QCD at Short Distances: Jets and Factorization

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I. A review of high energy jets and their factorization properties

(not very formal)

II. An introduction to all-orders analysis in perturbative QCD

(more formal)

II. An introduction to all-orders analysis in perturbative QCD

1. Using asymptomatic freedom: infrared safety

2. Infrared safety for cross sections: Jets and Event Shapes

3. Factorization and soft radiation

4. Cancellation and the importance of remainders

Appendix: The transition to perturbative QCD: Pinch surfaces, power counting and Ward identities

1. Using aymptotic freedom: infrared safety

- To use perturbation theory in QCD, would like to choose the renormalization scale μ 'as large as possible' to make $\alpha_s(\mu)$ as small as possible.

- But how small is possible?

- A "typical" (dimensionless) cross section, define $Q^2 = s_{12}$ and $x_{ij} = s_{ij}/Q^2$,

$$\sigma\left(\frac{Q^2}{\mu^2}, x_{ij}, \frac{m_i^2}{\mu^2}, \alpha_s(\mu)\right) = \sum_{n=1}^{\infty} a_n\left(\frac{Q^2}{\mu^2}, x_{ij}, \frac{m_i^2}{\mu^2}\right)\alpha_s^n(\mu)$$

  with m_i^2 all fixed masses – external, quark, gluon (=0!)

- Generically, the a_n depend logarithmically on their arguments, so a choice of large μ results in large logs of m_i^2/μ^2.

- **But if we <u>could</u> find quantities that depend on $m_i's$ only through powers, $(m_i/\mu)^p, p > 0$, the large-μ limit would exist.**

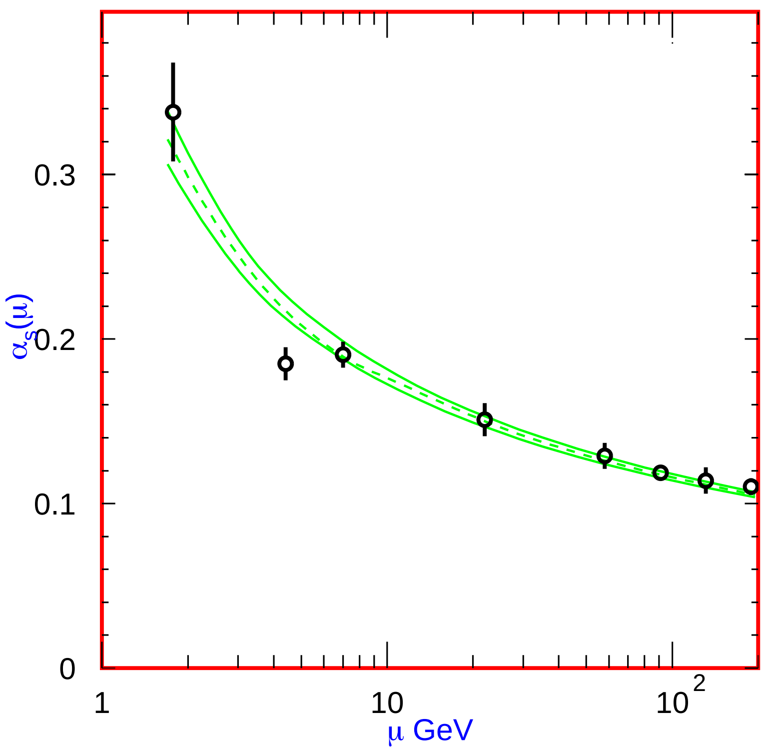
$$\sigma\left(\frac{Q^2}{\mu^2}, x_{ij}, \frac{m_i^2}{\mu^2}, \alpha_s(\mu)\right) = \sum_{n=1}^{\infty} a_n\left(\frac{Q}{\mu}, x_{ij}\right) \alpha_s^n(\mu) + \mathcal{O}\left(\left[\frac{m_i^2}{\mu^2}\right]^p\right)$$

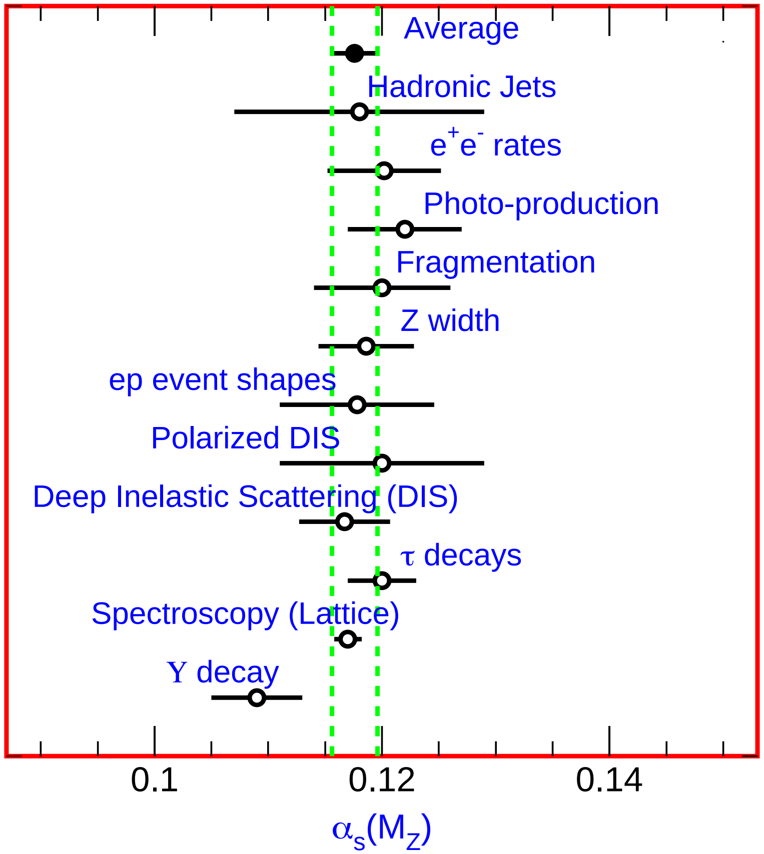
- **Such quantities are called infrared (IR) safe.**

- **Measure $\sigma \to$ solve for α_s. Allows observation of the running coupling.**

- **Most of pQCD is the computation of IR safe quantities.**

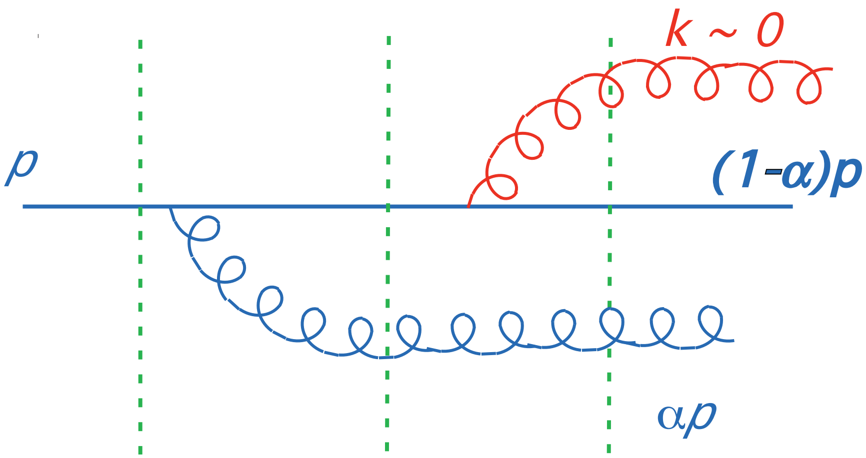
- **Consistency of $\alpha_s(\mu)$ found as above at various momentum scales**
  Each comes from identifying an IR safe quantity, computing it and comparing the result to experiment. **(Particle Data Group)**





- **To find IR safe quantities, need to understand where the low-mass logs come from.**

- To analyze diagrams, we generally think of $m \to 0$ limit in m/Q. Gives "IR" logs.
- Generic source of IR (soft and collinear) logarithms:



