The initial stages of heavy ion collisions

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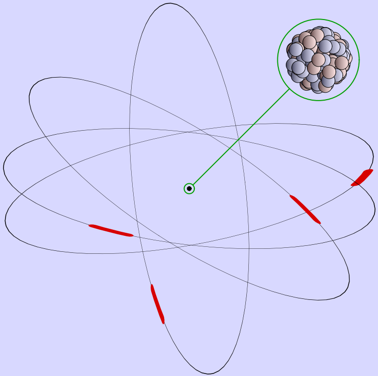
Heavy Ion Collisions

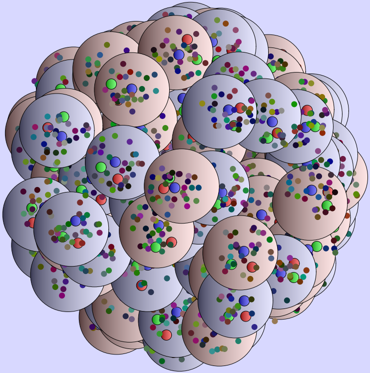
From atoms to nuclei, to quarks and gluons

10^{-10} m : atom (*99.98%* of the mass is in the nucleus)



From atoms to nuclei, to quarks and gluons

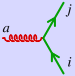
$< 10^{-15}$ m : quarks + gluons



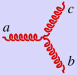
Quarks and gluons

Strong interactions : Quantum Chromo-Dynamics

- Matter : quarks ; Interaction carriers : gluons



$\sim \ g\,(t^a)_{ij}$



$\sim \ g\,(T^a)_{bc}$

- i, j : quark colors ; a, b, c : gluon colors
- $(t^a)_{ij}$: 3×3 SU(3) matrix ; $(T^a)_{bc}$: 8×8 SU(3) matrix

Lagrangian

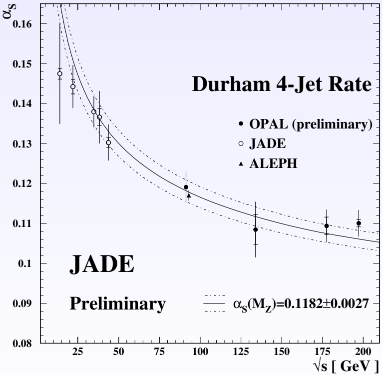
$$\mathcal{L} = -\frac{1}{4}F^2 + \sum_f \overline{\psi}_f (i\not{D} - m_f)\psi_f$$

