Baryons, nuclei and neutron stars as solitons in chiral fluid

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based on collaboration with

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rediscovering skyrmions

contents

- motivation - the (near) BPS Skyrme model
- the BPS limit
  - BPS property
  - perfect fluid
  - microscopic (FT) and macro thermodynamics
    - non-barotropic fluid
    - emergent ω and σ mesons $\rightarrow$ two (hidden) meson fluid
    - BPS as exact $\omega - \sigma$ balance
- applications
  - binding energies of nuclei
  - neutron stars
    - non-uniques EoS $\rightarrow$ inverse TOV (?)
- future
    - temperature
    - SU(3)
    - near BPS

fundamental result

there is a limit of Skyrme type actions in which the model has two properties

- **BPS**

  classically zero binding energies
  physical (small) binding energies: semiclassical quantization
  Coulomb interaction
  isospin breaking

- **perfect fluid field theory**

  $T_{\mu\nu}$ in a perfect fluid form
  SDiff symmetry
  Euler fluid formulation
  FT (micro) d.o.f. = thermodynamical (macro) functions

  Physically well motivated idealization of the nuclear matter
  weakly modified by the (small) non-BPS part

- **solvablity**

  exact solutions
  exact, analytical EoS and thermodynamics
  a way beyond MF limit in nuclear matter

motivation

- **the Skyrme framework** Skyrme (61)

  pionic EFT of

  - baryons and nuclei $\to$ emergent objects: **solitons**
    extended, non-perturbative
  - nuclear matter
  - with applications to neutron stars
    $\to$ complementary to lattice

- support form $N_c \to \infty$ limit t'Hooft (83), Witten (84)
  - chiral effective meson/baryon theory
  - primary d.o.f. are mesons
  - baryons (nuclei) are realized as solitons

  - simplest case (two flavors): $U(x) = e^{i\vec{\pi}\vec{\sigma}} \in SU(2)$
  - $\vec{\pi}$ - pions
  - topological charge = baryon number

$$U : \mathbb{R}^3 \cup \{\infty\} \cong \mathbb{S}^3 \ni \vec{x} \to U(\vec{x}) \in SU(2) \cong \mathbb{S}^3$$

$$\pi_3(\mathbb{S}^3) = \mathbb{Z}$$

- **what is the proper action?**

motivation

- **"Derivation" of the effective chiral soliton theory**
  - chiral perturbation theory Gasser,Leutwyler (84)
  - derivative expansion at $N_C \to \infty$ Simic (85), Aitchinson (86)

$$\mathcal{L} = \underbrace{\mathcal{L}_2 + \mathcal{L}_4}_{\mathcal{L}_{skyrme}} + \mathcal{L}_0 \; + \; \text{higher order terms}$$

Skyrme (61), Adkins, Nappi, Witten (85)

$$\mathcal{L}_2 = -\lambda_2 \, \mathrm{Tr}\,(L_\mu L^\mu), \quad \mathcal{L}_4 = \lambda_4 \, \mathrm{Tr}\,([L_\mu, L_\nu]^2), \quad L_\mu = U^\dagger \partial_\mu U$$

- infinitely many terms: no hierarchy
- destabilizing terms $\to$ false vacuum expansion?
- higher time derivative terms
- solitons $\to$ need a non-perturbative expansion

- complicated theoretically and computationally

motivation

- Skyrme (minimal) model
  - Lorentz inv.
  - standard Hamiltonian
  - max. first time derivative squared

$$\mathcal{L} = \underbrace{\mathcal{L}_0 + \underbrace{\lambda_2\mathcal{L}_2 + \lambda_4\mathcal{L}_4}_{\text{massless } \mathcal{L}_{skyrme}}}_{\text{massive } \mathcal{L}_{skyrme}} + \lambda_6\mathcal{L}_6$$

$$\mathcal{L}_6 = -\mathbb{B}_\mu\mathbb{B}^\mu, \quad \mathbb{B}^\mu = \frac{1}{24\pi^2}\mathrm{Tr}\left(\epsilon^{\mu\nu\rho\sigma}L_\nu L_\rho L_\sigma\right)$$

- success
  - baryon physics Adkins, Nappi, Witten (84)....Praszalowicz, Nowak, Rho...
  - deuteron, light nuclei $\rightarrow$ iso-rotational spectra $\rightarrow$ SCQ correct Braaten, Carson, Manton, Rho....
  - ^{12}C and Hoyle states Manton, Liu (14)
- difficulties
  - **unphysical binding energies** Sutcliffe et. al. (97), (02), (05), (06), (10)
  - **crystal state of matter** Klebanov (85), Battye, Sutcliffe et. al. (06)

problematic for (heavy) nuclei and nuclear matter $\rightarrow$ neutron stars

Is it at all possible to describe
baryons, atomic nuclei and nuclear matter (neutron stars)
by one universal chiral solitonic theory?

motivation

- **the near BPS Skyrme model**
  a usual Skyrme theory with a particular relation between the coupling constants

$$\mathcal{L} = \epsilon \underbrace{\left(\tilde{\mathcal{L}}_0 + \lambda_2 \mathcal{L}_2 + \lambda_4 \mathcal{L}_4\right)}_{\text{massive } \mathcal{L}_{skyrme}} + \underbrace{\lambda_6 \mathcal{L}_6 + \mathcal{L}_0}_{\mathcal{L}_{BPS}}$$

  ϵ is a **small** parameter
  why important and interesting?