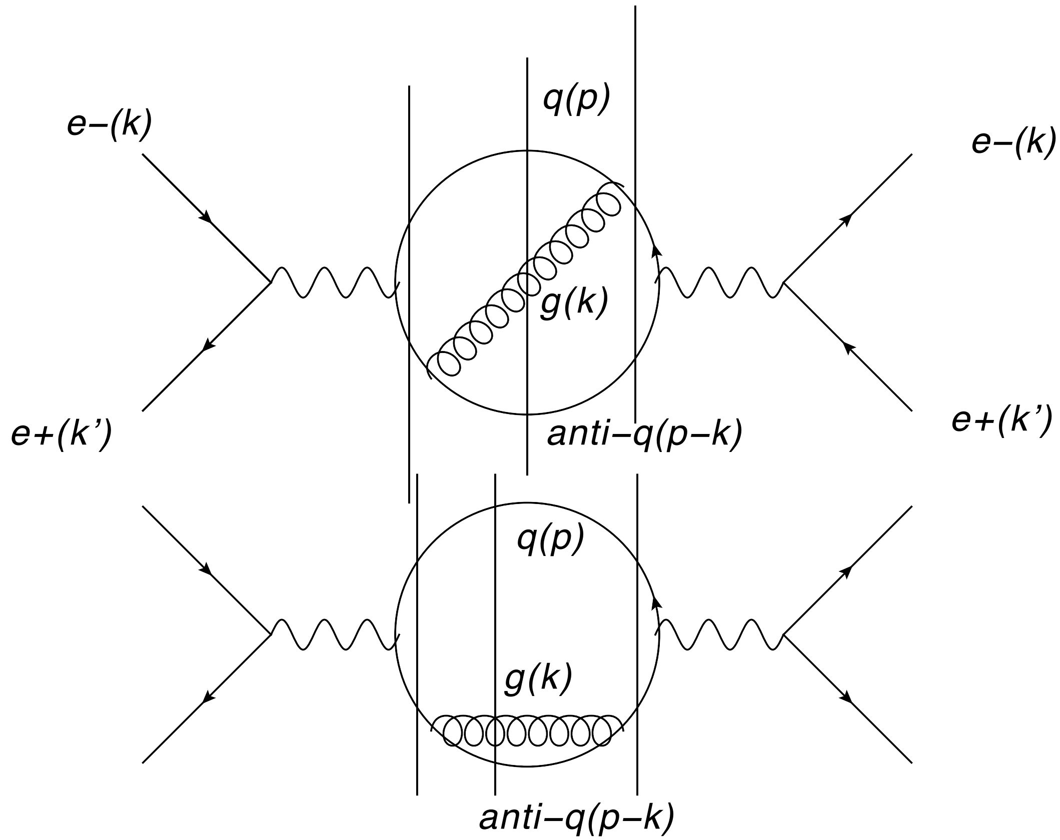
- IR logs come from degenerate states: Uncertainty principle $\Delta E \to 0 \Leftrightarrow \Delta t \to \infty$.
- After a while, noncollinear particles are too separated to interact. For soft emission and collinear splitting it's "never too late". But these processes don't change the flow of energy ...
- This will follow from a general analysis of Feynman diagrams, and leads to IR safety of jet cross sections.
- For IR safety, sum over degenerate final states in perturbation theory, and don't ask how many particles of each kind we have. This requires us to introduce another regularization, this time for IR behavior.

- **The IR regulated theory is like QCD at short distances, but is better-behaved at long distances.**

- **IR-regulated QCD <u>not the same as QCD except</u> for IR safe quantities.**

- **Similar considerations apply to factorized cross sections and amplitudes.**

- See how it works for the total e^+e^- annihilation cross section to order α_s. Lowest order is $2 \to 2, \sigma_2^{(0)} \equiv \sigma_{\rm LO}$, σ_3 starts at order α_s.
- The cross section, shown in "cut diagram" organization (amplitude and complex conjugate)



– **Each final state (cut) is divergent for m_g, m_q $\to 0$. So try and regularize, then combine final states. Two representative choices:**

– **i) Gluon mass regularization:** $1/k^2 \to 1/(k^2 - m_G)^2$

$$\sigma_3^{(m_G)} = \sigma_{\rm LO} \frac{4}{3}\frac{\alpha_s}{\pi}\left(2\ln^2\frac{Q}{m_g} - 3\ln\frac{Q}{m_g} - \frac{\pi^2}{6} + \frac{5}{2}\right)$$
$$\sigma_2^{(m_G)} = \sigma_{\rm LO}\left[1 - \frac{4}{3}\frac{\alpha_s}{\pi}\left(2\ln^2\frac{Q}{m_g} - 3\ln\frac{Q}{m_g} - \frac{\pi^2}{6} + \frac{7}{4}\right)\right]$$

which gives

$$\sigma_{\rm tot} = \sigma_2^{(m_G)} + \sigma_3^{(m_G)} = \sigma_{\rm LO}\left[1 + \frac{\alpha_s}{\pi}\right]$$

– **Pretty simple! (Cancellation of virtual (σ_2) and real (σ_3) gluon diagrams.)**

– ii) Dimensional regularization: change the area of a sphere of radius R

$$4\pi R^2 \Rightarrow (4\pi)^{(1-\varepsilon)} \frac{\Gamma(1-\varepsilon)}{\Gamma(2(1-\varepsilon))} R^{2-2\varepsilon}$$

with $\varepsilon = 2 - D/2$ in D dimensions, and the coupling $g_s \to g_s \mu^{\varepsilon}$.

– Do the integrals this way, and get

$$\begin{aligned}
\sigma_3^{(\varepsilon)} &= \sigma_{\mathrm{LO}} \frac{4}{3}\frac{\alpha_s}{\pi} \left(\frac{(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} \right) \left(\frac{4\pi\mu^2}{Q^2} \right)^{\varepsilon} \\
&\times \left(\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - \frac{\pi^2}{2} + \frac{19}{4} \right) \\
\sigma_2^{(\varepsilon)} &= \sigma_{\mathrm{LO}} \left[1 - \frac{4}{3}\frac{\alpha_s}{\pi} \left(\frac{(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} \right) \left(\frac{4\pi\mu^2}{Q^2} \right)^{\varepsilon} \right. \\
&\left. \times \left(\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - \frac{\pi^2}{2} + 4 \right) \right]
\end{aligned}$$

which gives again

$$\sigma_{\mathrm{tot}} = \sigma_2^{(m_G)} + \sigma_3^{(m_G)} = \sigma_0 \left[1 + \frac{\alpha_s}{\pi} \right]$$

- **This illustrates IR Safety: σ_2 and σ_3 depend on regulator, but their sum does not.**

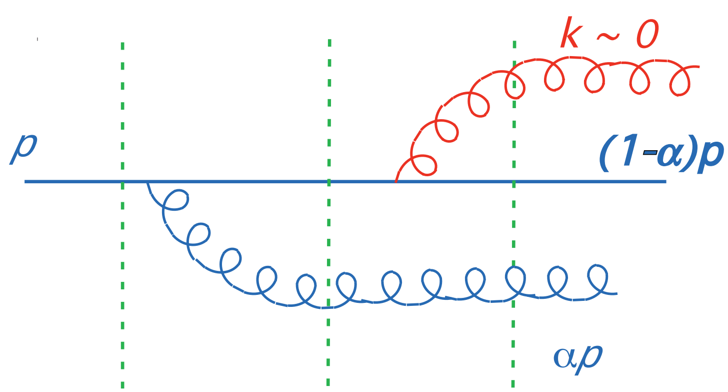
- **Generalized IR safety: sum over all states with the same flow of energy into the final state. Introduce** *IR safe weight* "$e(\{p_i\})$"

$$\frac{d\sigma}{de} = \sum_n \int_{PS(n)} |M(\{p_i\})|^2 \delta\left(e(\{p_i\}) - e\right)$$

with

$$e(\ldots p_i \ldots p_{j-1}, \alpha p_i, p_{j+1} \ldots) = e(\ldots (1+\alpha) p_i \ldots p_{j-1}, p_{j+1} \ldots)$$

- **Neglect long times in the initial state for the moment and see how this works in** e^+e^- **annihilation: event shapes and jet cross sections. Again . . .**



The transition to perturbative QCD: Pinch surfaces, power counting and Ward identities (See Appendix)

- **B. To generalize the above and prove QCD properties like IR Safety of jet & related of cross sections for e^+e^- to all orders, and**
- **C. Provide a basis for factorization in ep, pp inclusive cross sections and exclusive amplitudes**

- **What we need (all orders):**

  A Method to identify infrared sensitivity in PT: "physical pictures" (see part I)

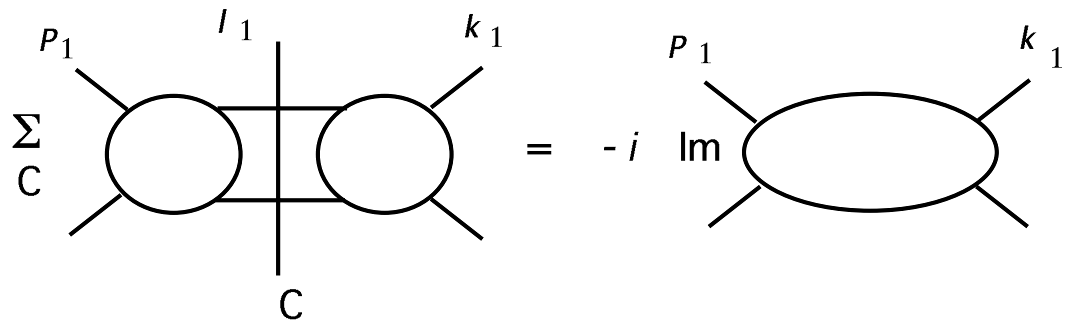
  B Method to identify IR finiteness/divergence: "power counting"

  – **These are outlined in an appendix for those interested in some (not all!) details. Our example: factorization for elastic amplitudes for parton-parton scattering in ϕ^4 and QCD. The roles of Ward identities and Wilson lines. The purpose is to describe, not perform, explicit calculations and to describe the ultimate justification of factorization.**

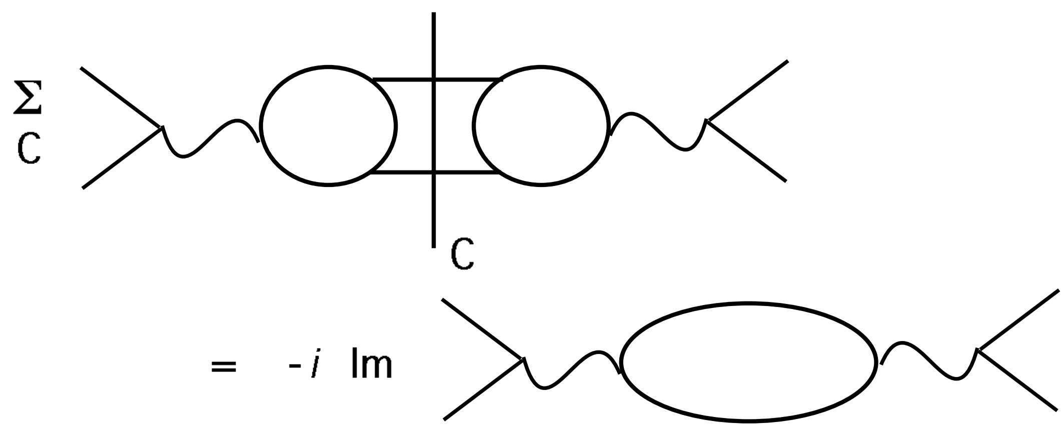
2. Infrared safety for cross sections: jets and event shapes

