathscr{D}$ by the permutations $(p_2 \leftrightarrow p_3)$, $(p_5 \leftrightarrow p_6)$ and $(p_2 \leftrightarrow p_3, p_5 \leftrightarrow p_6)$, respectively, of the momentum variables in $\mathscr{D}$. The individual components of the vector $\mathscr{D}$ represent the sums of all contributions proportional to the appropriately chosen eleven basis color factors. The matrices K, which are the suitable sums over the color indices of products of the color bases, contain two independent structures, proportional to $N^4(N^2-1)$ and $N^2(N^2-1)$, respectively (N is the number of colors, $N=3$ for QCD):

$$K = \tfrac{1}{8} g^8 N^4 (N^2-1) K^{(4)} + \tfrac{1}{2} g^8 N^2 (N^2-1) K^{(2)}. \quad (7)$$

Here g denotes the gauge coupling constant. The matrices $K^{(4)}$ and $K^{(2)}$ are given in table 1. The vector $\mathscr{D}$ is related to the thirty-three diagrams $D^{\mathrm{G}}(I=1-33)$ for two-gluon to four-scalar scattering, eleven diagrams $D^{\mathrm{F}}(I=1-11)$ for two-fermion to four-scalar scattering and sixteen diagrams $D^{\mathrm{S}}(I=1-16)$ for two-scalar to four-scalar scattering, in the following way:

$$\mathscr{D}_0 = \frac{2 s_{14}}{\sqrt{|s_{15} s_{45} s_{16} s_{46}|}\, s_{23} s_{56}} \{ t_{123}^2 C^{\mathrm{G}} \cdot D_0^{\mathrm{G}} - 4 s_{14} t_{123} E(p_5+p_6, p_6) C^{\mathrm{F}} \cdot D_0^{\mathrm{F}} - 2 s_{14} G(p_5+p_6, p_5+p_6) C^{\mathrm{S}} \cdot D_0^{\mathrm{S}} \},$$

$$\mathscr{D}_2 = \frac{s_{56}}{s_{23}} C^{\mathrm{G}} \cdot D_2^{\mathrm{G}}, \quad (8)$$

where the constant matrices $C^{\mathrm{G}}(11 \times 33)$, $C^{\mathrm{F}}(11 \times 11)$ and $C^{\mathrm{S}}(11 \times 16)$ are given in table 2. The Lorentz invariants s_{ij} and t_{ijk} are defined as $s_{ij} = (p_i + p_j)^2$, $t_{ijk} = (p_i + p_j + p_k)^2$ and the complex functions E and G are given by

$$E(p_i, p_j) = \tfrac{1}{2}\{(p_1 p_4)(p_i p_j) - (p_1 p_i)(p_j p_4) - (p_1 p_j)(p_i p_4) + i \varepsilon_{\mu\nu\rho\lambda} p_1^\mu p_i^\nu p_j^\rho p_4^\lambda\}/(p_1 p_4),$$

$$G(p_i, p_j) = E(p_i, p_5) E(p_j, p_6), \quad (9)$$

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TABLE 1
Matrices $K(I, J)[I = 1\text{-}11,\ J = 1\text{-}11]$.

Matrix $K^{(4)}$											Matrix $K^{(2)}$										
8	4	−2	2	−1	2	0	1	0	0	−1	0	0	0	0	0	0	0	0	3	3	−3
4	8	−1	1	−1	0	2	1	0	1	−1	0	0	0	0	0	0	0	0	3	3	0
−2	−1	8	4	4	1	1	2	2	1	2	0	0	0	0	0	0	0	0	0	0	0
2	1	4	8	2	−1	−1	4	1	1	1	0	0	0	0	0	0	0	0	0	0	0
−1	−1	4	2	8	1	2	4	−2	−1	4	0	0	0	0	0	0	0	0	0	0	0
2	0	1	−1	1	8	4	−1	0	1	0	0	0	0	0	0	0	0	0	3	3	0
0	2	1	−1	2	4	8	−2	0	0	0	0	0	0	0	0	0	0	0	3	3	0
1	1	2	4	4	−1	−2	8	−1	−1	2	0	0	0	0	0	0	0	0	0	0	0
0	0	2	1	−2	0	0	−1	8	4	−2	3	3	0	0	0	3	3	0	0	0	0
0	1	1	1	−1	1	0	−1	4	8	−1	3	3	0	0	0	3	3	0	0	0	0
−1	−1	2	1	4	0	0	2	−2	−1	8	−3	0	0	0	0	0	0	0	0	0	0

Matrix $K^{(4)}_{\rho}$											Matrix $K^{(2)}_{\rho}$										
0	0	0	0	1	1	0	1	1	0	−1	3	3	0	3	0	0	0	3	0	0	0
0	0	0	0	2	0	1	1	2	1	−2	3	3	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	1	1	0	1	1	0	0	3	0	0	3	0	0	3	3	0
0	0	0	0	1	0	0	2	0	1	0	3	0	0	0	0	3	0	0	0	0	0
1	2	0	1	0	1	2	2	0	0	2	0	0	0	0	0	0	0	0	3	3	0
1	0	1	0	1	4	2	0	0	0	−1	0	0	3	3	0	0	0	0	3	3	0
0	1	1	0	2	2	4	0	0	0	−2	0	0	0	0	0	0	0	0	0	3	0
1	1	1	2	2	0	0	4	0	0	0	3	0	0	0	0	0	0	0	3	0	0
1	2	0	0	0	0	0	0	0	2	−1	0	0	3	0	3	3	0	3	0	0	0
0	1	1	1	0	0	0	0	2	4	0	0	0	3	0	3	3	3	0	0	0	0
−1	−2	1	0	2	−1	−2	0	−1	0	4	0	0	0	0	0	0	0	0	0	0	0