- **the BPS Skyrme model**: $\epsilon \to 0$ limit

$$\mathcal{L}_{BPS} = \lambda_6 \mathcal{L}_6 + \mathcal{L}_0$$

  - **BPS**
  - **perfect fluid field theory** for any B
    - not a gas of *weakly* interacting skyrmions Kalbermann (97), Jaikumar et al.(07)

    Physically well motivated idealization of the nuclear matter
    weakly modified by the (small) non-BPS part

  - **solvablity**
    very simple solvable model which covers the main features of nuclear matter $\to$ hard core of Skyrme-type EFT

motivation

- **the near BPS Skyrme model**

  Skyrme theory in a new perspective

  - separation of d.o.f.

$$\mathcal{L} = \epsilon\left(\tilde{\mathcal{L}}_0 + \lambda_2\mathcal{L}_2 + \lambda_4\mathcal{L}_4\right) \quad + \quad \mathcal{L}_{BPS}$$

perturbative
explicit pions
kinetic + two body int.

non-perturbative
coherent d.o.f.
topological term
hidden (emergent) ω and σ

surface
shape

far attractive int.

bulk
SDiff symmetry
perfect fluid
BPS: exact $\omega - \sigma$ balance

  - some (not all!) properties/observable of the near BPS action are dominated by the BPS part

$\Downarrow$

let's do the BPS model to learn about nuclear matter

BPS property - binding energies

BPS property - binding energies

- topological bound Adam, Sanchez, Wereszczynski (10), Speight (10)

$$
\begin{aligned}
E_{06} &= \int d^3x \left(\lambda^2 \pi^4 \mathbb{B}_0^2 + \mu^2 \mathcal{U} \right) \\
&= \int d^3x \left(\lambda \pi^2 \mathbb{B}_0 \pm \mu \sqrt{\mathcal{U}} \right)^2 \mp \int d^3x \lambda \mu \pi^2 \sqrt{\mathcal{U}} \mathbb{B}_0 \\
&\geq \int d^3x \lambda \mu \pi^2 \sqrt{\mathcal{U}} \mathbb{B}_0 = 2\pi^2 \lambda \mu < \sqrt{\mathcal{U}} >_{S^3} |B|
\end{aligned}
$$

- the bound is saturated $\Rightarrow$ BPS equation

$$\lambda \pi^2 \mathbb{B}_0 \pm \mu \sqrt{\mathcal{U}} = 0$$

- **zero binding energy** $E = \lambda \mu C |B|$

- symmetries
  - ∞ many target space symmetries: subgroup of SDiff($\mathbb{S}^3$)
  - ∞ many conservation laws $\Rightarrow$ **generalized integrability**

  - static energy: ∞ many base space symmetries SDiff($\mathbb{R}^3$)
  $\Rightarrow$ symmetries of **incompressible fluid**
  $\Rightarrow$ ∞ many BPS solutions

BPS property - binding energies

- semiclassical quantization
  - collective coordinate quantization of spin and isospin

$$U(t,\vec{x}) = \mathrm{A}(t) U_0(R_{\mathrm{B}}(t)\vec{x}) \mathrm{A}^{\dagger}(t), \ \mathrm{A}, \mathrm{B} \in SU(2), \ R_{\mathrm{B}} \in SO(3)$$

  - promote $\mathrm{A}(t), \mathrm{B}(t)$ to quantum mechanical variables
  - Finkelstein-Rubinstain constrains

$$E_{rot} = \frac{105}{512\sqrt{2}\pi} \frac{\hbar^2}{\lambda^2 \left(\frac{\mu}{\lambda n}\right)^{1/3}} \left(\frac{j(j+1)}{n^2} + \frac{4|i_3|(|i_3|+1)}{3n^2+1} \right)$$

- Coulomb energies

$$E_{\mathrm{C}} = \frac{1}{2\varepsilon_0} \int d^3x d^3x' \frac{\rho(\vec{r})\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}$$

  - $\rho(\vec{r})$ expectation value of charge density operator

$$E_{\mathrm{C}} = \frac{1}{\sqrt{2}\pi\varepsilon_0} \left(\frac{\mu}{\lambda n}\right)^{1/3} \left(\frac{128}{315\pi^2} n^2 + \frac{245}{1536} n\, i_3 + \frac{805}{5148} i_3^2 + \frac{7}{429} \frac{i_3^2}{(1+3n^2)^2} \right)$$

- isospin breaking

$$E_{\mathrm{I}} = a_{\mathrm{I}} i_3 \quad \text{where} \quad a_{\mathrm{I}} < 0 \quad \Leftrightarrow \quad M_{\mathrm{n}} > M_{\mathrm{p}}$$

