Anisotropic Hydrodynamics
Lecture 1

Michael Strickland
Kent State University

2014 Cracow School of Theoretical Physics
QCD meets experiment

Three Lecture Plan

Lecture 1

- Motivation and Introduction
- Transport Theory Primer
- Hydro from Transport
- Tensor Basis
- Ideal Hydrodynamics
- Boost Invariance
- Bjorken Solution

Lecture 2

- 1^{st} and 2^{nd} Order Viscous Hydro
- Limitations of Viscous Hydro
- Spheroidal Distribution
- Anisotropic $T^{\mu\nu}$
- LO anisotropic Hydro (aHydro) Equations
- Connection to Viscous Hydro
- 2+1d LO spheroidal aHydro

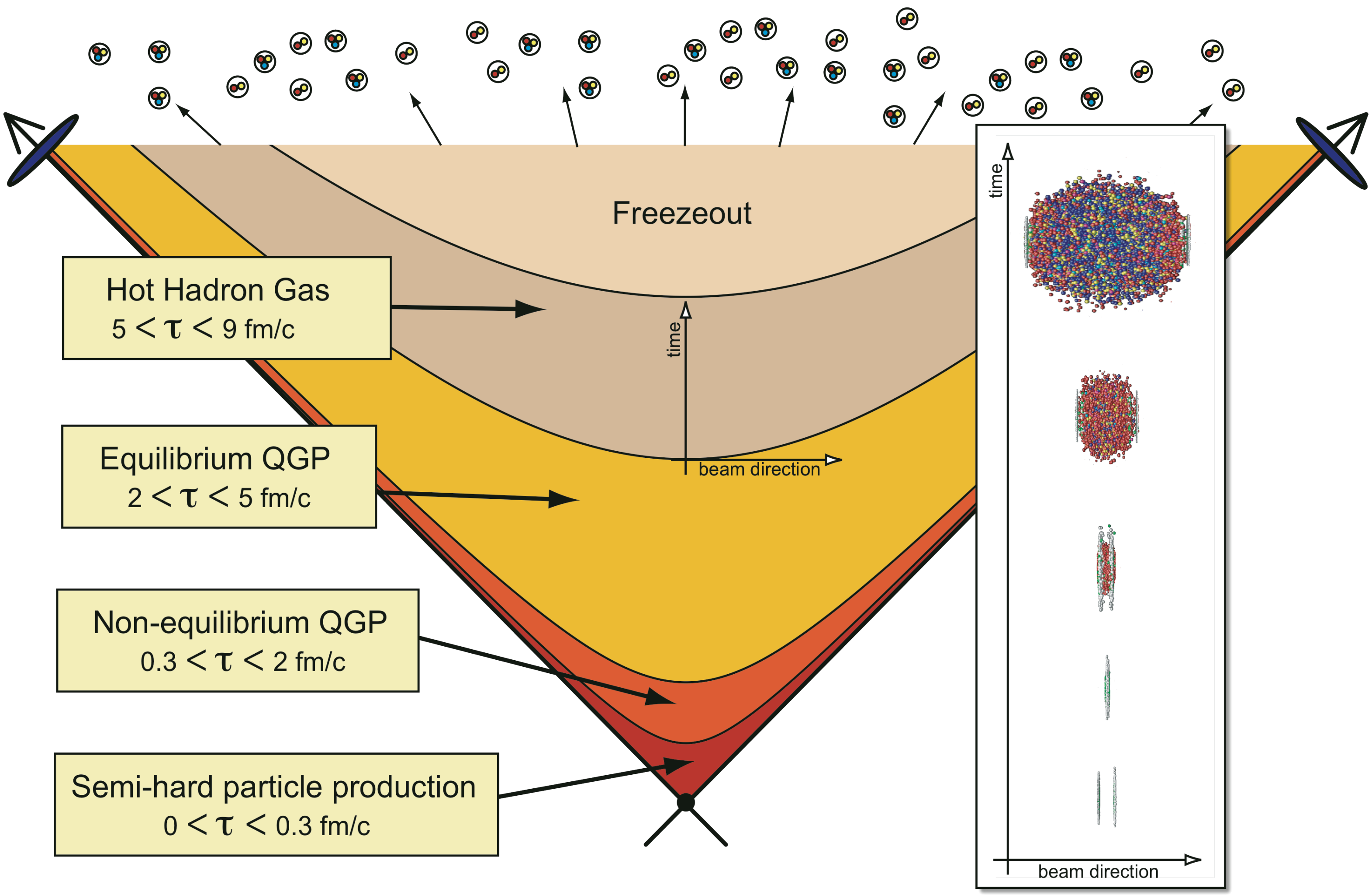
Lecture 3

- Exact solution of RTA Boltzmann EQ
- 2^{nd} order spheroidal anisotropic hydrodynamics
- Ellipsoidal anisotropic hydrodynamics for a system of massive particles
- Phenomenology (General)
- Heavy quarkonium suppression

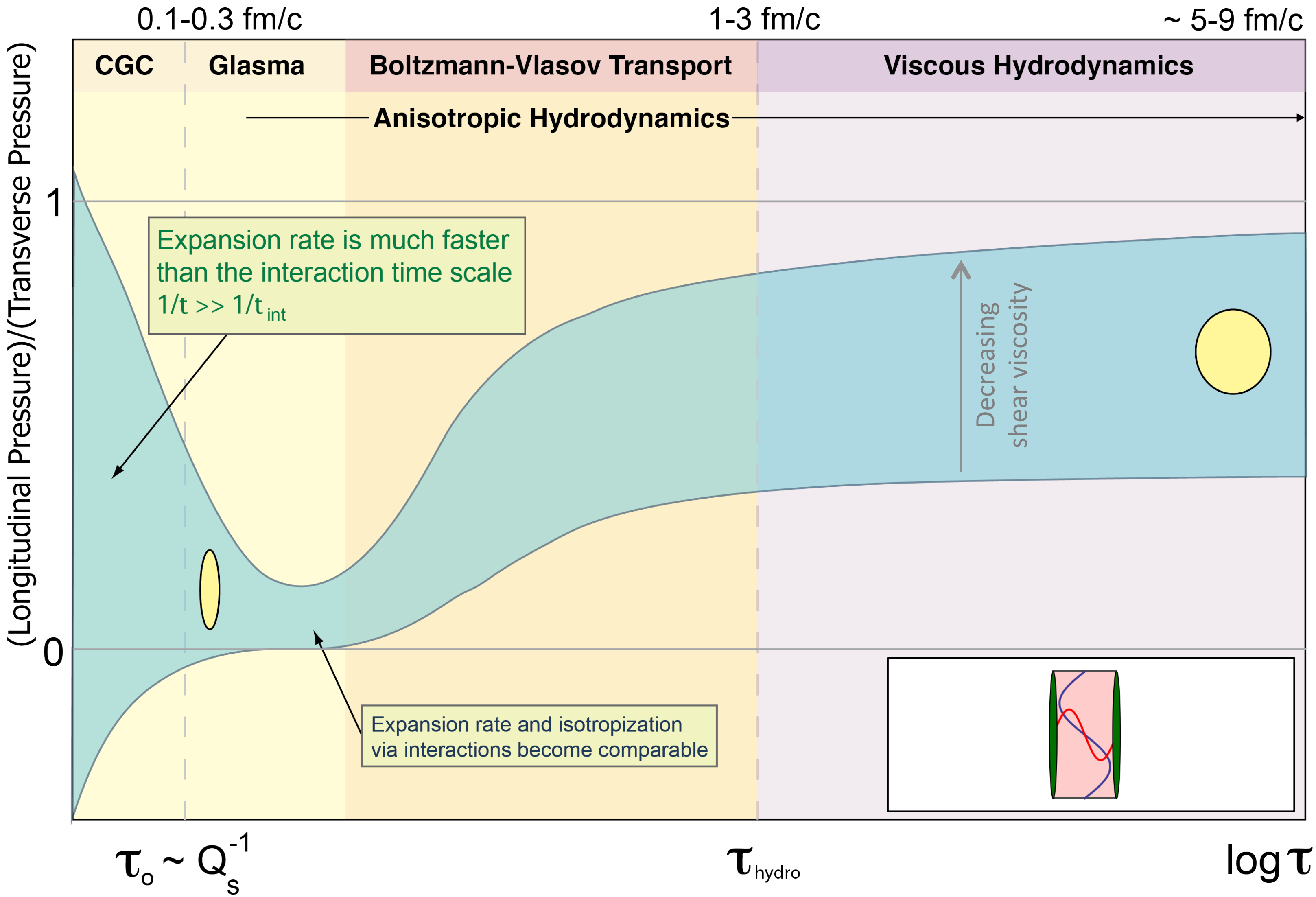
Motivation

- Relativistic viscous hydrodynamical modeling for heavy ion collisions at RHIC and LHC is ubiquitous
- Application is justified a priori by the smallness of the shear viscosity of the plasma (relative to the entropy density)
- The canonical way to derive viscous hydrodynamics relies on a linearization around an isotropic equilibrium state (local rest frame = LRF)
- However, the **QGP is not isotropic in LRF** → there are large corrections to ideal hydrodynamics primarily due to strong longitudinal expansion
- Alternative approach: Anisotropic hydrodynamics builds in momentum-space anisotropies in the LRF from the beginning
- The goal is to create a quantitatively reliable viscous-hydro-like code that more accurately describes:
    - Early time dynamics
    - Dynamics near the transverse edges of the overlap region
    - Temperature-dependent (and potentially large) η/S

LHC Heavy Ion Collision Timescales



QGP momentum anisotropy cartoon



Estimating Early-time Pressure Anisotropy

- CGC @ leading order predicts negative → approximately zero longitudinal pressure
- QGP scattering + plasma instabilities work to drive the system towards isotropy on the fm/c timescale, but don't seem to fully restore it
- Viscous hydrodynamics predicts early-time anisotropies ≤ 0.35 → 0.5 (see next slide)
- AdS-CFT dynamical calculations in the strong coupling limit predict anisotropies of ≤ 0.3 (discussion in three slides from now)

Estimating Anisotropy – Viscous hydro

- To get a feeling for the magnitude of pressure anisotropies to expect, let's consider the Navier-Stokes limit

$$\boxed{\left(\frac{\mathcal{P}_L}{\mathcal{P}_T}\right)_{\rm NS} = \frac{P_{\rm eq}+\pi^{zz}_{\rm NS}}{P_{\rm eq}+\pi^{xx}_{\rm NS}} = \frac{3\tau T - 16\bar\eta}{3\tau T + 8\bar\eta}}$$

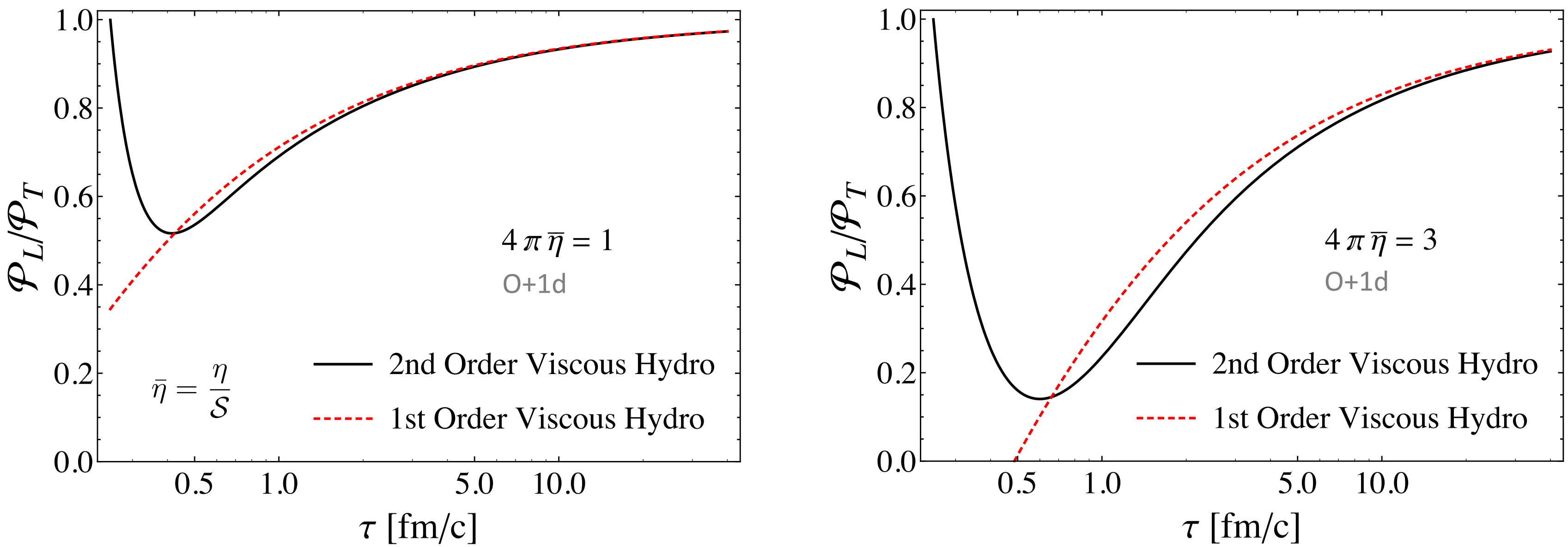
$$\boxed{\bar\eta = \frac{\eta}{\mathcal{S}}}$$

$$\boxed{\pi^{zz}_{\rm NS} = -2\pi^{xx}_{\rm NS} = -2\pi^{yy}_{\rm NS} = -4\eta/3\tau}$$

- P_L/P_T decreases with increasing η/S
- P_L/P_T decreases with decreasing T
- Assume $\eta/S = 1/4\pi$ in order to get an upper bound on the anisotropy
- Using RHIC initial conditions (T_0 = 400 MeV @ τ_0 = 0.5 fm/c) we obtain $P_L/P_T \leq 0.5$
- Using LHC initial conditions (T_0 = 600 MeV @ τ_0 = 0.25 fm/c) we obtain $P_L/P_T \leq 0.35$
- Negative P_L at large η/S or low temperatures!?

Estimating Anisotropy – Viscous hydro

- Navier-Stokes solution is "attractor" for the 2nd order solution
- τ_π sets timescale to approach Navier-Stokes evolution
- $\tau_\pi \sim 5\eta/(TS) \sim 0.1$ fm/c at LHC temperatures
- Assume isotropic LHC initial conditions T_0= 600 MeV @ τ_0 = 0.25 fm/c and solve for the 0+1d viscous hydro dynamics



Estimating Anisotropy – AdS/CFT

- In 0+1d case there are now numerical solutions of Einstein's equations to compare with.
  [Heller, Janik, and Witaszczyk, 1103.3452]

- They studied a wide variety of initial conditions and found a kind of universal lower bound for the thermalization time

$$\langle T_{\tau\tau}\rangle \equiv \varepsilon(\tau) \equiv N_c^2 \cdot \frac{3}{8}\pi^2 \cdot T_{eff}^4 \cdot$$

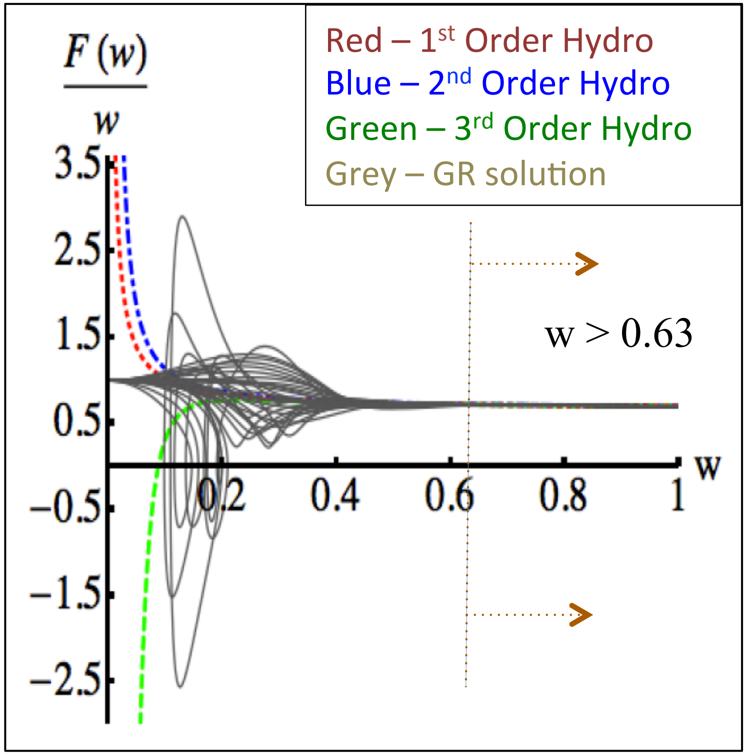
$$w = T_{eff} \cdot \tau$$

$$\frac{\tau}{w}\frac{d}{d\tau}w = \frac{F_{hydro}(w)}{w},$$

F_{hydro} known up to 3rd order hydro analytically

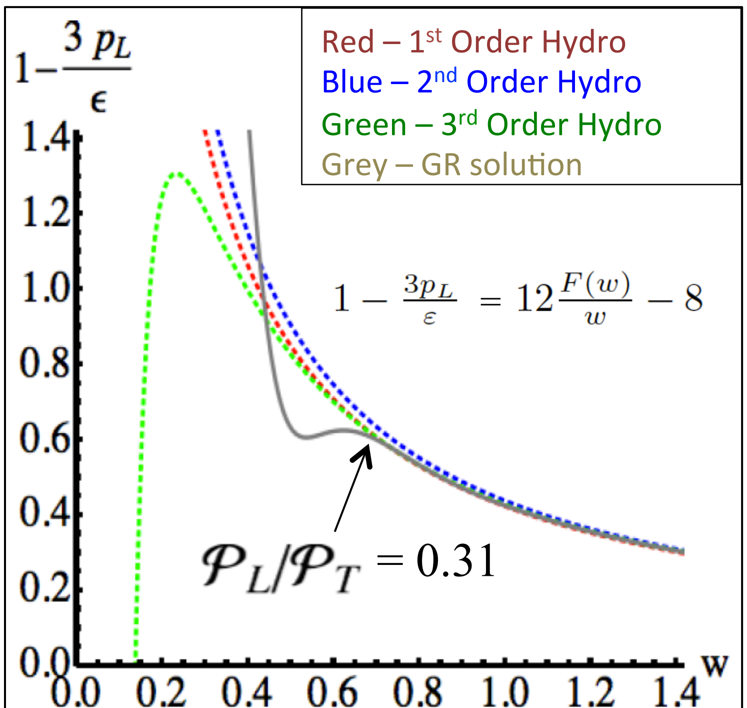
RHIC 200 GeV/nucleon:
T_0 = 350 MeV, τ_0 > 0.35 fm/c

LHC 2.76 TeV/nucleon:
T_0 = 600 MeV, τ_0 > 0.2 fm/c



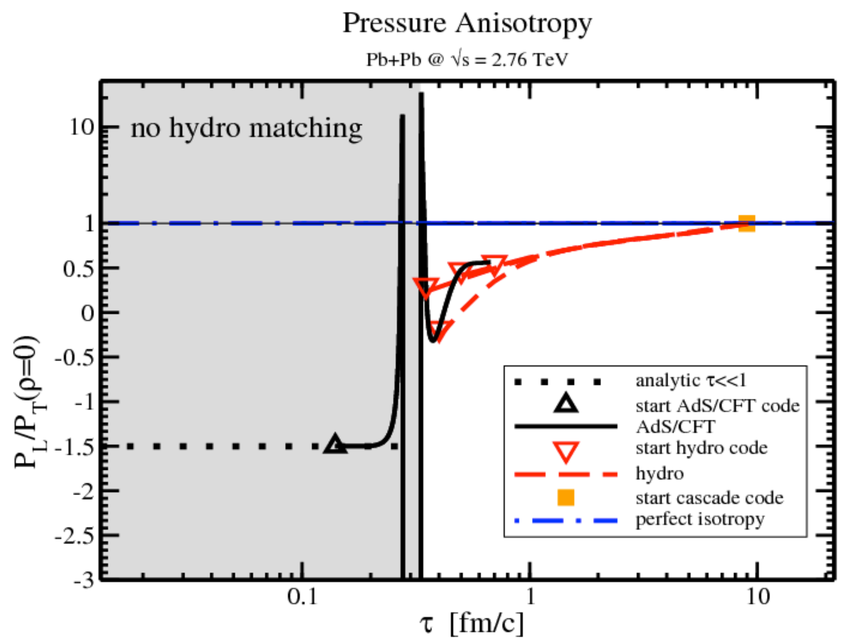
N=4 SUSY using AdS/CFT

However, at that time the system is not isotropic and remains anisotropic for the entirety of the evolution



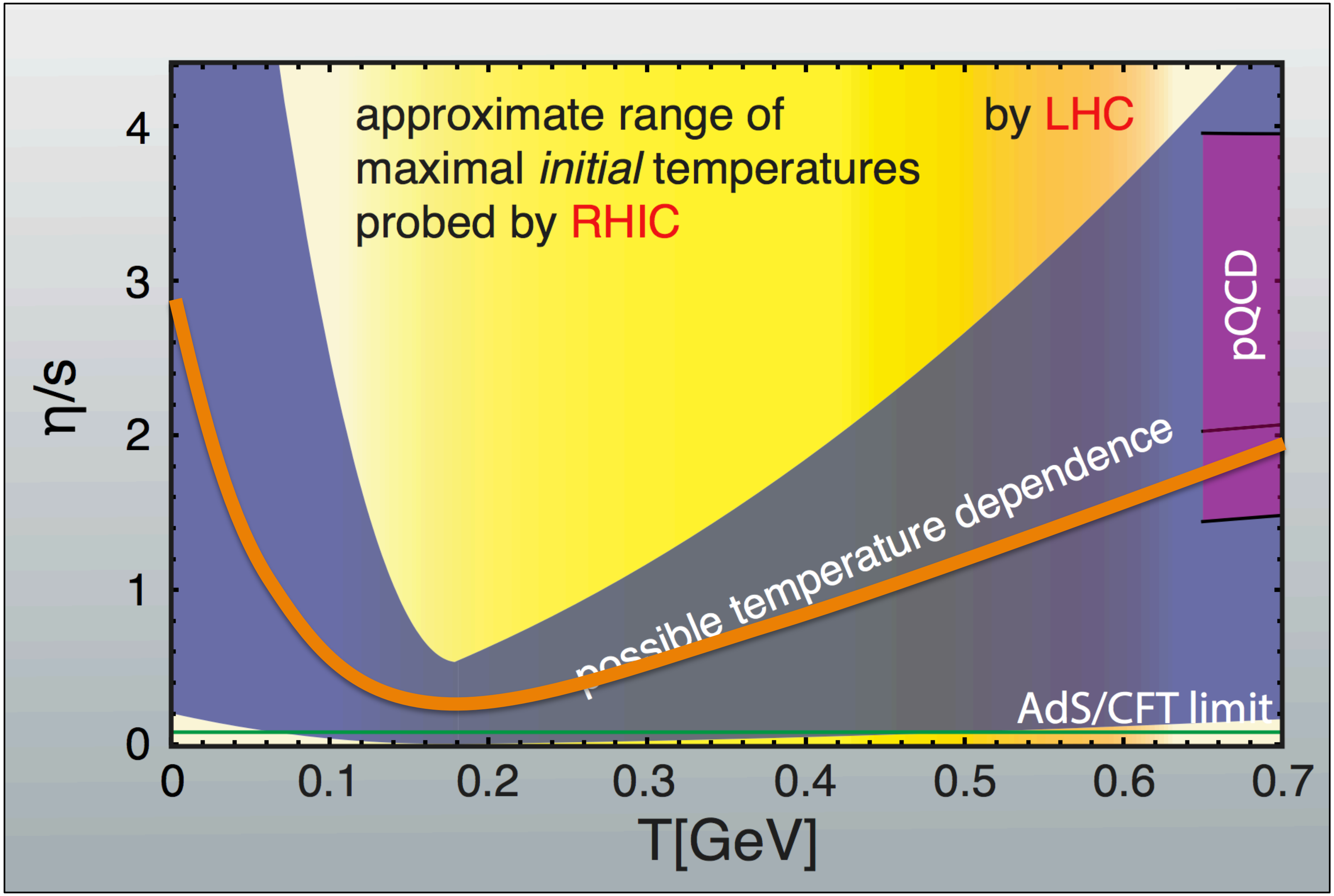
Another AdS/CFT numerical GR paper which includes transverse expansion reaches a similar conclusion

[van der Schee et al. 1307.2539]