- Free parameters : quark masses m_f, scale Λ_{QCD}

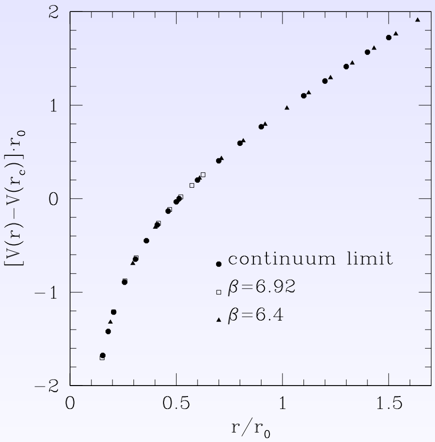
Asymptotic freedom

Running coupling : $\alpha_s = g^2/4\pi$

$$\alpha_s(E) = \frac{2\pi N_c}{(11N_c - 2N_f)\log(E/\Lambda_{QCD})}$$



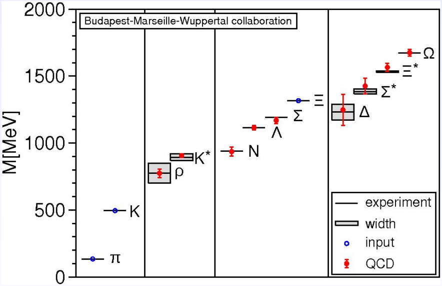
Color confinement

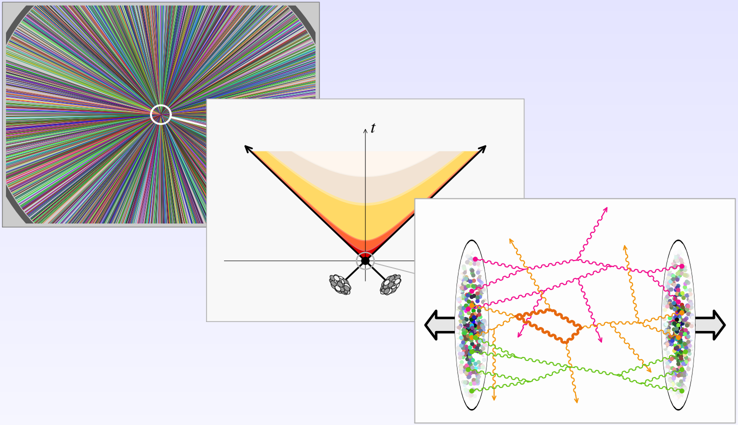


- The quark-antiquark potential increases linearly with distance

Color confinement

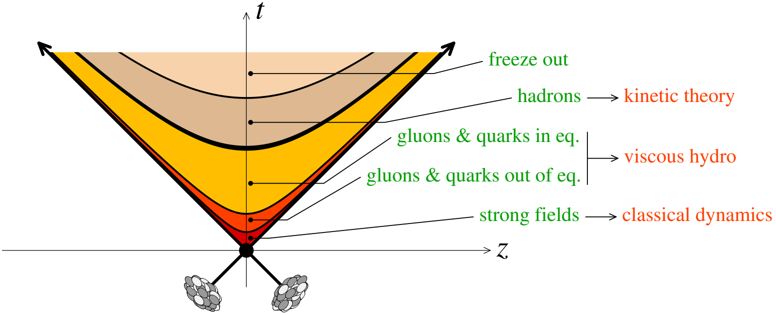
- In nature, we do not see free quarks and gluons (the closest we have to actual quarks and gluons are jets)
- Instead, we see hadrons (quark-gluon bound states):





What can be said about hadronic and nuclear collisions in terms of the underlying quarks and gluons degrees of freedom?

Stages of a nucleus-nucleus collision



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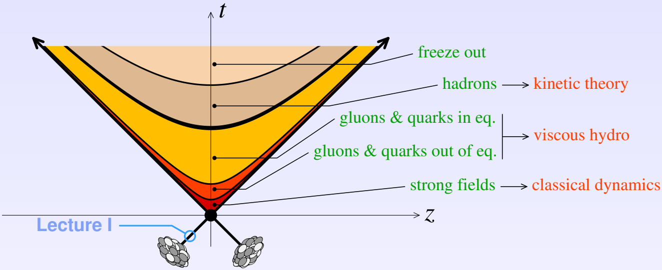
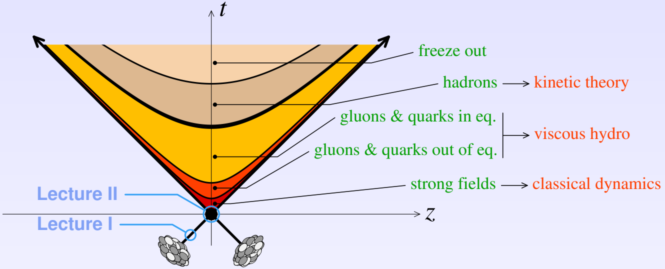


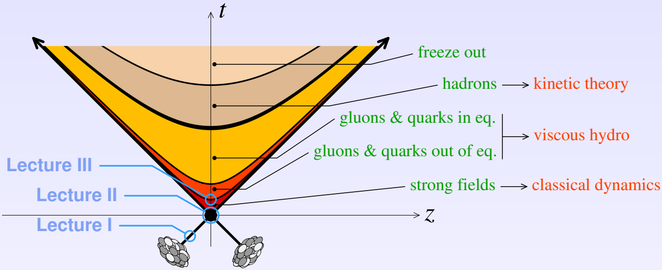
- Lecture I : Nucleon at high energy, Color Glass Condensate

Stages of a nucleus-nucleus collision



- Lecture I : Nucleon at high energy, Color Glass Condensate
- Lecture II : Collision, Factorization at high energy

Stages of a nucleus-nucleus collision



- Lecture I : Nucleon at high energy, Color Glass Condensate
- Lecture II : Collision, Factorization at high energy
- Lecture III : Evolution after the collision