- binding energy of nucleus $X = {}^A_Z X$, $N = A - Z$

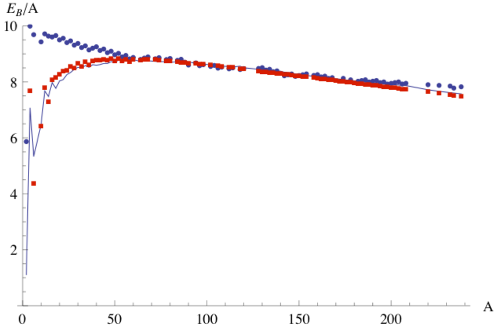
$$E_{\mathrm{B},X} = ZE_{\mathrm{p}} + NE_{\mathrm{n}} - E_X$$
$$E_X = E_{\mathrm{sol}} + E_{\mathrm{rot}} + E_{\mathrm{C}} + E_{\mathrm{I}}$$

- 3 free parameters $\lambda, \mu, a_{\mathrm{I}}$: fit to 3 nuclear masses

$$\begin{aligned} M_{\mathrm{p}} &= 938.272\ \mathrm{MeV} \\ M_{\mathrm{n}} - M_{\mathrm{p}} &= 1.29333\ \mathrm{MeV} \\ M({}^{138}_{56}\mathrm{Ba}) &= 137.905\ \mathrm{u} \quad \text{where} \quad \mathrm{u} = 931.494\ \mathrm{MeV} \end{aligned}$$

$$\begin{aligned} E_{\mathrm{B},X}(A,Z,j) = \quad & a_1 A + a_2 Z - a_3 A^{5/3} - a_4 A^{2/3} Z - a_5 A^{-1/3} Z^2 \\ & - a_6 \frac{A^{1/3}}{1+3A^2}(A-2Z) - a_7 \frac{A^{1/3}}{1+3A^2}(A-2Z)^2 \\ & - a_8 \frac{A^{-1/3}}{(1+3A^2)^2}(A-2Z)^2 - a_9 A^{-5/3} j(j+1) \end{aligned}$$

where a constants.



Binding energy per nucleon: BPS model (blue), Weizsäcker's formula (red), experimental values (solid line)

- axially symmetric solutions ⇒ **exact result**
- weakly depend on the potential

Heavy atomic nuclei (binding energies) can be described by a solitonic model Adam, Naya, Ssanchez, Wereszczynski (2013) PRL

Perfect fluid

perfect fluid

- **SDiff symmetries**
- **energy-momentum tensor of a perfect fluid**

$$T^{00} = \lambda^2 \pi^2 \mathbb{B}_0^2 + \nu^2 \mathcal{U} \equiv \varepsilon$$

$$T^{ij} = \delta^{ij} \left(\lambda^2 \pi^2 \mathbb{B}_0^2 - \nu^2 \mathcal{U} \right) \equiv \delta^{ij} P$$

  - *local* thermodynamical quantities
  - BPS eq. = zero pressure condition
  - e-m. conservation: $\partial_\mu T^{\mu\nu} = 0$
    static: $\partial_i T^{ij} = 0 \Rightarrow \partial_j P = 0 \Rightarrow P = const.$
  - constant pressure equation is a first integral of static EL eq.

$$\lambda^2 \pi^4 \mathcal{B}_0^2 - \nu^2 \mathcal{U} = P > 0$$

$$\lambda \pi^2 \mathcal{B}_0 = \pm \nu \sqrt{\mathcal{U} + \tilde{P}}, \quad \tilde{P} \equiv (P/\nu^2)$$

static non-BPS solutions with $P > 0$

perfect fluid - exact thermodynamics

- **energy density EoS**

$$\varepsilon - P = 2\nu^2\mathcal{U}$$

  - **non-barotropic chiral fluid** $\varepsilon \neq \varepsilon(P)$

    the step-function potential $\quad \varepsilon = P + 2\nu^2$

    no potential $\quad \varepsilon = P$

    high pressure limit - potential independent

$$\varepsilon = P$$

  - *on-shell* EoS

$$\varepsilon = \varepsilon(P, \vec{x})$$

  beyond mean-field thermodynamics:

$$P = const. \text{ but } \varepsilon \neq const.$$

perfect fluid - exact thermodynamics

- **particle (baryon) density EoS**

$$\rho_B = \mathcal{B}_0$$

  - generically non-constant (beyond MF)

$$\rho_B = \frac{\nu}{\lambda\pi^2}\sqrt{\mathcal{U} + \frac{P}{\nu^2}}$$

  - *on-shell*

$$\rho_B = \rho_B(P, \vec{x})$$

no universal $\varepsilon = \varepsilon(P)$, $\rho_B = \rho_B(P)$

universal relation - *off-shell* and *non-MF*

$$\varepsilon + P = 2\lambda^2\pi^4\rho_B^2$$

- **baryon chemical potential**

definition: $\varepsilon + P = \rho\mu \qquad \Rightarrow \qquad \mu_B = 2\lambda^2\pi^4\rho_B$

  - off-shell
  - universal, potential independent
  - non-MF (local)

perfect fluid - exact thermodynamics

- generically **exact (non-mean field) thermodynamics**
  - ϵ, ρ_B non-constant generically non-constant
  - non-barotropic fluid
  - no universal EoS