Matrix $K^{(4)}_{\sigma}$											Matrix $K^{(2)}_{\sigma}$										
4	2	0	2	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	3	0	−3
2	4	0	1	0	0	1	1	0	1	0	0	0	0	0	3	0	0	3	0	0	0
0	0	4	2	2	1	1	1	2	1	0	0	0	0	0	0	0	3	0	0	0	0
2	1	2	0	1	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	2	1	0	0	0	0	4	2	2	0	3	0	0	0	0	3	0	0	0	0
1	0	1	2	0	0	0	0	1	2	0	0	0	0	0	0	3	3	0	0	0	−3
0	1	1	1	0	0	0	0	2	4	0	0	0	3	0	3	3	3	0	0	0	0
1	1	1	0	0	0	0	0	2	0	1	0	3	0	0	0	0	0	3	0	0	0
0	0	2	1	4	1	2	2	0	0	−4	3	0	0	0	0	0	0	0	0	0	0
0	1	1	0	2	2	4	0	0	0	−2	0	0	0	0	0	0	0	0	0	3	0
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Matrix $K^{(4)}_{\tau}$											Matrix $K^{(2)}_{\tau}$										
0	1	−1	−1	1	1	0	1	2	0	0	3	3	0	0	0	3	3	0	0	0	0
1	0	−2	−1	2	0	1	1	4	2	0	3	3	0	0	0	3	3	0	0	0	0
−1	−2	0	0	0	1	1	1	−1	1	0	0	0	3	3	3	0	0	3	0	0	0
−1	−1	0	1	0	2	1	0	1	−1	0	0	0	3	3	3	0	0	3	0	0	3
1	2	0	0	1	−1	−1	0	−2	2	1	0	0	3	3	3	0	0	3	0	0	0
1	0	1	2	−1	0	1	−2	2	4	−1	3	3	0	0	0	3	3	0	0	0	−3
0	1	1	1	−1	1	0	−1	4	8	−1	3	3	0	0	0	3	3	0	0	0	0
1	1	1	0	0	−2	−1	0	2	−2	0	0	0	3	3	3	0	0	3	0	0	0
2	4	−1	1	−2	2	4	2	1	0	−2	0	0	0	0	0	0	0	0	3	3	0
0	2	1	−1	2	4	8	−2	0	0	0	0	0	0	0	0	0	0	0	3	3	0
0	0	0	0	1	−1	−1	0	−2	0	2	0	0	0	3	0	−3	0	0	0	0	0

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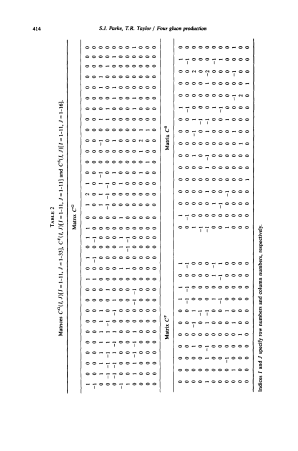

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where ε is the totally antisymmetric tensor, $\varepsilon_{xyzt} = 1$. For the future use, we define one more function,

$$F(p_i, p_j) = \{(p_1 p_4)(p_i p_j) + (p_1 p_i)(p_j p_4) - (p_1 p_j)(p_i p_4)\}/(p_1 p_4)\,. \qquad (10)$$

Note that when evaluating A_0 and A_2 at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions E, F and G on the momenta p_1, p_4, p_5, p_6.

The diagrams D_2^G are listed below:

$$D_2^G(1) = \frac{\delta_2}{s_{14}s_{25}s_{36}}\{[(p_2-p_5)(p_3-p_6)][(p_1-p_4)(p_3+p_6)] - [(p_2-p_5)(p_3+p_6)] \times [(p_1-p_4)(p_3-p_6)] + [(p_2+p_5)(p_3-p_6)][(p_1-p_4)(p_2-p_5)]\}\,,$$

$$D_2^G(2) = \frac{1}{s_{25}s_{36}}\{2E(p_2-p_5, p_3-p_6) - 2E(p_3-p_6, p_2-p_5) + \delta_2[(p_2-p_5)(p_3-p_6)]\}\,,$$

$$\begin{aligned} D_2^G(3) = \frac{4}{s_{25}s_{36}t_{125}}\{&[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2, p_3) \\ &-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2, p_6) \\ &-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5, p_3) \\ &+[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5, p_6) \\ &+[p_1(p_2-p_5)]E(p_3-p_6, p_3+p_6) - [p_4(p_3-p_6)]E(p_2+p_5, p_2-p_5) \\ &+\delta_2[p_1(p_2-p_5)][p_4(p_3-p_6)]\}\,, \end{aligned}$$

$$D_2^G(4) = \frac{-2}{s_{36}t_{125}}\{E(p_3-p_6, p_3+p_6) - \delta_2[p_4(p_3-p_6)]\}\,,$$

$$D_2^G(5) = \frac{-2}{s_{25}t_{125}}\{E(p_2+p_5, p_2-p_5) - \delta_2[p_1(p_2-p_5)]\}\,,$$

$$D_2^G(6) = \frac{\delta_2}{t_{125}}\,,$$

$$\begin{aligned} D_2^G(7) = \frac{4}{s_{12}s_{36}t_{125}}\{&[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2, p_3) \\ &-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2, p_6) - [p_4(p_3-p_6)]E(p_2, p_2-p_5)\}\,, \end{aligned}$$

$$\begin{aligned} D_2^G(8) = \frac{4}{s_{34}s_{25}t_{125}}\{&[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2, p_3) \\ &-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5, p_3) - [p_1(p_2-p_5)]E(p_3-p_6, p_3)\}\,, \end{aligned}$$

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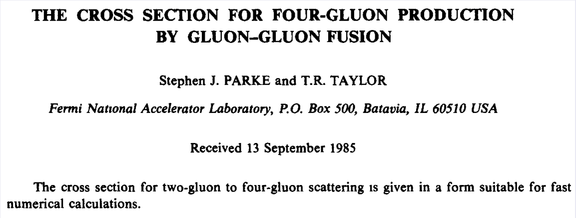