Terminology

- Weakly coupled : $g \ll 1$
- Strongly coupled : $g \gg 1$

- Weakly interacting : $g\mathcal{A} \ll 1 \qquad g^2 f(\mathbf{p}) \ll 1$

  $(2 \to 2) \gg (2 \to 3), (3 \to 2), \cdots$

- Strongly interacting : $g\mathcal{A} \sim 1 \qquad g^2 f(\mathbf{p}) \sim 1$

  $(2 \to 2) \sim (2 \to 3) \sim (3 \to 2) \sim \cdots$

Terminology

- Weakly coupled : $g \ll 1$
- Strongly [overlapped by box]

Strongly coupled	$\Rightarrow$	Strongly interacting
Weakly coupled	$\not\Rightarrow$	Weakly interacting

- Weakly interacting : $g\mathcal{A} \ll 1 \qquad g^2 f(\mathbf{p}) \ll 1$

  $$(2 \to 2) \gg (2 \to 3), (3 \to 2), \cdots$$

- Strongly interacting : $g\mathcal{A} \sim 1 \qquad g^2 f(\mathbf{p}) \sim 1$

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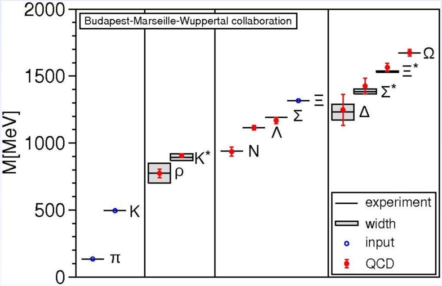
Parton model

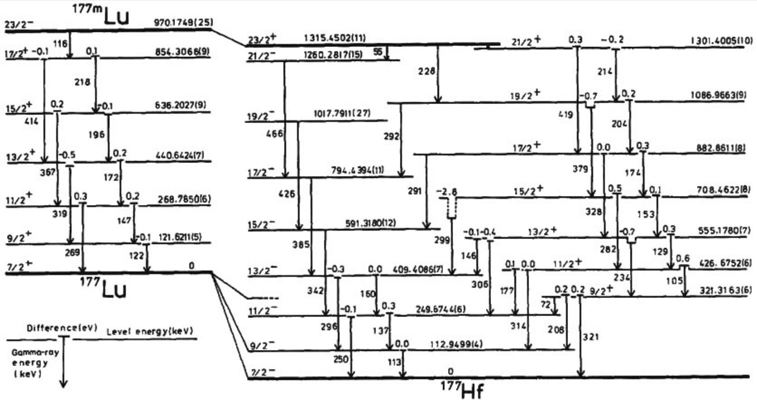
Hadronic spectrum

- In nature, we do not see free quarks and gluons (the closest we have to actual quarks and gluons are jets)
- Instead, we see hadrons (quark-gluon bound states):



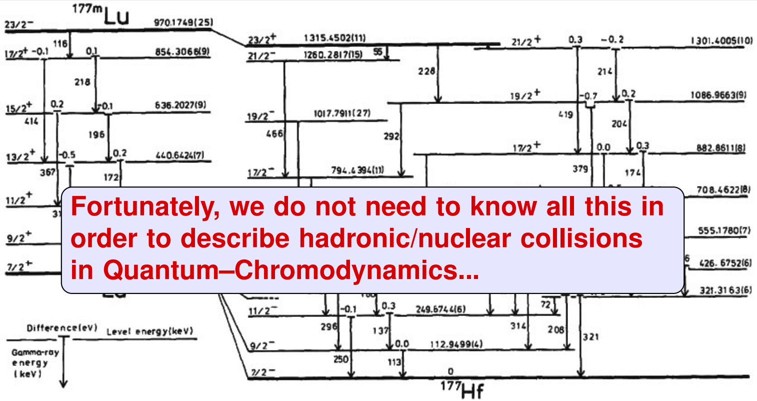
- The hadron spectrum is uniquely given by $\Lambda_{\rm QCD}, m_f$
- But this dependence is non-perturbative (it can now be obtained fairly accurately by lattice simulations)

Nuclear spectrum



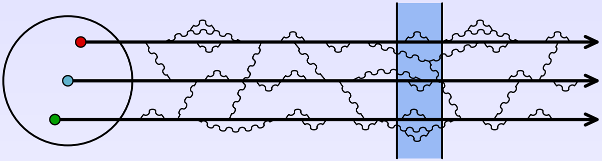
- But nuclear spectroscopy is at the moment out of reach of lattice QCD, even for the lightest nuclei

Nuclear spectrum

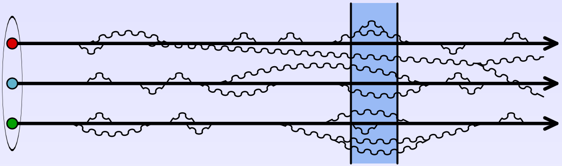


Fortunately, we do not need to know all this in order to describe hadronic/nuclear collisions in Quantum–Chromodynamics...