- mean-field limit
  - MF averages $\bar{\epsilon}$, $\bar{\rho}_B$

$$\bar{\epsilon} = \frac{E_{06}}{V}, \qquad \bar{\rho} = \frac{B}{V}$$

  - *universal* (geometrical) EoS
    - E_{06}, V, $\bar{\epsilon}$, $\bar{\rho}_B$
      - known as functions of P FT pressure
      - no need for solutions!
      - only $\mathcal{U}$ matters

$$\bar{\epsilon} = \bar{\epsilon}(P), \qquad \bar{\rho} = \bar{\rho}(P)$$

- FT pressure is the pressure

$$P = -\frac{dE_{06}}{dV}$$

micro (FT) thermodynamics = macro themrodynamics

example: **the step-function potential** $\mathcal{U} = \Theta\left(\mathrm{Tr}\left(1 - U\right)\right)$

- MF = non-MF
  - baryon chemical potential

$$\mu = 2\pi^4\lambda^2\bar{\rho}_B$$

  - pressure

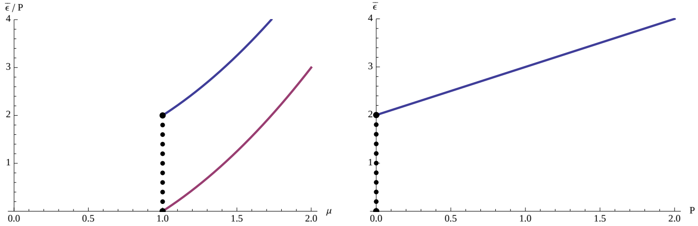
$$P = \frac{1}{4\pi^4\lambda^2}\mu^2 - \nu^2$$

  - energy density

$$\varepsilon = \frac{1}{4\pi^4\lambda^2}\mu^2 + \nu^2$$

  - EoS

$$\varepsilon = 2\nu^2 + P$$



gas-liquid phase transition

perfect fluid

- the BPS action is equivalent to the action of a field theoretical description of perfect fluid in an **Eulerian formulation**

  - particle trajectories $\vec{X}_n(t) \rightarrow$ fluid element trajectories (cont. limit) $\vec{X}(t,\vec{y})$
  $\vec{y}$ comoving fluid coordinates
  - the Eulerian formulation: dynamical variables

$$\rho(t,\vec{x}) = \rho_0 \int d^3y \, \delta^{(3)} \left(\vec{X}(t,\vec{y}) - \vec{x} \right)$$

$$\vec{v}(t,\vec{x}) = \rho^{-1}\vec{j}, \quad \vec{j} = \rho_0 \int d^3y \dot{\vec{X}} \, \delta^{(3)} \left(\vec{X}(t,\vec{y}) - \vec{x} \right)$$

  formulated on phys. space but constrained ($N = const$. etc.)

- FT realisation Brown (93), Dubovski et. al. (03), (13), Jackiw (04) de Boer et. al. (15)
  - y^a promoted to the dynamical fields
  $x^i = X^i(t,y^a) \rightarrow y^a = \phi^a(t,x^i)$
  - density $\rho(t,\vec{x}) = \rho_0 D, \quad D = \Omega(\phi^a)\det\left(\frac{\partial \phi^a}{\partial x^i}\right)$
  particle number

$$\mathcal{N}^\mu = \Omega \epsilon^{\mu\nu\rho\sigma} \epsilon_{abc} \partial_\nu \phi^a \partial_\rho \phi^b \partial\sigma \phi^c$$

  - four velocity

$$u^\mu = \frac{\mathcal{N}^\mu}{\sqrt{\mathcal{N}^\nu \mathcal{N}_\nu}} = \frac{1}{\sqrt{6}D} \Omega \epsilon^{\mu\nu\rho\sigma} \epsilon_{abc} \partial_\nu \phi^a \partial_\rho \phi^b \partial\sigma \phi^c$$

$$\mathcal{N}^\mu = \rho u^\mu \quad \Rightarrow \quad \rho = \sqrt{6}D$$

perfect fluid

- a perfect fluid action = chose a Lagrange density $F = F(\phi^a, \partial_\mu \phi^a)$
  - $F = F(\rho, g(\phi^a))$

$$S = \int d^4x F(\rho, g) \quad \Rightarrow \quad T^{\mu\nu} = (p+\epsilon)u^\mu u^\nu - p\eta^{\mu\nu}$$

where

$$\epsilon = -F(\rho, g), \quad p = \rho \frac{\partial \epsilon}{\partial \rho} - \epsilon$$

  - simplest $F = F(\rho) \Rightarrow$ **barotropic fluid** $\epsilon = \epsilon(p)$
  - general **non-barotropic**
    - interpretation: $g = s(\phi^a)$ entropy and $\mathcal{S}^\mu = s\mathcal{N}$ is entropy current

- **BPS Skyrme model**
  - $\mathcal{B}^\mu = \mathcal{N}^\mu$ i.e., the baryon current
  - fluid Lagrangian

$$F = -\lambda^2 \pi^4 \rho^2 - \nu^2 \mathcal{U}(\phi^a)$$

genuine non-barotropic fluid

$\rightarrow$ thermodynamical interpretation of $\mathcal{U}$?