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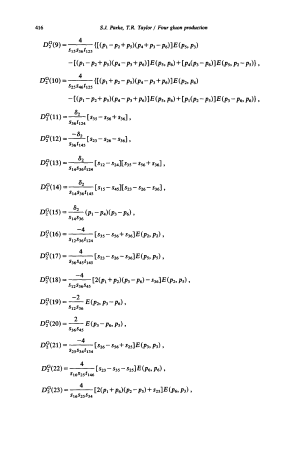

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$$D_2^G(9) = \frac{4}{s_{15}s_{36}t_{125}}\{[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5,p_3) - [(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6) + [p_4(p_3-p_6)]E(p_5,p_2-p_5)\}\,,$$

$$D_2^G(10) = \frac{4}{s_{25}s_{46}t_{125}}\{[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2,p_6) - [(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6) + [p_1(p_2-p_5)]E(p_3-p_6,p_6)\}\,,$$

$$D_2^G(11) = \frac{\delta_2}{s_{36}t_{124}}[s_{35}-s_{56}+s_{36}]\,,$$

$$D_2^G(12) = \frac{-\delta_2}{s_{36}t_{145}}[s_{23}-s_{26}-s_{36}]\,,$$

$$D_2^G(13) = \frac{\delta_2}{s_{14}s_{36}t_{124}}[s_{12}-s_{24}][s_{35}-s_{56}+s_{36}]\,,$$

$$D_2^G(14) = \frac{\delta_2}{s_{14}s_{36}t_{145}}[s_{15}-s_{45}][s_{23}-s_{26}-s_{36}]\,,$$

$$D_2^G(15) = \frac{\delta_2}{s_{14}s_{36}}(p_1-p_4)(p_3-p_6)\,,$$

$$D_2^G(16) = \frac{-4}{s_{12}s_{36}t_{124}}[s_{35}-s_{56}+s_{36}]E(p_2,p_2)\,,$$

$$D_2^G(17) = \frac{4}{s_{36}s_{45}t_{145}}[s_{23}-s_{26}-s_{36}]E(p_5,p_5)\,,$$

$$D_2^G(18) = \frac{-4}{s_{12}s_{36}s_{45}}[2(p_1+p_2)(p_3-p_6)-s_{36}]E(p_2,p_5)\,,$$

$$D_2^G(19) = \frac{-2}{s_{12}s_{36}}E(p_2,p_3-p_6)\,,$$

$$D_2^G(20) = \frac{2}{s_{36}s_{45}}E(p_3-p_6,p_5)\,,$$

$$D_2^G(21) = \frac{-4}{s_{25}s_{34}t_{134}}[s_{26}-s_{56}+s_{25}]E(p_3,p_3)\,,$$

$$D_2^G(22) = \frac{4}{s_{16}s_{25}t_{146}}[s_{23}-s_{35}-s_{25}]E(p_6,p_6)\,,$$

$$D_2^G(23) = \frac{4}{s_{16}s_{25}s_{34}}[2(p_1+p_6)(p_2-p_5)+s_{25}]E(p_6,p_3)\,,$$

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$$D_2^G(24) = \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$$

$$D_2^G(25) = \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_5),$$

$$D_2^G(26) = \frac{-2}{s_{12}t_{125}} E(p_2, p_2 - p_5),$$

$$D_2^G(27) = \frac{2}{s_{46}t_{125}} E(p_3 - p_6, p_6),$$

$$D_2^G(28) = \frac{2}{s_{15}t_{125}} E(p_5, p_2 - p_5),$$

$$D_2^G(29) = \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3),$$

$$D_2^G(30) = \frac{4}{s_{12}s_{34}t_{125}} [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6) - t_{125}] E(p_2, p_3),$$

$$D_2^G(31) = \frac{4}{s_{12}s_{46}t_{125}} [(p_1 + p_2 - p_5)(p_4 - p_3 + p_6) + t_{125}] E(p_2, p_6),$$

$$D_2^G(32) = \frac{4}{s_{15}s_{34}t_{125}} [(p_1 - p_2 + p_5)(p_4 + p_3 - p_6) + t_{125}] E(p_5, p_3),$$

$$D_2^G(33) = \frac{4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_5)(p_4 - p_3 + p_6) - t_{125}] E(p_5, p_6), \qquad (11)$$

where $\delta_2 = 1$.

The diagrams D_0^G are obtained from D_2^G by replacing δ_2 by $\delta_0 = 0$ and the functions $E(p_i, p_j)$ by $G(p_i, p_j)$.

The diagrams D_0^F are listed below:

$$D_0^F(1) = \frac{4}{s_{15}s_{34}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) + [F(p_6, p_3) + s_{34}]E(p_5, p_5)\},$$

$$D_0^F(2) = \frac{-4}{s_{16}s_{25}s_{34}} \{[F(p_6, p_2) + \tfrac{1}{2}s_{16}]E(p_3, p_5) + [F(p_2, p_3) + \tfrac{1}{2}s_{34}]E(p_6, p_5) - F(p_6, p_3)E(p_2, p_5)\},$$

$$D_0^F(3) = \frac{4}{s_{15}s_{36}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) - [F(p_3, p_6) - \tfrac{1}{2}s_{36} - \tfrac{1}{2}s_{34} + \tfrac{1}{2}s_{46}]E(p_5, p_5)\},$$