Cross sections, cut diagrams and generalized unitarity

– **Applications to cross sections**



– **Or for** e^+e^-**,**



– **Unitarity for a total cross section or decay rate:**

– **Unitarity in terms of $(S = 1 - iT)$**

$$TT^\dagger = -i(T - T^\dagger)\,.$$

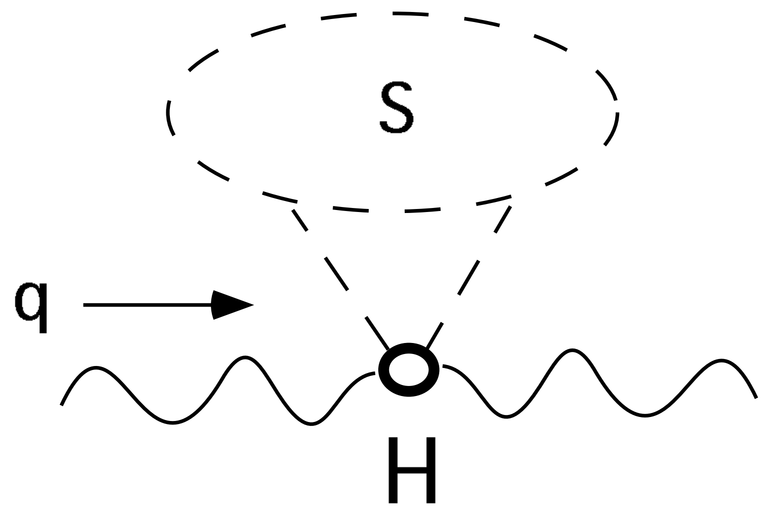
– **Infrared safety for inclusive annihilation and decay**

$$\sigma^{(\mathrm{tot})}_{e^+e^-}(q^2) = \frac{e^2}{q^2}\mathrm{Im}\,\pi(q^2)\,,$$

where the function π is defined in terms of the two-point correlation function of the relevant electroweak currents J_μ (with their couplings included) as

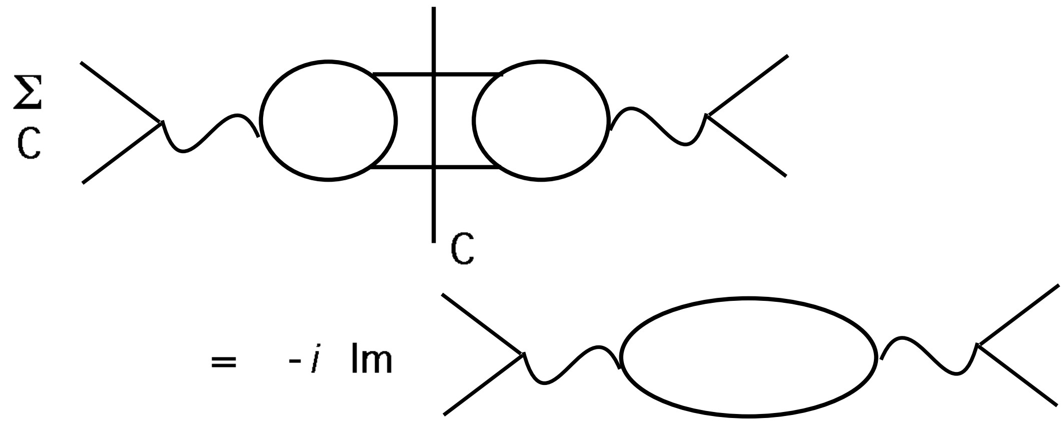
$$\pi(q^2)(q_\mu q_\nu - q^2 g_{\mu\nu}) = i \int d^4x\, e^{iqx} < 0|\; T\; J_\mu(x) J_\nu(0)\; |0 >$$

– **The only physical pictures for $\langle JJ \rangle$ and hence for π:**



– **Power counting confirms finiteness.**

– **But the method is much more general – unitarity holds point-by-point in** *spatial* **loop momenta** $\vec{l}$ **of relation**



$$\sum_{\text{all } C} G_C(p_i, k_j, \vec{l}) = 2 \ \mathrm{Im} \ \left(-iG(p_i, k_j, \vec{l})\right) \ .$$

– Proof (and the origin of jet analysis): Do the time integrals for a general amplitude in part I, and get time-ordered perturbation theory. This is equivalent to the sum over Feynman diagrams. The amplitude and its compilex conjugate are given by a sum over virtual states:

$$\begin{aligned}\sum_m \Gamma_m^* \Gamma_m &= \sum_{m=1}^{A} \prod_{j=m+1}^{A} \frac{1}{E_j - S_j - i\epsilon} \\ &\quad \times (2\pi)\delta(E_m - S_m) \prod_{i=1}^{m-1} \frac{1}{E_i - S_i + i\epsilon} \\ &= -i \left[- \prod_{j=1}^{A} \frac{1}{E_j - S_j + i\epsilon} + \prod_{j=1}^{A} \frac{1}{E_j - S_j - i\epsilon} \right]\end{aligned}$$

– From

$$i \left(\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} \right) = 2\pi\delta(x)$$

At the level of the loop integrands of TOPT.

$\Rightarrow$ the sum of cut diagrams has the same set of Landau equations as total cross section. Hence the sum at fixed loop momenta is infrared safe.

– Varieties of jet cross sections

– Cone size, energy in cone, cone direction

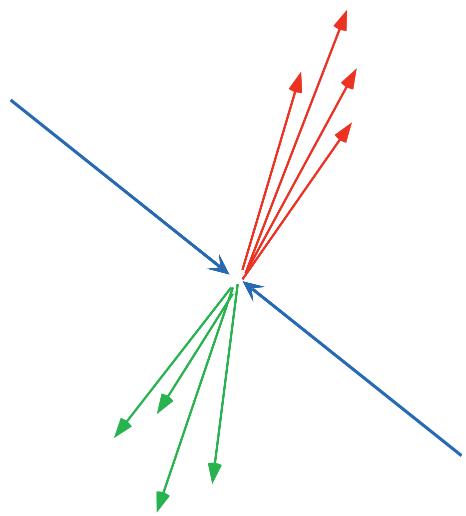
– Thrust Shape Variable ($T = 1$ for "back-to-back" jets)

$$\boldsymbol{T} = \frac{1}{s}\mathrm{max}_{\hat{n}} \sum_i |\hat{n} \cdot \vec{p}_i|$$

$$\boldsymbol{T} = \frac{1}{\sqrt{s}}\mathrm{max}_{\hat{n}} \sum_i |\hat{n} \cdot \vec{p}_i| \,,$$

– $e = 1 - T \Rightarrow 0 \leftrightarrow$ *two narrow jets*

$$e_T = 1 - T \sim \frac{M_R^2 + M_L^2}{Q^2}$$



– With $\hat{z}$ along thrust axis in "R" hemisphere:

$$e_T = \frac{1}{Q}\left[\sum_{i\in H_R} k_i^- + \sum_{i\in H_L} k_i^+\right] = \frac{1}{Q}\sum_{\text{all } i} k_{iT}\, \mathrm{e}^{-|\eta|}$$

– Example of generalized IR safe dijet weights

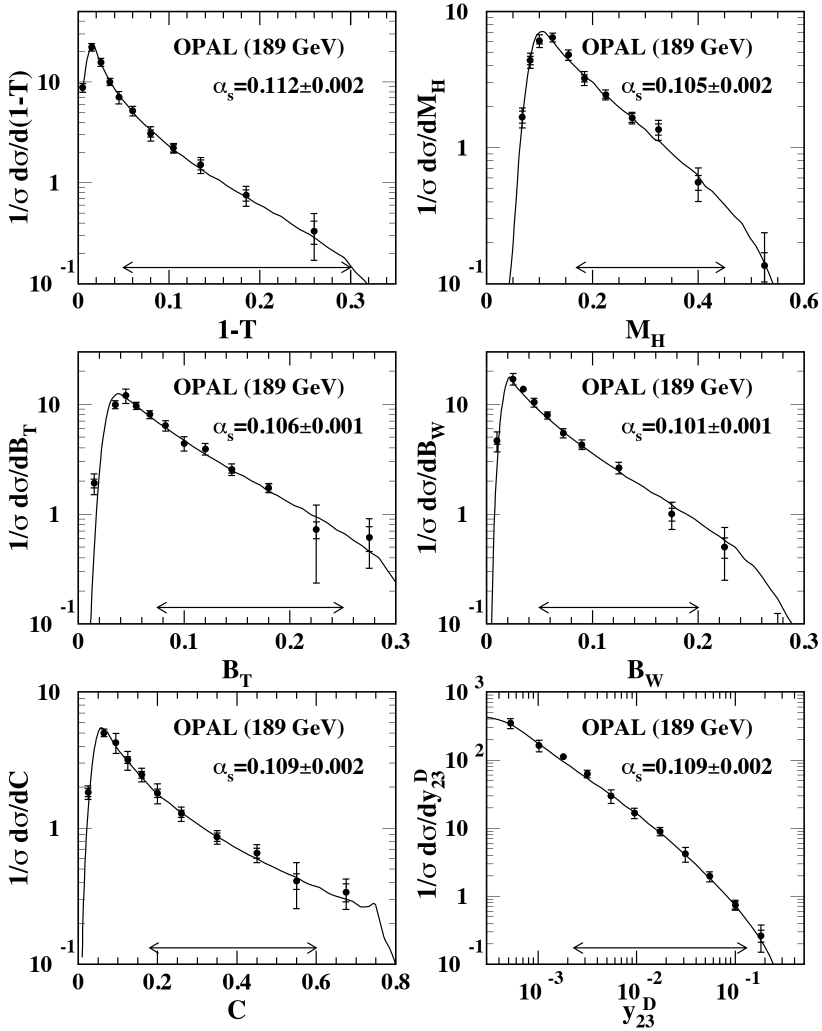
$$e_a = \frac{1}{Q}\sum_{\text{all } i} k_{iT}\, \mathrm{e}^{-|\eta|(1-a)}$$