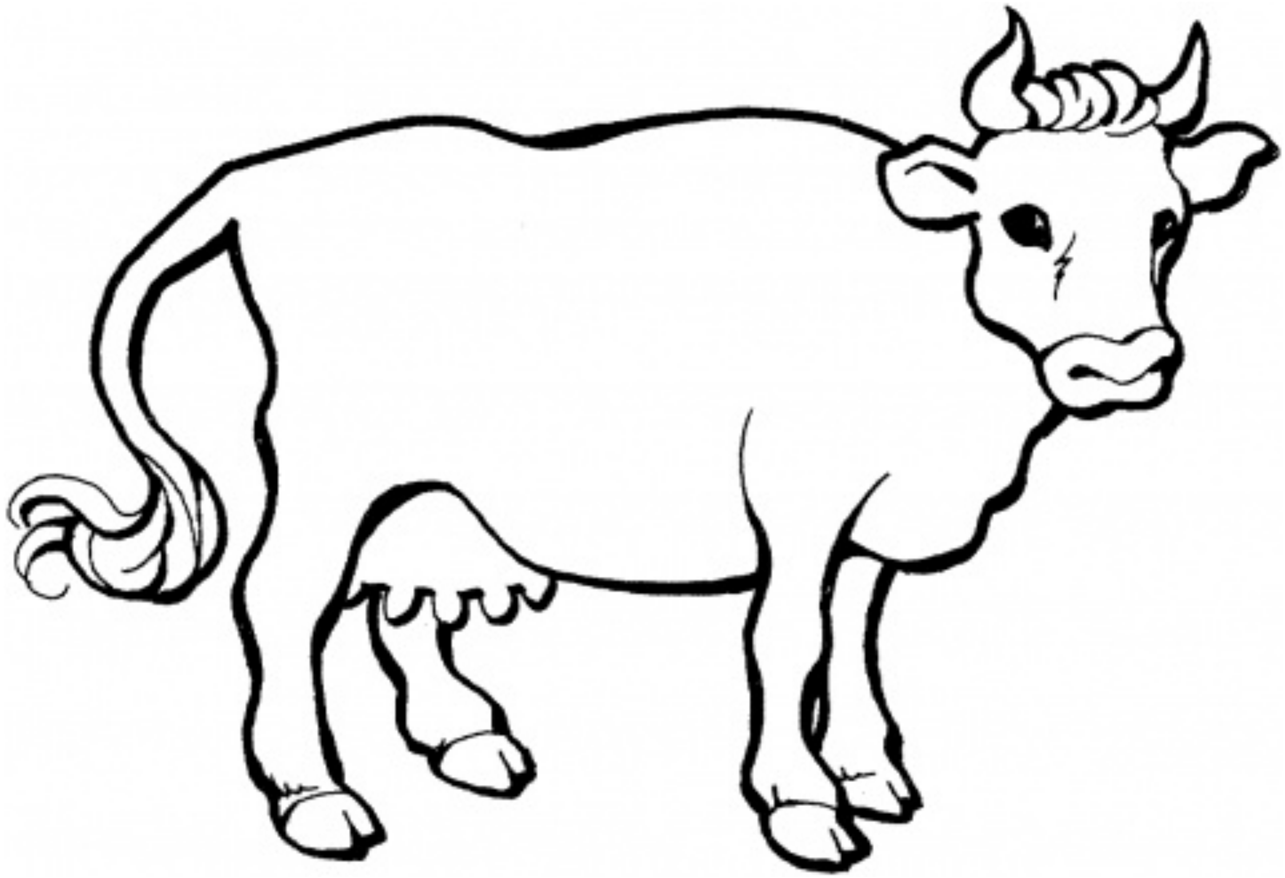


See also J. Casalderrey-Solana et al. arXiv: 1305.4919

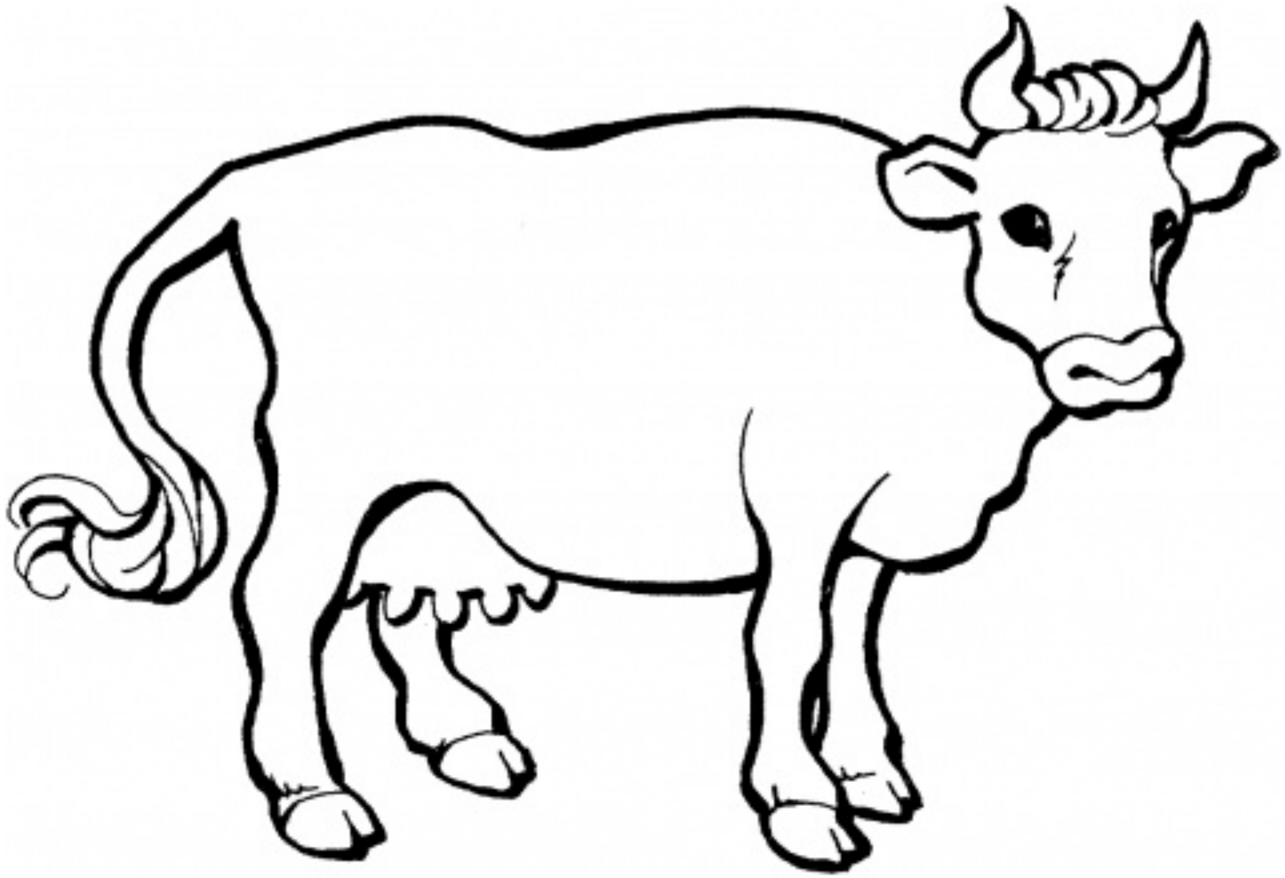
Temperature dependence of η/S



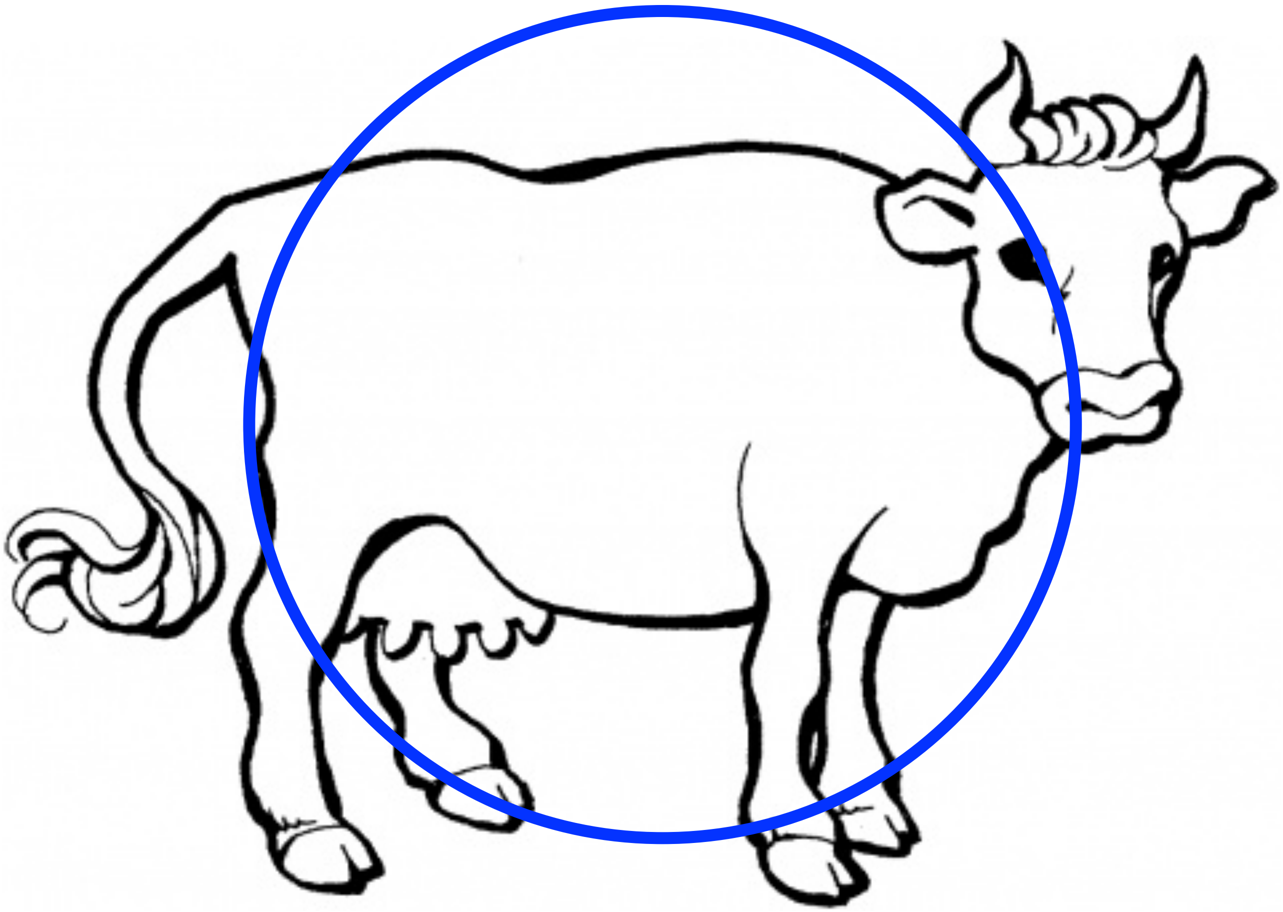
Physics 101



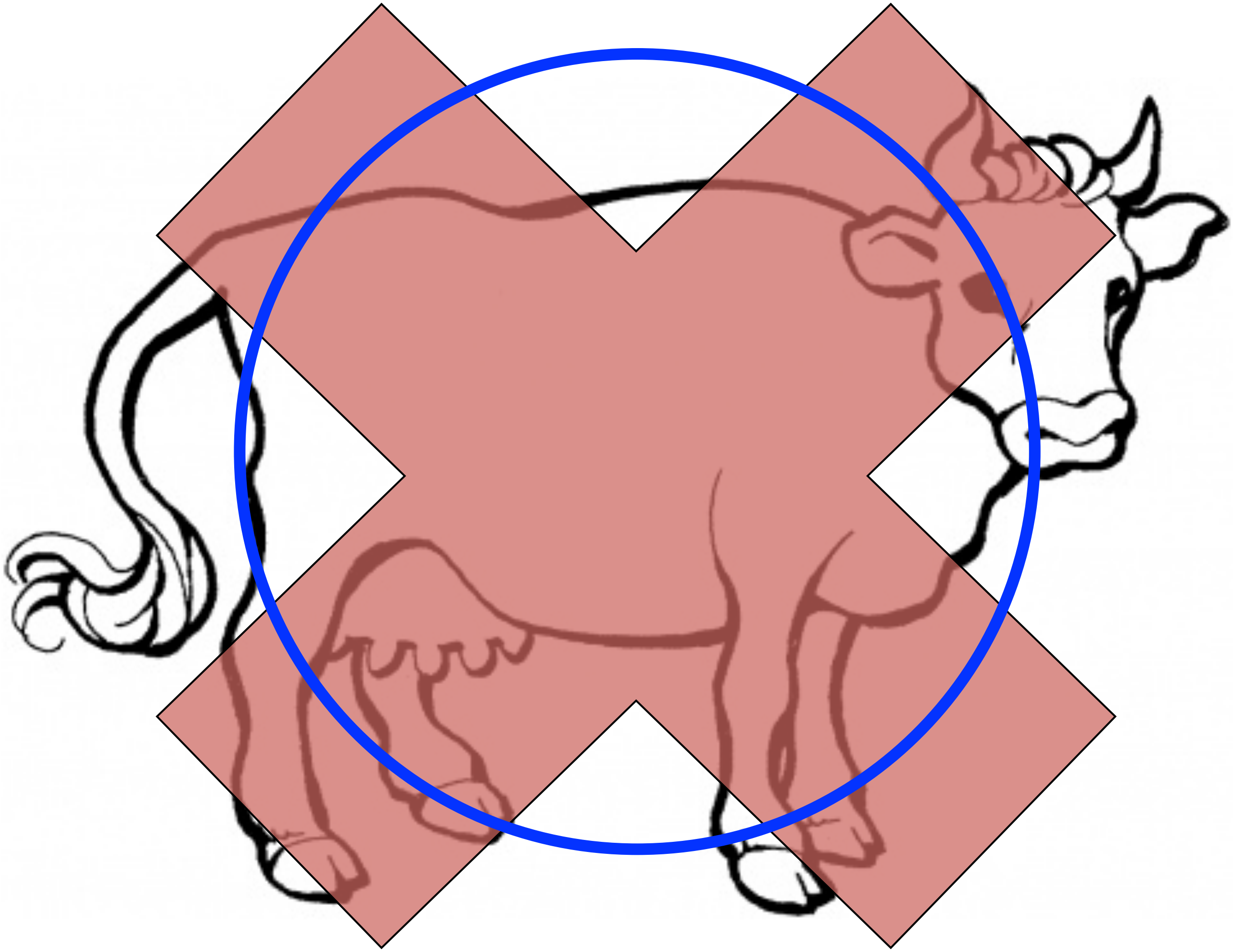
Cows are spheres?



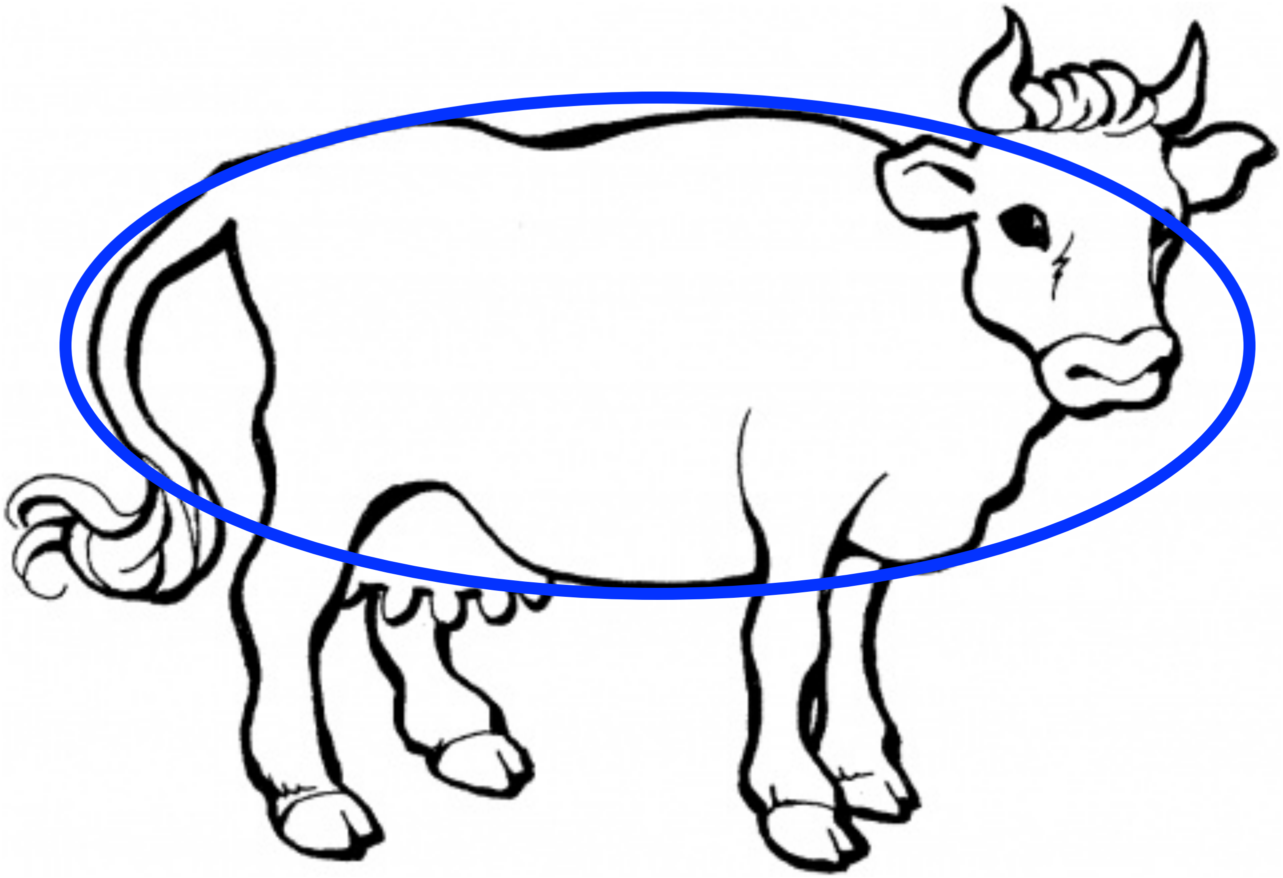
Cows are spheres?



Cows are <u>not</u> spheres



Cows are more like ellipsoids!



Anisotropic Hydrodynamics Basics

M. Martinez and MS, 1007.0889
W. Florkowski and R. Ryblewski, 1007.0130

Viscous Hydrodynamics Expansion

$$f(\tau,\mathbf{x},\mathbf{p}) = \underline{f_{\rm eq}(\mathbf{p},T(\tau,\mathbf{x}))} + \delta f$$

Isotropic in momentum space

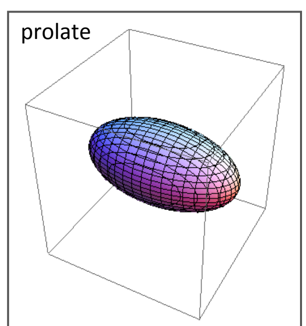
Anisotropic Hydrodynamics Expansion

$$f(\tau,\mathbf{x},\mathbf{p}) = f_{\rm aniso}(\mathbf{p},\underbrace{\Lambda(\tau,\mathbf{x})}_{T_\perp},\underbrace{\xi(\tau,\mathbf{x})}_{\rm anisotropy}\,) + \delta\tilde{f}$$

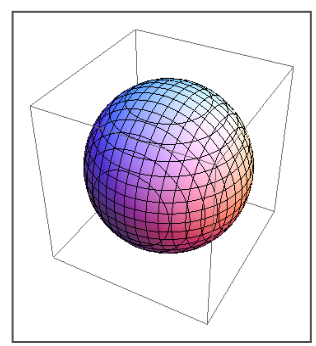
→ "Romatschke-Strickland" form in LRF

$$\boxed{f_{\rm aniso}^{LRF} = f_{\rm iso}\left(\frac{\sqrt{\mathbf{p}^2+\xi(\mathbf{x},\tau)p_z^2}}{\Lambda(\mathbf{x},\tau)}\right)}$$

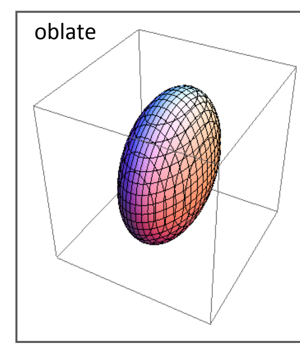
$$\boxed{\xi = \frac{\langle p_T^2\rangle}{2\langle p_L^2\rangle} - 1}$$



$-1 < \xi < 0$



$\xi = 0$



$\xi > 0$

- Transport Theory Primer -

Transport Theory Primer I

$$\boxed{f(t, \mathbf{x}, \mathbf{p}) \propto \frac{dN}{d^3p\, d^3x}}$$

f = one-particle distribution function
= # of on-shell particles per unit phase space

Liouville's Theorem (phase space volume conserved)

$$\boxed{\frac{df}{d\tau} = 0 \longrightarrow \frac{df}{d\tau} = \underbrace{\frac{dt}{d\tau}}_{\gamma} \frac{\partial f}{\partial t} + \underbrace{\frac{d\mathbf{x}}{d\tau}}_{\gamma \mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0}$$

Multiply by m : Use $\gamma m = E$ and $\mathbf{p} = \gamma m \mathbf{v}$ $\rightarrow E\, \partial_t f + \mathbf{p} \cdot \nabla f = 0$

$$\longrightarrow \boxed{p^\mu \partial_\mu f = 0}$$

Including the possibility of collisions we have

Boltzmann Equation $\boxed{p^\mu \partial_\mu f = -C[f]}$ ← Collisional Kernel

The Kinetic Energy-Momentum Tensor

- Starting from QFT and evaluating statistical averages $<T^{\mu\nu}>$.
- Write in terms of the one-particle distribution functions

$$\langle \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})\rangle = \frac{1}{e^{\beta(E-\mu)}-1} \equiv f_B\,,$$

$$\langle b_s^\dagger(\mathbf{k})b_s(\mathbf{k})\rangle = \frac{1}{e^{\beta(E-\mu)}+1} \equiv f_F^+(E,T,\mu)\,,$$

$$\langle d_s^\dagger(\mathbf{k})d_s(\mathbf{k})\rangle = \frac{1}{e^{\beta(E+\mu)}+1} \equiv f_F^-(E,T,\mu)\,,$$

- Energy momentum tensor can be expressed as invariant phase space integral

$$\boxed{T^{\mu\nu}(x) = \int dP\, p^\mu p^\nu f(x,p)}$$

$$\int dP \equiv \int \frac{d^4\mathbf{p}}{(2\pi)^3}\delta(p^\mu p_\mu - m^2)\,2\theta(p^0) = \int_{\mathbf{p}} \frac{1}{E}$$

Transport Theory Primer II

In equilibrium the phase space distribution is stationary
→ collision kernel vanishes for equilibrium distribution

$$\boxed{p^\mu \partial_\mu f_{\rm eq} = 0} \longrightarrow \boxed{p^\mu \partial_\mu f = -C[f]} \longrightarrow \boxed{C[f_{\rm eq}] = 0}$$

$$f_{\rm eq} = F\left(\frac{u_\mu p^\mu}{T}\right)$$

F can be Boltzmann distribution (classical eq)
or Bose-Einstein/Fermi-Dirac distribution (quantum eq)

- In the local rest frame (LRF) u = (1,0,0,0) and we have $u_\mu p^\mu = E_{\rm LRF}$ → particle energy in LRF.
- Since $u_\mu p^\mu$ is Lorentz invariant, it will be $E_{\rm LRF}$ in all frames

- Hydro From Transport -

Hydro from Transport

- Describe evolution of the system using the Boltzmann equation

$$\boxed{p^\alpha \partial_\alpha f = -C[f]}$$

- One can extract hydro equations from the Boltzmann equation by taking "moments" of the equation using the following integral operator

$$\boxed{\hat{I}^{\mu\nu\cdots\sigma}[F] = \int dP\, p^\mu p^\nu \cdots p^\sigma F}$$

$$\boxed{\int dP \equiv \int \frac{d^4\mathbf{p}}{(2\pi)^3} \delta(p^\mu p_\mu - m^2)\, 2\theta(p^0) = \int_\mathbf{p} \frac{1}{E}} \quad \boxed{\int_\mathbf{p} \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3}}$$

0^{th} Moment

Take the zeroth moment and rearrange a bit

$$\int dP\, p^\alpha \partial_\alpha f = -\int dP\, C[f]$$

$$\underbrace{\partial_\alpha \left(\int_{\mathbf{p}} v^\alpha f \right)}_{N^\alpha} = -\int dP\, C[f]$$

N^α : Particle Number and Current

$$\partial_\alpha N^\alpha = -\int dP\, C[f] \xrightarrow[\text{collisional kernel}]{\text{if number conserving}} \boxed{\partial_\alpha N^\alpha = 0}$$

Number conservation

If particle number changing processes in kernel, eg $2 \to 3$, RHS is nonzero

1^{st} Moment

Take the first moment and rearrange a bit

$$\int dP\, p^\beta (p^\alpha \partial_\alpha f) = -\int dP\, p^\beta C[f]$$

$$\partial_\alpha \underbrace{\left(\int dP\, p^\alpha p^\beta f \right)}_{T^{\alpha\beta}} = -\int dP\, p^\beta C[f]$$

$T^{\alpha\beta}$: Energy-Momentum Tensor

$$\boxed{\partial_\alpha T^{\alpha\beta} = -\int dP\, p^\beta C[f]} \xrightarrow[\text{collisional kernel}]{\text{if energy conserving}} \boxed{\partial_\alpha T^{\alpha\beta} = 0}$$

Energy-momentum conservation!

- Tensor Basis -

General Tensor Basis I

- Can span (flat) space-time with 4 four-vectors

$$
\begin{aligned}
X^\mu_{0,\mathrm{LRF}} &\equiv u^\mu_{\mathrm{LRF}} = (1,0,0,0) \\
X^\mu_{1,\mathrm{LRF}} &\equiv x^\mu_{\mathrm{LRF}} = (0,1,0,0) \\
X^\mu_{2,\mathrm{LRF}} &\equiv y^\mu_{\mathrm{LRF}} = (0,0,1,0) \\
X^\mu_{3,\mathrm{LRF}} &\equiv z^\mu_{\mathrm{LRF}} = (0,0,0,1)
\end{aligned}
$$

These vectors are mutually orthogonal in all frames.

$$
\begin{aligned}
u^\mu u_\mu &= 1 \\
u^\mu \mathbf{x}_\mu &= 0 \\
\mathbf{x}^\mu \mathbf{x}_\mu &= -1
\end{aligned}
$$

- The first four-vector (u^μ) will be identified as the fluid four-velocity
- Lab frame quantities are obtained using Lorentz boost corresponding to u^μ

Metric Tensor and Transverse Projector

- Can construct metric tensor with these vectors

$$g^{\mu\nu} = X_0^\mu X_0^\nu - \sum_{i=1}^{3} X_i^\mu X_i^\nu$$

- Transverse projector: projects out four-vector components perpendicular to u^μ

$$\Delta^{\mu\nu} = g^{\mu\nu} - X_0^\mu X_0^\nu = -\sum_{i=1}^{3} X_i^\mu X_i^\nu$$

$$\Delta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$u_\mu \Delta^{\mu\nu} = u_\nu \Delta^{\mu\nu} = 0$$

$$X_{i\mu} \Delta^{\mu\nu} = X_i^\nu$$

General Tensor Basis III

- $T^{\mu\nu}$ is a symmetric rank-2 tensor ($T^{\mu\nu} = T^{\nu\mu}$)
- A general symmetric rank-2 tensor can be written as

$$\boxed{c_{\alpha\beta} = c_{\beta\alpha}} \rightarrow \boxed{\text{10 independent components}}$$

$$\downarrow$$

$$\boxed{A^{\mu\nu}(t,\mathbf{x}) = c_{00} g^{\mu\nu} + \sum_{i=1}^{3} d_{ii} X_i^\mu X_i^\nu + \sum_{\substack{\alpha,\beta=0 \\ \alpha>\beta}}^{3} c_{\alpha\beta}(X_\alpha^\mu X_\beta^\nu + X_\beta^\mu X_\alpha^\nu)\,.}$$

$$\downarrow$$

$$\boxed{T^{\mu\nu}(t,\mathbf{x}) = t_{00} g^{\mu\nu} + \sum_{i=1}^{3} t_{ii} X_i^\mu X_i^\nu + \sum_{\substack{\alpha,\beta=0 \\ \alpha>\beta}}^{3} t_{\alpha\beta}(X_\alpha^\mu X_\beta^\nu + X_\beta^\mu X_\alpha^\nu)\,,}$$

- Ideal Hydrodynamics -

Number Conservation

In ideal hydrodynamics there can be a conserved (net) particle number/charge, e.g. baryon number = # of baryons - # of antibaryons.

$$\boxed{\partial_\mu N^\mu = 0}$$

+

$$N^\mu = n u^\mu + \underbrace{\sum_i c_i X_i^\mu}$$
$$= n u^\mu + \cancel{v^\mu}$$

Assume v^μ=0 for ideal hydro

$$n = \text{net charge density}$$
$$v^\mu = \Delta^\mu_\nu N^\nu = \text{diffusion current}$$

Diffusion current measures net flow of charge transverse to u^μ

$$\partial_\mu N^\mu = \underbrace{u^\mu \partial_\mu n}_{\text{Convective Derivative}} + n \underbrace{\partial_\mu u^\mu}_{\text{4-divergence}}$$
$$= Dn + n\theta$$

$\rightarrow$

$$\boxed{Dn + n\theta = 0}$$

(Eq 1)

Ideal Energy Momentum Tensor

In the LRF the ideal energy momentum tensor satisfies

$T^{00} = \mathcal{E}$	:	local energy density
$T^{ij} = \mathcal{P}\,\delta^{ij}$	:	local isotropic pressure
$T^{0i} = 0$	:	momentum flux in i^{th} direction

$$T^{\mu\nu}(t,\mathbf{x}) = t_{00} g^{\mu\nu} + \sum_{i=1}^{3} t_{ii} X_i^\mu X_i^\nu + \sum_{\substack{\alpha,\beta=0 \\ \alpha>\beta}}^{3} t_{\alpha\beta}(X_\alpha^\mu X_\beta^\nu + X_\beta^\mu X_\alpha^\nu)\,,$$

$$T^{00}_{\mathrm{LRF}} = \mathcal{E} = t_{00}$$

$$T^{ii}_{\mathrm{LRF}} = \mathcal{P} = -t_{00} + t_{ii}$$

$$t_{\alpha\beta} = 0 \text{ for all } \alpha \neq \beta.$$

$$\begin{aligned} T^{\mu\nu}(t,\mathbf{x}) &= \mathcal{E} g^{\mu\nu} + (\mathcal{P}+\mathcal{E}) \sum_{i=1}^{3} X_i^\mu X_i^\nu\,, \\ &= \mathcal{E} g^{\mu\nu} + (\mathcal{P}+\mathcal{E})(X_0^\mu X_0^\nu - g^{\mu\nu}) \\ &= (\mathcal{E}+\mathcal{P}) X_0^\mu X_0^\nu - \mathcal{P} g^{\mu\nu}\,, \end{aligned}$$

$$\boxed{T^{\mu\nu} = (\mathcal{E}+\mathcal{P}) u^\mu u^\nu - \mathcal{P} g^{\mu\nu}}$$

$$T^{\mu}{}_{\mu} = \mathcal{E} - 3\mathcal{P}$$

Conformal systems obey $T^{\mu}{}_{\mu} = 0$

Ideal Hydrodynamics Equations

$$\boxed{T^{\mu\nu} = (\mathcal{E} + \mathcal{P})u^\mu u^\nu - \mathcal{P} g^{\mu\nu}}$$

$+$

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

$$\downarrow$$

$$\partial_\mu(\mathcal{E} + \mathcal{P})u^\mu u^\nu + (\mathcal{E} + \mathcal{P})(u^\nu \partial_\mu u^\mu + u^\mu \partial_\mu u^\nu) - g^{\mu\nu}\partial_\mu \mathcal{P} = 0$$

Four equations ($\nu = 0, 1, 2,$ and 3). Standard way to proceed is to project out components along direction of u^ν using u_ν and perpendicular to u^ν using $\Delta^\alpha{}_\nu$

u_ν:

$$\underbrace{u^\mu \partial_\mu}_{\equiv D}(\mathcal{E} + \mathcal{P}) + (\mathcal{E} + \mathcal{P})(\underbrace{\partial_\mu u^\mu}_{\equiv \theta} + u^\mu \underbrace{u_\nu \partial_\mu u^\nu}_{= \frac{1}{2}\partial_\mu(u^\nu u_\nu) = 0}) - u^\mu \partial_\mu \mathcal{P} = 0$$

$$\longrightarrow \quad D\mathcal{E} + (\mathcal{E} + \mathcal{P})\theta = 0 \qquad \text{(Eq 2)}$$

Ideal Hydrodynamics Equations

Second Ideal Hydro EQ

$$\partial_\mu(\mathcal{E}+\mathcal{P})u^\mu u^\nu + (\mathcal{E}+\mathcal{P})(u^\nu\partial_\mu u^\mu + u^\mu\partial_\mu u^\nu) - g^{\mu\nu}\partial_\mu\mathcal{P} = 0$$

$\Delta^{\alpha}{}_{\nu}$:

Spatial Gradient $\nabla^\alpha \equiv \Delta^\alpha{}_\nu \partial^\nu = -\sum_{\beta=1}^{3} X^\alpha_\beta X_{\nu\beta}\partial^\nu$,

$$\boxed{\begin{aligned} Dn + n\theta &= 0 && \text{(Eq 1)} \\ D\mathcal{E} + (\mathcal{E}+\mathcal{P})\theta &= 0 && \text{(Eq 2)} \\ (\mathcal{E}+\mathcal{P})Du^i - \nabla^i\mathcal{P} &= 0 && \text{(Eq 3)} \end{aligned}}$$

$$D \equiv u^\mu \partial_\mu$$

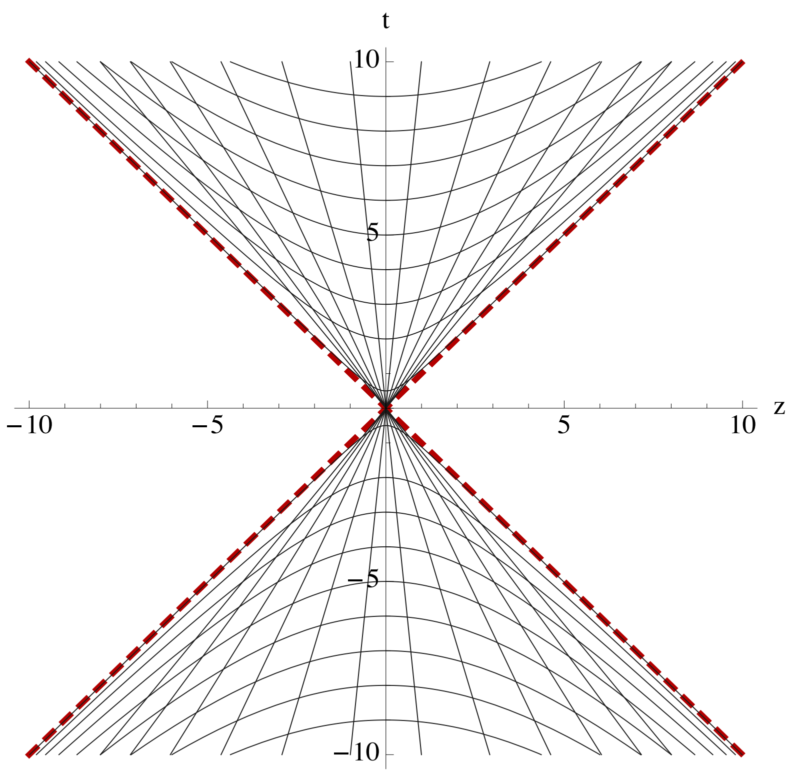
$$\theta \equiv \partial_\mu u^\mu$$

Degrees of Freedom
1 : Particle number
1 : Energy Density
1 : Pressure
3 : Independent components of u^μ
6 : Total

Equations
1 : (Eq 1)
1 : (Eq 2)
3 : (Eq 3) $i = 1, 2, 3$
1 : EQUATION OF STATE $T^\mu{}_\mu = \#$
6 : Total

Boost Invariance

- High-energy → <u>approximate</u> boost-invariance of the particle production
- Introduce proper time (τ) and rapidity (ς)



$$t = \tau \cosh \varsigma \, ,$$
$$z = \tau \sinh \varsigma \, ,$$

"Milne Coordinates"

$$\tau^2 = t^2 - z^2$$
$$\varsigma = \text{arctanh}(z/t)$$

$$(t, x, y, z) \to (\tau, x, y, \varsigma)$$

$$\mathcal{E}(t, x, y, z) \to \mathcal{E}(\tau, x, y)$$
$$\mathcal{P}(t, x, y, z) \to \mathcal{P}(\tau, x, y)$$
$$u^\mu = (u_0 \cosh \varsigma, u_x, u_y, u_0 \sinh \varsigma)$$
$$u_0^2 = 1 + u_x^2 + u_y^2$$

O+1d Boost Invariant → Bjorken Model

Additionally assume that the system is homogeneous in the x,y directions

$$\mathcal{P}(\tau,x,y) \to \mathcal{P}(\tau)$$

$$\mathcal{E}(\tau,x,y) \to \mathcal{E}(\tau)$$

$$u^\mu = (\cosh\varsigma, 0, 0, \sinh\varsigma) = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right)$$

$$u_\tau = 1$$
$$u_\varsigma = 0$$

$$D = u^\mu \partial_\mu = \partial_\tau \,,$$
$$\theta = \partial_\mu u^\mu = \frac{1}{\tau} \,.$$