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$$D_0^{\rm F}(4) = \frac{4}{s_{25}s_{34}t_{125}}\{F(p_2,p_3)E(p_5,p_5) - F(p_5,p_3)E(p_2,p_5) + [F(p_5,p_2) - \tfrac12 s_{25} - \tfrac12 s_{12} + \tfrac12 s_{15}]E(p_3,p_5)\}\,,$$

$$D_0^{\rm F}(5) = \frac{2}{s_{16}s_{25}t_{146}}[s_{35} - s_{23} + s_{25}]E(p_6,p_5)\,,$$

$$D_0^{\rm F}(6) = \frac{2}{s_{25}s_{34}t_{134}}[s_{56} - s_{26} - s_{25}]E(p_3,p_5)\,,$$

$$D_0^{\rm F}(7) = \frac{4}{s_{25}s_{36}t_{125}}\{[F(p_5,p_2) - \tfrac12 s_{25} - \tfrac12 s_{12} + \tfrac12 s_{15}]E(p_3,p_5) + [F(p_2,p_3) + \tfrac14 t_{125}]E(p_5,p_5) - [F(p_5,p_3) + \tfrac14 t_{125}]E(p_2,p_5)\}\,,$$

$$D_0^{\rm F}(8) = \frac{1}{s_{14}s_{36}}E(p_3 - p_6, p_5)\,,$$

$$D_0^{\rm F}(9) = \frac{2}{s_{14}s_{36}t_{124}}[s_{35} - s_{56} + s_{36}]E(p_2,p_5)\,,$$

$$D_0^{\rm F}(10) = \frac{2}{s_{14}s_{36}t_{145}}[s_{23} - s_{26} - s_{36}]E(p_5,p_5)\,,$$

$$D_0^{\rm F}(11) = \frac{1}{2s_{14}s_{25}s_{36}}\{[s_{23} + s_{35} - s_{26} - s_{56}]E(p_2 - p_5, p_5) - [s_{23} + s_{26} - s_{35} - s_{56}]E(p_3 - p_6, p_5) - [s_{23} + s_{56} - s_{35} - s_{26}]E(p_2 + p_5, p_5)\}\,. \qquad (12)$$

The diagrams $D_0^{\rm S}$ are listed below:

$$D_0^{\rm S}(1) = \frac{1}{s_{25}s_{36}t_{125}}[s_{34} - s_{46} + s_{36}][s_{12} - s_{15} - s_{25}]\,,$$

$$D_0^{\rm S}(2) = \frac{1}{s_{14}s_{36}t_{124}}[s_{12} - s_{24} - s_{14}][s_{35} - s_{56} + s_{36}]\,,$$

$$D_0^{\rm S}(3) = \frac{1}{s_{14}s_{36}t_{145}}[s_{15} - s_{45} + s_{14}][s_{23} - s_{26} - s_{36}]\,,$$

$$D_0^{\rm S}(4) = \frac{1}{s_{15}s_{36}t_{125}}[s_{15} + s_{25} - s_{12}][s_{34} - s_{46} + s_{36}]\,,$$

$$D_0^{\rm S}(5) = \frac{1}{s_{15}s_{34}t_{156}}[s_{56} - s_{15} - s_{16}][s_{23} - s_{24} - s_{34}]\,,$$

$$D_0^{\rm S}(6) = \frac{1}{s_{15}s_{34}t_{125}}[s_{46} - s_{34} - s_{36}][s_{12} - s_{25} - s_{15}]\,,$$

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$$D_0^S(7) = \frac{1}{s_{25}s_{34}t_{125}}[s_{36}-s_{46}+s_{34}][s_{12}-s_{15}-s_{25}]\,,$$

$$D_0^S(8) = \frac{1}{s_{16}s_{25}t_{146}}[s_{25}+s_{35}-s_{23}][s_{14}-s_{46}+s_{16}]\,,$$

$$D_0^S(9) = \frac{1}{s_{25}s_{34}t_{134}}[s_{14}+s_{34}-s_{13}][s_{26}-s_{56}+s_{25}]\,,$$

$$D_0^S(10) = \frac{1}{s_{25}s_{36}}(p_2-p_5)(p_3-p_6)\,,$$

$$D_0^S(11) = \frac{1}{s_{14}s_{36}}(p_1-p_4)(p_3-p_6)\,,$$

$$D_0^S(12) = \frac{1}{s_{16}s_{25}}(p_6-p_1)(p_2-p_5)\,,$$

$$D_0^S(13) = \frac{1}{s_{15}s_{34}}(p_5-p_1)(p_3-p_4)\,,$$

$$D_0^S(14) = \frac{1}{s_{25}s_{34}}(p_2-p_5)(p_3-p_4)\,,$$

$$\begin{aligned} D_0^S(15) = \frac{1}{s_{14}s_{25}s_{36}}\{&[(p_2+p_5)(p_3-p_6)][(p_1-p_4)(p_2-p_5)] \\ &+[(p_2-p_5)(p_3-p_6)][(p_1-p_4)(p_3+p_6)] \\ &+[(p_1+p_4)(p_2-p_5)][(p_1-p_4)(p_3-p_6)]\}\,, \end{aligned}$$

$$\begin{aligned} D_0^S(16) = \frac{2}{s_{16}s_{34}s_{25}}\{&[(p_2-p_5)(p_3+p_4)][(p_1-p_6)(p_3-p_4)] \\ &+[(p_1+p_6)(p_3-p_4)][(p_1-p_6)(p_2-p_5)] \\ &+[(p_1-p_6)(p_2+p_5)][(p_3-p_4)(p_2-p_5)]\}\,. \end{aligned} \qquad (13)$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using eqs. (11)–(13). The result is substituted to eq. (8) to obtain the vectors $\mathscr{D}_0$ and $\mathscr{D}_2$. After generating the vectors $\mathscr{D}_{0_\rho}$, $\mathscr{D}_{0_\sigma}$, $\mathscr{D}_{0_\tau}$, $\mathscr{D}_{2_\rho}$, $\mathscr{D}_{2_\sigma}$ and $\mathscr{D}_{2_\tau}$ by the appropriate permutations of momenta, eq. (6) is used to obtain the functions A_0 and A_2. Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates ten Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multi-gluon amplitudes are tested by checking the gauge invariance. Due to the specifics

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of our calculation, the most powerful test does not rely on the gauge symmetry, but on the appropriate permutation symmetries. The function $A_0(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under arbitrary permutations of the momenta (p_1, p_2, p_3) and separately, (p_4, p_5, p_6), whereas the function $A_2(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under the permutations of (p_1, p_2, p_3, p_4) and separately, (p_5, p_6). This test is extremely powerful, because the required permutation symmetries are hidden in our supersymmetry relations, eqs. (1) and (3), and in the structure of amplitudes involving different species of particles. Another, very important test relies on the absence of the double poles of the form $(s_{ij})^{-2}$ in the cross section, as required by general arguments based on the helicity conservation. Further, in the leading $(s_{ij})^{-1}$ pole approximation, the answer should reduce to the two goes to three cross section [3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