Complete thermodynamics (at $T = 0$) in a solvable solitonic model

Two (hidden) mesons fluid

- **the BPS Skyrme and the Walecka model**

$$\mathcal{L}_W = \mathcal{L}_N + \mathcal{L}_{\sigma,\omega} + \mathcal{L}_{int}, \qquad \mathcal{L}_N = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m_N + \mu\gamma^0\right)\psi$$

$$\mathcal{L}_{\sigma,\omega} = \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}m_\sigma^2\sigma^2 - \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu$$

$$\mathcal{L}_{int} = g_\sigma\bar{\psi}\sigma\psi + g_\omega\bar{\psi}\gamma^\mu\omega_\mu\psi + \ldots$$

$\rightarrow$ non perfect fluid form

- **mean-field $\rightarrow$ perfect fluid**

- compute the partition function the thermodynamical limit
$Z = \int e^{\int \mathcal{L}_W}$
- bosonic fields take their vacuum expectation values $\bar{\sigma}$, $\bar{\omega}_0$
- all derivative dependent terms disappear and the interactions are simplified to a mesonic background field seen by nucleons

$$\mathcal{L}_W = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m_N^* + \mu^*\gamma^0\right)\psi - \frac{1}{2}m_\sigma^2\bar{\sigma}^2 + \frac{1}{2}m_\omega^2\bar{\omega}_0^2$$

where

$$m_N^* = m_N - g_\sigma\bar{\sigma}, \quad \mu^* = \mu - g_\omega\bar{\omega}_0$$

- the baryon chemical potential (which enters in all thermodynamical relations) is still μ
- the effective chemical potential μ^* sets the Fermi energy of the "effective" free fermions

$$E_F^* = \mu^* = \sqrt{k_F^2 + (m_N^*)^2}$$

- condensates

$$\bar{\sigma} = \frac{g_\sigma}{m_\sigma^2}\rho_\sigma\,, \quad \bar{\omega}_0 = \frac{g_\omega}{m_\omega^2}\rho_B$$

- densities (T=0)

$$\bar{\rho}_B = \langle\psi^\dagger\psi\rangle = \frac{2k_F^3}{3\pi^2}$$

$$\rho_\sigma = \langle\bar{\psi}\psi\rangle = \frac{m_N^*}{\pi^2}\left[k_F E_F^* - (m_N^*)^2 \ln\frac{k_F + E_F^*}{m_N^*}\right]$$

- EoS

$$\bar{\varepsilon} = \frac{1}{2}\frac{g_\omega^2}{m_\omega^2}\bar{\rho}_B^2 + \frac{1}{2}\frac{g_\sigma^2}{m_\sigma^2}\bar{\rho}_\sigma^2 + \bar{\varepsilon}_N$$

$$P = \frac{1}{2}\frac{g_\omega^2}{m_\omega^2}\bar{\rho}_B^2 - \frac{1}{2}\frac{g_\sigma^2}{m_\sigma^2}\bar{\rho}_\sigma^2 + P_N$$

- large density / pressure limit: $\mu, P \to \infty$

$$\bar{\varepsilon} = \frac{1}{2}\frac{g_\omega^2}{m_\omega^2}\rho_B^2, \quad \bar{\varepsilon} = P \quad \Rightarrow \quad \pi^4\lambda^2 = \frac{1}{2}\frac{g_\omega^2}{m_\omega^2}$$

- large density / pressure limit again:

$$k_F \to \infty \;\Rightarrow\; \mu^* = k_F \;\text{ and }\; \mu^* \sim \rho_B^{1/3}$$

$$\mu = \mu^* + g_\omega\bar{\omega}_0 = \mu^* + \frac{g_\omega^2}{m_\omega^2}\rho_B = \frac{g_\omega^2}{m_\omega^2}\rho_B$$

- in BPS $\bar{\mu} = 2\pi^4\lambda^2\bar{\rho}_B \Rightarrow \pi^4\lambda^2 = \frac{1}{2}\frac{g_\omega^2}{m_\omega^2}$

- high density / pressure

  the BPS Skyrme and the Walecka model **coincide**

  - the same thermodynamics
  - relation between parameters

$$\pi^4 \lambda^2 = \frac{1}{2} \frac{g_\omega^2}{m_\omega^2}$$

  $m_\omega = 738$ MeV, $g_\omega^2/(4\pi) = 10 - 12 \Rightarrow \lambda = 9 - 11$ MeV fm^3

  - BPS: emergent (hidden) ω meson
  - reason

    - Walecka: ω meson (baryon current) dominating at high μ, P
    - BPS: based on the baryon current although in a different form
  - the local chemical potential

$$\mu = 2\lambda^2 \pi^4 \rho_B$$

  "ω meson always hidden in the (derivative part of the) BPS action"

  agreement is **generic** for any EFT with ω: $\rightarrow$ NJL

  Klähn, Fisher (15) $\rightarrow$ MIT + vector int.