Hydro Eqs

$$D\mathcal{E} + (\mathcal{E}+\mathcal{P})\theta = 0 \longrightarrow \partial_\tau \mathcal{E} + \frac{\mathcal{E}+\mathcal{P}}{\tau} = 0$$

$$(\mathcal{E}+\mathcal{P})Du^i - \nabla^i \mathcal{P} = 0$$
$$\downarrow$$
$$0 = 0$$

O+1d Boost Invariant → Bjorken Model

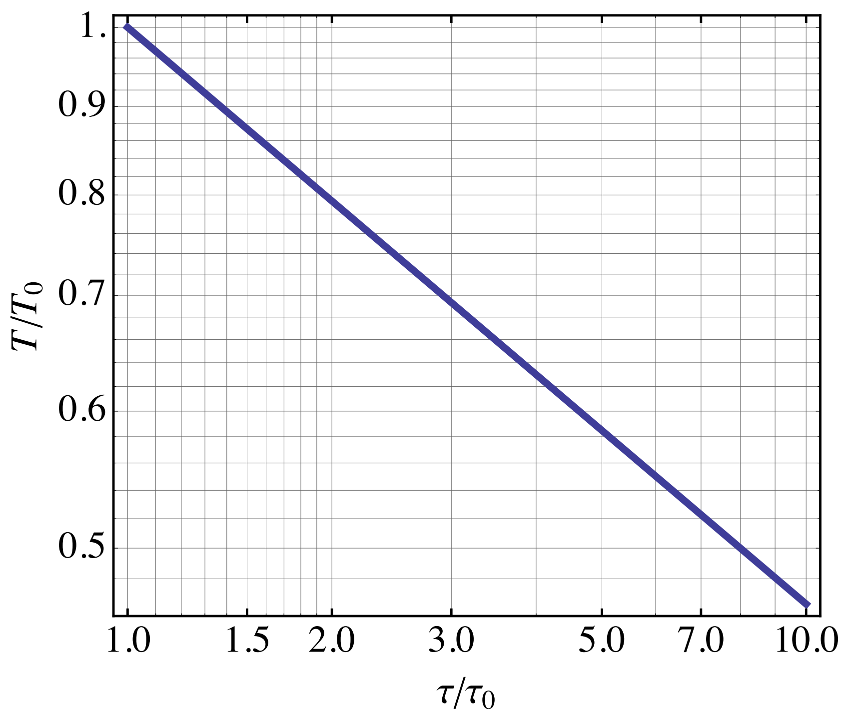
$$\boxed{{T^\mu}_\mu = 0 \longrightarrow \mathcal{E} = 3\mathcal{P}}$$

Ideal Equation of State

$$\boxed{\partial_\tau \mathcal{E} + \frac{\mathcal{E}+\mathcal{P}}{\tau} = 0 \longrightarrow \partial_\tau \mathcal{E} = -\frac{4}{3}\frac{\mathcal{E}}{\tau} \longrightarrow \mathcal{E} = \mathcal{E}_0 \left(\frac{\tau_0}{\tau}\right)^{4/3}}$$

$$\boxed{\mathcal{E} \propto T^4}$$

$$\boxed{T = T_0 \left(\frac{\tau_0}{\tau}\right)^{1/3}}$$



For $\mathcal{P} = c_s^2 \mathcal{E}$

$$\mathcal{E} = \mathcal{E}_0 \left(\frac{\tau_0}{\tau}\right)^{1+c_s^2}$$

Lecture 1 - Conclusions

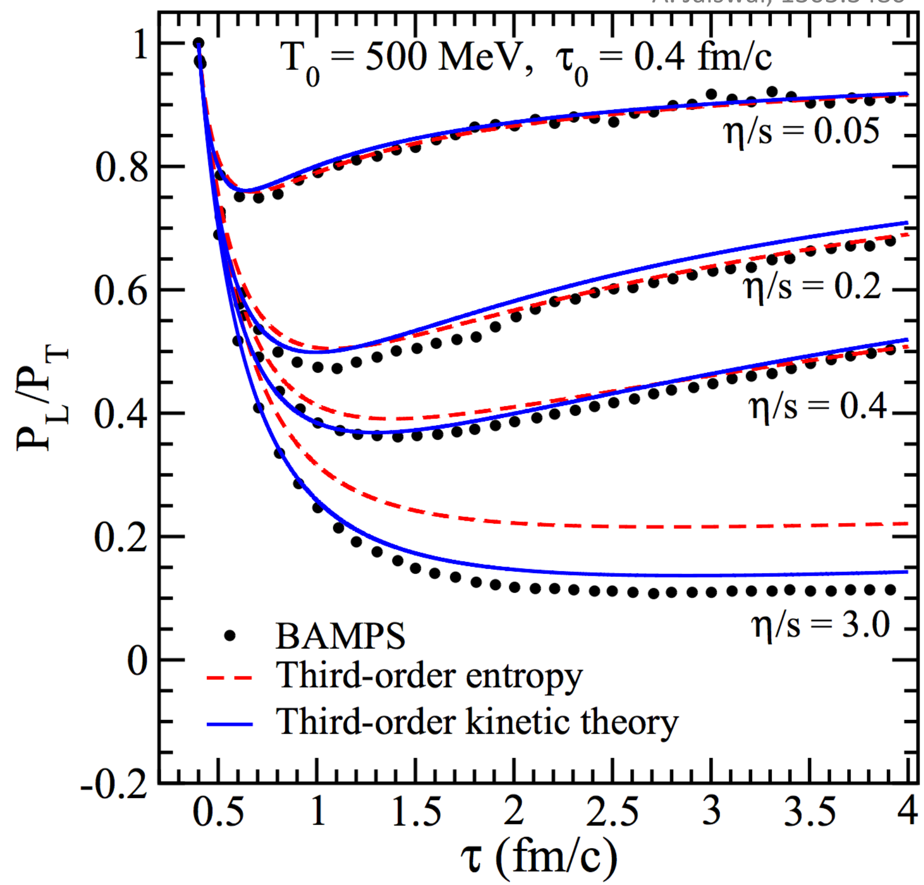
- 0^{th} and 1^{st} moments of Boltzmann equation → conservation laws
- Used a general tensor basis that can be extended to more general cases (next lecture)
- Used symmetries of system to restrict form of energy-momentum tensor
- For ideal hydro energy conservation equations are enough
- In the next lecture we allow deviations from the ideal form

Backup Slides

Kinetic theory vs Hydro

- Kinetic theory can be pushed beyond it's range of applicability and still has good agreement with hydrodynamic evolution!
- This is typical of a "good theory" in that, although it has some a priori limits, it can actually be applied further into the "forbidden zone" than one would naively guess.
- Right plot shows comparison of 3rd order viscous hydro results with a kinetic transport code with a tuned cross section.

A. Jaiswal, 1305.3480



BAMPS: A. El, Z. Xu and C. Greiner, Nucl. Phys. A 806, 287 (2008).

Come ye of little faith …

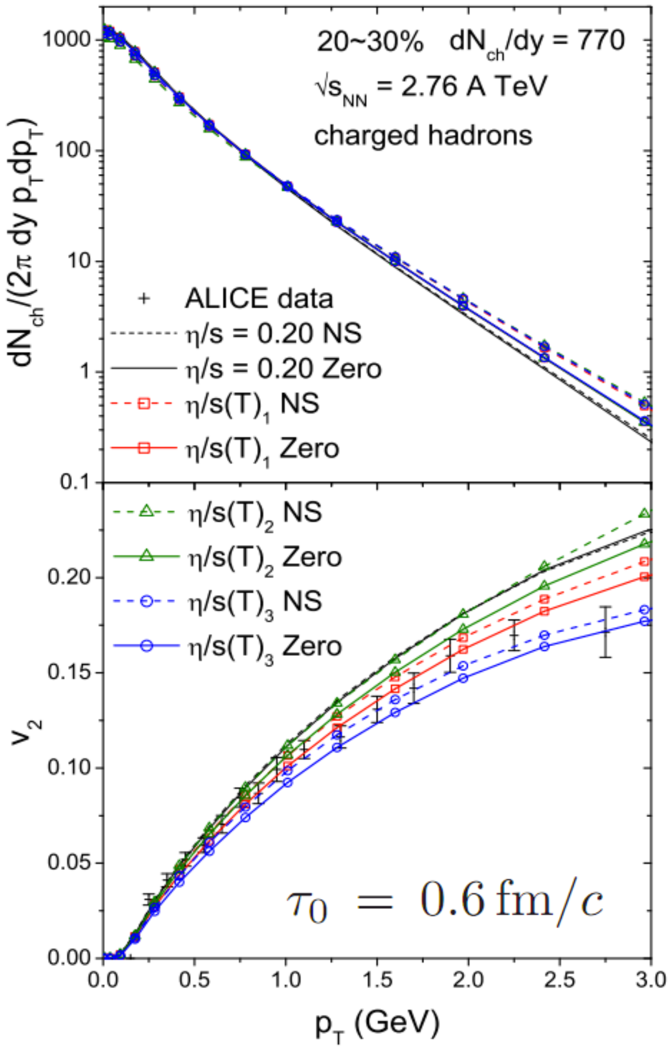
- For a long time it was taken as "gospel" that agreement with HIC experimental data for elliptic flow requires early isotropization at times on the order of 0.5 fm/c.

- Is this true?

Radial and elliptic flow in Pb+Pb collisions at the Large Hadron Collider from viscous hydrodynamics

arXiv:1105.3226v2 [nucl-th] 9 Sep 2011

Chun Shen,[1,*] Ulrich Heinz,[1,†] Pasi Huovinen,[2,‡] and Huichao Song[3,§]



η/s model	$\pi_0^{\mu\nu}$	s_0 (fm^{-3})	T_0 (MeV)
$\eta/s = 0.2$	0	191.6	427.9
	NS	172.4	413.9
$(\eta/s)_1(T)$	0	179.6	419.2
	NS	119.3	368.7
$(\eta/s)_2(T)$	0	179.6	419.2
	NS	115.6	365.1
$(\eta/s)_3(T)$	0	175.2	416.0
	NS	116.6	366.1

- Considered two different initial conditions for the shear tensor
- Isotropic and Navier-Stokes (NS)
- For NS at the initial time shown, the longitudinal pressure is negative!

Radial and elliptic flow in Pb+Pb collisions at the Large Hadron Collider from viscous hydrodynamics

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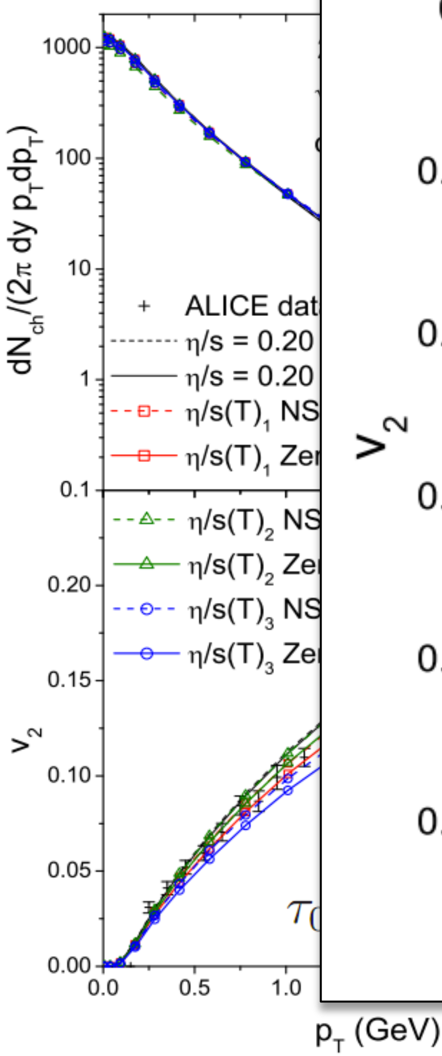
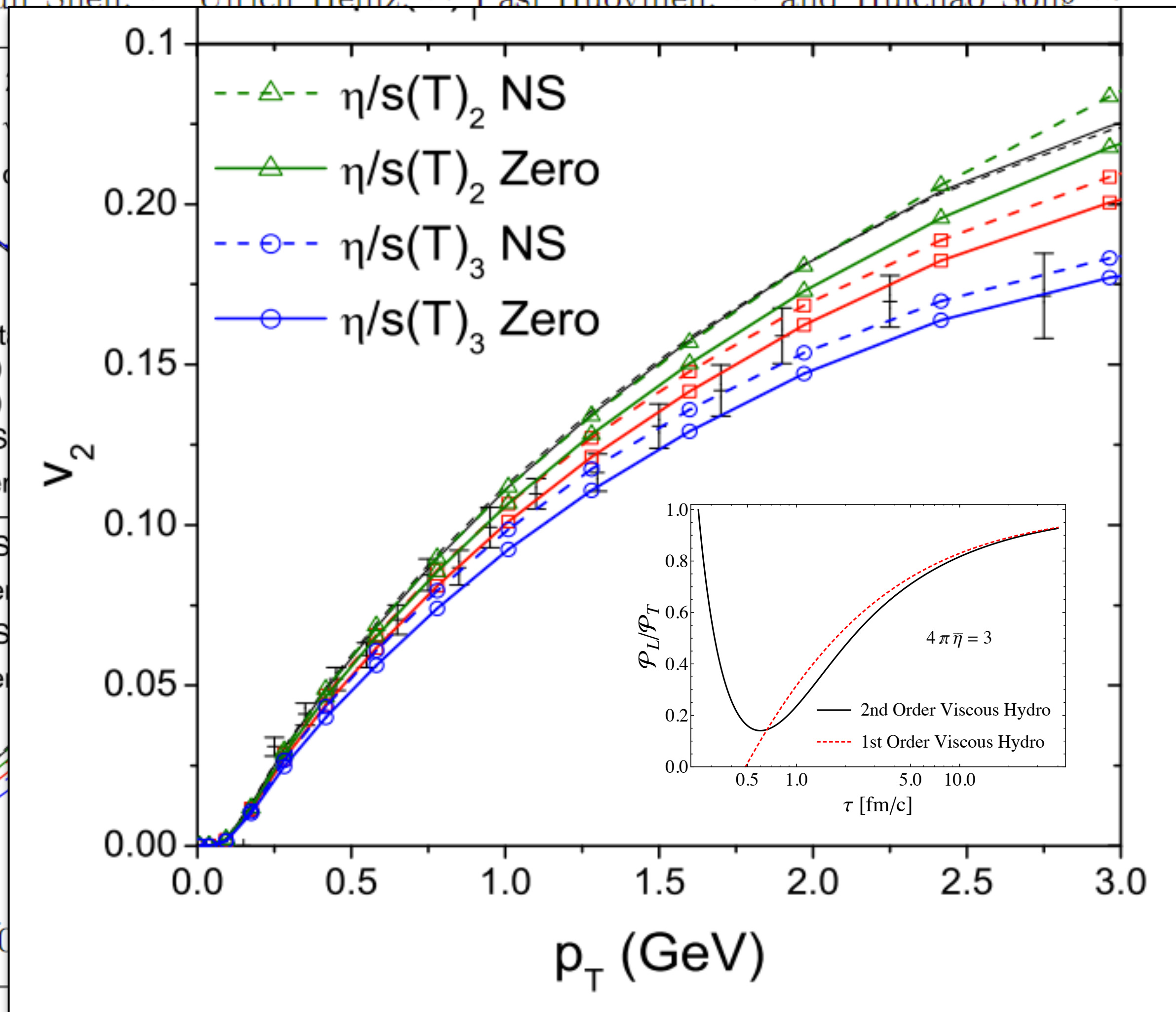




T_0 (MeV)
427.9
413.9
419.2
368.7
419.2
365.1
416.0
366.1

itial
nsor
s (NS)
own,

General Tensor Basis II

- We need an expression for $T^{\mu\nu}$
- A general rank-2 tensor can be written as

$$A^{\mu\nu}(t,\mathbf{x}) = \sum_{\alpha,\beta=0}^{3} c_{\alpha\beta} X^\mu_\alpha X^\nu_\beta \,,$$

$$g^{\mu\nu} = X^\mu_0 X^\nu_0 - \sum_{i=1}^{3} X^\mu_i X^\nu_i \,.$$

$$= c_{00} X^\mu_0 X^\nu_0 + \sum_{i=1}^{3} c_{ii} X^\mu_i X^\nu_i + \sum_{\substack{\alpha,\beta=0 \\ \alpha\neq\beta}}^{3} c_{\alpha\beta} X^\mu_\alpha X^\nu_\beta \,,$$

$$= c_{00} g^{\mu\nu} + \sum_{i=1}^{3} \underbrace{(c_{ii} + c_{00})}_{\equiv d_{ii}} X^\mu_i X^\nu_i + \sum_{\substack{\alpha,\beta=0 \\ \alpha\neq\beta}}^{3} c_{\alpha\beta} X^\mu_\alpha X^\nu_\beta \,,$$

Anisotropic Hydrodynamics Lecture 2

Michael Strickland
Kent State University

2014 Cracow School of Theoretical Physics
QCD meets experiment

Lecture 2

- 1st and 2nd Order Viscous Hydro
- Limitations of Viscous Hydro
- Spheroidal Distribution
- Anisotropic $T^{\mu\nu}$
- Anisotropic Hydro (aHydro) Equations
- 0+1d Limit and connection to Viscous Hydro
- 2+1d LO spheroidal aHydro

Ideal Hydrodynamics Equations

Second Ideal Hydro EQ

$$\partial_\mu(\mathcal{E}+\mathcal{P})u^\mu u^\nu + (\mathcal{E}+\mathcal{P})(u^\nu\partial_\mu u^\mu + u^\mu\partial_\mu u^\nu) - g^{\mu\nu}\partial_\mu\mathcal{P} = 0$$

$\Delta^\alpha{}_\nu$:

Spatial Gradient $\nabla^\alpha \equiv \Delta^\alpha{}_\nu\partial^\nu = -\sum_{\beta=1}^{3} X^\alpha_\beta X_{\nu\beta}\partial^\nu \, ,$

$$Dn + n\theta = 0 \qquad \text{(Eq 1)}$$
$$D\mathcal{E} + (\mathcal{E}+\mathcal{P})\theta = 0 \qquad \text{(Eq 2)}$$
$$(\mathcal{E}+\mathcal{P})Du^i - \nabla^i\mathcal{P} = 0 \qquad \text{(Eq 3)}$$

$$D \equiv u^\mu\partial_\mu$$
$$\theta \equiv \partial_\mu u^\mu$$

Degrees of Freedom
1 : Particle number
1 : Energy Density
1 : Pressure
3 : Independent components of u^μ
6 : Total

Equations
1 : (Eq 1)
1 : (Eq 2)
3 : (Eq 3) $i = 1, 2, 3$
1 : EQUATION OF STATE $T^\mu{}_\mu = \#$
6 : Total

The Stress Tensor I

$$\boxed{T^{\mu\nu} = T^{\mu\nu}_{\rm ideal} + \Pi^{\mu\nu}}$$

$$A_{(\mu}B_{\nu)} = \frac{1}{2}\left(A_\mu B_\nu + A_\nu B_\mu\right)$$

$$D\mathcal{E} + (\mathcal{E}+\mathcal{P})\theta - \Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = 0$$
$$(\mathcal{E}+\mathcal{P})Du^i - \nabla^i\mathcal{P} + \Delta^i{}_\nu\partial_\mu\Pi^{\mu\nu} = 0$$

Gives four equations which depend on $\Pi^{\mu\nu}$

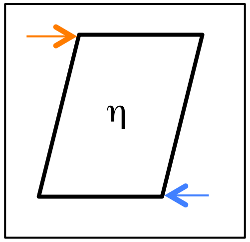
Can decompose $\Pi^{\mu\nu}$ into a traceless part ($\pi^{\mu\nu}$) and a remainder (Φ)

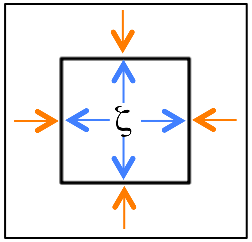
Approximation: 1st order in gradients of u^ν → Relativistic Navier-Stokes

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Delta^{\mu\nu}\Phi \qquad \pi^\mu{}_\mu = 0$$
$$\pi^{\mu\nu} = \eta\nabla^{\langle\mu}u^{\nu\rangle} \qquad \Phi = \zeta\nabla_\alpha u^\alpha$$

η = Shear Viscosity

ζ = Bulk Viscosity





$$\nabla^{\langle\mu}u^{\nu\rangle} \equiv 2\nabla^{(\mu}u^{\nu)} - \frac{2}{3}\Delta^{\mu\nu}\nabla_\alpha u^\alpha$$

Angle brackets project out traceless symmetric part

The Stress Tensor II

$$\boxed{T^{\mu\nu} = T^{\mu\nu}_{\rm ideal} + \Pi^{\mu\nu}}$$

$$D\mathcal{E} + (\mathcal{E}+\mathcal{P})\theta - \Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = 0$$

$$(\mathcal{E}+\mathcal{P})Du^i - \nabla^i\mathcal{P} + \Delta^i{}_\nu\partial_\mu\Pi^{\mu\nu} = 0$$

$$u_\mu T^{\mu\nu} = \mathcal{E}u^\nu \longrightarrow u_\mu\Pi^{\mu\nu} = 0$$ Landau frame

$$\Pi^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Pi^{xx} & \Pi^{xy} & \Pi^{xz} \\ 0 & \cdot & \Pi^{yy} & \Pi^{yz} \\ 0 & \cdot & \cdot & \Pi^{zz} \end{pmatrix}$$

- How many unknowns?
- 6 DOF in $\Pi^{\mu\nu}$ (5 $\pi^{\mu\nu}$ + 1 Φ)
- 3 DOF in number flow ν^i
- 6 DOF in Ideal Hydro contribution (n, E, P, and u^i)
- 15 DOF Total
- 5 ideal equations + 1 EOS
- Need 9 more equations!
- Only 6 if $\nu^i = 0$. ☺
- Can use the 2nd moment of the kinetic equations

The Kinetic Energy-Momentum Tensor

- Starting from QFT and evaluating statistical averages $<T^{\mu\nu}>$.
- Write in terms of the one-particle distribution functions

$$\langle \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})\rangle = \frac{1}{e^{\beta(E-\mu)}-1} \equiv f_B\,,$$

$$\langle b_s^\dagger(\mathbf{k})b_s(\mathbf{k})\rangle = \frac{1}{e^{\beta(E-\mu)}+1} \equiv f_F^+(E,T,\mu)\,,$$

$$\langle d_s^\dagger(\mathbf{k})d_s(\mathbf{k})\rangle = \frac{1}{e^{\beta(E+\mu)}+1} \equiv f_F^-(E,T,\mu)\,,$$

- Energy momentum tensor can be expressed as invariant phase space integral

$$\boxed{T^{\mu\nu}(x) = \int dP\, p^\mu p^\nu f(x,p)}$$

$$\int dP \equiv \int \frac{d^4\mathbf{p}}{(2\pi)^3}\delta(p^\mu p_\mu - m^2)\,2\theta(p^0) = \int_{\mathbf{p}} \frac{1}{E}$$

Connection to Viscous Hydro

For small departures from equilibrium we can linearize

$$\boxed{f(x,p) = f_{\rm eq}\!\left(\frac{p^\mu u_\mu}{T}\right)(1+\delta f(x,p))}$$

$$\boxed{\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm ideal} + \int dP\; p^\mu p^\nu f_{\rm eq}\,\delta f \\ &\equiv T^{\mu\nu}_{\rm ideal} + \Pi^{\mu\nu} \end{aligned}} \longrightarrow \boxed{\Pi^{\mu\nu} = \int dP\; p^\mu p^\nu f_{\rm eq}\,\delta f}$$

For viscous hydro one expands δf in a gradient expansion: $n^{\rm th}$ order in gradients → $n^{\rm th}$-order viscous Hydro

- $1^{\rm st}$ order Hydro : Relativistic Navier-Stokes (parabolic diff eqs → acausal) [e.g. Eckart and Landau-Lifshitz]
- $2^{\rm nd}$ order Hydro : Including quadratic gradients fixes causality problem; hyperbolic diff eqs
  [e.g. Israel-Stewart]
- …

1^{st} Order Hydro

- Expand kinetic equations to first order in gradients.

Approximation: 1^{st} order in gradients of u^ν → Relativistic Navier-Stokes

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Delta^{\mu\nu}\Phi \qquad \pi^{\mu}{}_{\mu} = 0$$

$$\pi^{\mu\nu} = \eta\nabla^{\langle\mu}u^{\nu\rangle} \qquad \Phi = \zeta\nabla_\alpha u^\alpha$$

η = Shear Viscosity

ζ = Bulk Viscosity

$$\nabla^{\langle\mu}u^{\nu\rangle} \equiv 2\nabla^{(\mu}u^{\nu)} - \frac{2}{3}\Delta^{\mu\nu}\nabla_\alpha u^\alpha$$

- For now, I will ignore the bulk viscosity (it will come back later!)
- If $f_{\rm eq}$ is a Boltzmann distribution one finds

$$\boxed{f(x,p) = f_{\rm eq}\left(\frac{p^\mu u_\mu}{T}\right)\left[1 + \frac{p^\alpha p^\beta \pi_{\alpha\beta}}{2(\mathcal{E}+\mathcal{P})T^2}\right]}$$

1st Order Hydro – 0+1d

$$u^\mu = (\cosh\varsigma, 0, 0, \sinh\varsigma) = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right)$$

$$\rightarrow \pi^{\mu\nu} = \eta \nabla^{\langle\mu} u^{\nu\rangle}$$

$$\nabla^{\langle\mu} u^{\nu\rangle} \equiv 2\nabla^{(\mu} u^{\nu)} - \frac{2}{3}\Delta^{\mu\nu}\nabla_\alpha u^\alpha$$

$$\pi^{xx} = \eta\left(2\underbrace{\nabla^{(x} u^{x)}}_{0} - \frac{2}{3}\underbrace{\Delta^{xx}}_{-1}\underbrace{\partial_\mu u^\mu}_{1/\tau}\right) = \frac{2\eta}{3\tau} = \pi^{yy}$$

$$\pi^{zz} = -(\pi^{xx} + \pi^{yy}) = -\frac{4\eta}{3\tau}$$

$$\mathcal{P}_T \equiv \mathcal{P}_{\rm eq} + \pi^{xx} = \mathcal{P}_{\rm eq} + \frac{2\eta}{3\tau}$$

$$\mathcal{P}_L \equiv \mathcal{P}_{\rm eq} + \pi^{zz} = \mathcal{P}_{\rm eq} - \frac{4\eta}{3\tau}$$

$$\boxed{\mathcal{P}_T \neq \mathcal{P}_L}$$

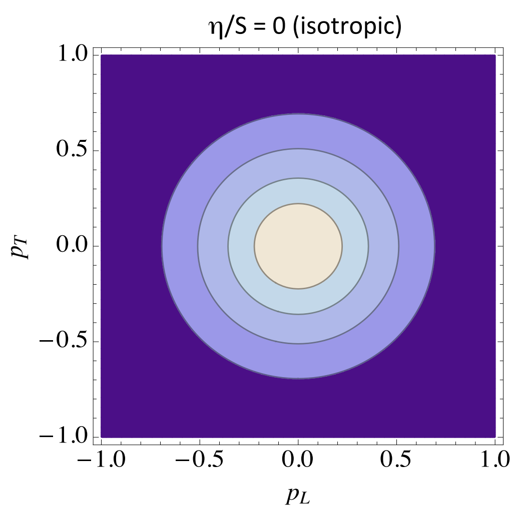
Longitudinal pressure can become negative!