* $e_0 = e_T$ is $1 - T$
* e_1 is jet broadening B

– y_{cut} k_T Cluster Algorithm: $y_{ij} > y_{\text{cut}}$,

$$y_{ij} = 2\text{min}\left(E_i^2, E_j^2\right)(1 - \cos\theta_{ij})$$

– **From OPAL at LEP: PT (+ power-suppressed nonperturbative corrections) vs. jet shapes**



– **Cluster variables for hadronic collisions:**

$$d_{ij} = \min\left(k_{ti}^{2p}\,,\,k_{tj}^{2p}\right)\frac{\Delta_{ij}^2}{R^2}$$

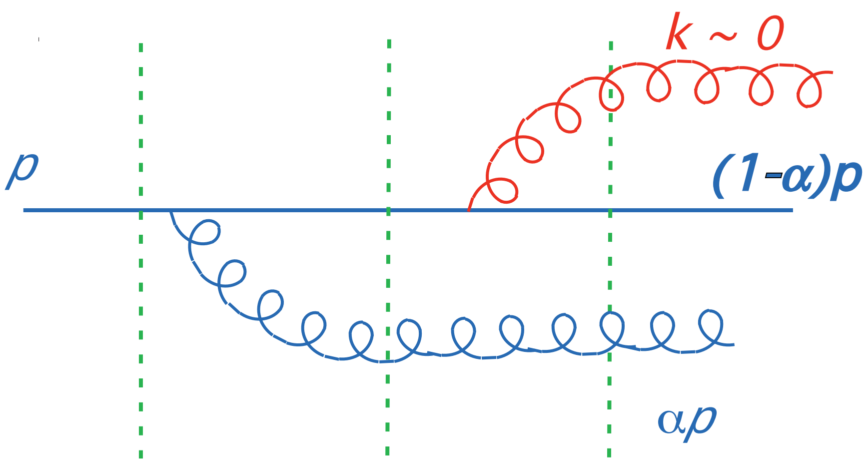
$\Delta_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$. R **is an adjustable parameter.**

– **The "classic" choices:**

- $p = 1$ **"k_t algorithm:**
- $p = 0$ **"Cambridge/Aachen"**
- $p = -1$ **"anti-k_t"**

Summarize: what makes a cross section infrared safe?

– **Independence of long-time interactions:**



More specifically: should depend on only the flow of energy into the final state. This implies independence of collinear re-arrangements and soft parton emisssion.

But if we prepare one or two particles in the initial state (as in DIS or proton-proton scattering), we will always be sensitive to long time behavior inside these particles. The parton model suggests what to do: factorize.

Summary continued: what is infrared safety?

- QCD perturbation theory gives self-consistent predictions for a quantity C when C:
  - is dominated by short-distance dynamics in the infrared-regulated theory;
  - remains finite when the regulation is taken away.
- Parton-hadron duality: supplement by identification of parton & hadron multiplicities

3. Factorization and soft radiation

$$Q^2\sigma_{\rm phys}(Q,m,f) \;=\; \omega_{\rm SD}(Q/\mu,\alpha_s(\mu),f)\otimes f_{\rm LD}(\mu,m) + \mathcal{O}\left(\frac{1}{Q^p}\right)$$

μ = factorization scale;
m= IR scale **(m *may* be perturbative)**

– **"New physics" in $\omega_{\rm SD}$; $f_{\rm LD}$ "universal"**

– **Almost all collider applications. Enables us to compute the Energy-transfer-dependence in $|\langle Q, \text{out}|A+B, \text{in}\rangle|^2$.**

– **But again, requires a smooth weight for final states.**

Resummation?

– **Whenever there is factorization, there is evolution**

$$0 = \mu \frac{d}{d\mu} \ln \sigma_{\rm phys}(Q, m)$$

$$\mu \frac{d \ln f}{d\mu} = -P(\alpha_s(\mu)) = -\mu \frac{d \ln \omega}{d\mu}$$

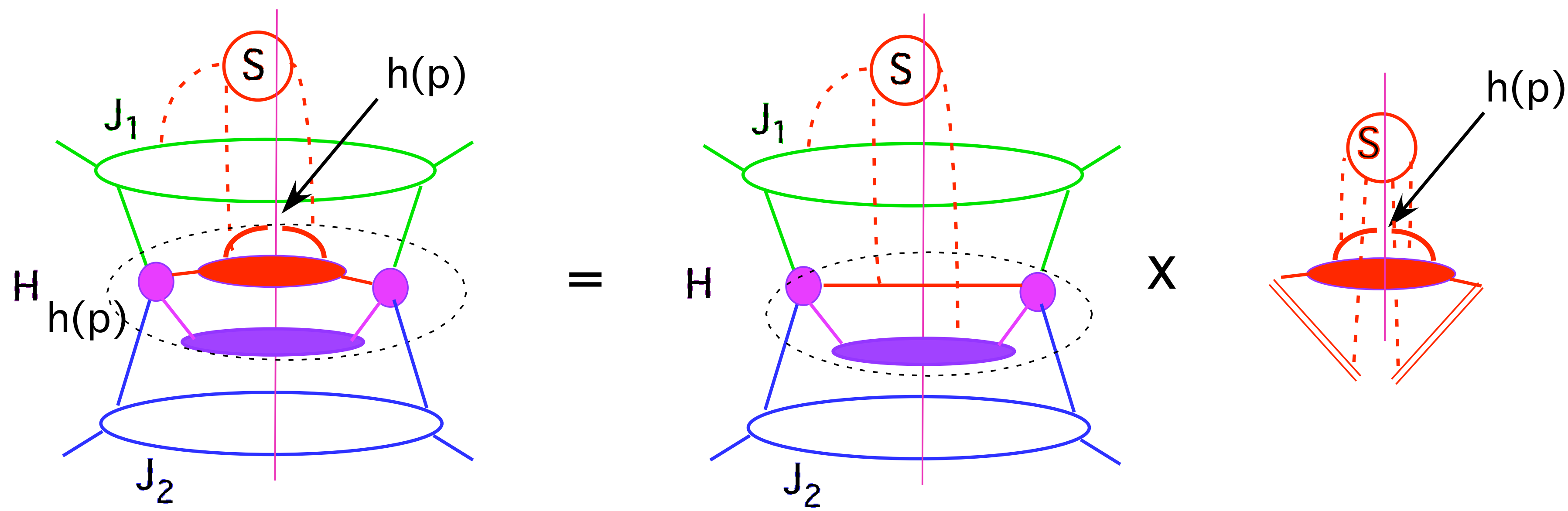
– **Wherever there is evolution there is resummation,**

$$\sigma_{\rm phys}(Q, m) = \sigma_{\rm phys}(q, m) \ \exp \left\{ \int_q^Q \frac{d\mu'}{\mu'} P\left(\alpha_s(\mu')\right) \right\}$$