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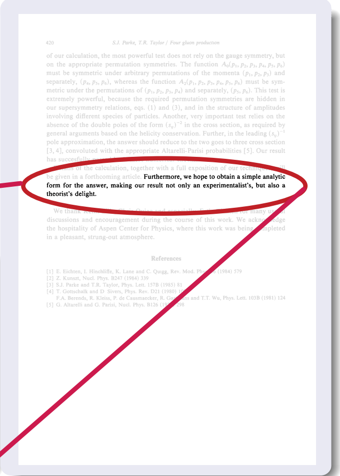

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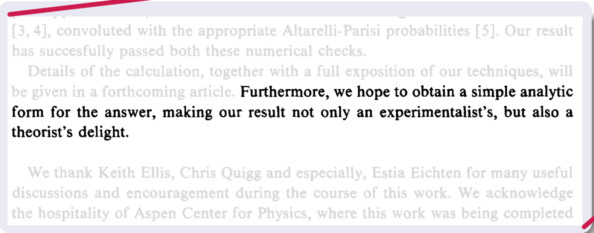

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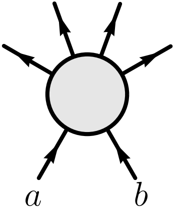

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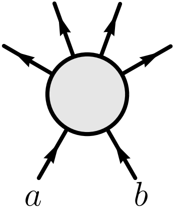

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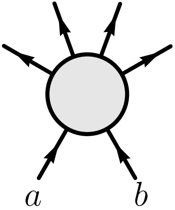

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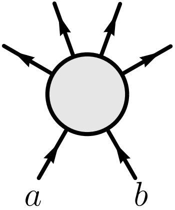

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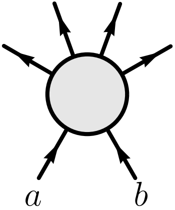

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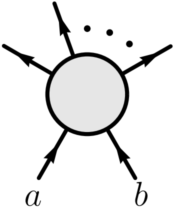

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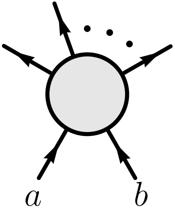

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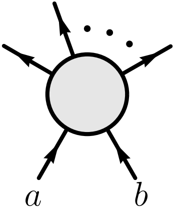

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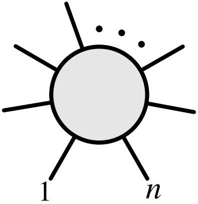
$\mathcal{A}_n \equiv$ 

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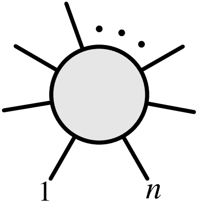

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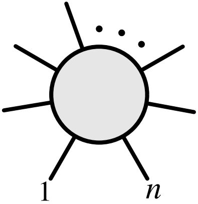
$\mathcal{A}_n \equiv$ 

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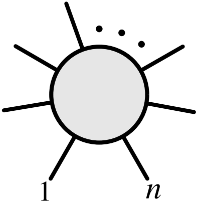
$\mathcal{A}_n \equiv$ 

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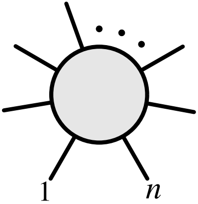

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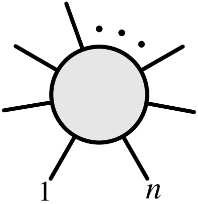
$\mathcal{A}_n \equiv$ 

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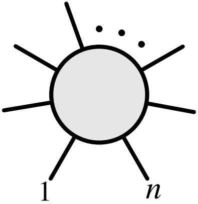
$\mathcal{A}_n \equiv$ 

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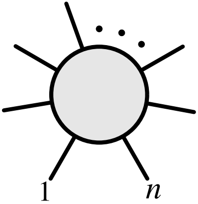
$\mathcal{A}_n \equiv$ 

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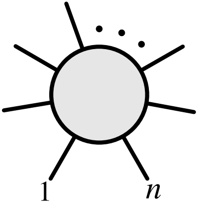
$\mathcal{A}_n \equiv$ 

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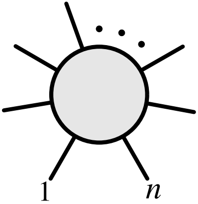
$\mathcal{A}_n \equiv$ 

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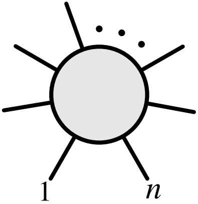
$\mathcal{A}_n \equiv$ 

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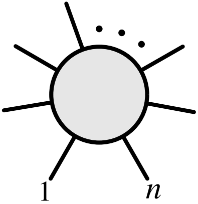
$\mathcal{A}_n \equiv$ 

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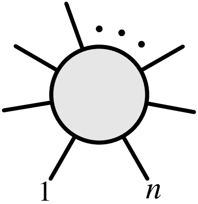
$\mathcal{A}_n \equiv$ 

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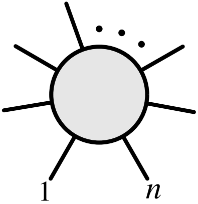
$\mathcal{A}_n \equiv$ 

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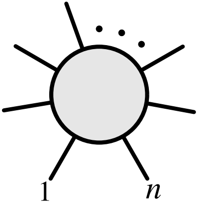
$\mathcal{A}_n \equiv$ 

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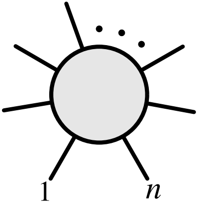