- low (saturation) density / pressure

  - *sigma* meson dominates but "polluted" by the fermion contribution

  work in progress

Two mesons fluid: ω, σ in the BPS Skyrme

- hidden d.o.f.

$$
\begin{aligned}
\mathcal{L}_{BPS} &= -\pi^4\lambda^2\mathcal{B}_\mu\mathcal{B}^\mu - \nu^2\mathcal{U} \\
&= \mathcal{L}_\omega(U) \quad + \quad \mathcal{L}_\sigma(U) \\
&= -\pi^4\lambda^2\rho_B^2 \quad - \quad \nu^2 G(\rho_\sigma)
\end{aligned}
$$

emergent objects (as baryons) in a mesonic fluid

- two mesonic fluids
  - strongly interacting!
  - coupled via pions
- exact balance in the BPS Skyrme
  - attractive σ channel cancels completely repulsive ω channel
  - in Walecka (and others) by a suitable choice of parameters
- near BPS model $\epsilon \neq 0$
  - no exact cancelation $\to$ non-zero binding energies

Neutron stars

- **the BPS Skyrme model with gravity** Adam, Naya, Sanchez, Vazquez, Wereszczynski (15)

$$S_{06} = \int d^4x |g|^{\frac{1}{2}} \left(-\lambda^2 \pi^4 |g|^{-1} g_{\rho\sigma} \mathbb{B}^\rho \mathbb{B}^\sigma - \mu^2 \mathcal{U} \right)$$

- energy-momentum tensor

$$T^{\rho\sigma} = -2|g|^{-\frac{1}{2}} \frac{\delta}{\delta g_{\rho\sigma}} S_{06}$$
$$= 2\lambda^2\pi^4 |g|^{-1} \mathcal{B}^\rho \mathcal{B}^\sigma - \left(\lambda^2 \pi^4 |g|^{-1} g_{\pi\omega} \mathcal{B}^\pi \mathcal{B}^\omega - \mu^2 \mathcal{U} \right) g^{\rho\sigma}$$

- the energy-momentum tensor of **a perfect fluid**

$$T^{\rho\sigma} = (p+\rho) u^\rho u^\sigma - p g^{\rho\sigma}$$

where the four-velocity $u^\rho = \mathcal{B}^\rho / \sqrt{g_{\sigma\pi} \mathcal{B}^\sigma \mathcal{B}^\pi}$ and

$$\rho = \lambda^2 \pi^4 |g|^{-1} g_{\rho\sigma} \mathcal{B}^\rho \mathcal{B}^\sigma + \mu^2 \mathcal{U}$$
$$p = \lambda^2 \pi^4 |g|^{-1} g_{\rho\sigma} \mathcal{B}^\rho \mathcal{B}^\sigma - \mu^2 \mathcal{U}$$

- for a static case with diagonal metric $u^\rho = (\sqrt{g^{00}}, 0, 0, 0)$

$$T^{00} = \rho g^{00}, \quad T^{ij} = -p g^{ij}.$$

flat space case $\Rightarrow$ pressure must be constant (zero for BPS solutions, nonzero for non-BPS static solutions

$$D_\rho T^{\rho\sigma} \to \partial_i T^{ij} = \delta^{ij} \partial_i p = 0$$

- In general, ρ and p arbitrary functions of the space-time coordinates, $\Rightarrow$ **no universal equation of state** $p = p(\rho)$ **valid for all solutions**

- **Einstein equations**
  - static, spherically symmetric metric (Schwarzschild coordinates)

$$ds^2 = \mathbf{A}(r)dt^2 - \mathbf{B}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

  - axially symmetric ansatz for the Skyrme field with baryon number B

$$U = e^{i\xi\vec{n}\cdot\vec{\tau}}$$

$$\xi = \xi(r), \quad \vec{n} = (\sin\theta\cos B\phi, \sin\theta\sin B\phi, \cos\theta)$$

are compatible with the Einstein equations

$$G_{\rho\sigma} = \frac{\kappa^2}{2} T_{\rho\sigma}$$

FT + GR with *full backreaction* $\leftrightarrow$ TOV: fix EoS

Einstein equations

$$
\begin{aligned}
\frac{1}{r}\frac{\mathbf{B}'}{\mathbf{B}} &= -\frac{1}{r^2}(\mathbf{B}-1)+\frac{\kappa^2}{2}\mathbf{B}\rho \\
r\,(\mathbf{B}p)' &= \frac{1}{2}(1-\mathbf{B})\mathbf{B}(\rho+3p)+\frac{\kappa^2}{2}\mu^2 r^2\mathbf{B}^2\mathcal{U}(h)p \\
\frac{\mathbf{A}'}{\mathbf{A}} &= \frac{1}{r}(\mathbf{B}-1)+\frac{\kappa^2}{2}r\mathbf{B}p
\end{aligned}
$$

- **A**, **B** and ξ are functions of $r \Rightarrow p$ and ρ are functions of r

$$\rho = \frac{4B^2\lambda^2}{\mathbf{B}r^4}h(1-h)h_r^2+\mu^2\mathcal{U}(h), \quad p=\rho-2\mu^2\mathcal{U}(h)$$

  eliminate $r \Rightarrow$ **on-shell EoS** $p=p(\rho)$
  except the step-function potential $\Rightarrow$ **off-shell EoS** $\rho = p+2\mu^2$

- axially symmetric ansatz **is the correct one** because gravity straightens out all deviations from spherical symmetry