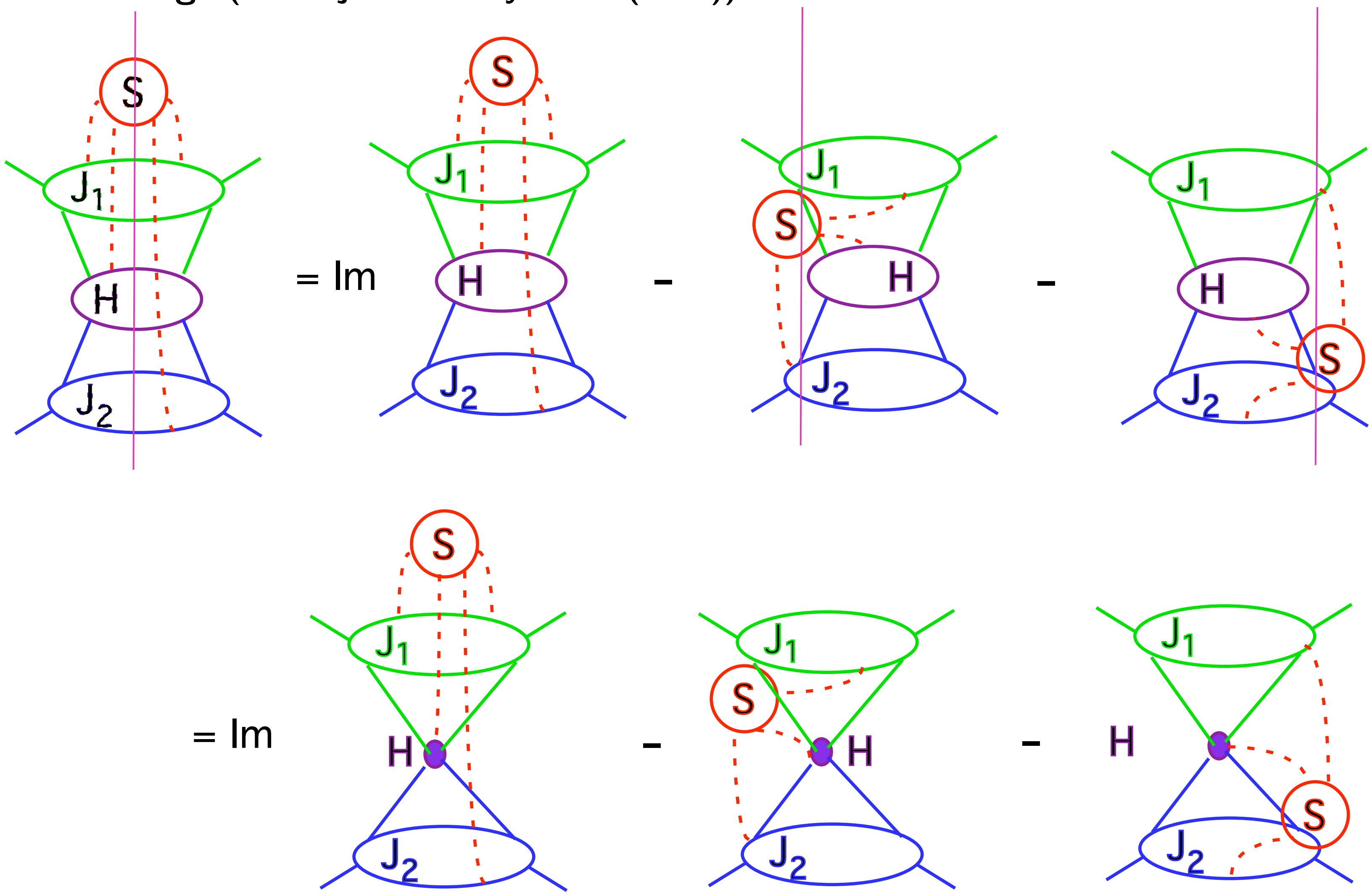
– **For example:** $\sigma_{\rm phys} = \tilde{F}_2(Q^2, N)$, **DIS moment.**

– **How factorization works in pQCD for jet substructure . . .**

– **Separation of soft quanta from fragmenting partons:**



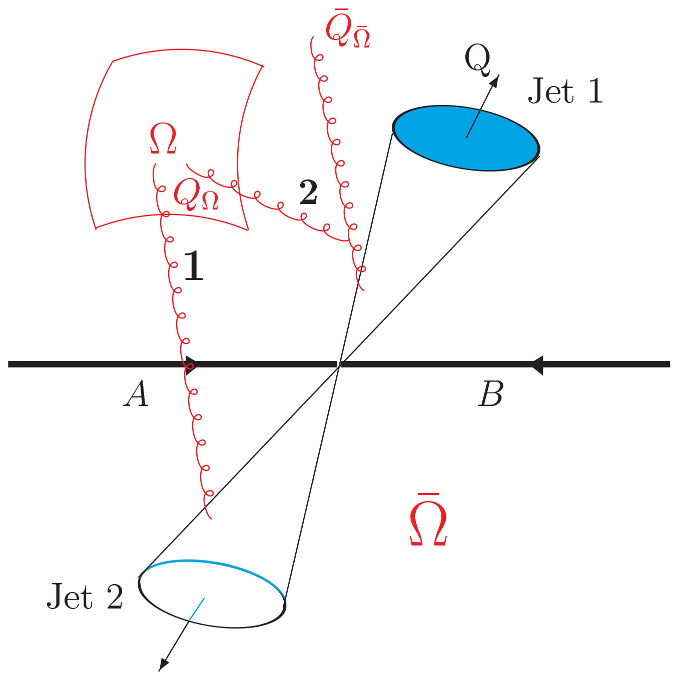
– The all-orders cancellation of soft singularities that connect initial and final states for single-particle inclusive and other short-distance cross sections in hadron-hadron scattering: (Sachrajda . . . Libby & GS (1978))



– all terms on RHS are power-suppressed because soft radiation is very unlikely to be emitted by lines that are far off-shell (H).

4. Cancellation and the importance of remainders

- But . . . all this assumes a full sum over soft radiation, with perturbative correction for hard tail of the distribution
- When we try to beat the soft radiation down, we have to do it carefully . . .
- Non-global logs: color and energy flow
  (Dasgupta & Salam (2001))



- Simplest cases: 2 jets. Measure distribution of energy into some angular region, Ω: $\Sigma_\Omega(E)$

– **Choices for Cross Section:**

– **a) Correlation with event shape τ_a . . .**

– **b) Inclusive in $\bar{\Omega}$ $\rightarrow$ Number of jets not fixed.**

– **Choice (a) fixes the number of jets, and we retain factorization.**
(C.F. Berger, Kúcs, GS (2003), Dokshitzer, Marchesini (2003))

– Choice (b): Number of jets not fixed: nonlinear evolution

(Banfi, Marchesini, Smye (2002)) LL in E/Q, large-N_c

$$\begin{aligned}\partial_\Delta \Sigma_{ab}(E) &= -\partial_\Delta R_{ab}\, \Sigma_{ab}(E) \\ &\quad + \int_{k \text{ not in } \Omega} dN_{ab\to k}\, \left(\Sigma_{ak}\Sigma_{kb} - \Sigma_{ab}\right)\end{aligned}$$

$$dN_{ab\to k} = \frac{d\Omega_k}{4\pi}\, \frac{\beta_a \cdot \beta_b}{\beta_k \cdot \beta_b\, \beta_k \cdot \beta_a} \qquad R_{ab} = \int_E^Q \frac{dE'}{E'}\, \int_\Omega dN_{ab\to k}$$

– Origin of the nonlinearity

– $\partial_\Delta = E\partial_E$

– The derivative ∂_E requires a "hard" gluon k.

– The new hard gluon acts as new, recoil-less source.

– In the large-N limit, $\bar{q}(a)G(k)q(b)$ is equivalent to $\to \bar{q}(a)q(k) \oplus \bar{q}(k)q(a)$, two dipoles.

– Familiar as the BK equation for dipole scattering.

Summary

– We have a good understanding of short distance processes in QCD at large momentum transfer whose definitions are "inclusive enough".

– This has opened the way to many exact higher order and resumed cross sections.

– One frontier is given by cross sections in which radiation is "vetoed" for large regions of phase space. Here we open the door to quantum mechanical interference between final state and initial state interactions even in very hard interactions, so that prerequisites for factorization may fail. This uncovers quantum mechanical histories that are wiped out in factoring cross sections.

– Another is the turn-on of medium induced radiation and collisions.

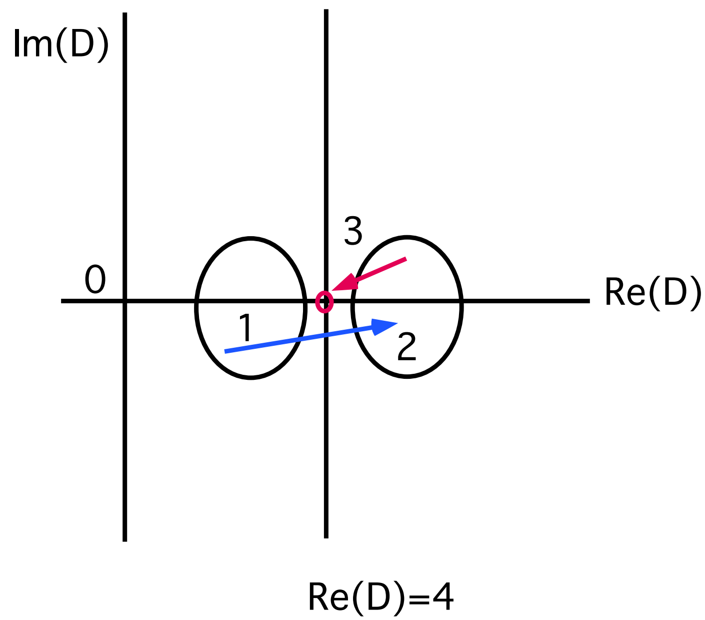
Appendix A. Outline for Dimensional Regularization

1. $$\mathcal{L}_{QCD} \rightarrow G^{(reg)}(p_1, \ldots p_n)\,, \quad D < 4$$
$$\rightarrow G^{(ren)}(p_1, \ldots p_n)\,, \quad D < 4 + \Delta$$

2. $$\rightarrow S^{(unphys)}(p_1, \ldots p_n)\,, \quad 4 < D < 4 + \Delta$$
$$\rightarrow \tau^{(unphys)}(p_1, \ldots p_n)\,, \quad 4 < D < 4 + \Delta$$

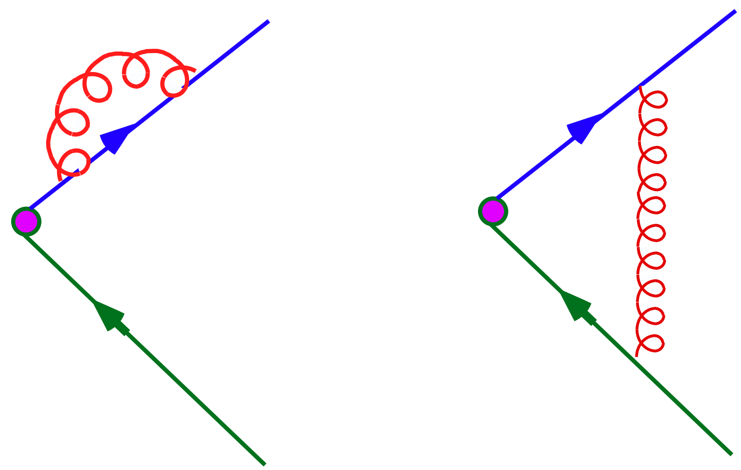
3. $$\rightarrow \tau^{(phys)}(p_1, \ldots p_n)\,, \quad D = 4$$

The D-plane



Appendix B. Physical Pictures from Feynman Diagrams

– **Example: one-loop quark EM form factor**



$$\Gamma_\mu(q^2,\varepsilon) = -ie\mu^\varepsilon\ \bar{u}\ (p_1)\gamma_\mu v(p_2)\ \rho(q^2,\varepsilon)$$

$$\rho(q^2,\varepsilon) \ = \ -\frac{\alpha_s}{2\pi}C_F\left(\frac{4\pi\mu^2}{-q^2-i\epsilon}\right)^\varepsilon \ \frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} \times\left\{\frac{1}{(-\varepsilon)^2}-\frac{3}{2(-\varepsilon)}+4\right\}$$

– **No UV counterterm necessary (QED Ward identity)**

– **Finding the IR ($1/(-\varepsilon)$) poles . . .**

– IR pole requires: (i) on-shell lines; (ii) pinch of momentum contours

– Also integral “singular enough” (power counting)

– Critereon for (i) and (ii): on-shell lines describe a physical process

– Free propagation between vertices in space time

– Start with examples, then generalize . . .