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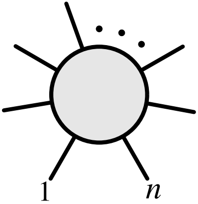
$\mathcal{A}_n \equiv$ 

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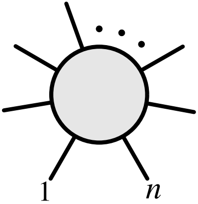
$\mathcal{A}_n \equiv$ 

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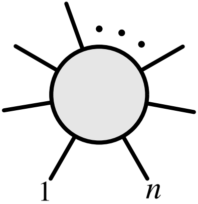

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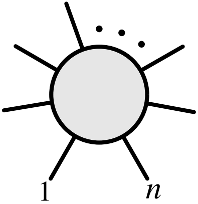

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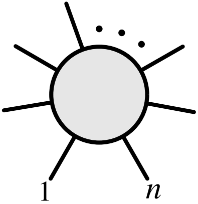

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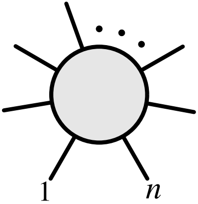
$\mathcal{A}_n \equiv$ 

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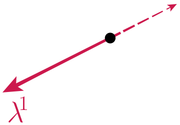

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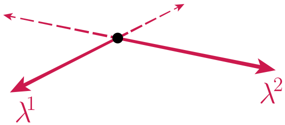

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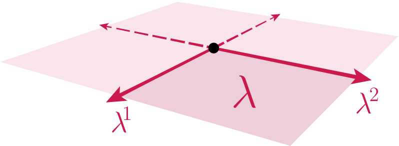

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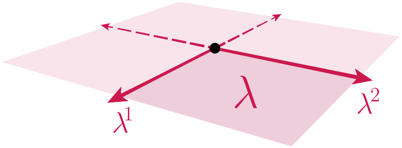

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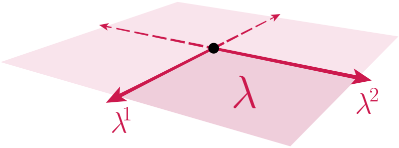

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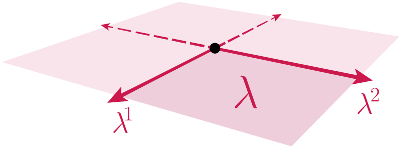

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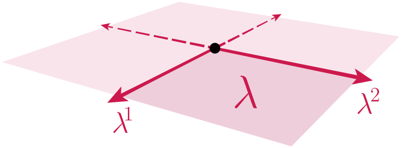

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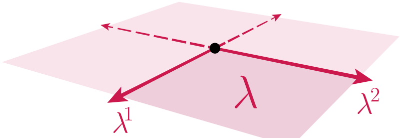

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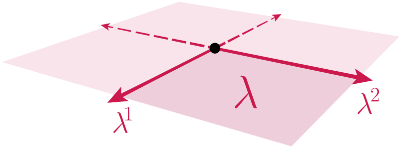

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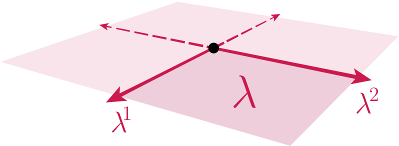

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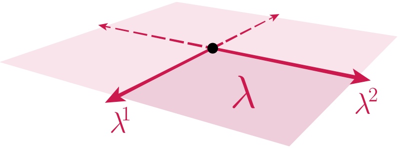

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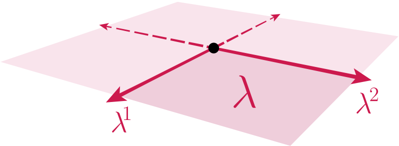

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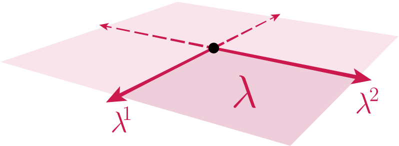

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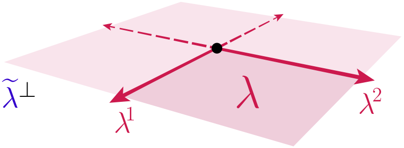

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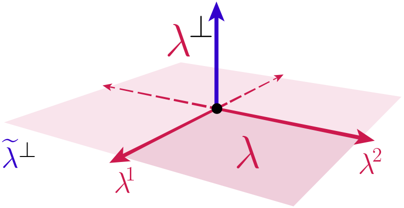

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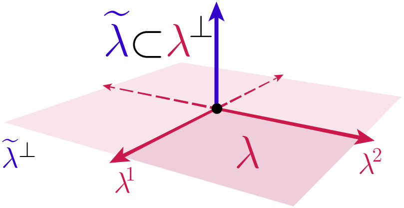

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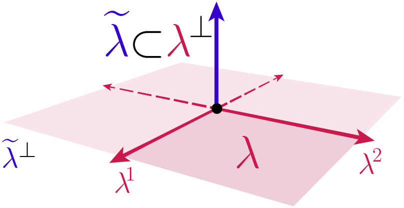

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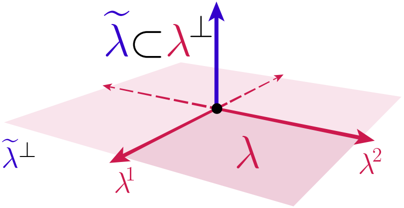

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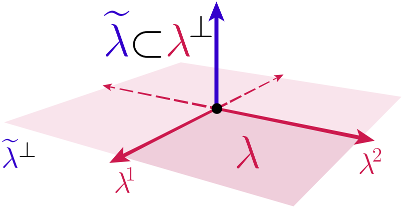

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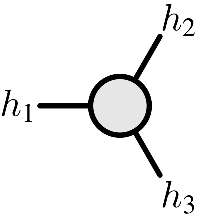

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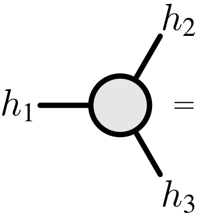