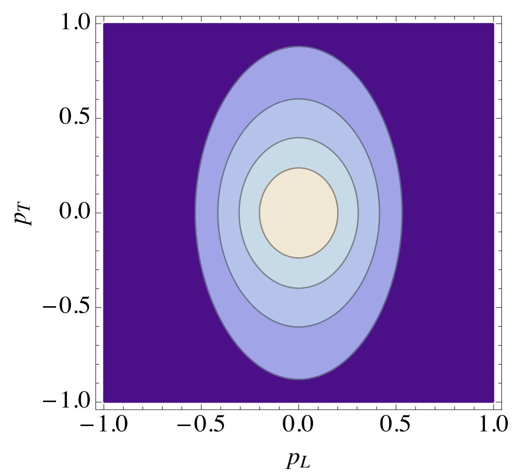
1st Order Hydro – 0+1d

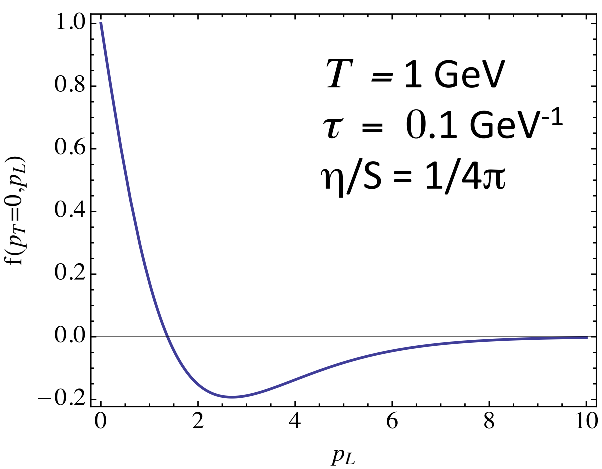
Additionally one finds for the first order distribution function

$$f(x,p) = f_{\rm eq}\left(\frac{p^\mu u_\mu}{T}\right)\left[1 + \frac{p^\alpha p^\beta \pi_{\alpha\beta}}{2(\mathcal{E}+\mathcal{P})T^2}\right] \longrightarrow f_{\rm eq}\left(\frac{E}{T}\right)\left[1 + \frac{\eta}{\mathcal{S}}\frac{p_x^2+p_y^2-2p_z^2}{3\tau T^3}\right]$$

- Distribution function becomes anisotropic in momentum space
- There are also regions where $f(x,p) < 0$
- Anisotropy and regions of negativity increase as τ or T decrease OR η/S increases







1st Order Hydro – 0+1d

Additionally one finds for the first order distribution function

$$\boxed{f(x,p) = f_{\rm eq}\left(\frac{p^\mu u_\mu}{T}\right)\left[1 + \frac{p^\alpha p^\beta \pi_{\alpha\beta}}{2(\mathcal{E}+\mathcal{P})T^2}\right] \longrightarrow f_{\rm eq}\left(\frac{E}{T}\right)\left[1 + \frac{\eta}{\mathcal{S}}\frac{p_x^2+p_y^2-2p_z^2}{3\tau T^3}\right]}$$

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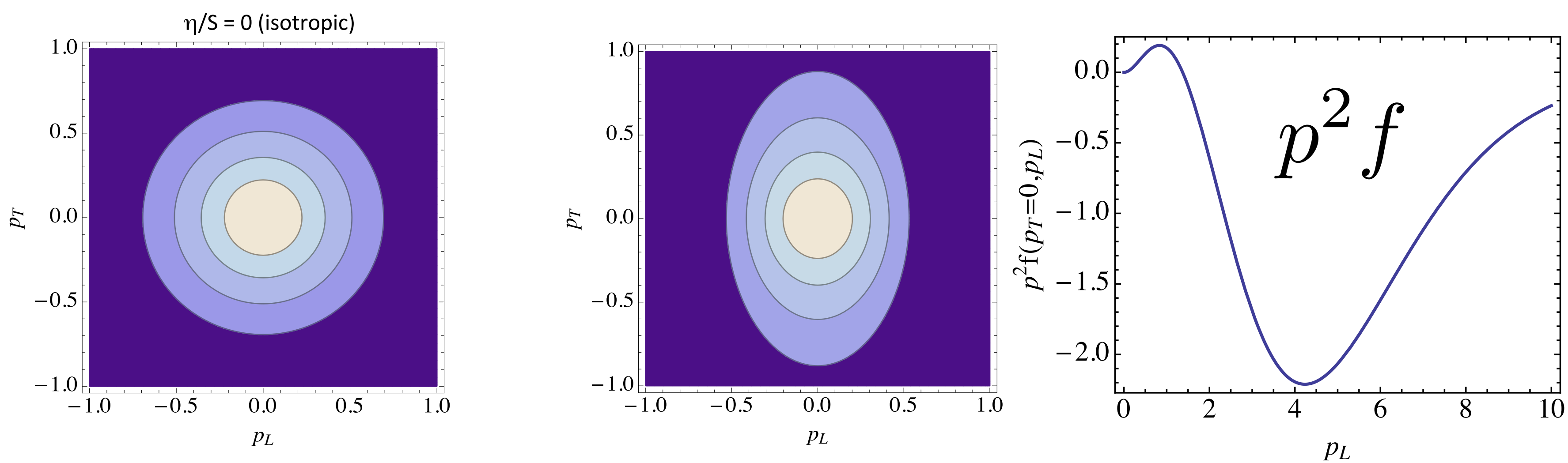


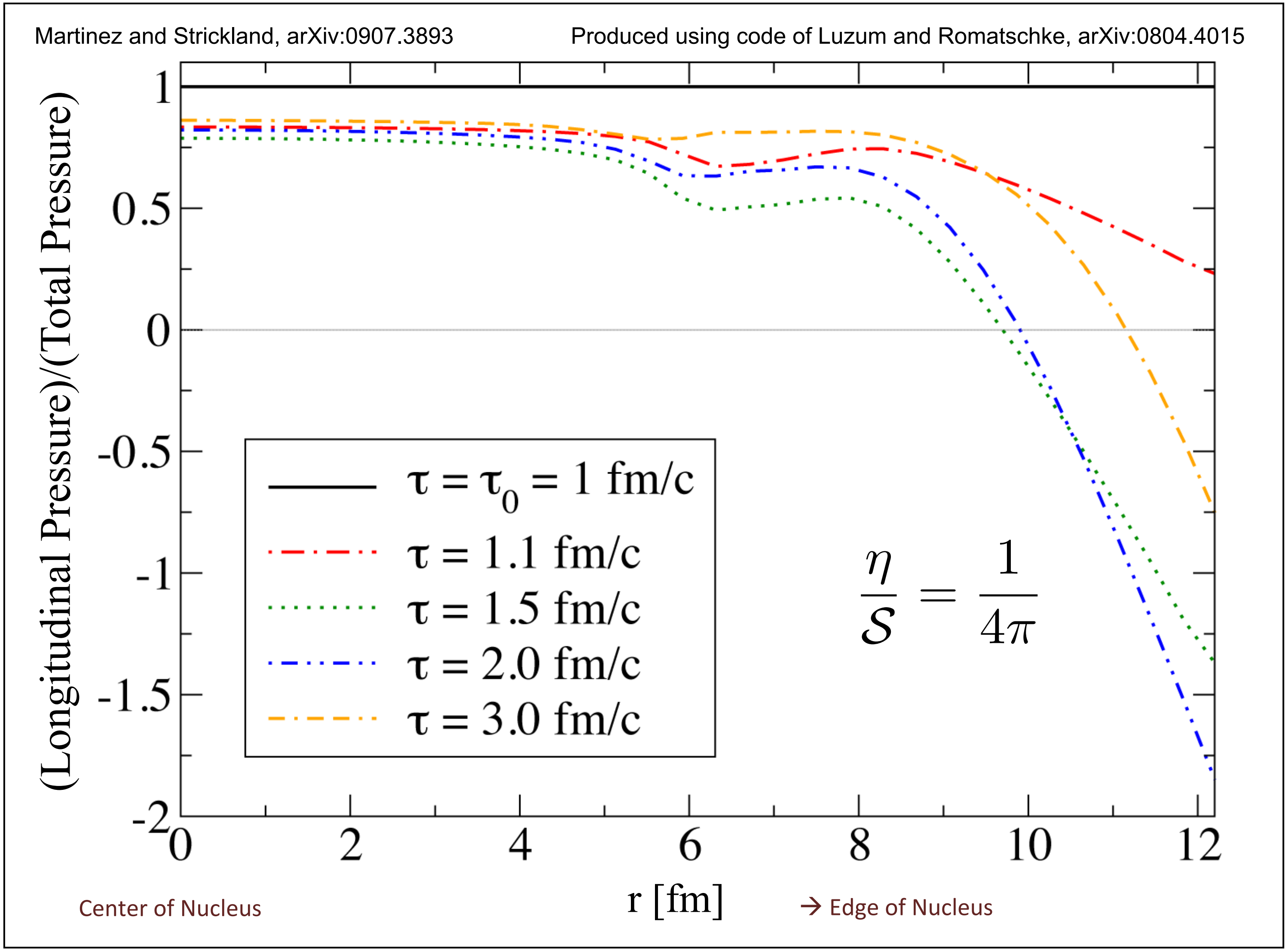
2^{nd} Order Hydro

- Expand 2^{nd} moment of the Boltzmann Eq to 2^{nd} order in gradients

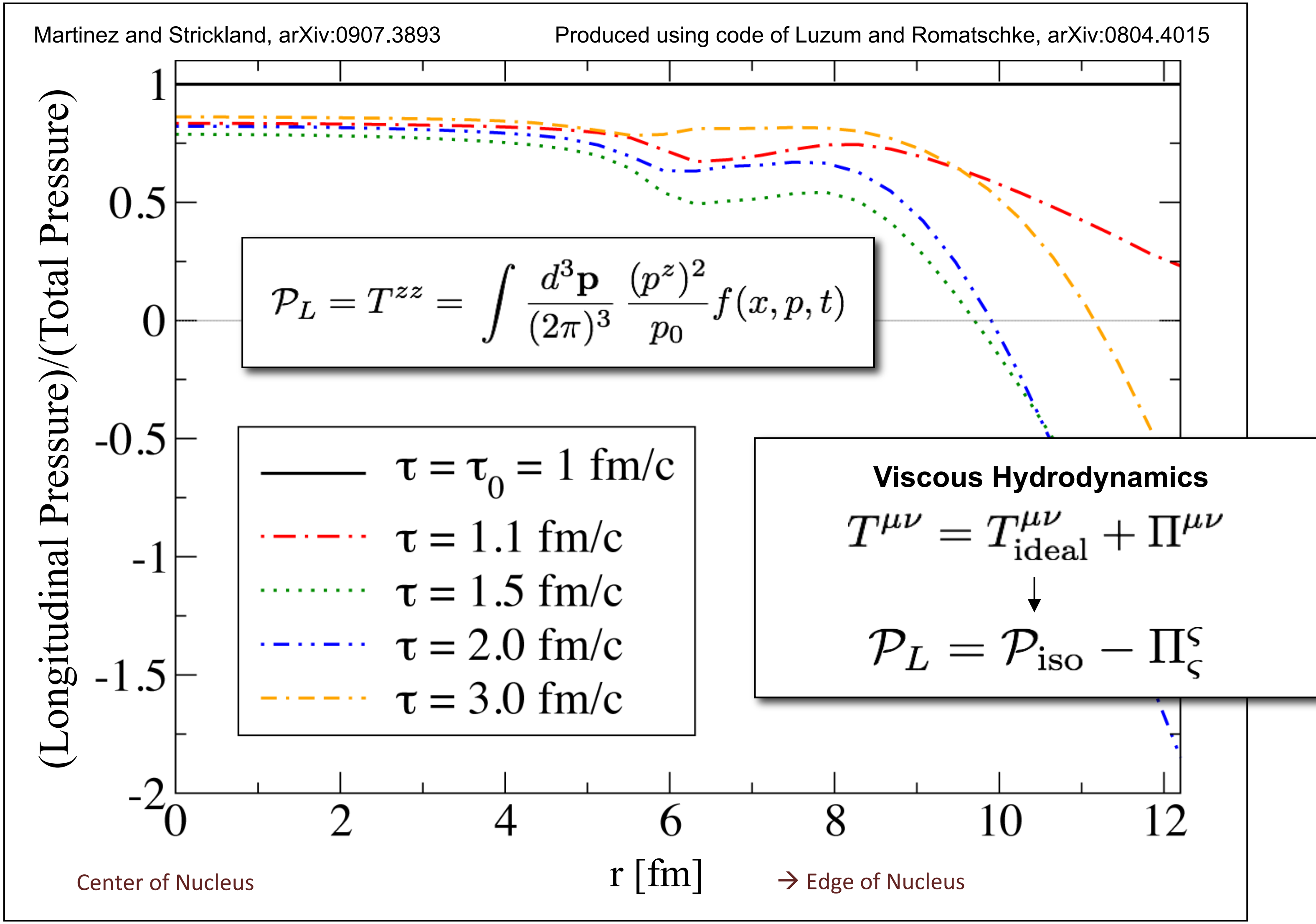
$$\boxed{\pi^{\mu\nu} + \tau_\pi \left[\Delta^\mu_\alpha \Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha - 2\pi^{\phi(\mu}\Omega^{\nu)}{}_\phi + \frac{\pi^{\phi<\mu}\pi^{\nu>}_\phi}{2\eta} \right] = \eta \nabla^{<\mu}u^{\nu>}}$$

- New structure called "vorticity" appears $\boxed{\Omega_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)}$
- New time scale called the shear relaxation time, τ_π, appears
- Equations are now causal!
- τ_π sets timescale to approach Navier-Stokes evolution
- $\tau_\pi \sim 5\eta/(TS) \sim$ 0.1 - 0.4 fm/c at RHIC/LHC temperatures
- If we set $\tau_\pi = 0$ above, we recover the Navier-Stokes limit

2^{nd} Order Hydro Results - Strong Coupling



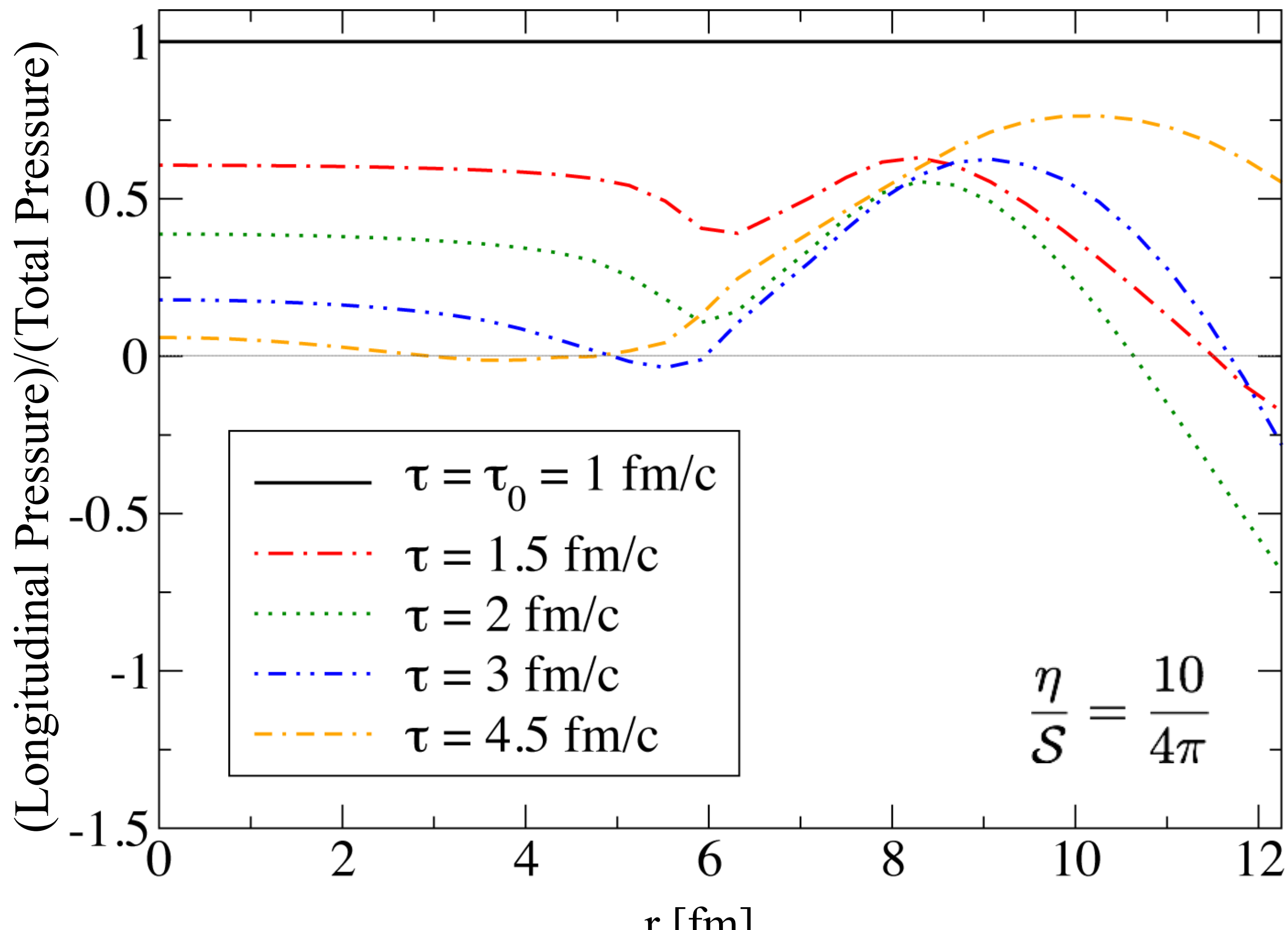
2^{nd} Order Hydro Results - Strong Coupling



2^{nd} Order Hydro Results - Weak Coupling

Martinez and Strickland, arXiv:0907.3893

Produced using code of Luzum and Romatschke, arXiv:0804.4015



Center of Nucleus → Edge of Nucleus

Anisotropic Hydrodynamics Basics

M. Martinez and MS, 1007.0889
W. Florkowski and R. Ryblewski, 1007.0130

Viscous Hydrodynamics Expansion

$$f(\tau,\mathbf{x},\mathbf{p}) = \underbrace{f_{\rm eq}(\mathbf{p},T(\tau,\mathbf{x}))}_{\text{Isotropic in momentum space}} + \delta f$$

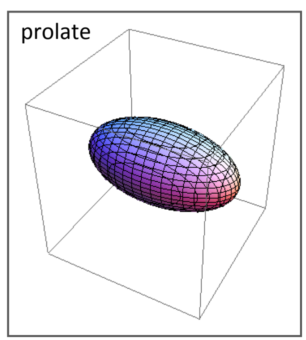
Anisotropic Hydrodynamics Expansion

$$f(\tau,\mathbf{x},\mathbf{p}) = f_{\rm aniso}(\mathbf{p},\underbrace{\Lambda(\tau,\mathbf{x})}_{T_\perp},\underbrace{\xi(\tau,\mathbf{x})}_{\rm anisotropy}) + \delta\tilde{f}$$

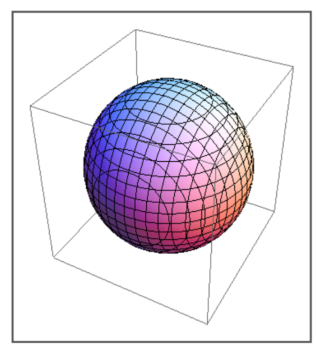
→ "Romatschke-Strickland" form in LRF

$$\boxed{f^{LRF}_{\rm aniso} = f_{\rm iso}\left(\frac{\sqrt{\mathbf{p}^2+\xi(\mathbf{x},\tau)p_z^2}}{\Lambda(\mathbf{x},\tau)}\right)}$$

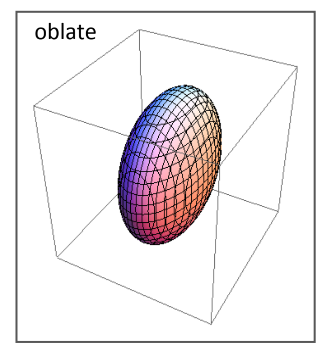
$$\boxed{\xi = \frac{\langle p_T^2\rangle}{2\langle p_L^2\rangle} - 1}$$



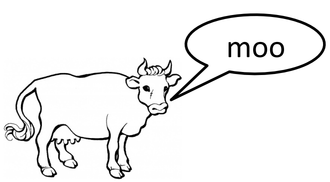
$-1 < \xi < 0$



$\xi = 0$



$\xi > 0$



- Leading Order (LO) Anisotropic Hydrodynamics -

Anisotropic Hydrodynamics Basics

M. Martinez and MS, 1007.0889
W. Florkowski and R. Ryblewski, 1007.0130

Viscous Hydrodynamics Expansion

$$f(\tau,\mathbf{x},\mathbf{p}) = \underbrace{f_{\rm eq}(\mathbf{p},T(\tau,\mathbf{x}))}_{\text{Isotropic in momentum space}} + \delta f$$

Anisotropic Hydrodynamics Expansion

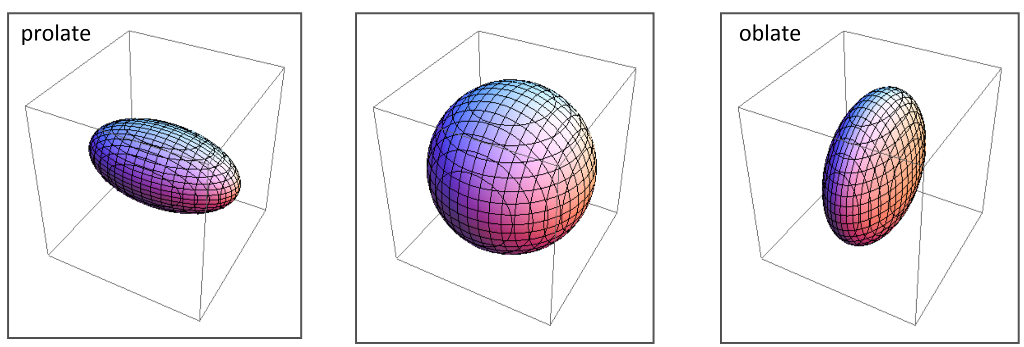
$$f(\tau,\mathbf{x},\mathbf{p}) = f_{\rm aniso}(\mathbf{p},\underbrace{\Lambda(\tau,\mathbf{x})}_{T_\perp},\underbrace{\xi(\tau,\mathbf{x})}_{\text{anisotropy}}\,) + \cancel{\delta\tilde{f}}$$

First, let's consider what happens when we ignore this term…

→ "Romatschke-Strickland" form in LRF

$$\boxed{f^{LRF}_{\rm aniso} = f_{\rm iso}\left(\frac{\sqrt{\mathbf{p}^2+\xi(\mathbf{x},\tau)p_z^2}}{\Lambda(\mathbf{x},\tau)}\right)}$$

$$\boxed{\xi = \frac{\langle p_T^2\rangle}{2\langle p_L^2\rangle} - 1}$$



$-1 < \xi < 0$ $\quad$ $\xi = 0$ $\quad$ $\xi > 0$

Why spheroidal form at LO?

- What is special about this form at leading order? Can I choose any background distribution I like as the expansion point?

$$f_{\rm aniso}^{LRF} = f_{\rm iso}\left(\frac{\sqrt{\mathbf{p}^2 + \xi(\mathbf{x},\tau)p_z^2}}{\Lambda(\mathbf{x},\tau)}\right)$$

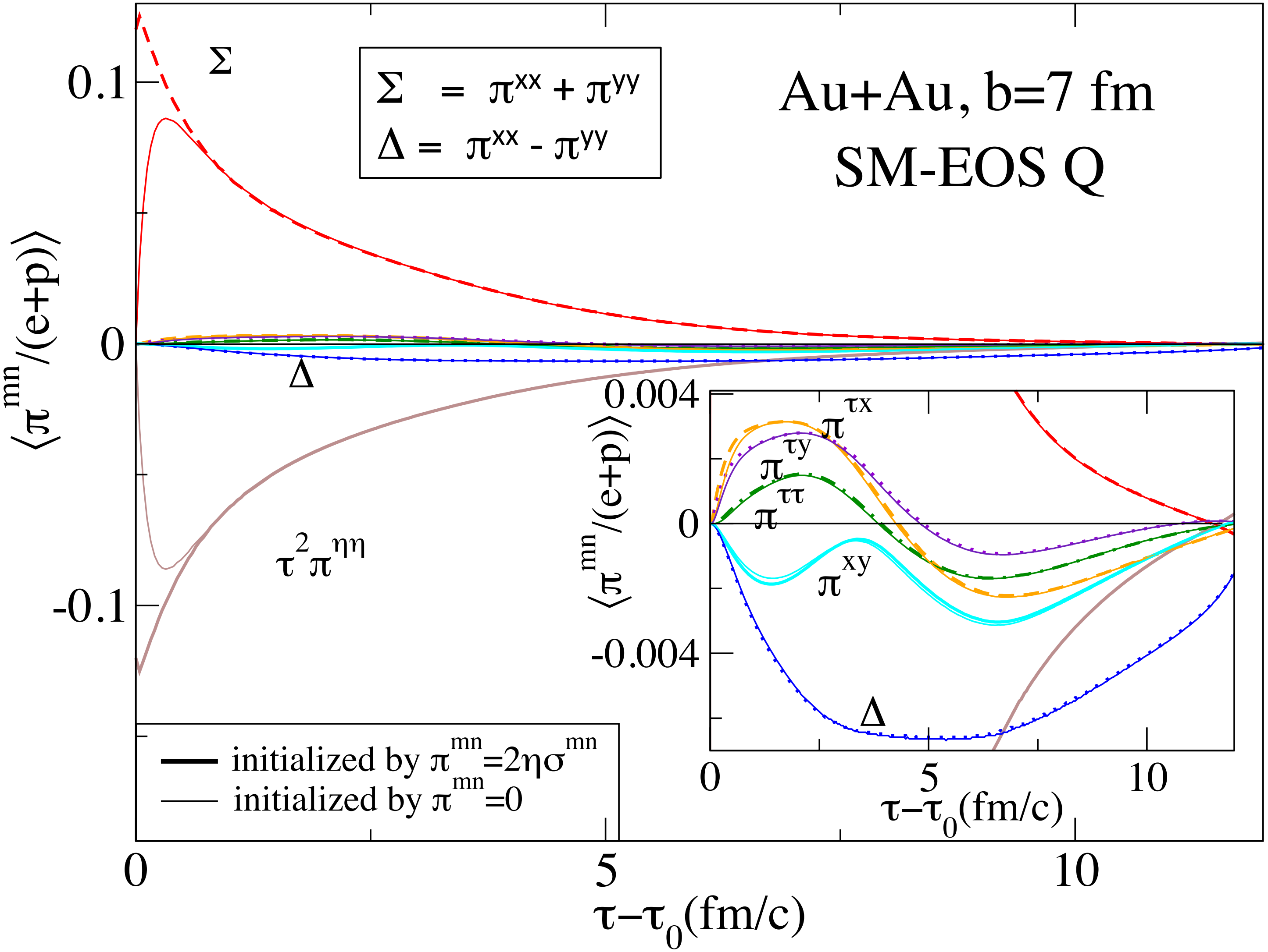
- Obviously can describe the ideal hydro limit when $\xi=0$ ($\Lambda \rightarrow$ T)
- For longitudinal (0+1d) free streaming, the Boltzmann equation can be solved analytically → LRF distribution function is of spheroidal form with

$$\xi_{\rm FS}(\tau) = (1+\xi_0)\left(\frac{\tau}{\tau_0}\right)^2 - 1$$

- Could also use ellipsoidal form etc (more discussion on this point later), but the spheroidal form is the simplest form that captures the largest components of the energy-momentum tensor (see next slide)
- Since $f_{iso} \geq 0$, the one-particle distribution function and pressures are guaranteed to be ≥ 0 (not true in viscous hydro)

Hints from Viscous Hydro

H. Song, PhD Dissertation, 0908.3656

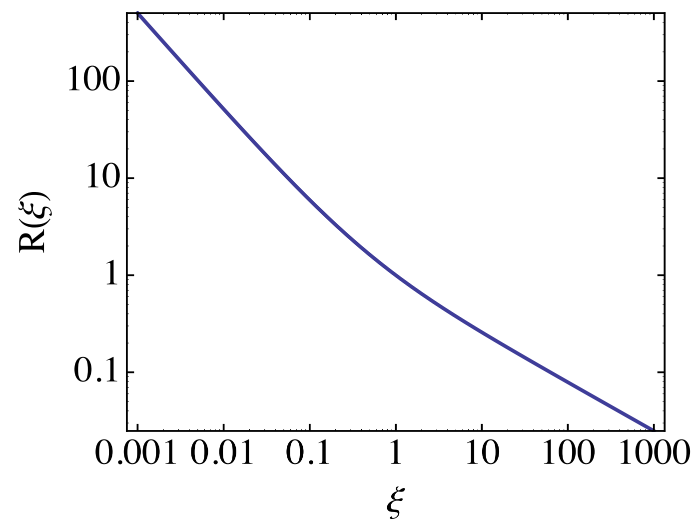


LO Spheroidal Distribution

- Consider a conformal system to start with
- In the conformal (massless) limit all bulk observables factorize into a product of two functions

$$n(\Lambda,\xi) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} f_{\rm aniso} = \frac{n_{\rm iso}(\Lambda)}{\sqrt{1+\xi}}$$
$$\mathcal{E}(\Lambda,\xi) = T^{\tau\tau} = \mathcal{R}(\xi)\,\mathcal{E}_{\rm iso}(\Lambda)$$
$$\mathcal{P}_\perp(\Lambda,\xi) = \frac{1}{2}\left(T^{xx}+T^{yy}\right) = \mathcal{R}_\perp(\xi)\mathcal{P}_{\rm iso}(\Lambda)$$
$$\mathcal{P}_L(\Lambda,\xi) = -T^{\varsigma}_{\varsigma} = \mathcal{R}_{\rm L}(\xi)\mathcal{P}_{\rm iso}(\Lambda)$$

$$\mathcal{R}(\xi) \equiv \frac{1}{2}\left(\frac{1}{1+\xi} + \frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)$$
$$\mathcal{R}_\perp(\xi) \equiv \frac{3}{2\xi}\left(\frac{1+(\xi^2-1)\mathcal{R}(\xi)}{\xi+1}\right)$$
$$\mathcal{R}_L(\xi) \equiv \frac{3}{\xi}\left(\frac{(\xi+1)\mathcal{R}(\xi)-1}{\xi+1}\right)$$



Azimuthally symmetric $T^{\mu\nu}$

$$T^{\mu\nu}(t,\mathbf{x}) = t_{00} g^{\mu\nu} + \sum_{i=1}^{3} t_{ii} X_i^\mu X_i^\nu + \sum_{\substack{\alpha,\beta=0 \\ \alpha>\beta}}^{3} t_{\alpha\beta}(X_\alpha^\mu X_\beta^\nu + X_\beta^\mu X_\alpha^\nu)\,,$$

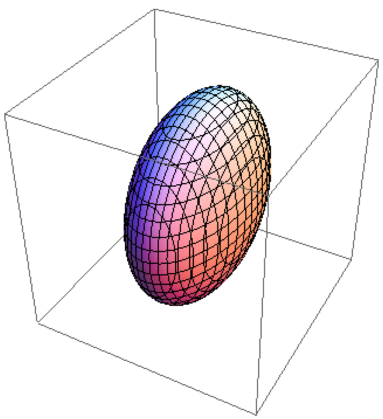
$$T^{00}_{\rm LRF} = \mathcal{E} = t_{00}\,,$$

$$T^{xx}_{\rm LRF} = \mathcal{P}_\perp = -t_{00} + t_{11}\,,$$

$$T^{yy}_{\rm LRF} = \mathcal{P}_\perp = -t_{00} + t_{22}\,,$$

$$T^{zz}_{\rm LRF} = \mathcal{P}_L = -t_{00} + t_{33}\,,$$

Assume, at leading order, rotational symmetry around p_z-axis in LRF



$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P}_\perp) u^\mu u^\nu - \mathcal{P}_\perp g^{\mu\nu} + (\mathcal{P}_L - \mathcal{P}_\perp) z^\mu z^\nu\,,$$

L. Satarov et al, hep-ph/0611099 R. Ryblewski and W. Florkowski, 1103.1260 M. Martinez, R. Ryblewski, and MS, 1204.1473

0+1d case – new Bjorken eqs

0th Moment of Boltzmann EQ

M. Martinez and MS, 1007.0889

$$\partial_\alpha N^\alpha \neq 0$$

$$\boxed{\frac{1}{1+\xi}\partial_\tau \xi - \frac{2}{\tau} - 6\,\partial_\tau \log\Lambda = 2\Gamma\left[1 - \mathcal{R}^{3/4}(\xi)\sqrt{1+\xi}\right]}$$

1st Moment of Boltzmann EQ

$$\partial_\alpha T^{\alpha\beta} = 0$$

$$\boxed{\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau \xi + 4\,\partial_\tau \log\Lambda = \frac{1}{\tau}\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1\right]}$$

Where (original M-S prescription)

$$\boxed{\Gamma = \frac{2T(\tau)}{5\bar{\eta}} = \frac{2\mathcal{R}^{1/4}(\xi)\Lambda}{5\bar{\eta}}}$$

$$\boxed{\begin{aligned}\mathcal{R}(\xi) &\equiv \frac{1}{2}\left(\frac{1}{1+\xi} + \frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)\\ \mathcal{E}(\Lambda,\xi) &= \mathcal{R}(\xi)\mathcal{E}_{\rm iso}(\Lambda)\end{aligned}}$$

Linearized Equations

If we expand the energy-momentum tensor to linear order in the anisotropy parameter and match to 2nd-order viscous hydro, we find

$$\frac{\Pi}{\mathcal{E}_{\rm eq}} = \frac{8}{45}\xi + \mathcal{O}(\xi^2)$$

If we similarly expand the coupled nonlinear differential equations to lowest order in the anisotropy parameter and rewrite in terms of the shear using the relation above, we obtain