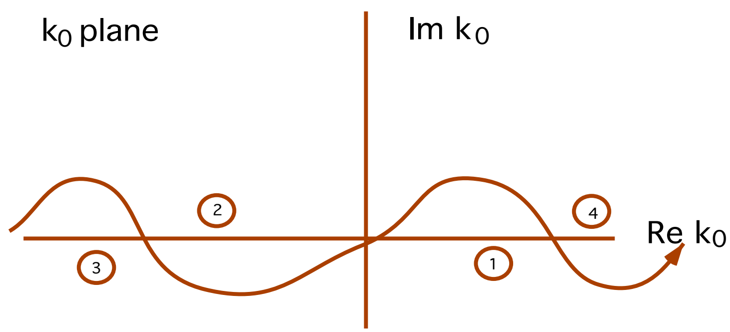
– **One-loop self energy with momentum p, when $p = (p_0, \vec{p}),\ p_0 = |\vec{p}|$:**

$$\pi(p^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon}$$

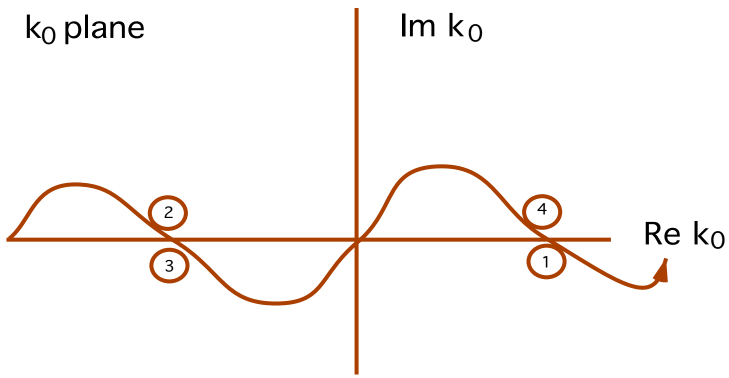
$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{2\pi} \frac{1}{[(k^0 - |\vec{k}| + i\epsilon)(k^0 + |\vec{k}| - i\epsilon)]}$$

$$\times \frac{1}{[(k^0 - p^0 - |\vec{k} - \vec{p}| + i\epsilon)(k^0 - p^0 + |\vec{k} - \vec{p}| - i\epsilon)]}$$

– **Label poles 1 . . . 4 in order. For generic $\vec{k}$, k^0 integral can avoid poles by continuation:**



– **But when $\vec{k} \to x\vec{p}$, pairs (1,4) and (2,3) pinch the contour at $k^0 = xp^0$:**



– **For example, pole 1 at**

$$k^0 = |x\vec{p}| - i\epsilon = xp^0 - i\epsilon$$

is "pinched" by pole 4:

$$k^0 = p^0 - |x\vec{p} - \vec{p}| + i\epsilon = p^0 \ (1 - (1 - x)) = xp^0 + i\epsilon$$

– **Notice it only works for $0 < x < 1$, but now we're really sensitive to on-shell behavior. This particular point is (pretty obviously) "collinear".**

– **Next example: triangle, neglecting the numerator (using Feynman parameterization)**

$$I_\Delta \;=\; 2\int \frac{d^nk}{(2\pi)^n}\;\int_0^1\;\frac{d\alpha_1 d\alpha_2 d\alpha_3\,\delta(1-\Sigma_{i=1}^3\,\alpha_i)}{D^3} \tag{1}$$

where

$$D \;=\; \alpha_1 k^2 + \alpha_2(p_1-k)^2 + \alpha_3(p_2+k)^2 + i\epsilon$$

– D **quadratic: Solutions for each k^μ must coincide:**

$$\frac{\partial}{\partial k^\mu}\,D(\alpha_i,k^\mu,p_a) = 0$$

– **This gives the Landau equations . . .**

$$\alpha_1 k + \alpha_2(p_1-k)\;+\;\alpha_3(p_2+k) = 0$$

– **A "physical" reinterpretation . . .**

$$\alpha_1 k + \alpha_2(p_1 - k) \; + \; \alpha_3(p_2 + k) = 0$$

$$\Updownarrow \tag{2}$$

$$\Delta t_1 v_k + \Delta t_2 v_{p_1-k} \; + \; \Delta t_3 v_{p_2+k}$$

– **Interpretation of "times"** $\delta t_q = \alpha q^0$ **and "velocities"** $v_q^\mu = q^\mu / q^0$

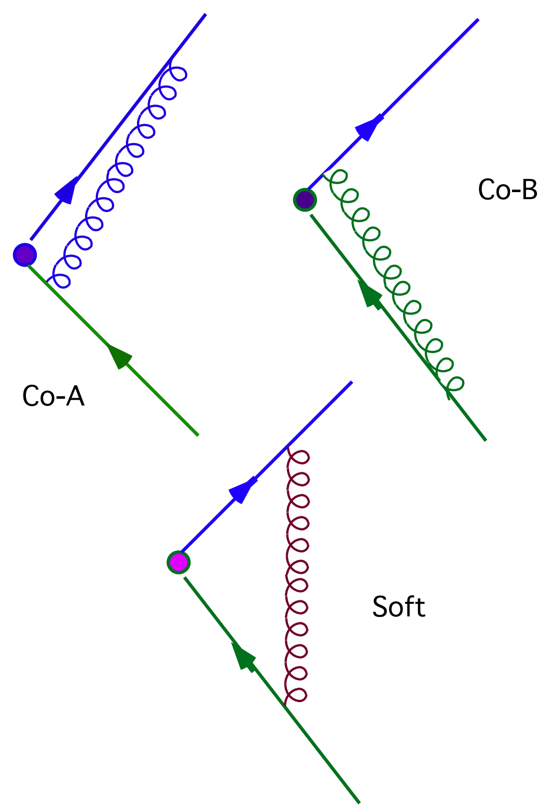
– **Solutions: "soft":** $k^\mu = 0\,, \quad (\alpha_2/\alpha_1) = (\alpha_3/\alpha_1) = 0$

– **and "collinear-A" and "collinear-B"**

$$k \;=\; \zeta p_1\,, \quad \alpha_3 = 0\,, \quad \alpha_1\zeta = \alpha_2(1-\zeta)$$

$$k \;=\; -\zeta' p_2\,, \quad \alpha_2 = 0\,, \quad \alpha_1\zeta' = \alpha_3(1-\zeta')$$

Portraits of “Co-A”, “Co-B” and “soft”:

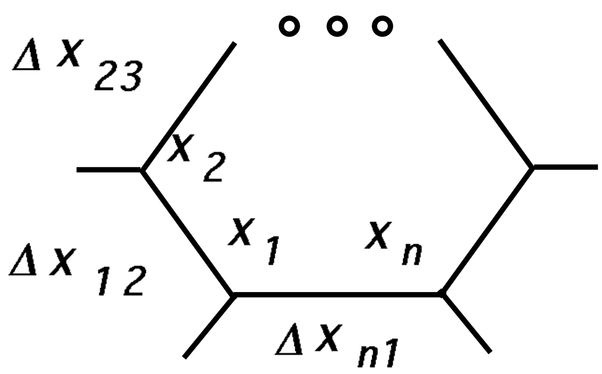


– **Generalization is easy:**

– **Landau equations & physical pictures for arbitrary diagrams**

$$\text{either } \ell_i^2 = m_i^2, \text{ or } \alpha_i = 0,$$

$$\text{and } \sum_{i \text{ in loop } s} \alpha_i \ell_i \epsilon_{is} = 0$$



$$\Delta x_{12} + \Delta x_{23} + \ldots + \Delta x_{n1} = 0$$

Appendix C:. Power Counting at Pinch Surfaces

– **Example: soft region for triangle with massless scalar quarks**

$$\Delta_{\rm soft} = \int_{\rm soft} d^Dk \frac{(2p_1+k)^\mu(-2p_2+k)_\mu}{(2p_1^+k^- - k_T^2 + i\epsilon)\,(-2p_2^-k^+ - k_T^2 + i\epsilon)\,(2k^+k^- - k_T^2 + i\epsilon)}$$