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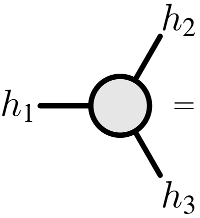

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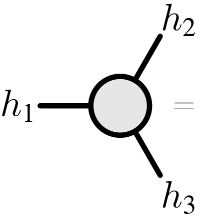

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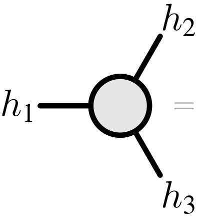

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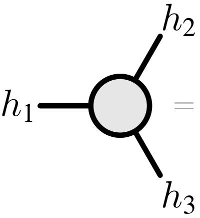

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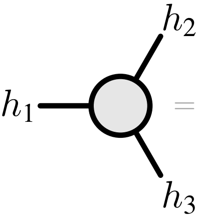

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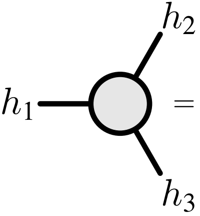

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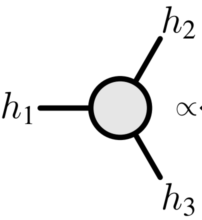

$$\propto \begin{cases} f(\lambda_1, \lambda_2, \lambda_3) \\ \\ f(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3) \end{cases}$$

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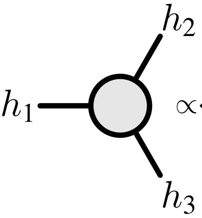

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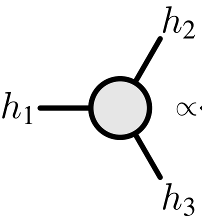

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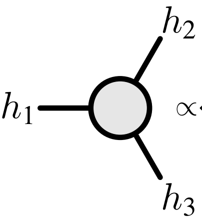

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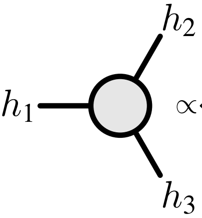

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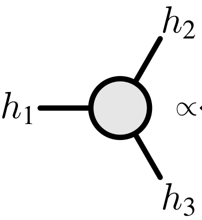

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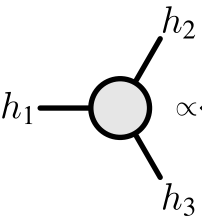

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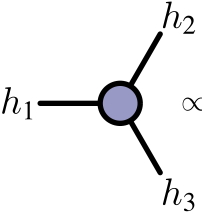

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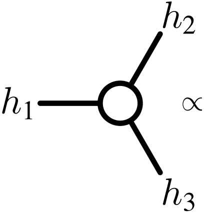

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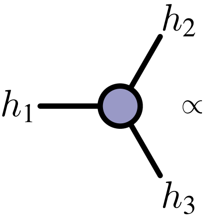

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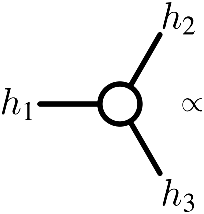

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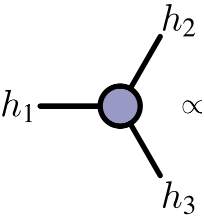

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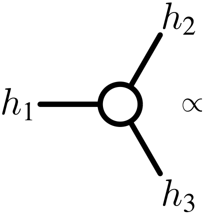

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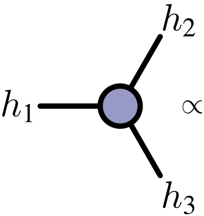

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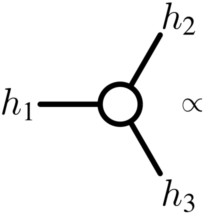

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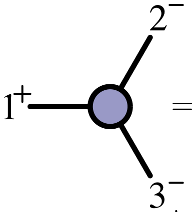

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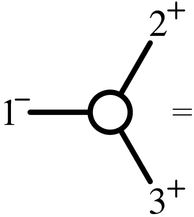

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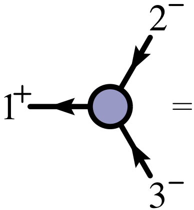

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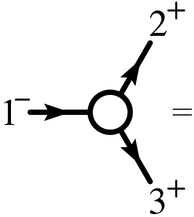

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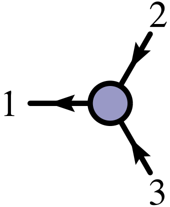

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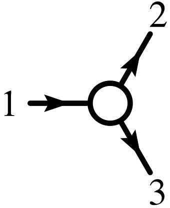

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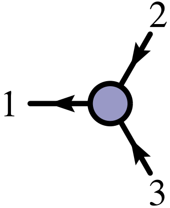

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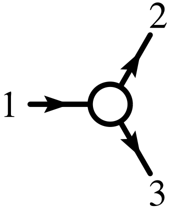

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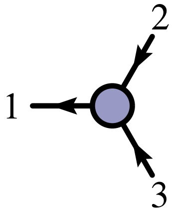

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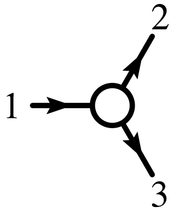

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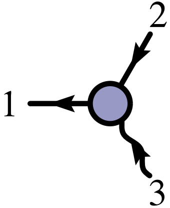

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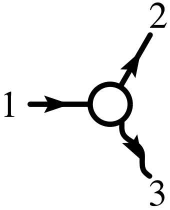

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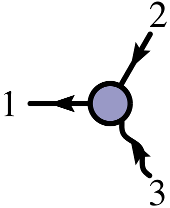
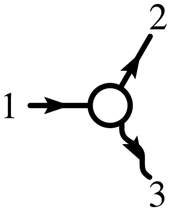


$$= \quad f^{q_1,q_2,q_3}\frac{\langle 3\,1\rangle\langle 2\,3\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda}) \equiv \mathcal{A}_3\left(+\tfrac{1}{2},-\tfrac{1}{2},-\right)$$



$$= \quad f^{q_1,q_2,q_3}\frac{[3\,1][2\,3]^3}{[1\,2]\,[2\,3]\,[3\,1]}\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda}) \equiv \mathcal{A}_3\left(-\tfrac{1}{2},+\tfrac{1}{2},+\right)$$