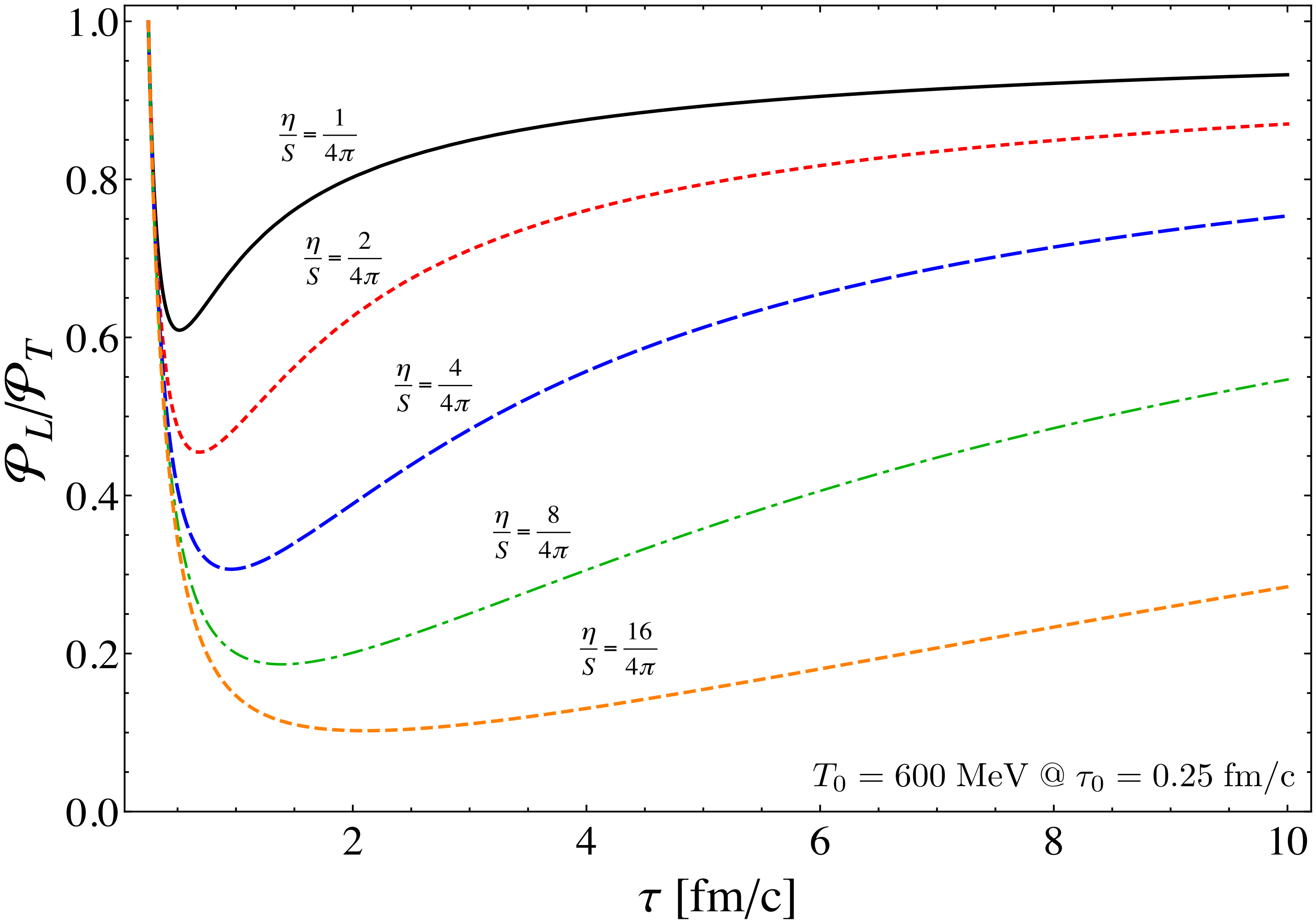
$$\boxed{\begin{aligned} \partial_\tau \mathcal{E} &= -\frac{\mathcal{E}+\mathcal{P}}{\tau} + \frac{\Pi}{\tau} \\ \partial_\tau \Pi &= -\frac{\Pi}{\tau_\pi} + \frac{4}{3}\frac{\eta}{\tau_\pi \tau} - \frac{4}{3}\frac{\Pi}{\tau} \end{aligned}} \qquad \boxed{\begin{aligned} \Gamma &= \frac{2}{\tau_\pi} \\ \tau_\pi &= \frac{5}{4}\frac{\eta}{\mathcal{P}} \end{aligned}}$$

- **Reproduces 2nd-order viscous hydro in the small anisotropy limit!**
- **<u>Also correctly describes the free streaming limit!</u> (not shown here)**

Pressure Anisotropy

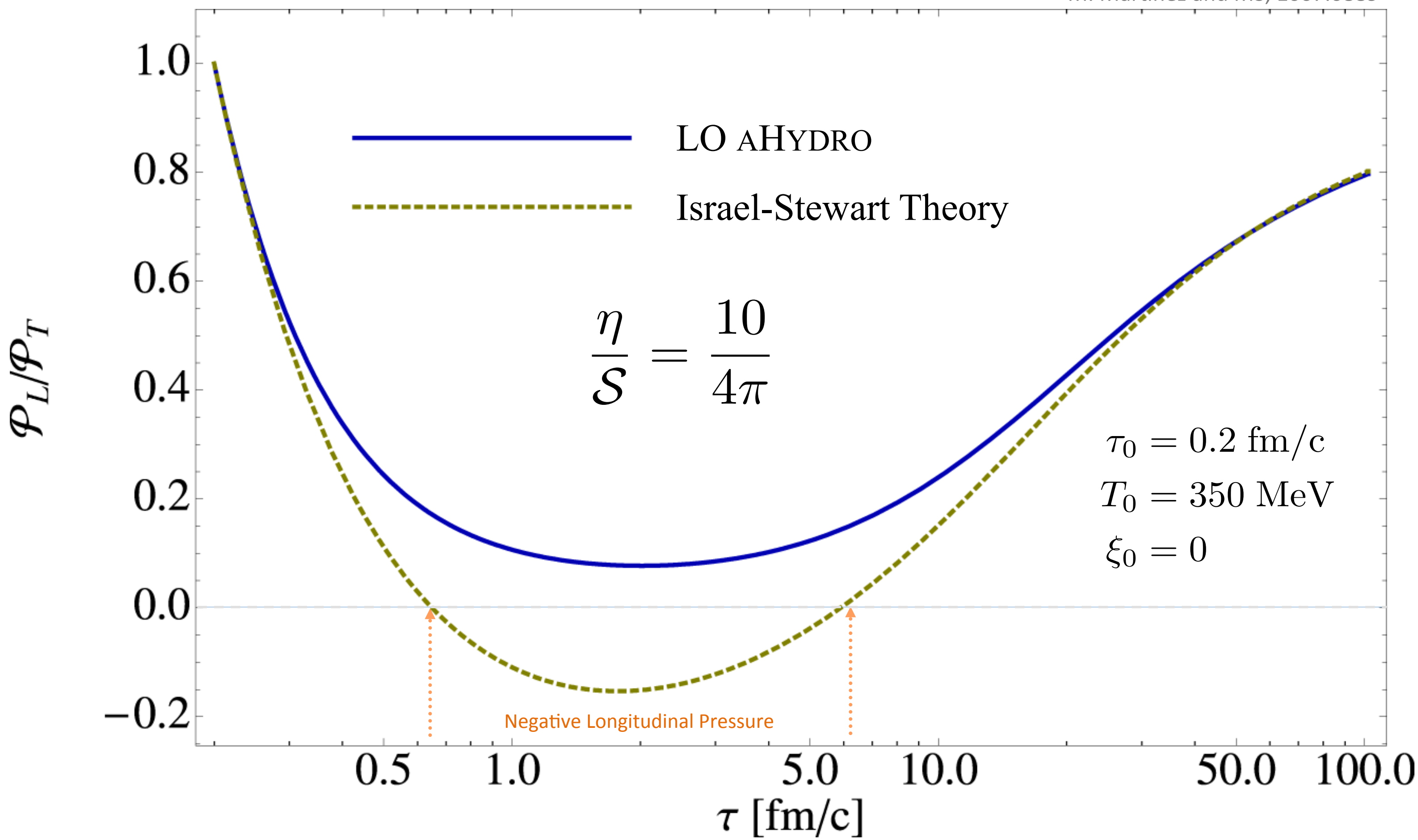
$$\boxed{\frac{1}{1+\xi}\partial_\tau \xi - \frac{2}{\tau} - 6\,\partial_\tau \log\Lambda = 2\Gamma\left[1-\mathcal{R}^{3/4}(\xi)\sqrt{1+\xi}\right]}$$

$$\boxed{\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau\xi + 4\,\partial_\tau\log\Lambda = \frac{1}{\tau}\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1\right]}$$



Viscous Hydro vs LO aHydro

M. Martinez and MS, 1007.0889



Including Transverse Dynamics

W. Florkowski and R. Ryblewski, 1103.1260
M. Martinez, R. Ryblewski, and MS, 1204.1473

- Allowing variables to depend on x and y while still assuming boost-invariance, we obtain the "2+1d" dimensional AHYDRO equations
- Conformal system → four equations for four variables u_x, u_y, ξ, and Λ.

0th moment

$$\boxed{Dn + n\theta = J_0 \,.}$$

$$D \equiv u^\mu \partial_\mu \,,$$
$$\theta \equiv \partial_\mu u^\mu \,,$$

$$u_0 = \sqrt{1 + u_x^2 + u_y^2}$$

1st moment

$$D\mathcal{E} + (\mathcal{E} + \mathcal{P}_\perp)\theta + (\mathcal{P}_L - \mathcal{P}_\perp)\frac{u_0}{\tau} = 0 \,,$$
$$(\mathcal{E} + \mathcal{P}_\perp)Du_x + \partial_x \mathcal{P}_\perp + u_x D\mathcal{P}_\perp + (\mathcal{P}_\perp - \mathcal{P}_L)\frac{u_0 u_x}{\tau} = 0 \,,$$
$$(\mathcal{E} + \mathcal{P}_\perp)Du_y + \partial_y \mathcal{P}_\perp + u_y D\mathcal{P}_\perp + (\mathcal{P}_\perp - \mathcal{P}_L)\frac{u_0 u_y}{\tau} = 0 \,.$$

2+1d Numerical Methods

- Fixed 2d lattice, centered differences in space
- For smooth initial conditions and finite η, standard 4th order Runge-Kutta evolution suffices
- For fluctuating initial conditions two evolution algorithms:

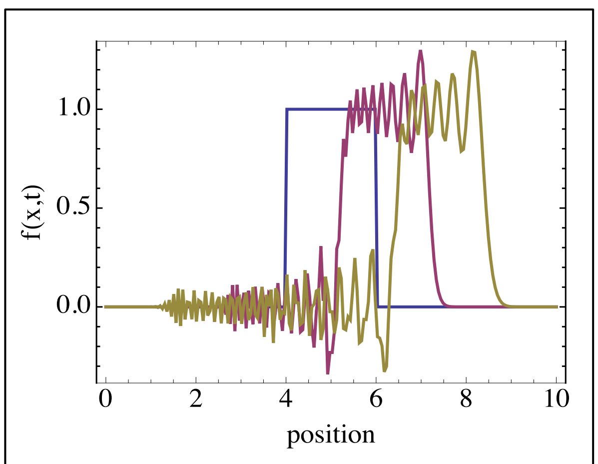
  (1) Weighted LAX (small fraction of spatial average admixed)
  (2) Kurganov-Tadmor (MUSCL) Algorithm

  MUSCL = Monotone Upstream-centered Schemes for Conservation Laws

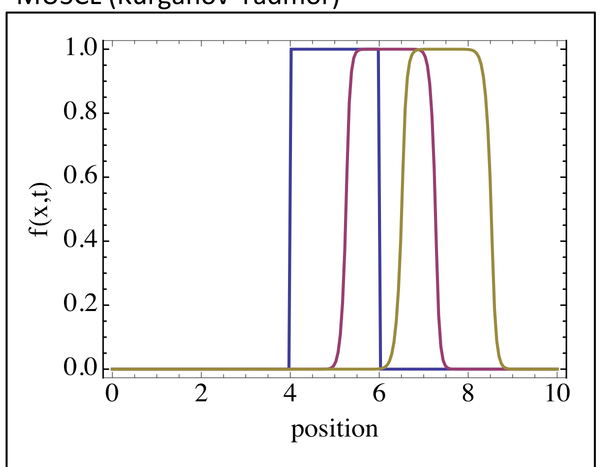
---

e.g. advective equation $(\partial_t + \partial_x)u(x,t) = 0$

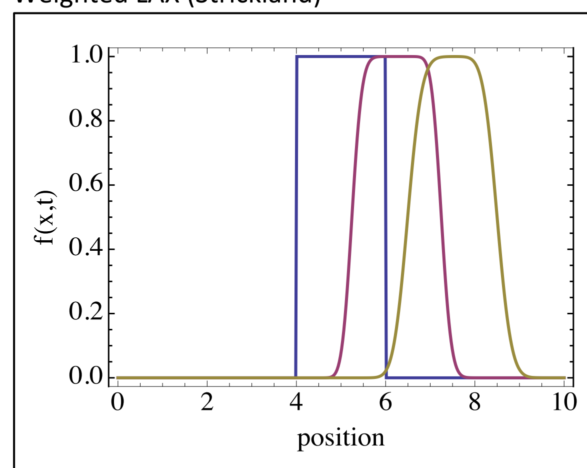
Naïve centered difference scheme



MUSCL (Kurganov-Tadmor)



Weighted LAX (Strickland)

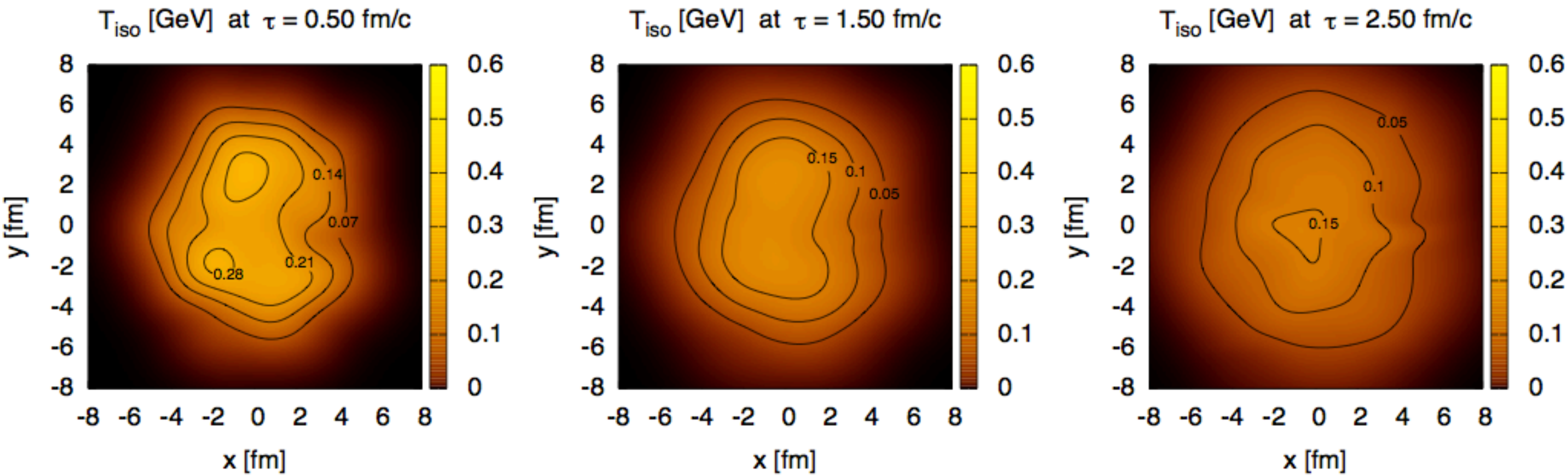


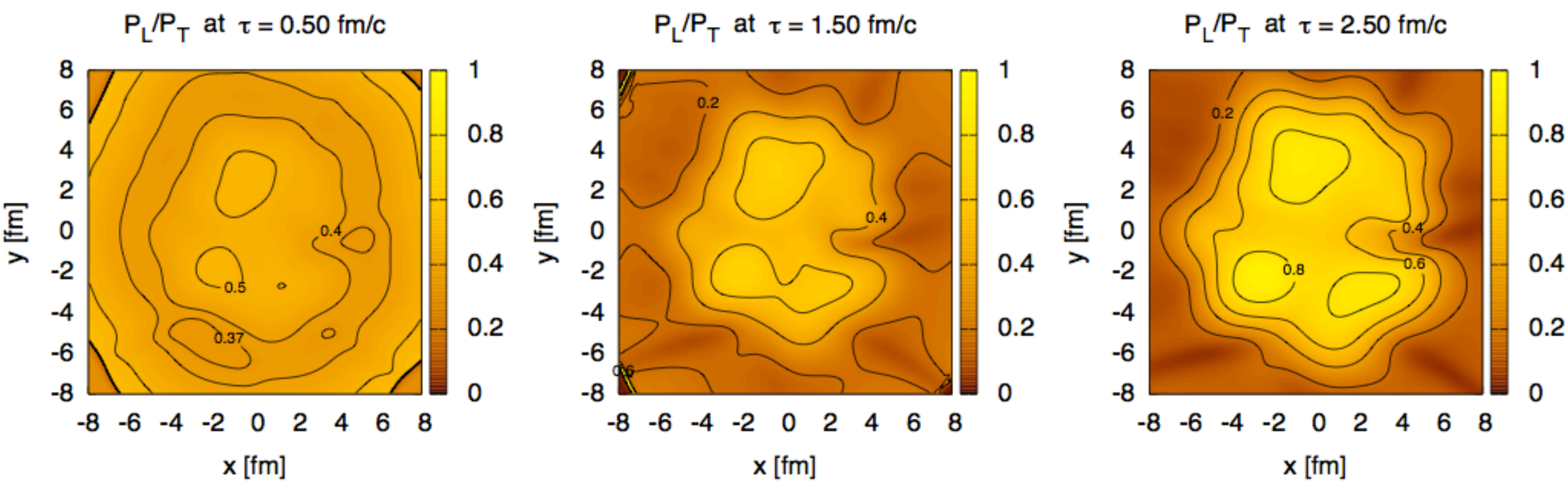
Transverse Dynamics

M. Martinez, R. Ryblewski, and MS, 1204.1473

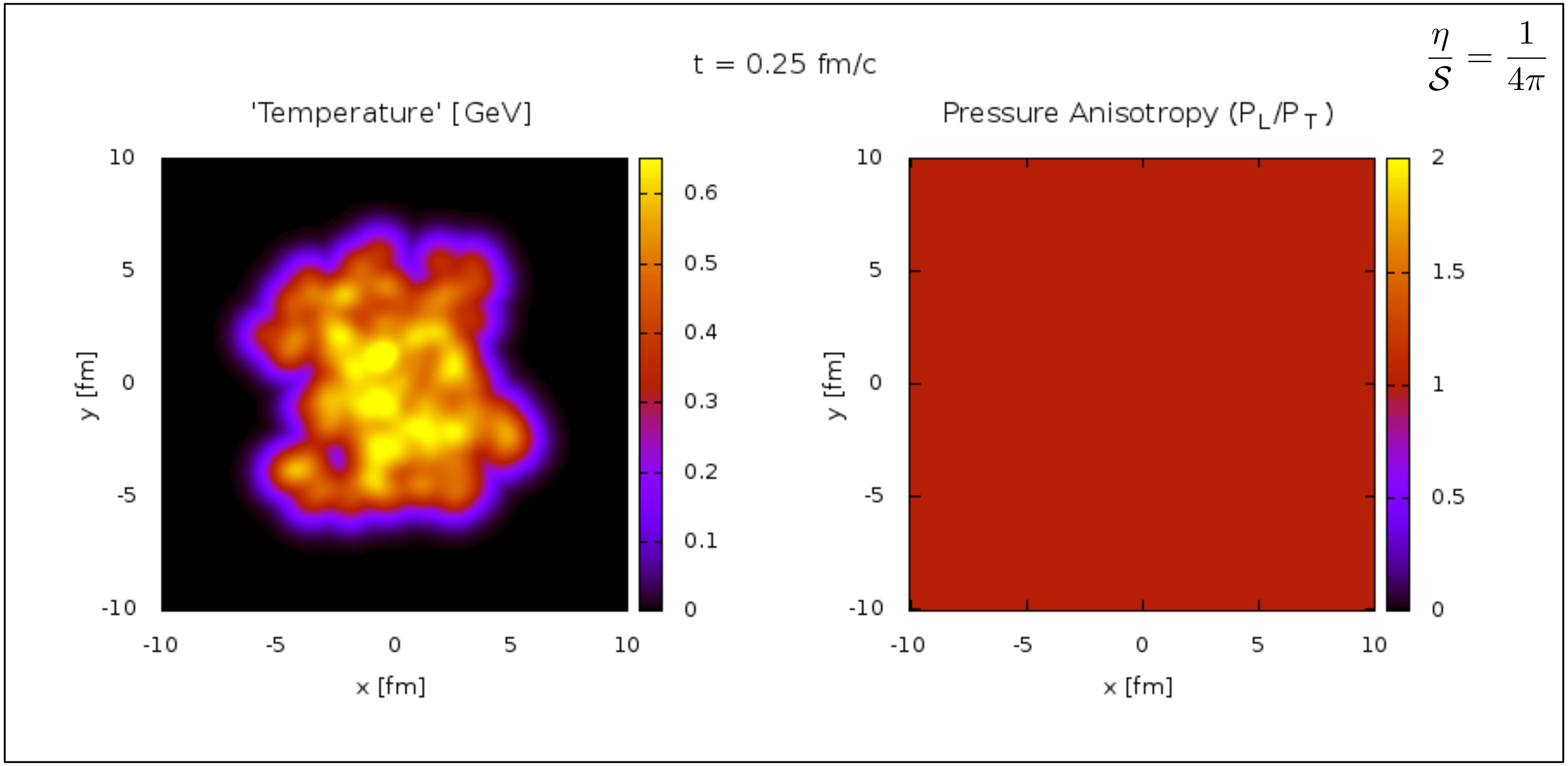
Pb-Pb @ 2.76 TeV
T_0 = 600 MeV
τ_0 = 0.25 fm/c
b = 7 fm

$$\frac{\eta}{\mathcal{S}} = \frac{1}{4\pi}$$





Spatiotemporal Evolution



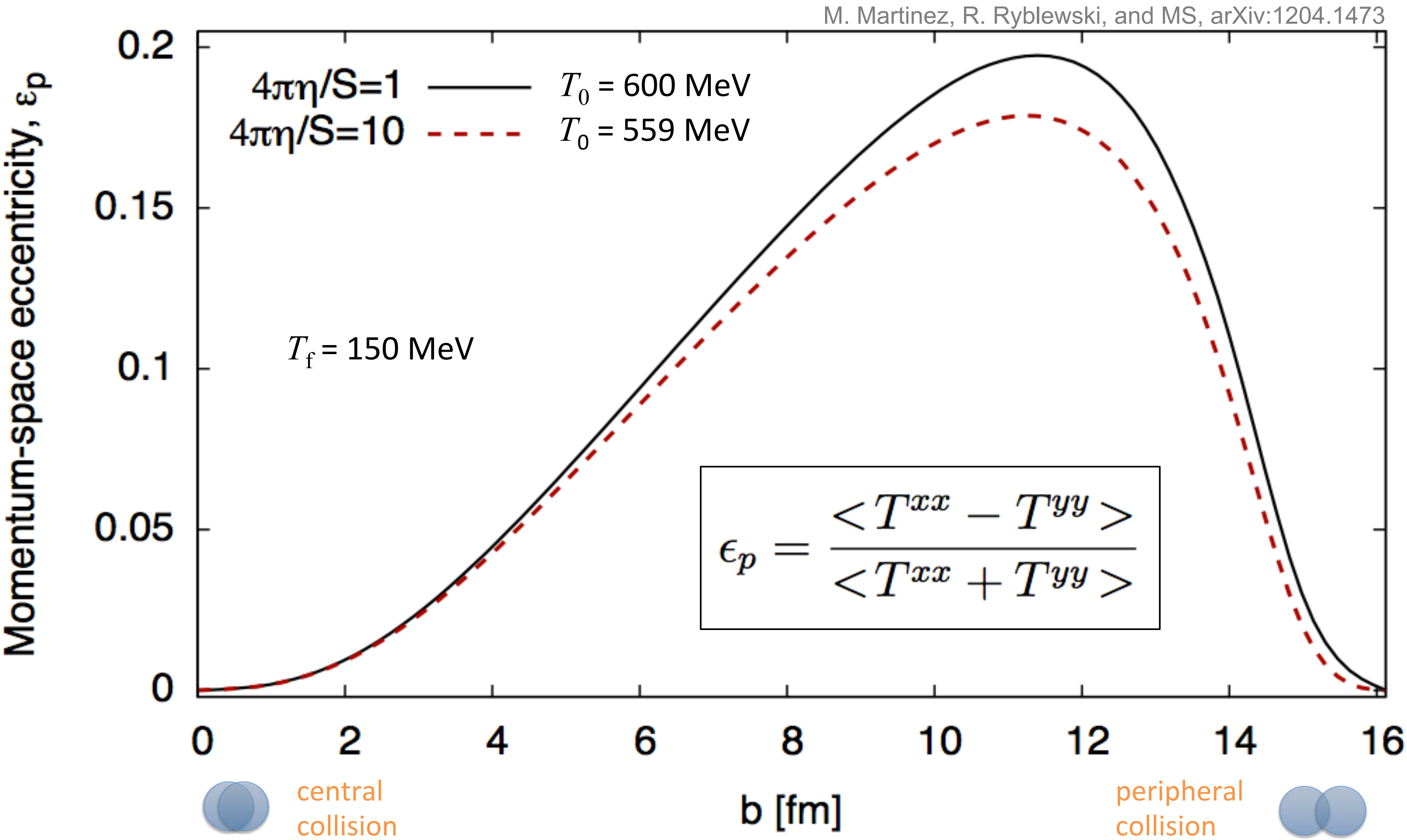
- Pb-Pb, b = 7 fm collision with Monte-Carlo Glauber initial conditions
  T_0 = 600 MeV @ τ_0 = 0.25 fm/c
- Left panel shows temperature and right shows pressure anisotropy

Lecture 2 - Conclusions

- 0^{th}, 1^{st}, and 2^{nd} moments of Boltzmann equation → viscous hydrodynamics
- 1^{st} order in gradients → Relativistic Navier-Stokes
- Anisotropies in momentum space appear
- aHydro introduces momentum-space anisotropies from the beginning
- aHydro reproduces 2^{nd} order viscous hydro equations for small η/S, but can better describe large η/S case
- Pressures and distribution function guaranteed to be >= 0 using LO aHydro!

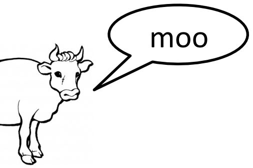
Backup Slides

Collective Flow



Anisotropic Hydrodynamics
Lecture 3

Michael Strickland
Kent State University



2014 Cracow School of Theoretical Physics
QCD meets experiment

Cracow School of Theoretical Physics

KENT STATE UNIVERSITY

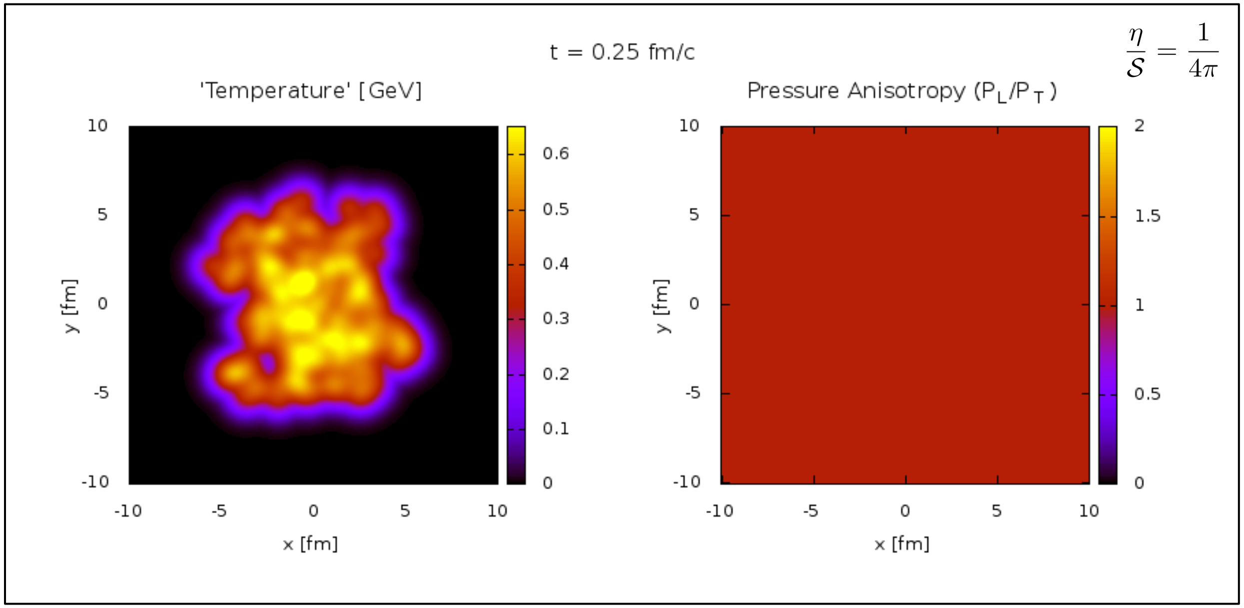
JET Collaboration

HIC for FAIR
Helmholtz International Center

Lecture 3

- Exact solution of RTA Boltzmann EQ
- 2nd order spheroidal anisotropic hydrodynamics
- Ellipsoidal anisotropic hydrodynamics for a system of massive particles
- Phenomenology (General)
- Heavy quarkonium suppression

How can we test the model?



Experimental data will most likely never be able to discern the difference due to statistical and systematic errors etc, so I'm happy to continue with my model until you _prove_ that your model is better.