– **Rescale: $k^\mu = \lambda\,\kappa^\mu$ & insert unity**

$$1_{\rm soft} \equiv \int_0^{\lambda_{\rm max}} d\lambda^2\,\delta\left(\lambda^2 - \sum_\mu k_\mu^2\right)$$

– **Which gives . . .**

$$\Delta_{\rm soft} = \int_0^{\lambda_{\rm max}} d\lambda^2 \lambda^D \int\ d^n\kappa\ (4p_1^+p_2^- + \mathcal{O}(\lambda))\ \lambda^{-2}\,\delta\left(1 - \sum_\mu \kappa_\mu^2\right)$$

$$\times \frac{1}{\lambda(2p_1^+\kappa^+ + \mathcal{O}(\lambda) + i\epsilon)\,\lambda(-2p_2^-\kappa^+ + \mathcal{O}(\lambda) + i\epsilon)\,\lambda^2(2\kappa^+\kappa^- - \kappa_T^2 + i\epsilon)}$$

$$\sim\ 2\int_0^{\lambda_{\rm max}} \frac{d\lambda}{\lambda^{5-D}}\ \int d^D\kappa\ \frac{\delta\left(1 - \sum_\mu \kappa_\mu^2\right)}{(\kappa^+ + i\epsilon)\,(-\kappa^- + i\epsilon)(2\kappa^+\kappa^- - \kappa_T^2 + i\epsilon)}$$

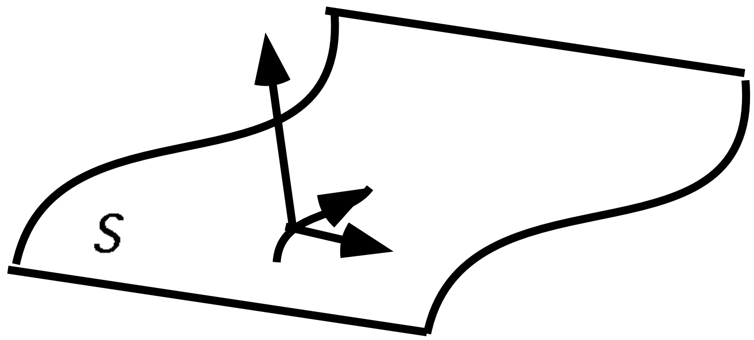
– **λ integral $\rightarrow 1/(4-D) = 1/2\varepsilon$ pole**

– **Remaining integral: pinches at** $\kappa^\pm = \kappa_T^2 = 0$ **(& hence $\kappa^\mp = 1$)**

– **Soft tail of collinear regions give the double poles we saw above.**

3. General pinch surface analysis

– Consider a pinch surface γ of graph G: ℓ_b internal; κ_a normal



– Could be dimensionally regularized (D; $\varepsilon = 2 - D/2$)

– The integral near surface γ looks like

$$G_\gamma(Q) = \int_{\mathcal{O}(Q)} \prod_b d\ell_b \int_\gamma \prod_{a=1}^{\mathcal{D}_\gamma(\varepsilon)} d\kappa_a \, \frac{n(\kappa_a, \ell_b, Q)}{\prod_j d_j(\kappa_a, \ell_b, Q)}$$

- Where Q represents external momenta, and n numerator factors (things like $\not{k}$)

– **Scaling:** $\kappa_a = \lambda_\gamma \, \kappa'_a$

$$1_\gamma = \int_0^{\lambda_\gamma^{\max 2}} d\lambda_\gamma^2 \, \delta \left(\lambda_\gamma^2 - \sum_a \kappa_a^2 \right)$$

$$\begin{aligned} n(\kappa_a, \ell_b, Q) &= \lambda_\gamma^{N_n} \left[\bar{n}(\kappa'_a, \ell_b, Q) + \mathcal{O}(\lambda) \right] \\ d_j(\kappa_a, \ell_b, Q) &= \lambda_\gamma^{N_j} \left[\bar{d}_j(\kappa'_a, \ell_b, Q) + \mathcal{O}(\lambda) \right] \end{aligned}$$

Gives . . .

$$G_\gamma(Q) \;=\; 2 \int_0^{\lambda_\gamma^{\max}} d\lambda_\gamma \; \lambda_\gamma^{p_\gamma - 1} \; \Delta_\gamma(Q)$$

where the "homogeneous integral" is

$$\Delta_\gamma(Q) = \int_{\mathcal{O}(Q)} \prod_b d\ell_b \int_\gamma \prod_{a=1}^{\mathcal{D}_\gamma(\varepsilon)} d\kappa'_a \; \frac{\bar{n}(\kappa'_a, \ell_b, Q)}{\prod_j \bar{d}_j(\kappa'_a, \ell_b, Q)} \; \delta \left(1 - \sum_a {\kappa'_a}^2 \right)$$

and the convergence/divergence is determined by

$$p_\gamma \;=\; \mathcal{D}_\gamma(\varepsilon) + N_n - \sum_j N_j$$

- **Simply the volume of the normal variable, plus numerator suppression minus denominator enhancement. If $p_\gamma > 0$, pQCD is applicable near this singular surface. We don't have to look for cancelation or factorization.**

Repeat for all γ of G

– If pinch surfaces of $\Delta_\gamma(Q)$ are already counted, provides bounds on G and classifies IR poles in dimensional regularization

– This is the case for $\mathrm{e}^+\mathrm{e}^-$ annihilation with massless quarks

– Massive case is similar, but more regions because mass provides an extra scale

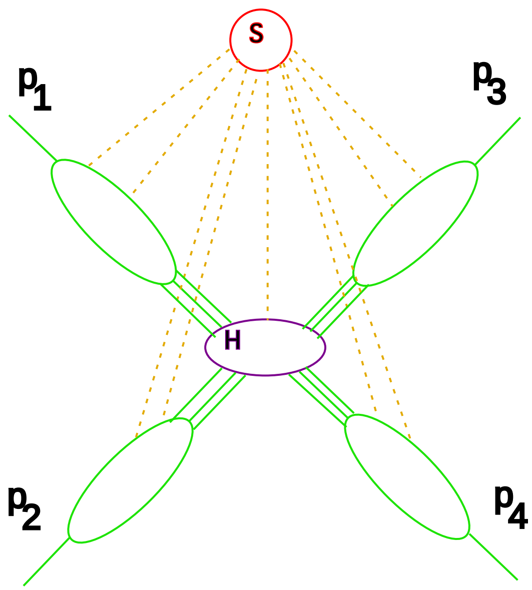
– As long as $p_\gamma > 0$, integrals are infrared safe.

– The search for long-distance behavior is the search for $p_\gamma = 0$.

Appendix D: Illustration

– Of special interest: elastic scattering $2 \to n$ (illustrate with $2 \to 2$), with Q c.m. energy and θ^* the c.m. scattering angle

– The most general pinch surface with a connected set of off-shell (hard) lines



– Sets of collinear, finite-energy on-shell lines ("jets") connected to the hard scattering, joined by sets of zero momentum lines.

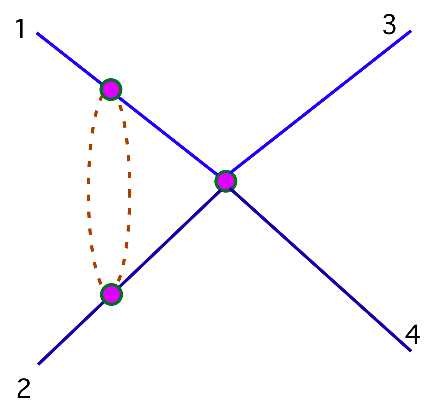
– But which of these will give logarithmic power counting?

- **Normal coordinates:**

  – **2 per loop collinear to each external momentum p: $l_\perp^2$ and $p \cdot l$ for each loop l. Then all propagators in the p-jet $\sim \lambda$.**

  – **4 per loop for "soft" momenta, $l^0 \ldots l^3$: Then all propagators in $S \sim \lambda^2$**

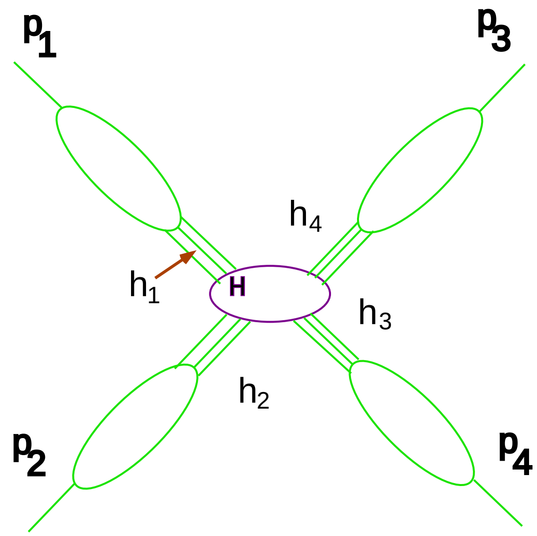
- **A ϕ^4 example with two soft lines:**



  – **Power counting: two soft loops, two soft lines and two jet lines: $p = 2(4) - 2(2) - 2(1) = 2$, finite.**

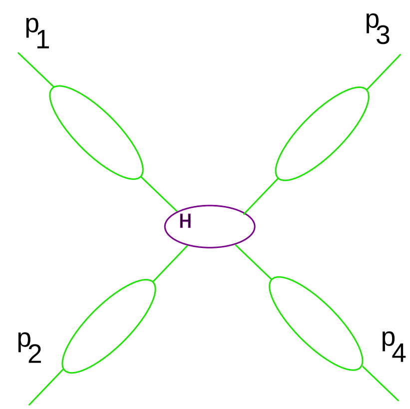
  – **Readily generalizes to all loop order in ϕ^4.**