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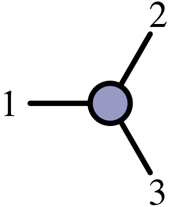
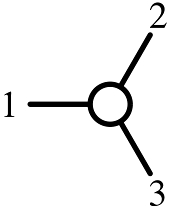


$$= f^{q_1,q_2,q_3}\frac{\delta^{2\times4}(\lambda\cdot\widetilde{\eta})}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda}) \equiv \mathcal{A}_3^{(2)}$$



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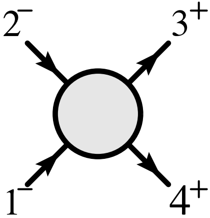

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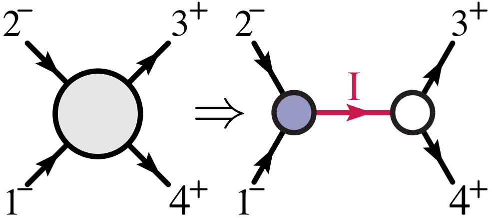

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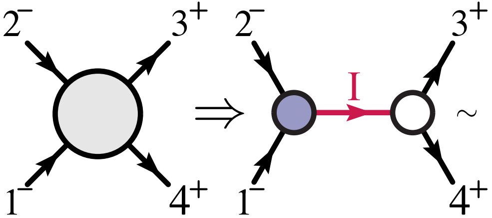

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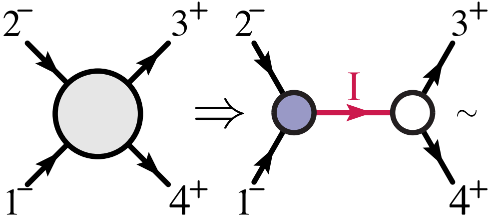

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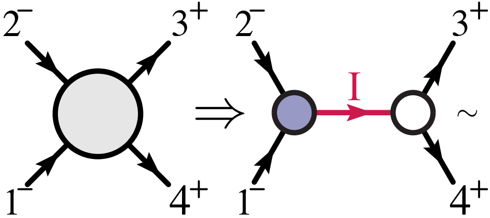

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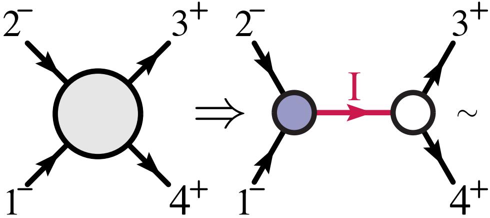

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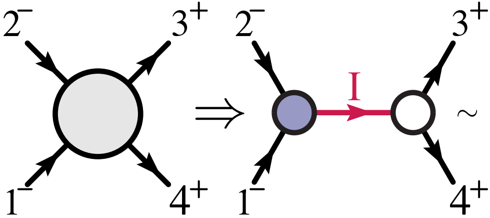

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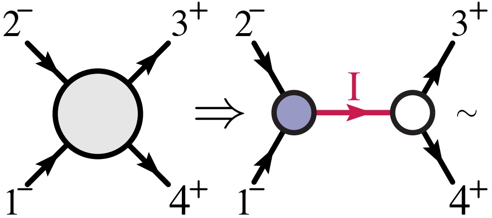

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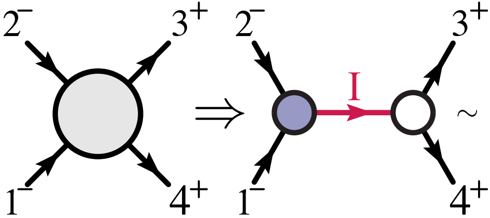

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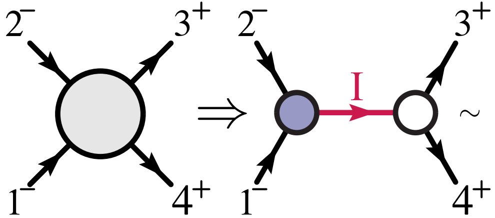

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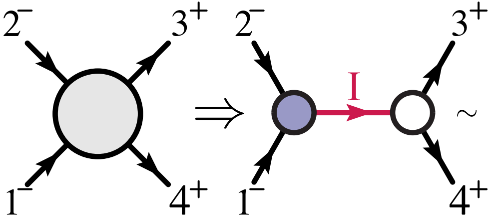

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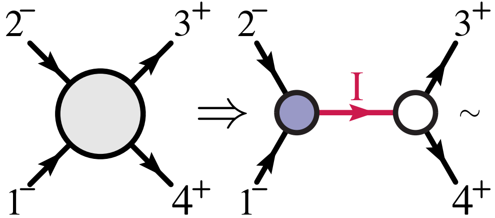

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Using Cauchy's theorem to relate the three factorization channels to each other, Benincasa and Cachazo prove in [arXiv:0705.4305] following:

- $\sigma=1$: the coupling constants satisfy a Jacobi identity!

$$f^{q_1,q_2,\bullet}f^{\bullet,q_3,q_4}+f^{q_2,q_3,\bullet}f^{\bullet,q_1,q_4}+f^{q_3,q_1,\bullet}f^{\bullet,q_2,q_4}=0.$$

  - *whatever* quantum numbers distinguish mutually interacting spin-1 particles, they form the **adjoint representation** of a Lie algebra!

- $\sigma=2$: multiple spin-2 particles can always be decomposed into mutually non-interacting sectors—*there is at most one graviton*!
  - the coupling strength of any spin-2 particle to itself must be the same as its coupling to any other field—*the equivalence principle*!