M. Strickland

0+1d Exact Solution

- Simple model: Boost-invariant transversally homogeneous Boltzmann equation in relaxation time approximation (RTA)
- Many results in this model, so we can compare with the literature
- Can be used to test different approximation schemes

Boltzmann EQ $$p^\mu \partial_\mu f(x,p) = C[f(x,p)]$$

RTA $$C[f] = \frac{p_\mu u^\mu}{\tau_{\rm eq}} \Big[f_{\rm eq}\Big(p_\mu u^\mu, T(x)\Big) - f(x,p) \Big]$$

Massless Particles
W. Florkowski, R. Ryblewski, and MS, 1304.0665 and 1305.7234

Massive Particles
W. Florkowski, E. Maksymiuk, R. Ryblewski, and MS, 1402.7348

Solution for the energy density (massless particle case)

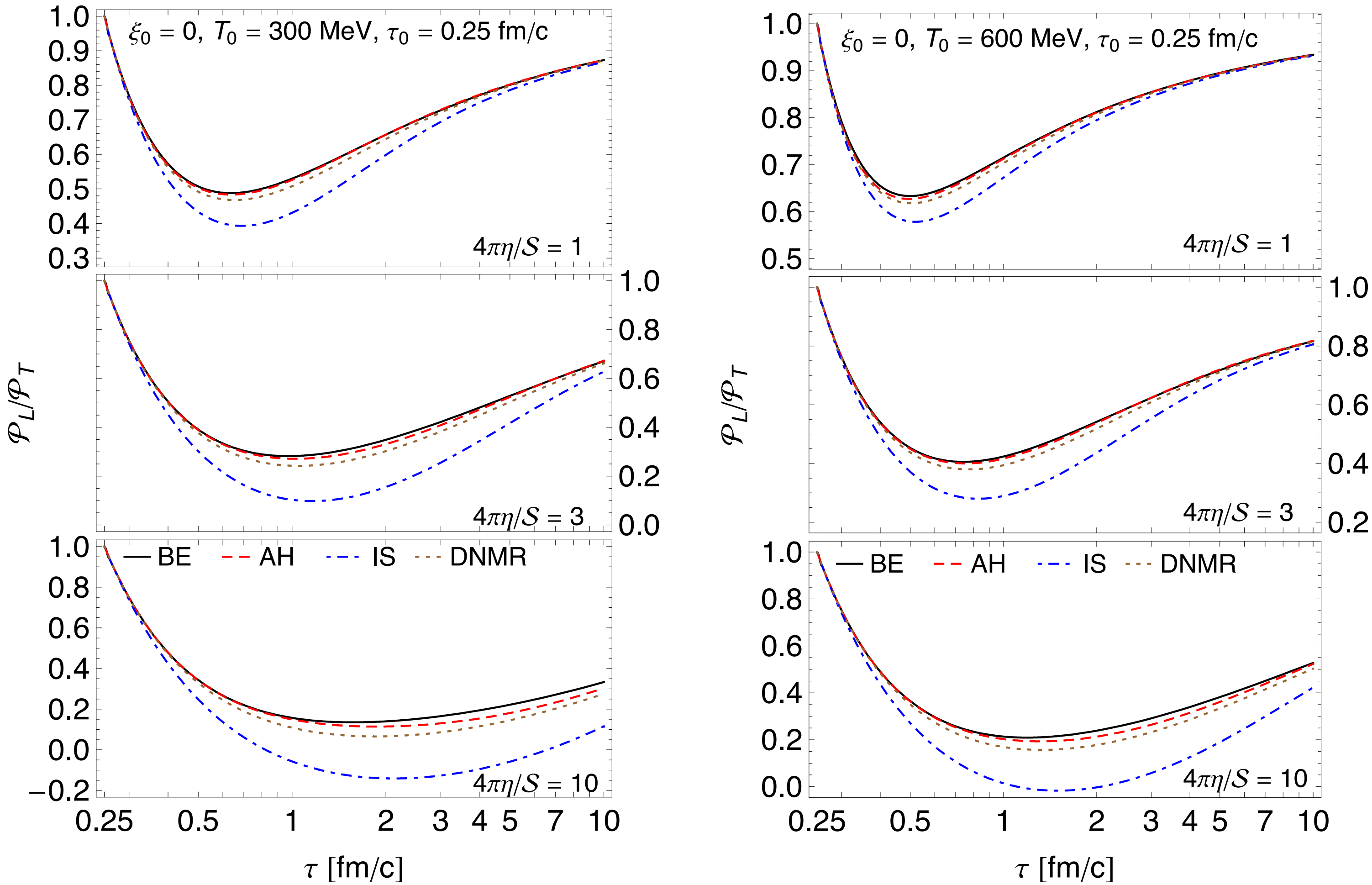
$$\bar{\mathcal{E}}(\tau) = D(\tau,\tau_0)\,\frac{\mathcal{R}\big(\xi_{\rm FS}(\tau)\big)}{\mathcal{R}\left(\xi_0\right)} + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')}\, D(\tau,\tau')\,\bar{\mathcal{E}}(\tau')\,\mathcal{R}\bigg(\Big(\frac{\tau}{\tau'}\Big)^2 - 1\bigg)$$

Time-dependent relaxation time	$\tau_{\rm eq}(\tau) = \dfrac{5\bar\eta}{T(\tau)}$	Damping Function	$D(\tau_2,\tau_1) = \exp\Big[-\int_{\tau_1}^{\tau_2} d\tau\, \tau_{\rm eq}^{-1}(\tau)\Big]$

See talk by R. Ryblewksi for more details

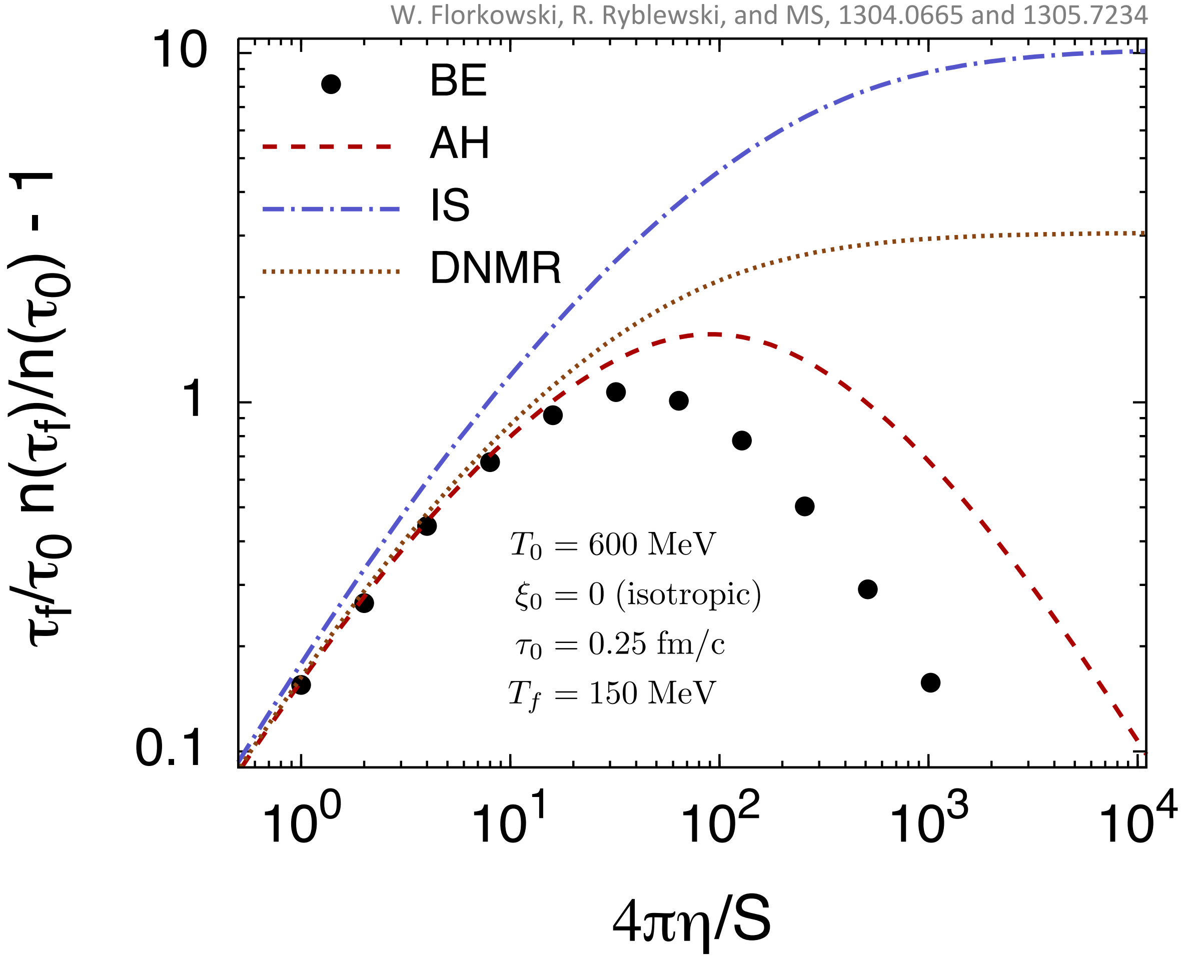
0+1d Exact Solution

W. Florkowski, R. Ryblewski, and MS, 1304.0665 and 1305.7234



BE = Exact Solution AH = aHydro IS = Israel Stewart DNMR = Denicol et al, Phys. Rev. D 85, 114047 (2012)

0+1d Exact Solution



- Particle (entropy) production vanishes in two limits: ideal hydro and free streaming limits
- For conformal (massless) systems, the number density is proportional to entropy density
- **NLO spheroidal aHydro does even better! (comparison coming in a few slides)**

NLO Anisotropic Hydrodynamics

NLO (spheroidal) aHydro

Viscous Hydrodynamics Expansion

$$f(\tau,\mathbf{x},\mathbf{p}) = \underbrace{f_{\rm eq}(\mathbf{p},T(\tau,\mathbf{x}))}_{\text{Isotropic in momentum space}} + \delta f$$

Anisotropic Hydrodynamics Expansion

Now let's treat this term "perturbatively"

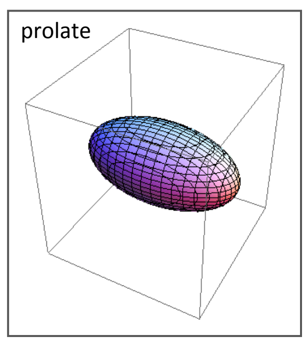
$$f(\tau,\mathbf{x},\mathbf{p}) = f_{\rm aniso}(\mathbf{p},\underbrace{\Lambda(\tau,\mathbf{x})}_{T_\perp},\underbrace{\xi(\tau,\mathbf{x})}_{\text{anisotropy}}) + \delta\tilde{f}$$

[D. Bazow, U. Heinz, and MS, 1311.6720]

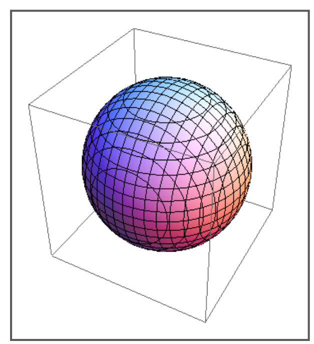
→ "Romatschke-Strickland" form in LRF

$$\boxed{f^{LRF}_{\rm aniso} = f_{\rm iso}\left(\frac{\sqrt{\mathbf{p}^2+\xi(\mathbf{x},\tau)p_z^2}}{\Lambda(\mathbf{x},\tau)}\right)}$$

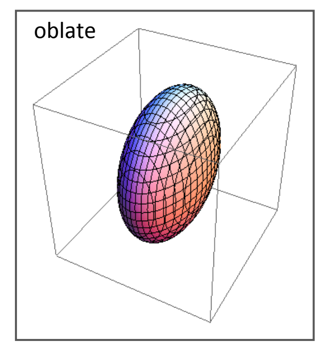
$$\boxed{\xi = \frac{\langle p_T^2\rangle}{2\langle p_L^2\rangle} - 1}$$



$-1 < \xi < 0$



$\xi = 0$



$\xi > 0$

NLO Anisotropic Hydrodynamics

- Treat LO term "non-perturbatively" assuming spheroidal "RS" form but couple it to the dissipative currents
- Treat corrections $\delta\tilde{f}$ "perturbatively" → viscous aHydro (vaHydro)
- Use the very impressive method of Denicol et al [1202.4551] adapted to an anisotropic background
- Complete and orthogonal relativistic polynomial basis + systematic expansion in Knudsen number and (modified) inverse Reynolds number
- For the results I show today, we used the "Grad 14-moment" approximation.
- This corresponds to a particular finite-element polynomial basis for the linearized corrections in momentum space.
- This basis can be extended to higher orders with some work.

Resulting Equations

Skipping over the gory details the final 14-moment approximation result is

$$\dot{\mathcal{N}} = -\mathcal{N}\theta - \boxed{\partial_\mu \tilde{V}^\mu} + \mathcal{C}.$$

[D. Bazow, U. Heinz, and MS, 1311.6720]

$$\dot{\mathcal{E}} + (\mathcal{E}+\mathcal{P}_\perp+\boxed{\tilde{\Pi}})\theta + (\mathcal{P}_\mathrm{L}-\mathcal{P}_\perp)\frac{u_0}{\tau} + \boxed{u_\nu \partial_\mu \tilde{\pi}^{\mu\nu}} = 0,$$

$$(\mathcal{E}+\mathcal{P}_\perp+\boxed{\tilde{\Pi}})\dot{u}_x + \partial_x(\mathcal{P}_\perp+\boxed{\tilde{\Pi}}) + u_x(\dot{\mathcal{P}}_\perp+\boxed{\dot{\tilde{\Pi}}}) + (\mathcal{P}_\perp-\mathcal{P}_\mathrm{L})\frac{u_0 u_x}{\tau} - \boxed{\Delta^{1\nu}\partial^\mu \tilde{\pi}_{\mu\nu}} = 0,$$

$$(\mathcal{E}+\mathcal{P}_\perp+\boxed{\tilde{\Pi}})\dot{u}_y + \partial_y(\mathcal{P}_\perp+\boxed{\tilde{\Pi}}) + u_y(\dot{\mathcal{P}}_\perp+\boxed{\dot{\tilde{\Pi}}}) + (\mathcal{P}_\perp-\mathcal{P}_\mathrm{L})\frac{u_0 u_y}{\tau} - \boxed{\Delta^{2\nu}\partial^\mu \tilde{\pi}_{\mu\nu}} = 0,$$

$$\begin{aligned}
\dot{\tilde{\Pi}} &= -\frac{\dot{\tilde{\gamma}}_r^\Pi}{\tilde{\gamma}_r^\Pi}\tilde{\Pi} + \frac{1}{\tilde{\gamma}_r^\Pi}\mathcal{C}_{r-1} + \mathcal{W}_r + \mathcal{U}_r^{\mu\nu}\nabla_\mu u_\nu \\
&\quad + \lambda_{\Pi\pi}^r \tilde{\pi}^{\mu\nu}\sigma_{\mu\nu} + \tau_{\Pi V}^r \tilde{V}^\mu \dot{u}_\mu - \frac{1}{\tilde{\gamma}_r^\Pi}\nabla_\mu\left(\tilde{\gamma}_{r-1}^V \tilde{V}^\mu\right) - \delta_{\Pi\Pi}^r \tilde{\Pi}\theta \\
\dot{\tilde{V}}^{\langle\mu\rangle} &= -\frac{\dot{\tilde{\gamma}}_r^V}{\tilde{\gamma}_r^V}\tilde{V}^\mu + \frac{1}{\tilde{\gamma}_r^V}\mathcal{C}_{r-1}^{\langle\mu\rangle} + \mathcal{Z}_r^\mu - \tilde{V}^\nu \omega_\nu{}^\mu + \delta_{VV}^r \tilde{V}^\mu\theta - \Delta^\mu{}_\lambda \frac{1}{\tilde{\gamma}_r^V}\nabla_\nu\left(\tilde{\gamma}_{r-1}^\pi \tilde{\pi}^{\nu\lambda}\right) \\
&\quad + \tau_{q\pi}^r \tilde{\pi}^{\mu\nu}\dot{u}_\nu + \lambda_{VV}^r \tilde{V}_\nu \sigma^{\nu\mu} + \tau_{q\Pi}^r \tilde{\Pi}\dot{u}^\mu + \ell_{q\Pi}^r \nabla^\mu \tilde{\Pi} + \tilde{\Pi}\mathcal{O}^\mu, \\
\dot{\tilde{\pi}}^{\langle\mu\nu\rangle} &= -\frac{\dot{\tilde{\gamma}}_r^\pi}{\tilde{\gamma}_r^\pi}\tilde{\pi}^{\mu\nu} + \mathcal{T}^{\langle\mu}V^{\nu\rangle} + \frac{1}{\tilde{\gamma}_r^\pi}\mathcal{C}_{r-1}^{\langle\mu\nu\rangle} + \mathcal{K}_r^{\mu\nu} + \mathcal{L}_r^{\mu\nu} + \mathcal{H}_r^{\mu\nu\lambda}\dot{z}_\lambda + \mathcal{Q}_r^{\mu\nu\lambda\alpha}\nabla_\lambda u_\alpha + \mathcal{X}_r^{\mu\nu\lambda}u^\alpha \nabla_\lambda z_\alpha \\
&\quad - 2\lambda_{\pi\pi}^r \tilde{\pi}_\alpha{}^{\langle\mu}\sigma^{\nu\rangle\alpha} + 2\tilde{\pi}^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda + 2\lambda_{\pi\Pi}^r \tilde{\Pi}\sigma^{\mu\nu} + 2\lambda_{\pi V}^r \nabla^{\langle\mu}\tilde{V}^{\nu\rangle} + 2\tau_{\pi V}^r \tilde{V}^{\langle\mu}\dot{u}^{\nu\rangle} - 2\delta_{\pi\pi}^r \tilde{\pi}^{\mu\nu}\theta.
\end{aligned}$$

- Orange-boxed terms are new
- Dot indicates a convective derivative
- Complicated bits in last two equations correspond to dissipative "forces" and anisotropic transport coefficients

(2+1)-dimensional Equations

For conformal boost-invariant systems assuming no gradients in the chemical potential and using an RTA collisional kernel, the equations reduce to

[D. Bazow, U. Heinz, and MS, 1311.6720]

$$\frac{\dot\xi}{1+\xi} - 6\frac{\dot\Lambda}{\Lambda} - 2\theta = 2\Gamma\left(1-\sqrt{1+\xi}\,\mathcal{R}^{3/4}(\xi)\right)$$

$$\mathcal{R}'\dot\xi + 4\mathcal{R}\frac{\dot\Lambda}{\Lambda} = -\left(\mathcal{R}+\frac{1}{3}\mathcal{R}_\perp\right)\theta_\perp - \left(\mathcal{R}+\frac{1}{3}\mathcal{R}_L\right)\frac{u_0}{\tau} + \frac{\tilde\pi^{\mu\nu}\sigma_{\mu\nu}}{\mathcal{E}_0(\Lambda)},$$

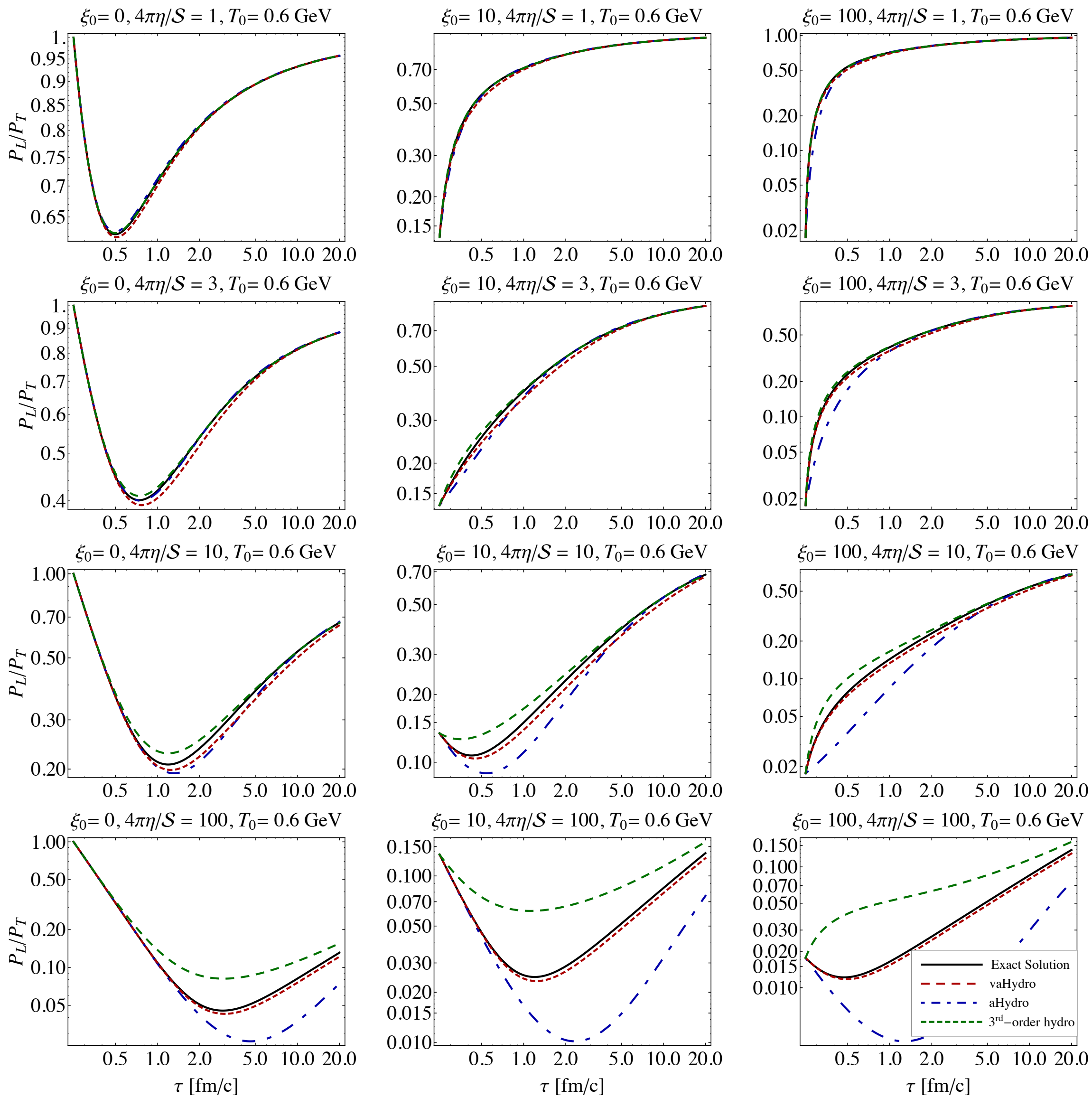
$$[3\mathcal{R}+\mathcal{R}_\perp]\,\dot u_\perp = -\mathcal{R}'_\perp\partial_\perp\xi - 4\mathcal{R}_\perp\frac{\partial_\perp\Lambda}{\Lambda} - u_\perp\left(\mathcal{R}'_\perp\dot\xi+4\mathcal{R}_\perp\frac{\dot\Lambda}{\Lambda}\right)$$
$$-u_\perp(\mathcal{R}_\perp-\mathcal{R}_L)\frac{u_0}{\tau} + \frac{3}{\mathcal{E}_0(\Lambda)}\left(\frac{u_x\Delta^1{}_\nu + u_y\Delta^2{}_\nu}{u_\perp}\right)\partial_\mu\tilde\pi^{\mu\nu},$$

$$[3\mathcal{R}+\mathcal{R}_\perp]\,u_\perp\dot\phi_u = -\mathcal{R}'_\perp D_\perp\xi - 4\mathcal{R}_\perp\frac{D_\perp\Lambda}{\Lambda} - \frac{3}{\mathcal{E}_0(\Lambda)}\left(\frac{u_y\partial_\mu\tilde\pi^{\mu1} - u_x\partial_\mu\tilde\pi^{\mu2}}{u_\perp}\right).$$

$$\dot{\tilde\pi}^{\mu\nu} = -2\dot u_\alpha\tilde\pi^{\alpha(\mu}u^{\nu)} - \Gamma\Big[\big(\mathcal{P}(\Lambda,\xi)-\mathcal{P}_\perp(\Lambda,\xi)\big)\Delta^{\mu\nu} + \big(\mathcal{P}_{\rm L}(\Lambda,\xi)-\mathcal{P}_\perp(\Lambda,\xi)\big)z^\mu z^\nu + \tilde\pi^{\mu\nu}\Big]$$
$$+\mathcal{K}_0^{\mu\nu} + \mathcal{L}_0^{\mu\nu} + \mathcal{H}_0^{\mu\nu\lambda}\dot z_\lambda + \mathcal{Q}_0^{\mu\nu\lambda\alpha}\nabla_\lambda u_\alpha + \mathcal{X}_0^{\mu\nu\lambda}u^\alpha\nabla_\lambda z_\alpha - 2\lambda^0_{\pi\pi}\tilde\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + 2\tilde\pi^{\lambda\langle\mu}\omega^{\nu\rangle}_\lambda - 2\delta^0_{\pi\pi}\tilde\pi^{\mu\nu}\theta,$$

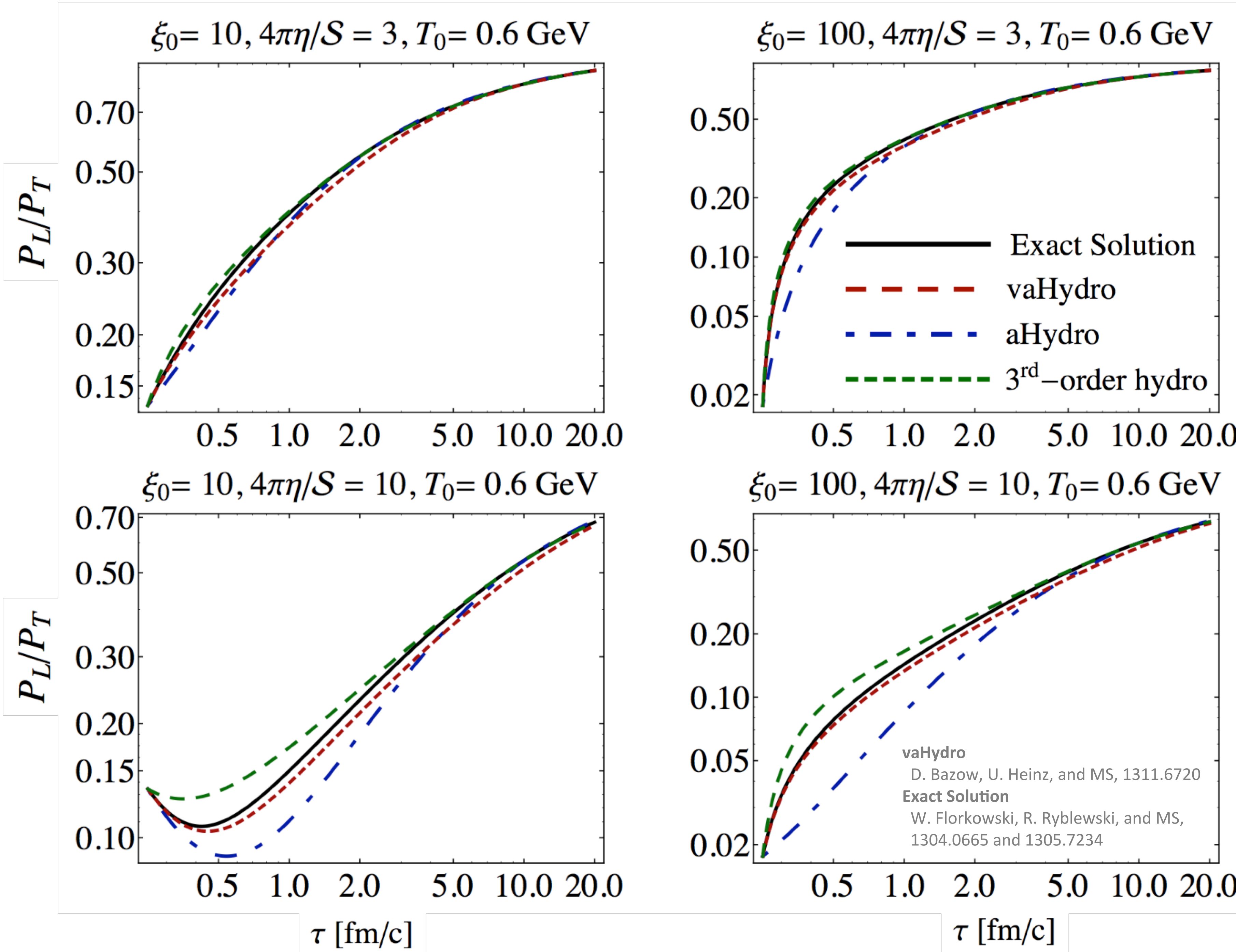
A prime indicates a derivative with respect to ξ.

Pressure Ratio Comparisons



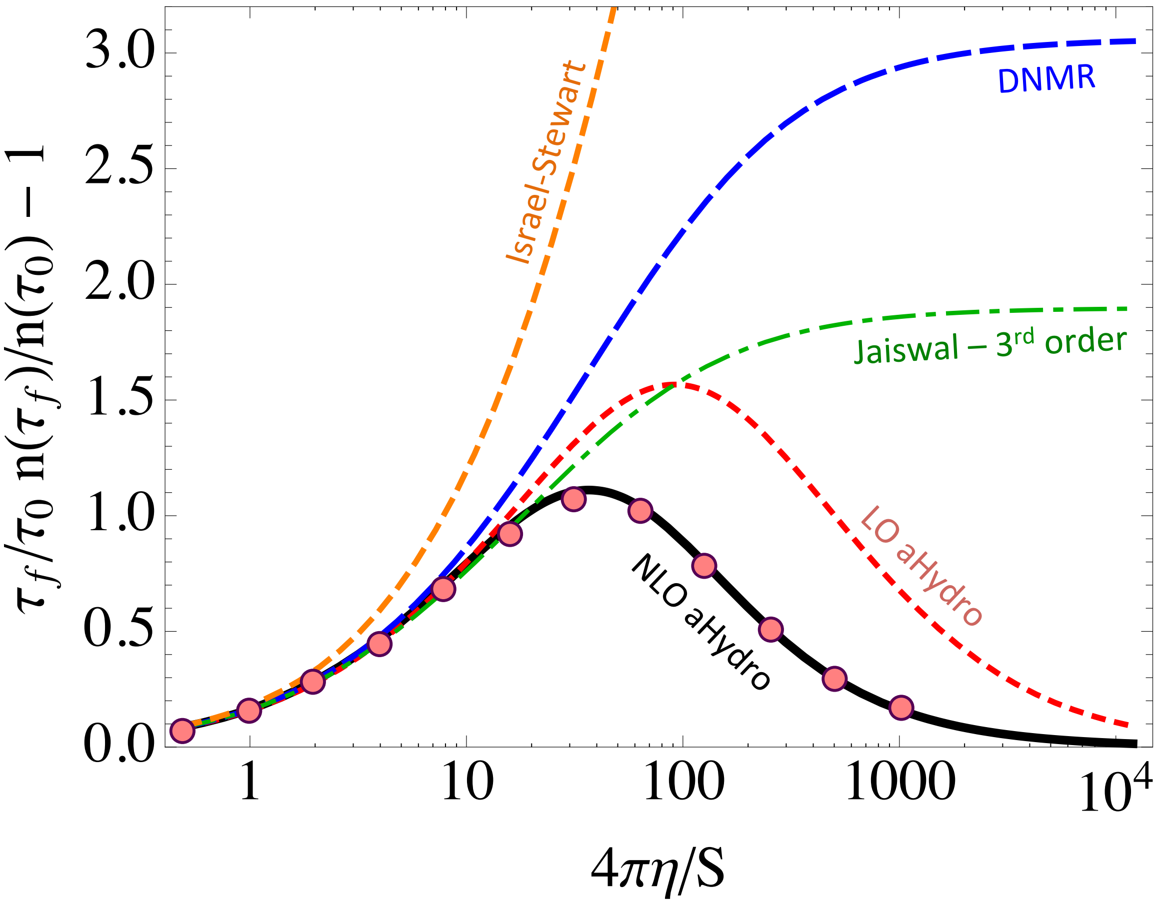
[D. Bazow, U. Heinz, and MS, 1311.6720]

- Panels show ratio of longitudinal to transverse pressure
- T_0 = 600 MeV @ τ_0 = 0.25 fm/c
- Left to right is increasing initial momentum-space anisotropy
- Top to bottom is increasing η/S
- Black line is the exact solution
- Red dashed line is the NLO aHydro approximation (vaHydro)
- Blue dot-dashed line is the aHydro approximation
- Green dashed line is a third-order Chapman-Enskog-like viscous hydrodynamics approximation [A. Jaiswal, 1305.3480]
- As we can see from these plots NLO aHydro does quite well even in extreme conditions!