- **parameters fitting $B = 1$ sector**
  two parameters in the model λ and μ
  - $\mathbf{m} \equiv \lambda\mu$ has the dimensions of mass (energy)
    fitting to the mass of helium (to avoid contributions from (iso)spin excitations)
    $$E_{06} = B\bar{m}_{\rm N}, \quad \bar{m}_{\rm N} = m_{\rm He}/4 = 931.75 \text{ MeV}$$
  - $\mathbf{l} \equiv (\lambda/\mu)^{1/3}$ has the dimensions of length
    typical potentials $\Rightarrow$ compacton $\Rightarrow$ finite geom. volume $V = \frac{4\pi}{3}R^3$
    $\Rightarrow$ radius of skyrmion $R = r_0 B^{1/3}$
    fitting the radius to the nucleon radius
    $$r_0 = r_N = 1.25\text{fm}$$
  - particular potentials $\mathcal{U}_\pi, \mathcal{U}_\pi^2$, where
    $$\mathcal{U}_\pi = 1 - \cos\xi$$

$$\mathcal{U}_\pi : \quad E_{06} = \frac{64\sqrt{2}\pi}{15}B\lambda\mu, \quad R = \sqrt{2}\left(\frac{\lambda B}{\mu}\right)^{\frac{1}{3}} \quad \Rightarrow \quad \mathbf{m} = 49,15 \text{ MeV}, \quad \mathbf{l} = 0,884 \text{ fm}$$

$$\mathcal{U}_\pi^2 : \quad E_{06} = 2\pi^2 B\lambda\mu, \quad R = \left(\frac{3\pi B}{2}\right)^{\frac{1}{3}}\left(\frac{\lambda}{\mu}\right)^{\frac{1}{3}} \quad \Rightarrow \quad \mathbf{m} = 47.20 \text{ MeV}, \quad \mathbf{l} = 0.746 \text{ fm}$$

- **results: maximal mass and radius**

$$\mathcal{U}_\pi : \quad n_{\max} = 5.005, \quad M_{\max} = 3.734 M_\odot, \quad R_{\max} = 18.458 \text{ km},$$

$$\mathcal{U}_\pi^2 : \quad n_{\max} = 3.271, \quad M_{\max} = 2.4388 M_\odot, \quad R_{\max} = 16.801 \text{ km}.$$

where $n \equiv (B/B_\odot) = (B\bar{m}_{\rm N}/M_\odot)$

$\mathcal{U}_\pi^2$ - very good agreement with the data

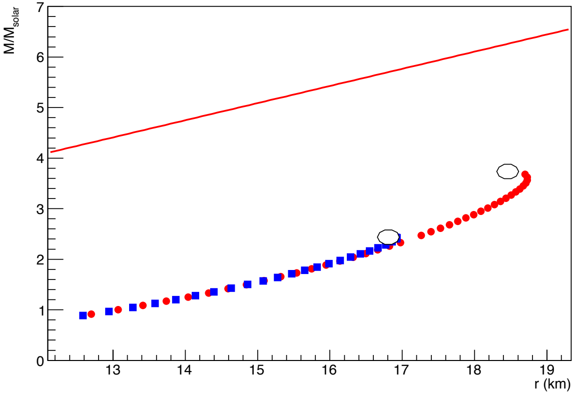
$M \sim 2M_\odot$ are firmly established
indications for masses up to about $2.5M_\odot$

the expected range of about $R \sim$ 10-20 km
$\rightarrow$ less precise (near BPS; proper radius, radiation radius)

extrapolation form $B = 1$ to $B \sim 10^{57}$

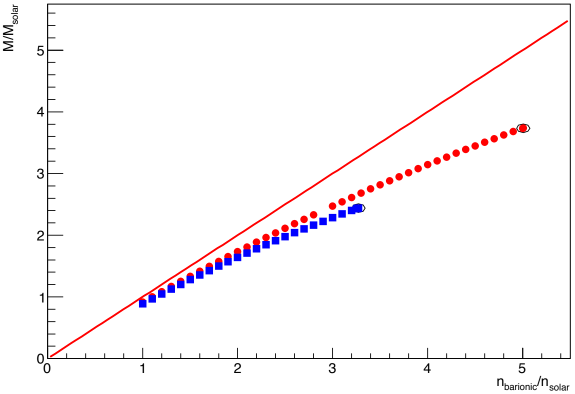
with only **2** parameters of the model

- **results: mass-radius relation**



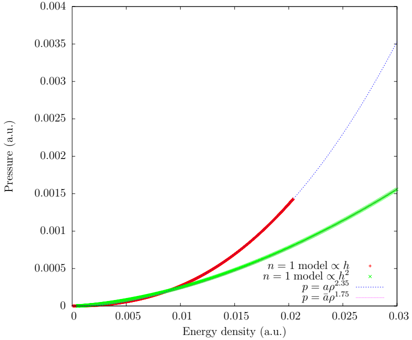
Red potential $\mathcal{U}_\pi$, blue potential $\mathcal{U}_\pi^2$. Maximum values are indicated by circles.