- So for ϕ^4 can use collinear lines alone . . . forget about "S"



- Collinear power counting for ϕ^4 jet

$$
\begin{aligned}
p &= 2L - N = 2(L-N) + N \\
L &= N - (v_4 + 1) + 1 \\
2N &= 4v_4 - 1 + h \\
&\Downarrow \\
p &= \left(\frac{h-1}{2}\right) \quad \text{only self energies give } p = 0!
\end{aligned}
$$



- **A factorized S-matrix amplitude. The jets are all identical:**

$$A(Q,\theta^*,m) = H(Q/\mu,\theta^*,g(\mu))\ \left[\, J(m/\mu,g(\mu))\,\right]^4$$

- **Each J can be extended to the full matrix element:**

$$J = \langle 0|\phi(0)|p\rangle$$

- **Now invoke $\mu dA/d\mu = 0$ and use separation of variables and chain rule:**

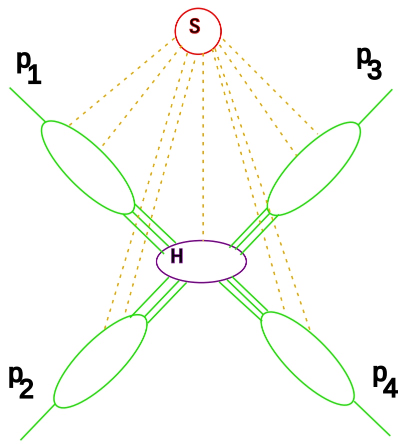
$$Q\frac{d\ln H}{dQ} = -\,\gamma(g(\mu)) = 4\mu\frac{d\ln J}{d\mu} \tag{3}$$

because renormalization scale μ is the only variable held in common. $\gamma(a)$ is IR safe because it can be found from the hard scattering.

- **Lesson: factorization determines energy dependence.**

Elastic scattering of gauge theory partons

- Certainly soft and collinear singularities (recall 1-loop example) so we need the general case



- Let's do power counting with the same normal variables

- **For gauge theories, the scaling of numerators is important**
  - **Any 3-gluon vertex internal to a jet gives a vector l^μ, order λ^0 near pinch surface.**
  - **Any fermion line in the jet gives a Dirac matrix $\not{l}$ order λ^0 near pinch surface.**
  - **Rule: The number of factors of λ^0, "jet" momentum in the numerator equals the number of 3-point vertices in the jet function in that jet.**
  - **The scalar product of two such momenta is order λ**

- **Observation: because the amplitude is a Lorentz scalar, pairs of jet momenta appear only scalar products with each other – unless they are contracted gluon propagators in S or they are contracted with vertices in H.**

- **For jet i: $N_n^{(i)} = (1/2)\left(v_3^{(i)} - n_L^{(i)} - n_s^{(i)}\right)$ where $v_3^{(i)}$ is the number of 3-point vertices in the jet subdiagram.**

- For simplicity, pure collinear power counting for gauge theory jet

$$p = 2L - N + \frac{v_3 - v_L}{2} = 2(L - N) + N + \frac{v_3 - v_L}{2}$$

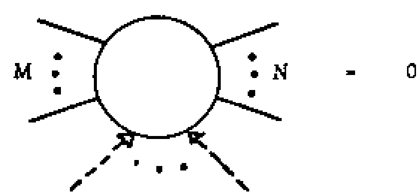
$$L = N - (v_4 + v_3 + 1) + 1$$

$$2N = 4v_4 + 3v_3 - 1 + h$$

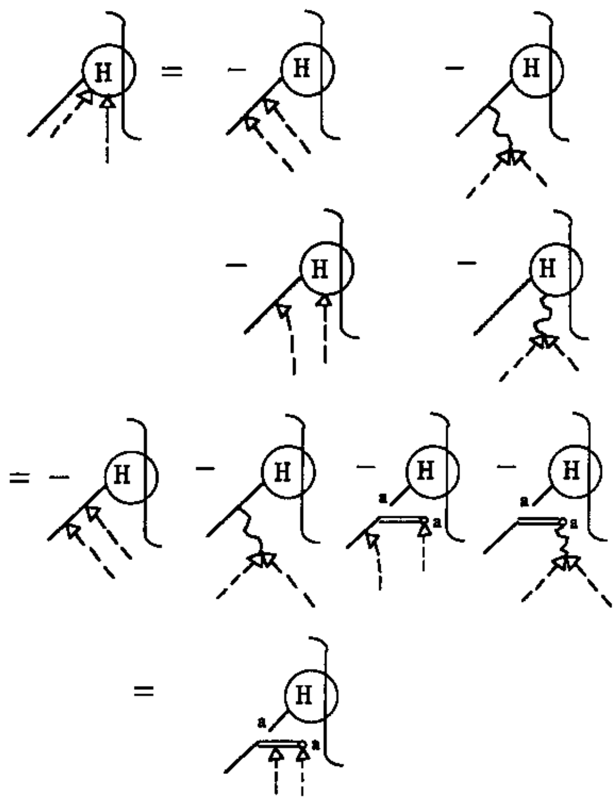
$$\Downarrow$$

$$p = \left(\frac{(h - v_L) - 1}{2}\right)$$

- Result: only one parton connecting any jet to the hard part is NOT a gluon that is contracted with the jet momentum. Such gluons are longitudinally- or scalar-polarized. This is an unphysical polarization. As such we know a lot about them . . .
- The basic Ward identity that decouples longitudinally polarized gluons from amplitudes involving physical partons, states 'M' and 'N' here:



- The Ward identity result requires a sum over all diagrams. For the leading pinch surfaces we don't have all diagrams, but we can prove that the result is insensitive to the details of the hard scattering. In fact, longitudinally-polarized gluons in one jet don't even know the directions of the other jets. Here's how it works. Basic steps of the inductive proof (Collins, Soper, S. 1989)...



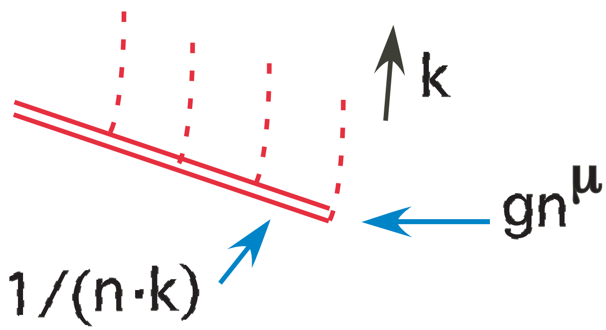
- The double line is a "gauge link", in this case from the position of the physical parton to infinity.

- The gauge link [a.k.a. Wilson line, path ordered expential, nonabelian phase, eikonal line] in x^- direction $(n^\mu = \delta_{\mu -})$ is defined by

$$W_n^{(A)}(\infty, x^-) = P \exp\left[-ig \int_0^\infty n \cdot A^{(adj)}\left((x^- + \lambda)n\right)\right]$$

- To the jet, all that's left of the rest of the world is a gluon source!

- The vector n^μ is arbitrary so long as it is not proportional to the jet momentum.

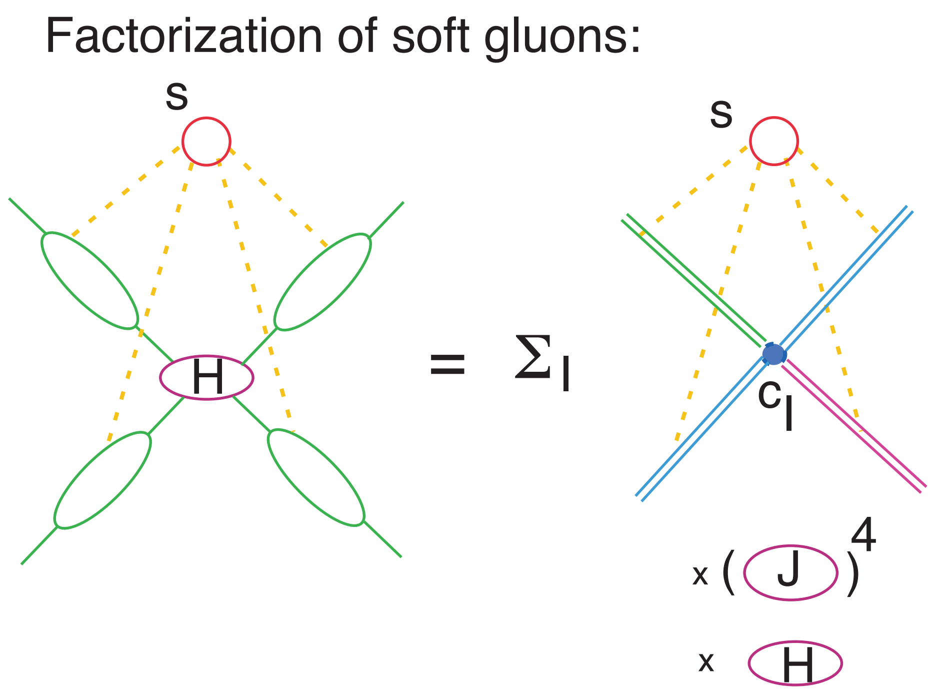


- The matrix element of the jet function:

$$\langle 0 | \Phi^{(A)}(\infty, 0) \phi(0) | p \rangle$$

- A common feature of all factorized long-distance functions, including parton distributions.

- **The complete amplitude also requires the soft gluons, which remember only the directions and charges of the jets:**



- **In a full treatment, we need to carefully avoid double counting between the soft gluon function and the jet functions. As applied to cross sections, this specifies whether the eikonals that define jet functions point toward the past or future.**