$\xi_0 = 10, 4\pi\eta/\mathcal{S} = 3, T_0 = 0.6$ GeV
$\xi_0 = 100, 4\pi\eta/\mathcal{S} = 3, T_0 = 0.6$ GeV
$\xi_0 = 10, 4\pi\eta/\mathcal{S} = 10, T_0 = 0.6$ GeV
$\xi_0 = 100, 4\pi\eta/\mathcal{S} = 10, T_0 = 0.6$ GeV
P_L/P_T
τ [fm/c]
Exact Solution
vaHydro
aHydro
3rd–order hydro
vaHydro
D. Bazow, U. Heinz, and MS, 1311.6720
Exact Solution
W. Florkowski, R. Ryblewski, and MS,
1304.0665 and 1305.7234

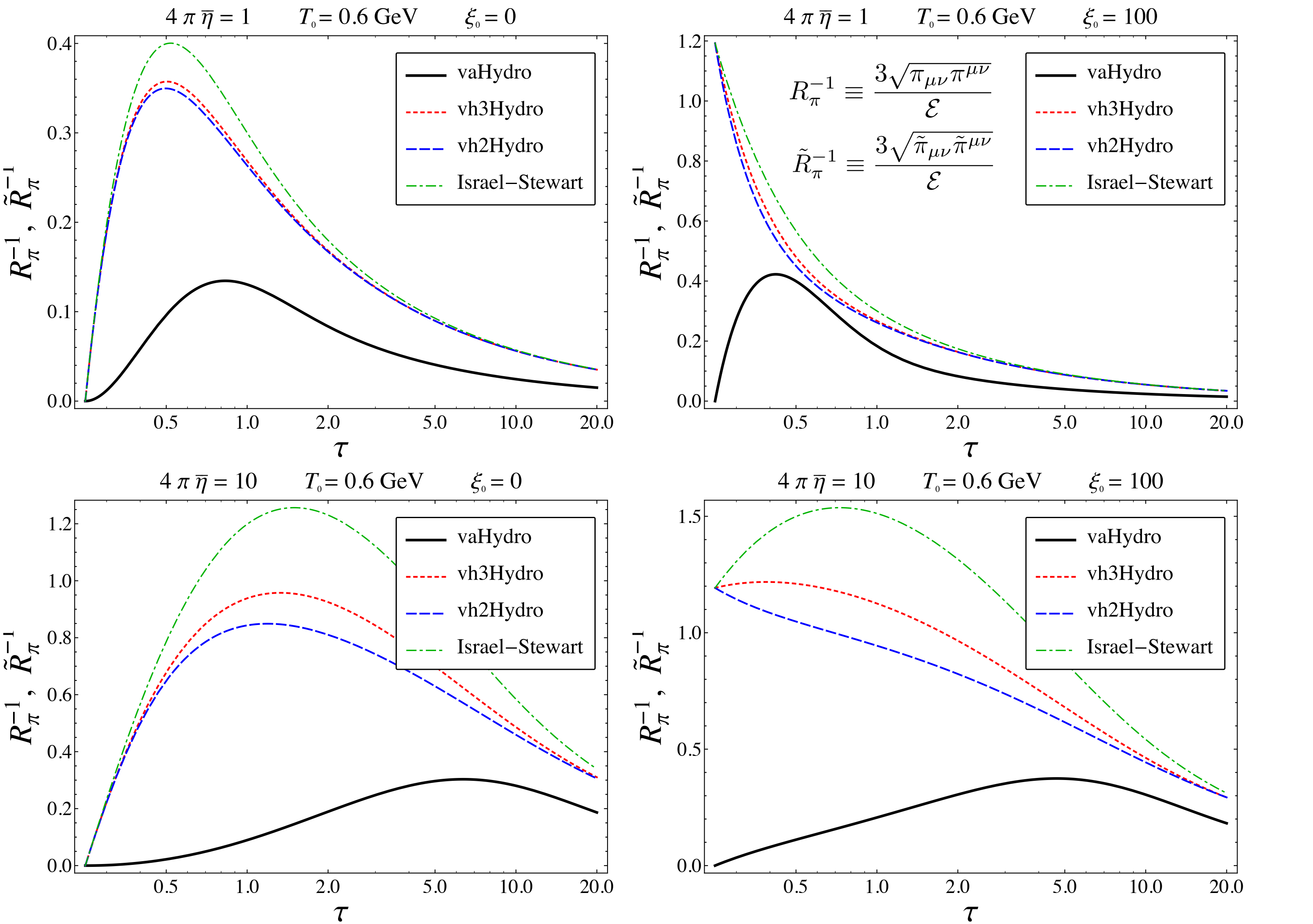
Entropy Generation

[D. Bazow, U. Heinz, and MS, 1311.6720]



- Number (entropy) production vanishes in two limits: ideal hydrodynamic and free streaming limits
- In the conformal model which we are testing with, number density is proportional to entropy density

Reynolds Number Comparison



LO Ellipsoidal aHydro

Conformal System: W. Florkowski and L. Tinti, 1312.6614
Non-Conformal System: M. Nopoush, R. Ryblewski, and MS, 1405.1355

- An alternative approach to expanding perturbatively is to try to include as much of the physics as possible in the LO distribution function ansatz.
- To start with, one might consider having <u>two anisotropy parameters</u>

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}_x x^\mu x^\nu + \mathcal{P}_y y^\mu y^\nu + \mathcal{P}_z z^\mu z^\nu$$

- For conformal systems, only two are needed since the third can be absorbed by a rescaling (three are needed for non-conformal case)
- The new starting point for the distribution function is then a kind of generalized Romatschke-Strickland form

$$f(x,p) = f_{\rm iso}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu}(x) p^\nu}}{\lambda(x)}, \frac{\mu(x)}{\lambda(x)} \right) + \delta\tilde{f}(x,p)$$

LO Ellipsoidal aHydro (Massive)

M. Nopoush, R. Ryblewski, and MS, 1405.1355

- In order to have a non-ideal EoS in the kinetic approach one way to proceed is to use a phenomenological massive quasiparticle model.
- This complicates life a bit, because finite mass breaks the conformal invariance
- In what I show today, I will assume that mass is a constant (will be allowed to depend on the temperature soon…)

$$f(x,p) = f_{\rm iso}\left(\frac{1}{\lambda}\sqrt{p_\mu \Xi^{\mu\nu} p_\nu}\right) = f_{\rm iso}\left(\frac{1}{\lambda}\sqrt{\sum_i \frac{p_i^2}{\alpha_i^2} + m^2}\right)$$

$$\alpha_i \equiv (1+\xi_i+\Phi)^{-1/2}$$

$$\boxed{\begin{aligned} \Xi^{\mu\nu} &= u^\mu u^\nu + \xi^{\mu\nu} - \Delta^{\mu\nu}\Phi \\ \xi^{\mu\nu} &= \mathrm{diag}(0, \boldsymbol{\xi}) \\ \xi^{\mu}{}_{\mu} &= 0 \longrightarrow \xi_x + \xi_y + \xi_z = 0 \end{aligned}}$$

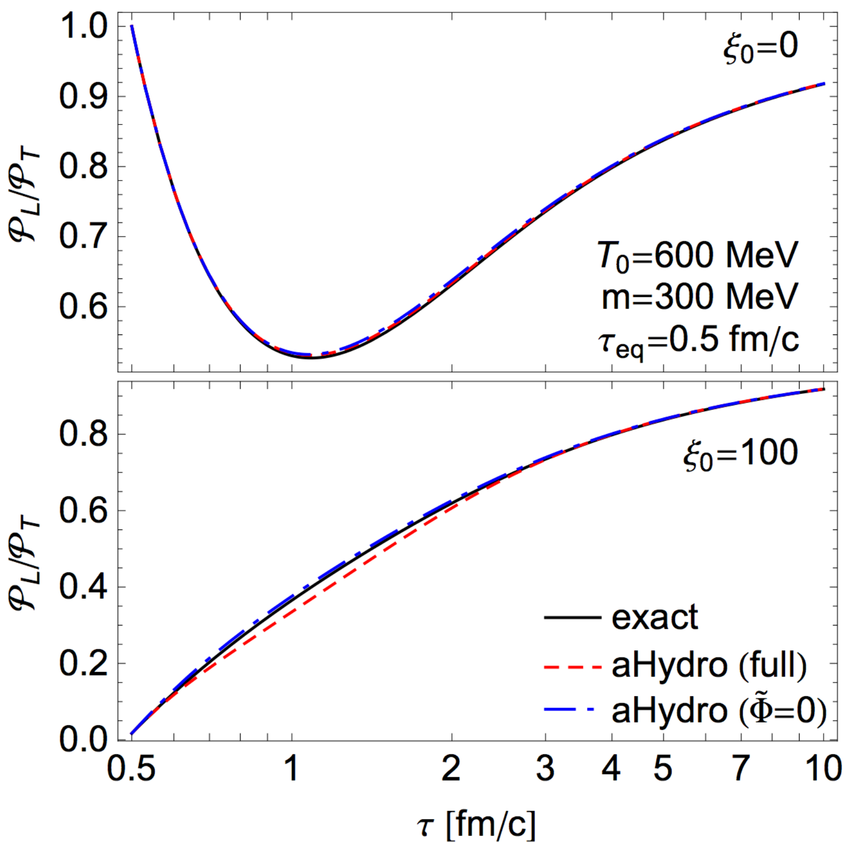
The field Φ is the analog of the bulk pressure correction in second-order viscous hydro.

LO Ellipsoidal aHydro (Massive)

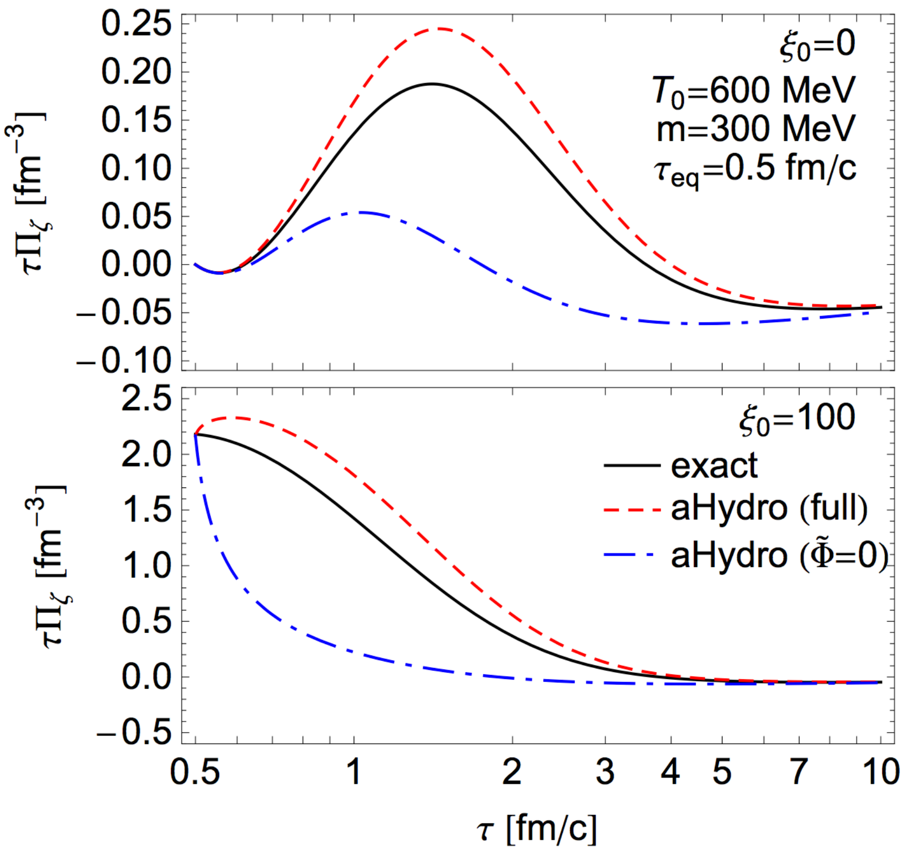
M. Nopoush, R. Ryblewski, and MS, 1405.1355

Traditional viscous hydrodynamics approaches like Israel-Stewart do not properly take into account the coupling between shear and bulk corrections [see talk by R. Ryblewski]. When one introduces a bulk degree of freedom into aHydro this is automatically taken into account.

0+1d Pressure Anisotropy



0+1d Bulk Pressure



LO Ellipsoidal aHydro (Massive)

M. Nopoush, R. Ryblewski, and MS, 1405.1355

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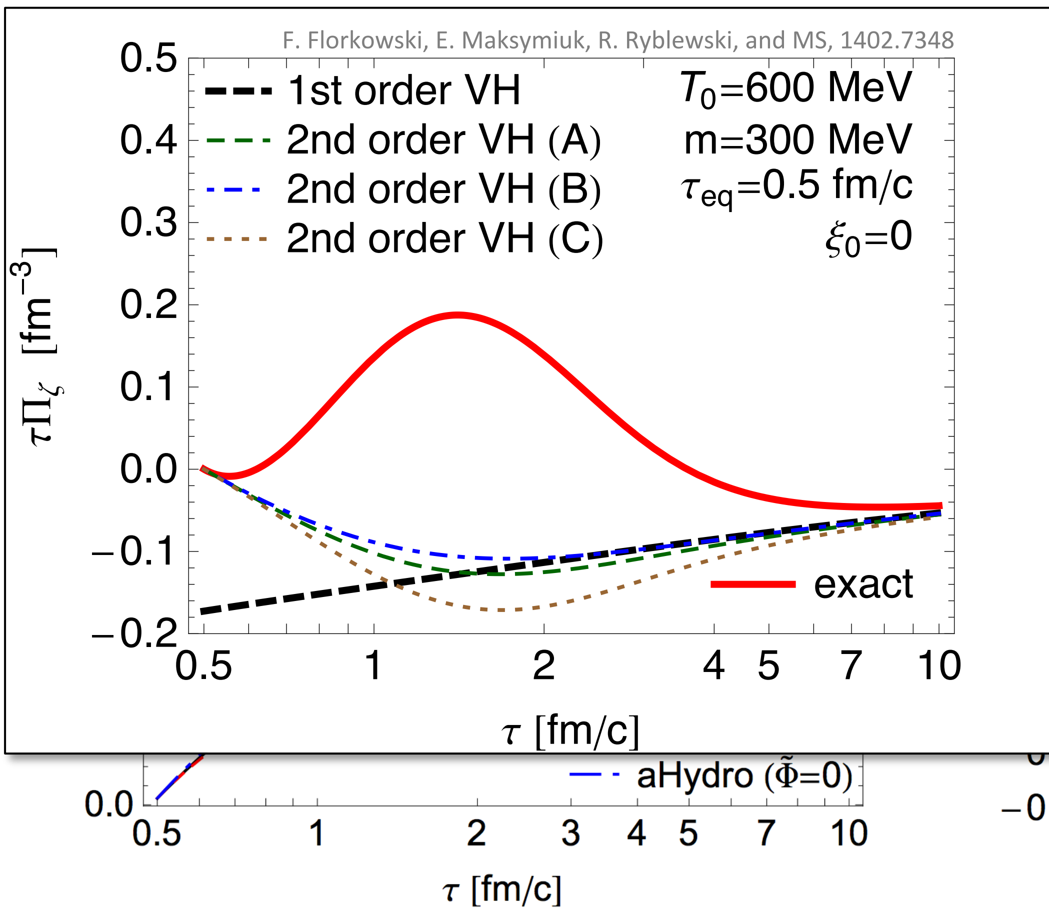
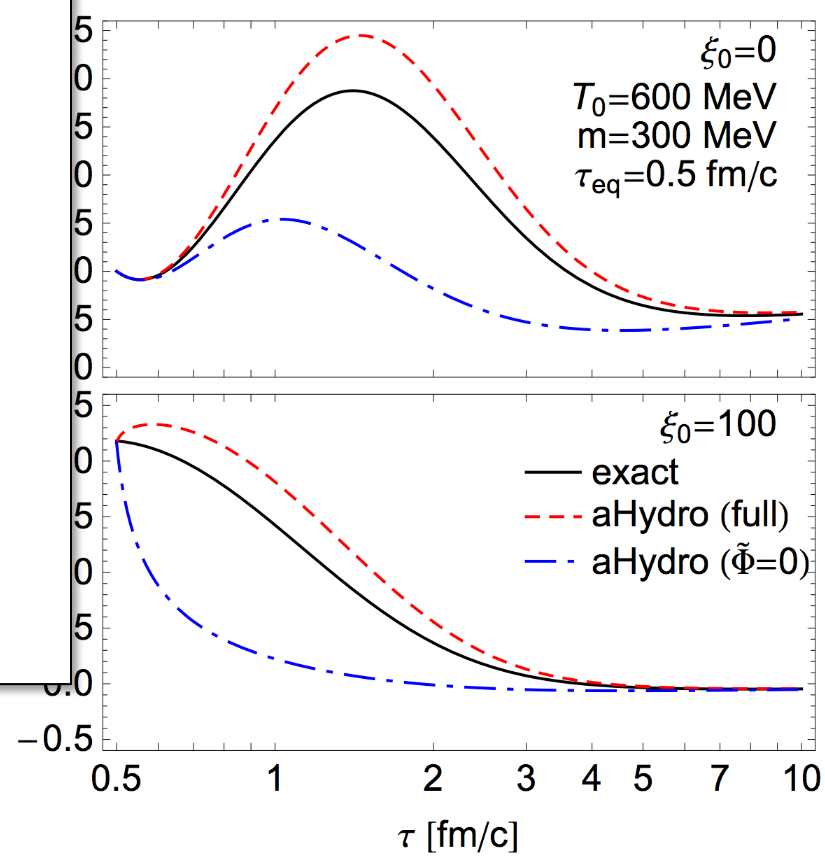


0+1d Bulk Pressure



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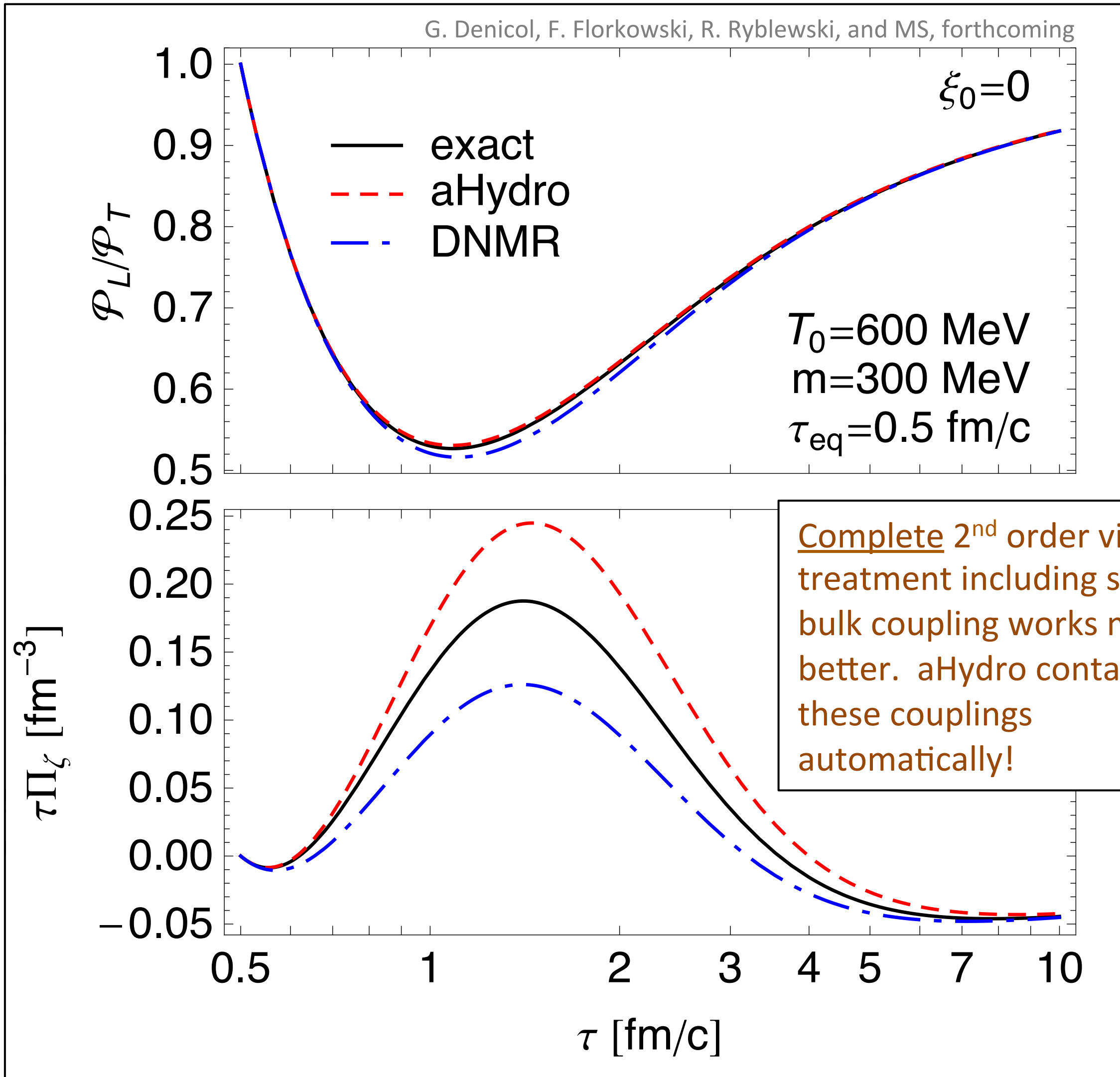
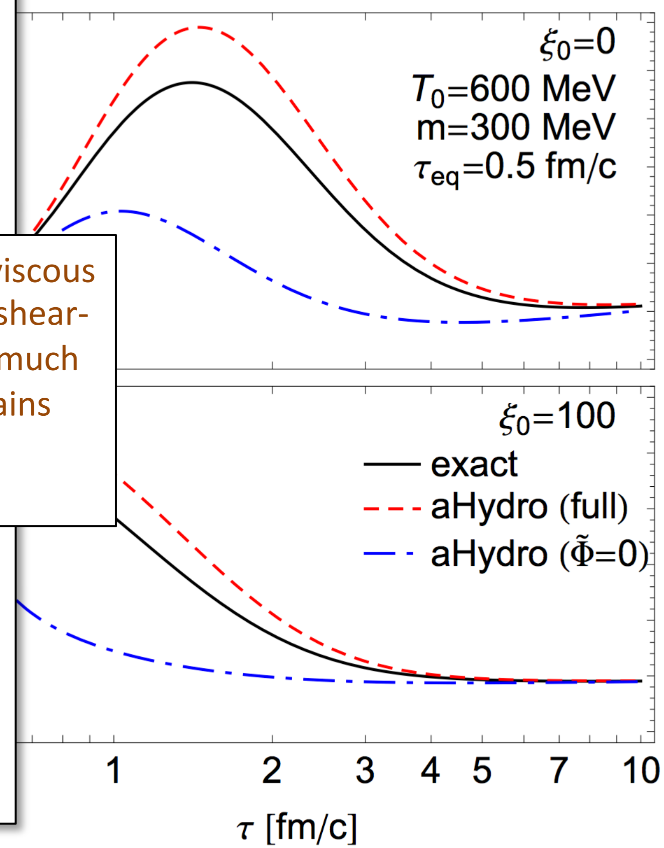


Complete 2nd order viscous treatment including shear-bulk coupling works much better. aHydro contains these couplings automatically!

0+1d Bulk Pressure



- Phenomenology -

Highly-anisotropic hydrodynamics in 3+1 space-time dimensions *

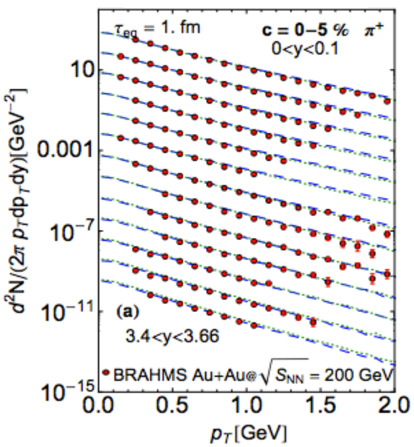
Radoslaw Ryblewski[1,†] and Wojciech Florkowski[2,1,‡]

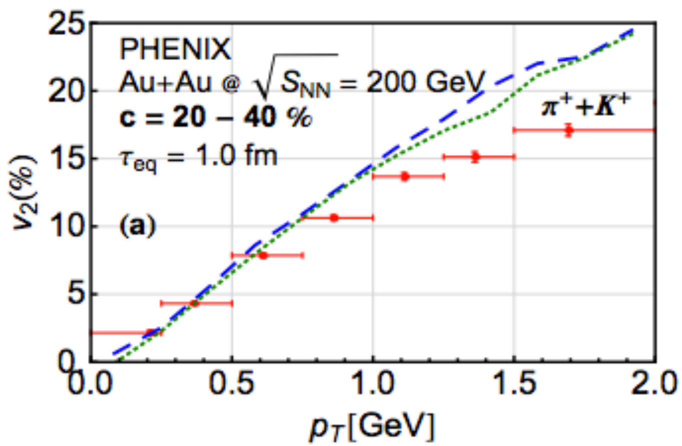
[1] *The H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, PL-31342 Kraków, Poland*

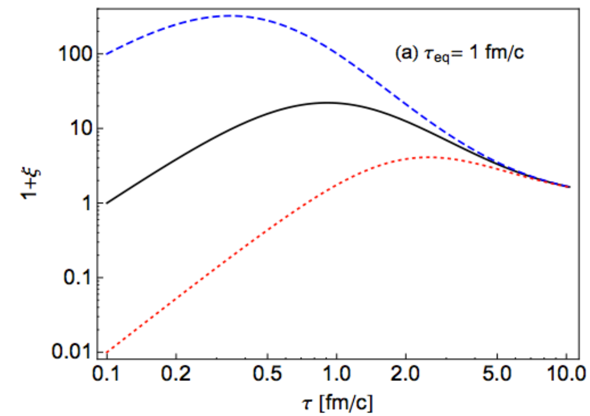
[2] *Institute of Physics, Jan Kochanowski University, PL-25406 Kielce, Poland*

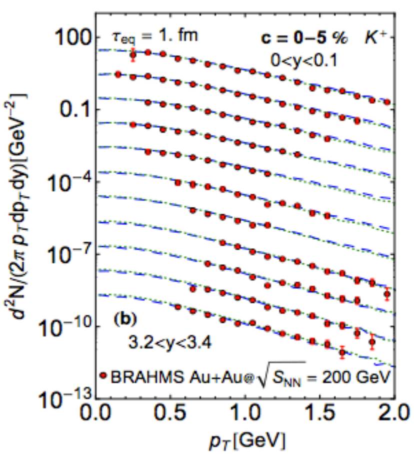
(Dated: April 10, 2012)

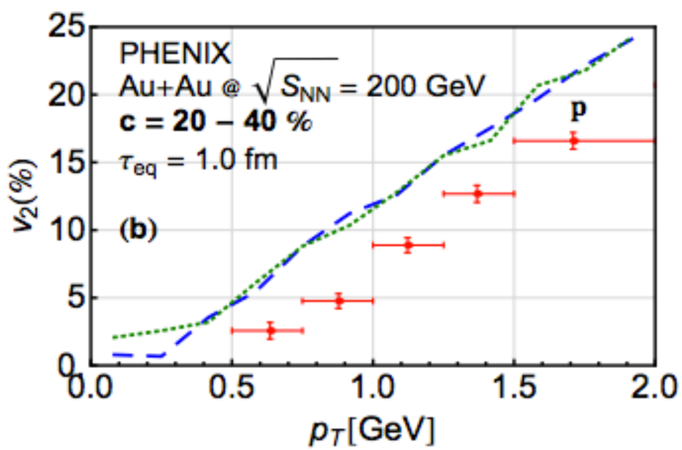
arXiv:1204.2624v1









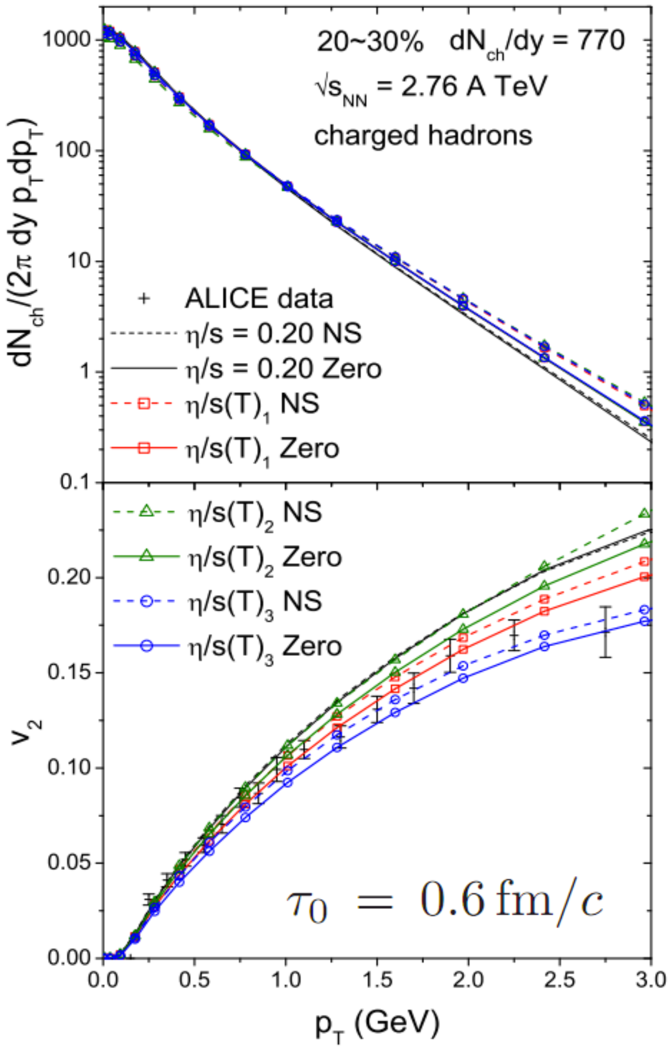


Flow results seem to be insensitive to even large persistent momentum-space anisotropies!