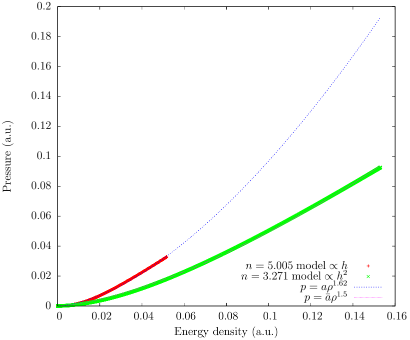
- **results: mass-baryon number relation - binding energies**



Red potential $\mathcal{U}_\pi$, blue potential $\mathcal{U}_\pi^2$. Maximum values are indicated by circles.

- **results: EoS**





Equation of state for $n = 1$ and $n_{\max}$. Symbol plus ($+$ red): potential $\mathcal{U}_\pi$. Symbol cross ($\times$ green): potential $\mathcal{U}_\pi^2$. Dotted lines: corresponding fit functions.

numerical fit to a power law $p = a\rho^b$

a=a(M) and **b=b(M)** $\Rightarrow$ **no universal EoS**

- M dependence governed by the potential $\mathcal{U}$

each neutron star has its own EoS

non-barotropic nuclear matter
strongly coupled 2-component meson fluid

$\Downarrow$

non-universal $M(R)$ relation

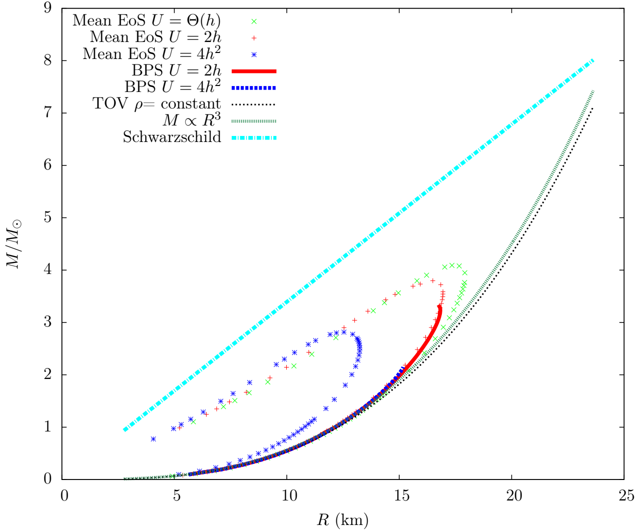
$\Downarrow$

reverse TOV problematic

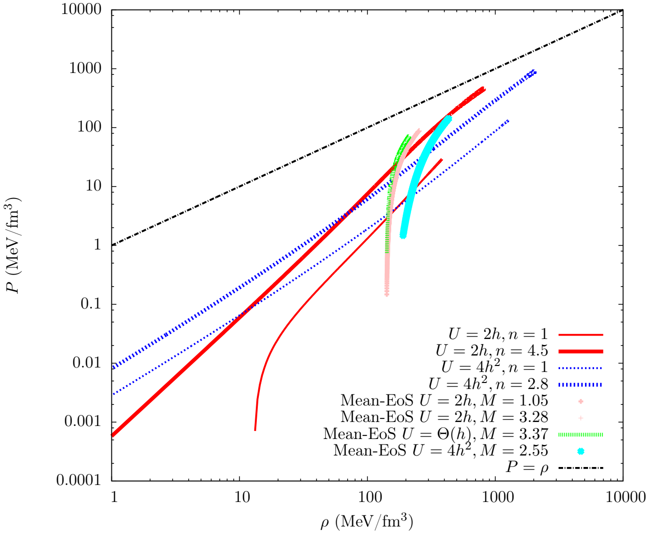
$$G_{\rho\sigma} = \frac{\kappa^2}{2} T_{\rho\sigma} \quad \Rightarrow \quad p = p(\rho) \text{ for nuclear matter}$$

Neutron stars: mean-field (TV) vs. full GR+FT

- **results: EoS - full GR+FT vs. mean-field**



- **results: EoS - full GR+FT vs. mean-field**



future

- **BPS**
  - $T > 0$
    - computation of the partition function
    - integration over the moduli *SDiff*
    - geodesic approximation Manton (93) but beyond Bradlow limit Bradlow (90) $\rightarrow$ dense matter limit
    - $\dim(SDiff) = \infty$! but finite volume
  - $SU(3)$
    - Callan-Klebanov
  - in-medium skyrmions
- **near-BPS**
  - valley of stability
    - right classical shape
  - structure of neutron stars
- **$B = 1$ and $B = 10^{57}$ sectors related: $\mathcal{U}$**
- **QCD reason for BPS/solvability/intgrability**

summary = fundamental result

there is a limit of Skyrme type actions in which the model has two properties

- **BPS**

  classically zero binding energies
  physical (small) binding energies: semiclassical quantization
  Coulomb interaction
  isospin breaking

- **perfect fluid field theory**

  $T_{\mu\nu}$ in a perfect fluid form
  SDiff symmetry
  Euler fluid formulation
  FT (micro) d.o.f. = thermodynamical (macro) functions

  Physically well motivated idealization of the nuclear matter
  weakly modified by the (small) non-BPS part

- **solvablity**

  exact solutions
  exact, analytical EoS and thermodynamics
  a way beyond MF limit in nuclear matter