Radial and elliptic flow in Pb+Pb collisions at the Large Hadron Collider from viscous hydrodynamics

arXiv:1105.3226v2 [nucl-th] 9 Sep 2011

Chun Shen,[1,*] Ulrich Heinz,[1,†] Pasi Huovinen,[2,‡] and Huichao Song[3,§]



η/s model	$\pi_0^{\mu\nu}$	s_0 (fm^{-3})	T_0 (MeV)
$\eta/s = 0.2$	0	191.6	427.9
	NS	172.4	413.9
$(\eta/s)_1(T)$	0	179.6	419.2
	NS	119.3	368.7
$(\eta/s)_2(T)$	0	179.6	419.2
	NS	115.6	365.1
$(\eta/s)_3(T)$	0	175.2	416.0
	NS	116.6	366.1

- Considered two different initial conditions for the shear tensor
- Isotropic and Navier-Stokes (NS)
- For NS at the initial time shown, the longitudinal pressure is negative!

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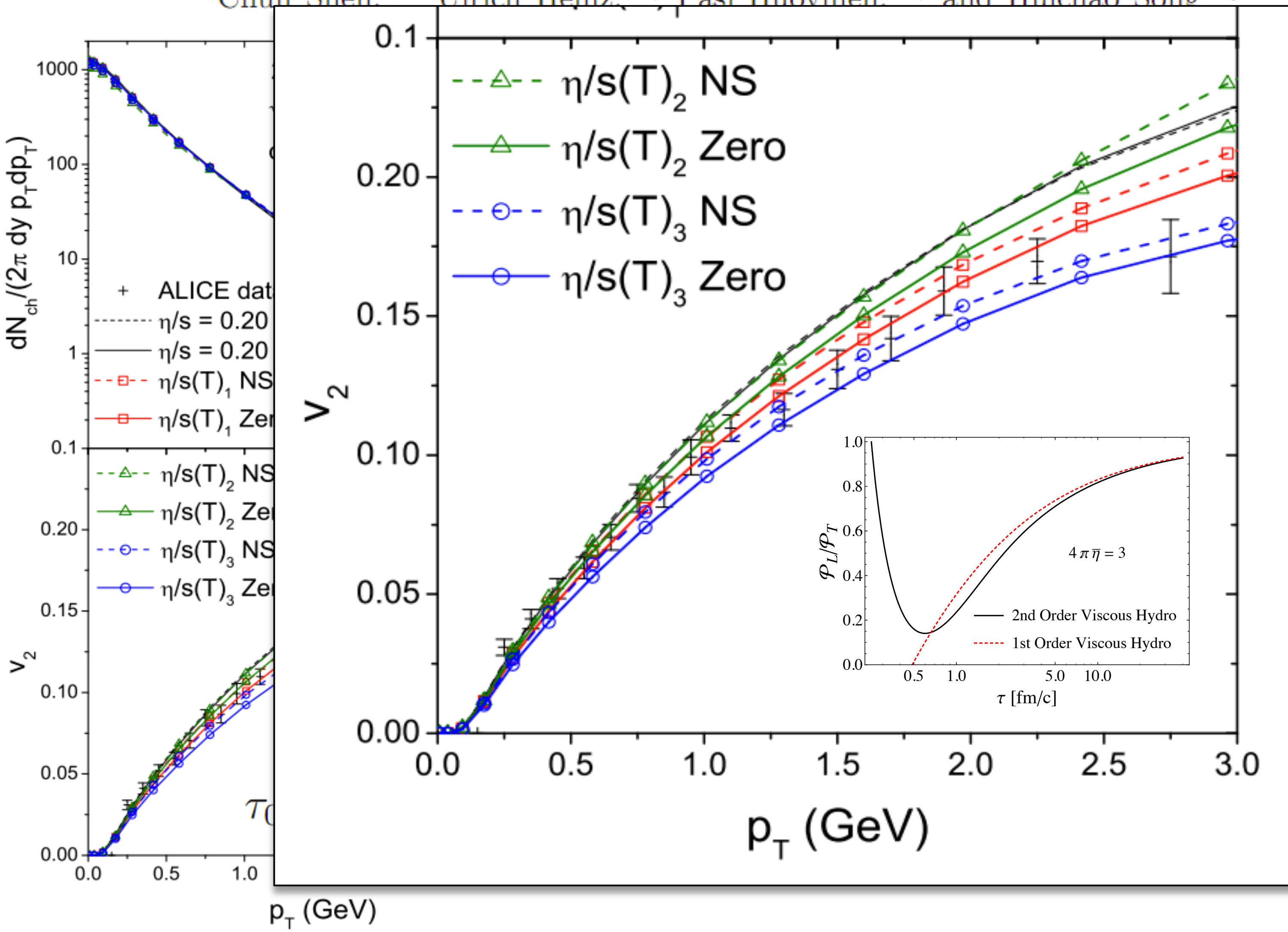

T_0 (MeV)
427.9
413.9
419.2
368.7
419.2
365.1
416.0
366.1

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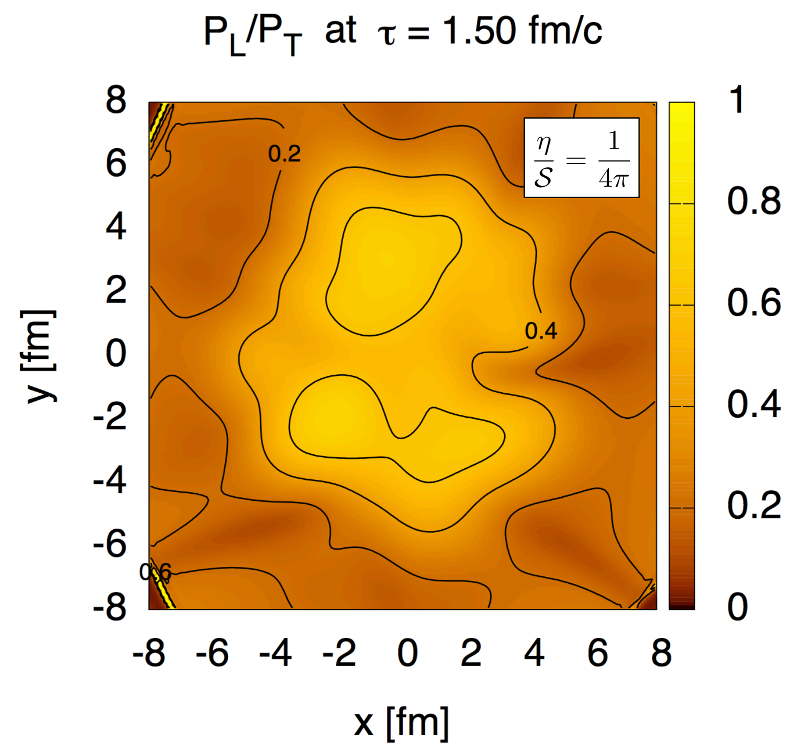
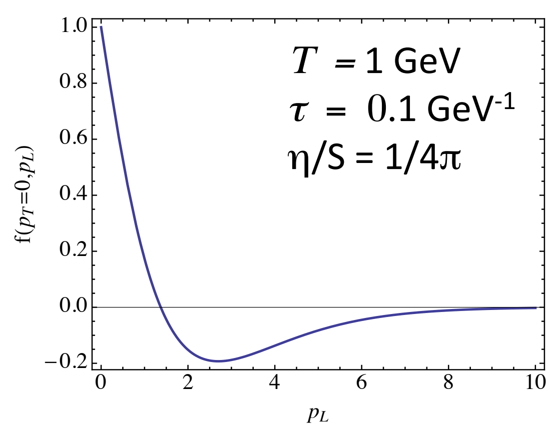
Some non-flow observables that are sensitive to anisotropies

- **Jet collisional and radiative energy loss**
  Romatschke & MS : hep-ph/0408275
  Dumitru, MS, et al: 0710.1223
  Schenke, MS, et al : 0810.1314
- **Photons**
  Schenke & MS: hep-ph/0611332
  McLerran & Schenke: 1403.7462
  Ipp et al: 0710.5700 (Polarization)
- **Dileptons**
  Martinez & MS: 0805.4552
  Martinez & MS: 0808.3969
- **Quarkonium suppression**
  MS: arXiv:1106.2571, arXiv:1112.2761
  Dumitru, Guo, & MS: 0711.4722
  Dumitru, Guo, & MS: 0903.4703



Viscous hydro (shear correction)

Dileptons & Photons
Dusling : 0803.1262
Dusling : 0903.1764
Dion et al : 1403.7462
Chen et al : 1403.7558, 1308.2111



Anisotropic Heavy Quark Potential

Using real-time formalism one can express potential in terms of *static* advanced, retarded, and Feynman propagators

$$\boxed{V(\mathbf{r},\xi) = -g^2 C_F \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,(e^{i\mathbf{p}\cdot\mathbf{r}} - 1)\frac{1}{2}\left(D^{*}{}^{L}_{R} + D^{*}{}^{L}_{A} + D^{*}{}^{L}_{F}\right)}$$

Real part can be written as

$$\mathrm{Re}[V(\mathbf{r},\xi)] = -g^2 C_F \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{r}} \frac{\mathbf{p}^2 + m_\alpha^2 + m_\gamma^2}{(\mathbf{p}^2 + m_\alpha^2 + m_\gamma^2)(\mathbf{p}^2 + m_\beta^2) - m_\delta^4}$$

With <u>direction-dependent masses</u>, e.g.

$$m_\alpha^2 = -\frac{m_D^2}{2p_\perp^2\sqrt{\xi}}\left(p_z^2 \arctan\sqrt{\xi} - \frac{p_z \mathbf{p}^2}{\sqrt{\mathbf{p}^2 + \xi p_\perp^2}} \arctan\frac{\sqrt{\xi}p_z}{\sqrt{\mathbf{p}^2 + \xi p_\perp^2}}\right)$$

Anisotropic potential calculation: Dumitru, Guo, and MS, 0711.4722 and 0903.4703
Gluon propagator in an anisotropic plasma: Romatschke and MS, hep-ph/0304092

Full anisotropic potential

- Debye-screened potential with a Debye mass that depends on the angle of the line between the quark-antiquark pair and the longitudinal direction

- The potential also has an imaginary part coming from the Landau damping of the exchanged gluon!

- This imaginary part also exists in the isotropic case
[Laine et al hep-ph/0611300]

$$V(r,\theta,\xi,p_{\rm hard}) = -C_F\alpha_s \frac{e^{-\mu(\theta,\xi,p_{\rm hard})r}}{r}$$

D Bazow and MS, 1112.2761; MS, 1106.2571.

$$\begin{aligned} V_{\rm R}(\mathbf{r}) = &-\frac{\alpha}{r}\,(1+\mu\,r)\exp\left(-\mu\,r\right) \\ &+\frac{2\sigma}{\mu}\left[1-\exp\left(-\mu\,r\right)\right] \\ &-\sigma\,r\;\exp(-\mu\,r)-\frac{0.8\,\sigma}{m_Q^2\,r} \end{aligned}$$

Dumitru, Guo, Mocsy, and MS, 0901.1998

$$V_{\rm I}(\mathbf{r}) = -C_F\alpha_s p_{\rm hard}\Big[\phi(\hat{r})-\xi\left(\psi_1(\hat{r},\theta)+\psi_2(\hat{r},\theta)\right)\Big]$$

Dumitru, Guo, and MS, 0711.4722 and 0903.4703

Solve the 3d Schrödinger EQ with complex-valued potential

MS and Yager-Elorriaga, 1101.4651; Margotta, MS, et al, 1101.4651

Obtain real and imaginary parts of the binding energies for the $\Upsilon(1s)$, $\Upsilon(2s)$, $\Upsilon(3s)$, χ_{b1}, χ_{b2}

Focus on Bottomonium – Why?

1. Bottom quarks ($m_b \approx 4.2$ GeV) are more massive than charm quarks ($m_c \approx 1.3$ GeV) and as a result the heavy quark effective theories underpinning phenomenological applications are on much surer footing.
2. Due to their higher mass, the effects of initial state nuclear suppression are expected to be smaller than for the charmonium states.
3. The masses of bottomonium states ($m_\Upsilon \approx 10$ GeV) are much higher than the temperatures ($T < 1$ GeV) generated in relativistic heavy ion collisions → bottomonia production will be dominated by initial hard scatterings.
4. Since bottom quarks and anti-quarks are relatively rare within the plasma, the probability for regeneration of bottomonium states through recombination is much smaller than for charm quarks.

Vacuum Quarkonia Spectra

J. Alford and MS, 1309.3003

Bottomonia

State	Name	Exp. [92]	Model	Rel. Err.
1^1S_0	$\eta_b(1S)$	9.398 GeV	9.398 GeV	0.001%
1^3S_1	$\Upsilon(1S)$	9.461 GeV	9.461 GeV	0.004%
1^3P_0	$\chi_{b0}(1P)$	9.859 GeV	9.869 GeV	0.21%
1^3P_1	$\chi_{b1}(1P)$	9.893 GeV		
1^3P_2	$\chi_{b2}(1P)$	9.912 GeV		
1^1P_1	$h_b(1P)$	9.899 GeV		
2^1S_0	$\eta_b(2S)$	9.999 GeV	9.977 GeV	0.22%
2^3S_1	$\Upsilon(2S)$	10.002 GeV	9.999 GeV	0.03%
2^3P_0	$\chi_{b0}(2P)$	10.232 GeV	10.246 GeV	0.05%
2^3P_1	$\chi_{b1}(2P)$	10.255 GeV		
2^3P_2	$\chi_{b2}(2P)$	10.269 GeV		
2^1P_1	$h_b(2P)$	-		
3^1S_0	$\eta_b(3S)$	-	10.344 GeV	-
3^3S_1	$\Upsilon(3S)$	10.355 GeV	10.358 GeV	0.03%

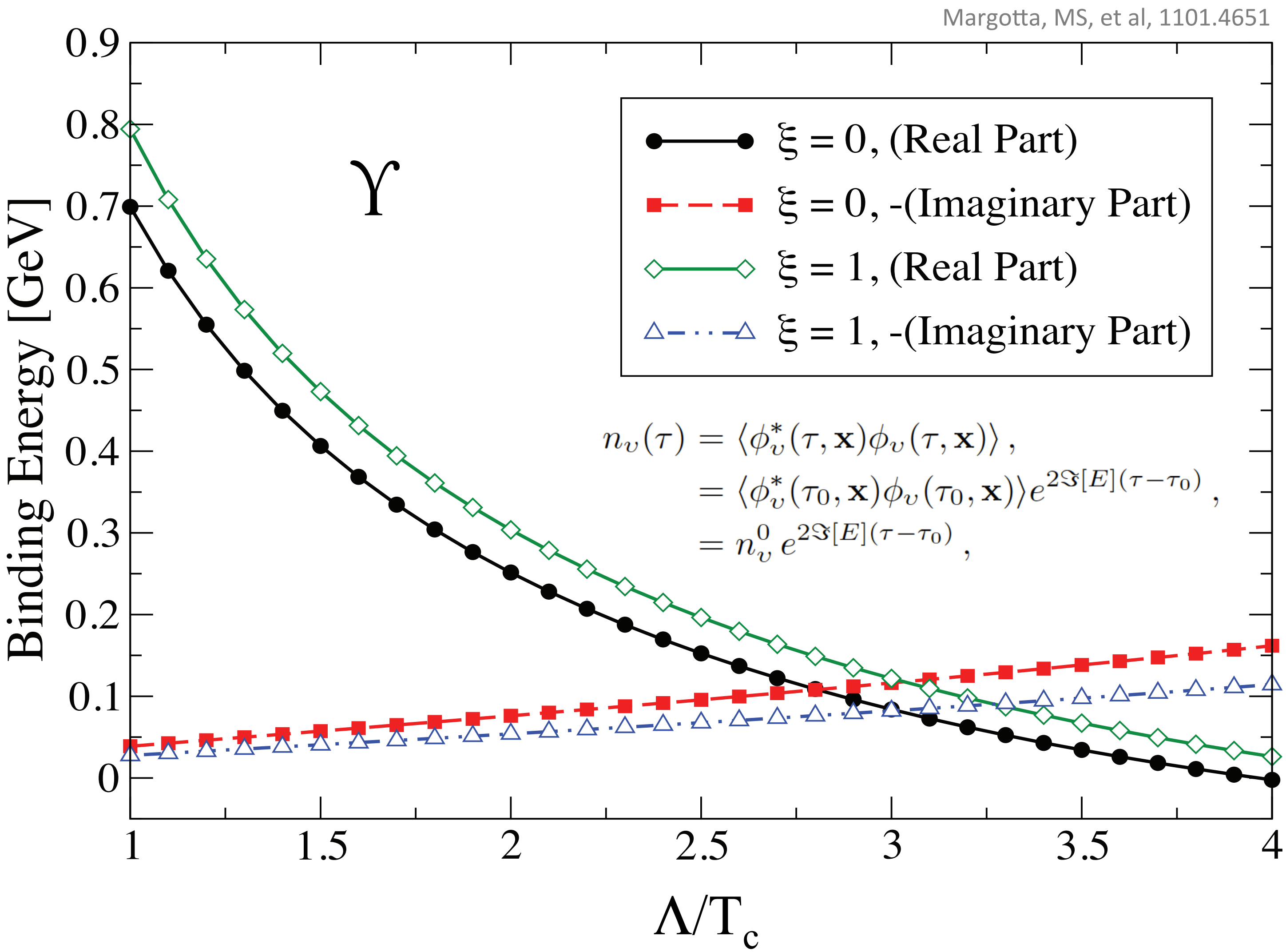
Charmonia

State	Name	Exp. [92]	Model	Rel. Error
1^1S_0	$\eta_c(1S)$	2.984 GeV	3.048 GeV	2.2%
1^3S_1	$J/\psi(1S)$	3.097 GeV	3.100 GeV	0.11%
2^1S_0	$\eta_c(2S)$	3.639 GeV	3.721 GeV	2.3%
2^3S_1	$J/\psi(2S)$	3.686 GeV	3.748 GeV	1.7%

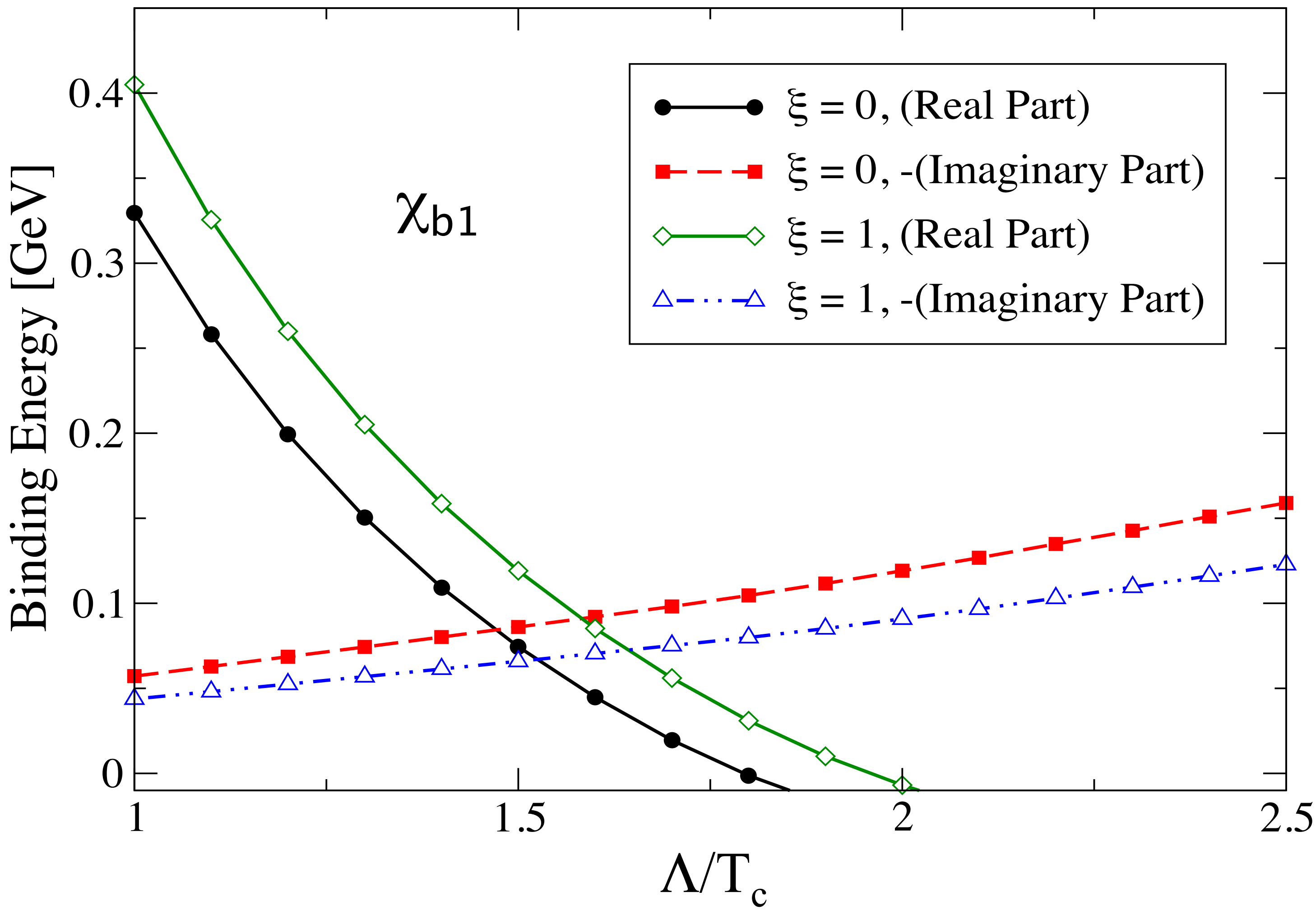
- With a simple pNRQCD potential model one can describe the known bottomonia state masses with a maximum error of 0.22%

- The situation with charmonia is a bit worse and one has to add lots of relativistic corrections with additional parameters.

Results for the Υ(1s) binding energy

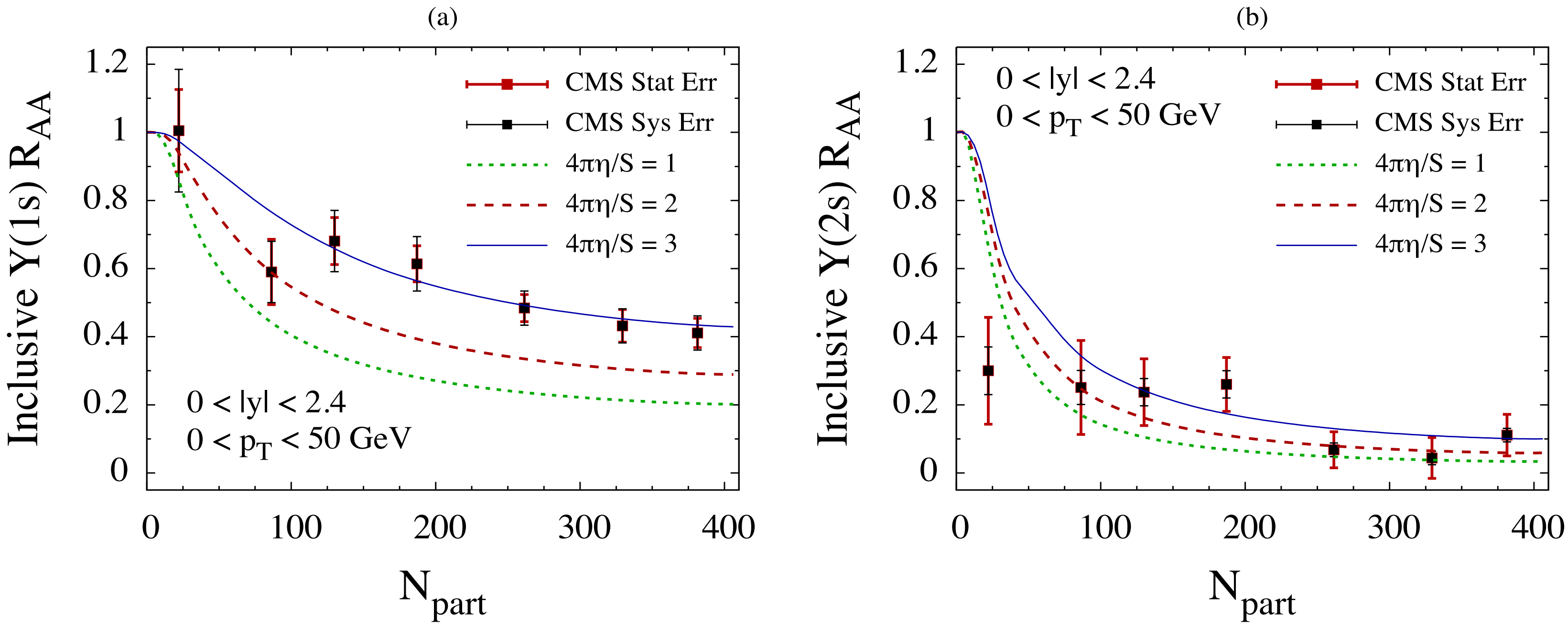


Results for the χ_{b1} binding energy

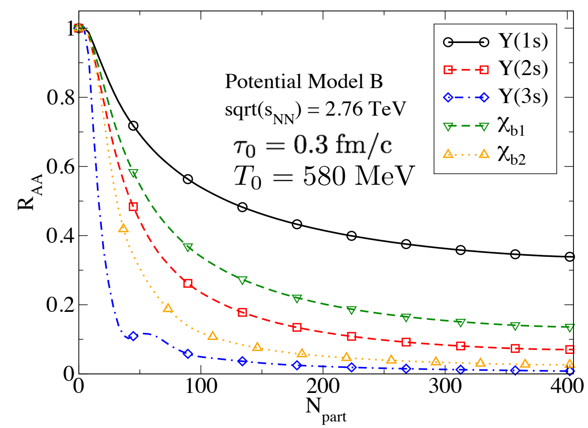


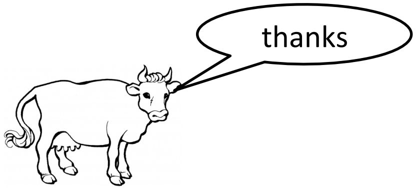
Inclusive Bottomonium Suppression

MS, arXiv:1207.5327; MS and D. Bazow, arXiv:1112.2761; MS arXiv:1106.2571



Compute inclusive ϒ(1s) and ϒ(2s) suppression including effects of feed-down, formation time, and aHydro evolution with anisotropic complex-valued quarkonium potential.





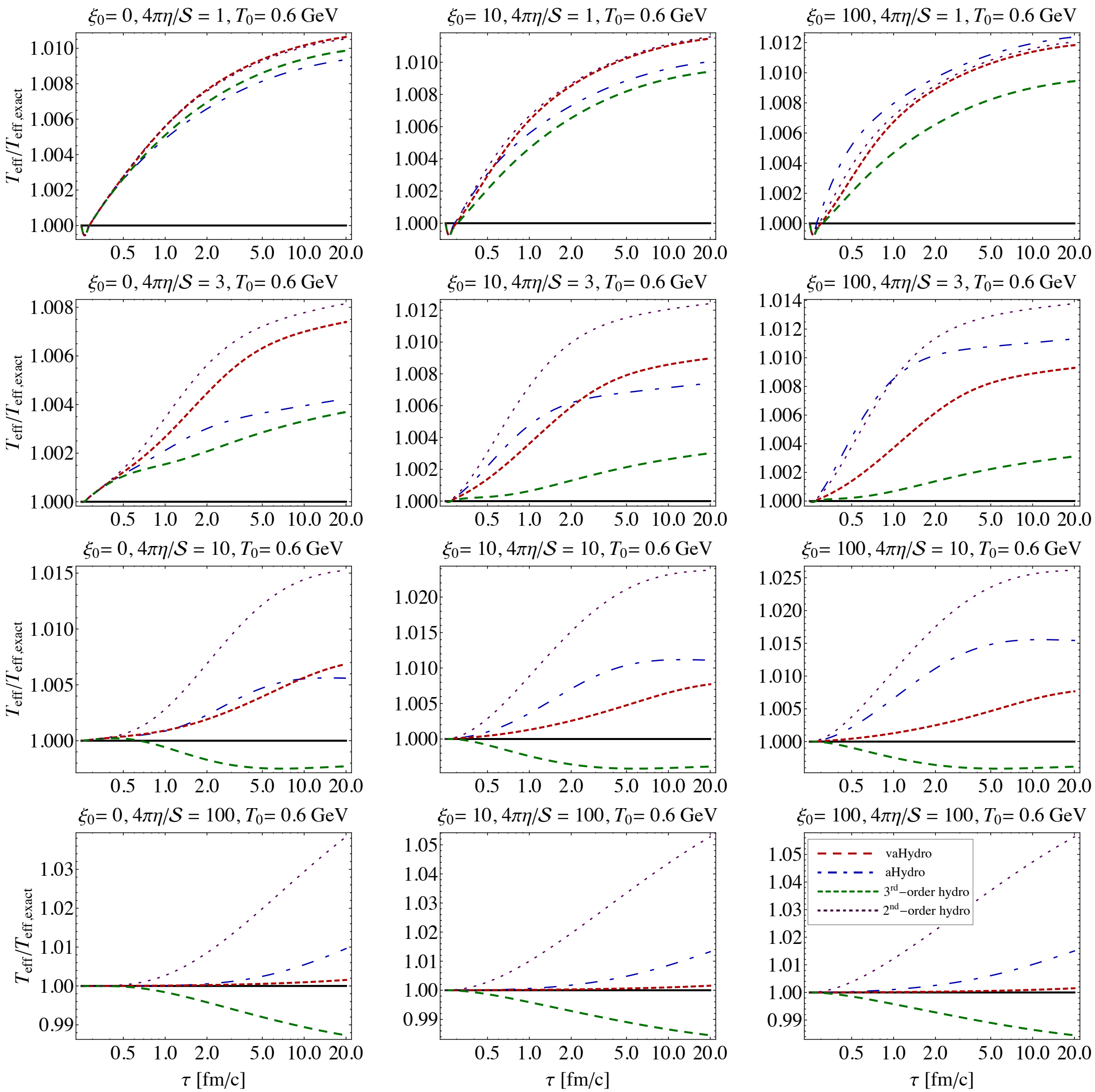
Lecture 3 - Conclusions

- Anisotropic hydrodynamics builds upon prior advances in relativistic hydrodynamics in an attempt to create an even more quantitatively reliable tool
- It incorporates some "facts of life" specific to the conditions generated in relativistic heavy ion collisions and, in doing so, optimizes the non-ideal hydrodynamics approach
- Having second-order anisotropic hydrodynamics (NLO aHydro) allows us to proceed to numerical modeling of heavy ion collisions
- The evolution of the matter (particularly at early times, near the transverse edges, or with large temperature-dependent η/S) should now be more reliably described
- Since we now know that the plasma is anisotropic, there needs to be serious reconsideration of the calculation of QGP signatures which traditionally have been computed assuming an isotropic thermal state.

Backup Slides

Effective Temperature Comparisons

[D. Bazow, U. Heinz, and MS, 1311.6720]



- But maybe I'm cheating and only showing you one measure? Let's check the temperature to make sure all is good…
- Panels show relative error in the effective temperature
- Same params as the previous slide etc.
- Once again, vaHydro "outperforms" all competitors
- That being said, one should note the scale on the axes here. All approximations considered are quite accurate for the effective temperature evolution.