- But nuclear spectroscopy is at the moment out of reach of lattice QCD, even for the lightest nuclei



- A nucleon at rest is a very complicated object...
- Contains valence quarks + fluctuations at all space-time scales smaller than its own size
- Only the fluctuations that are longer lived than the external probe participate in the interaction process
- Interactions are very complicated if the constituents of the nucleon have a non trivial dynamics over time-scales comparable to those of the probe



- Dilation of all internal time-scales for a high energy nucleon
- Interactions among constituents now take place over time-scales that are longer than the characteristic time-scale of the probe
  ▷ the constituents behave as if they were free
  ▷ the reaction sees a snapshot of the nucleon internals
- Many fluctuations live long enough to be seen by the probe. The nucleon appears denser at high energy (it contains more gluons)

What do we need in order to describe a hadronic collision?

- Provide a snapshot of the two projectiles
  - Flavor and color of each parton
  - Transverse position and momentum
- Since these properties are not know event-by-event, one should aim at a probabilistic description of the parton content of the projectiles

Why is this non trivial?

- In quantum mechanics, the transition probability from some hadronic states to the final state is expressed as :

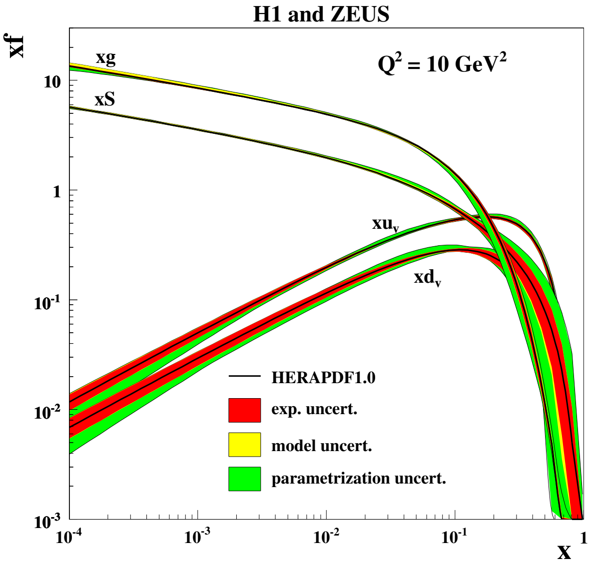
$$\begin{matrix}\text{transition probability}\\ \text{from hadrons to X}\end{matrix} \equiv \left| \sum \begin{matrix}\text{Amplitudes}\\ h_1 h_2 \rightarrow X\end{matrix} \right|^2$$

- The parton model assumes that we may be able to write it as :

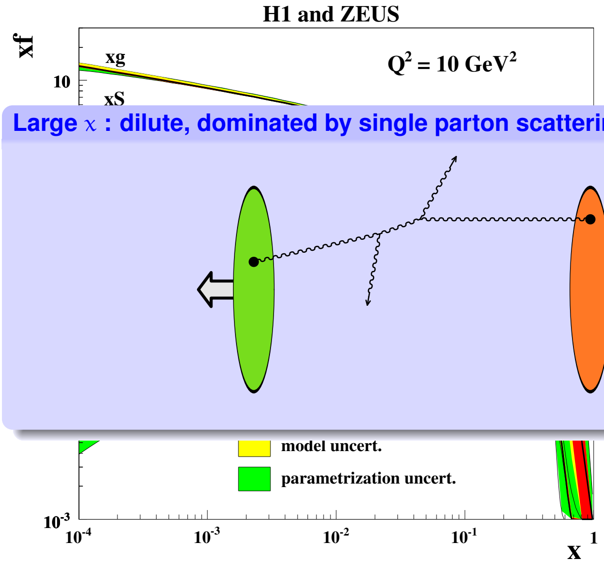
$$\begin{matrix}\text{transition probability}\\ \text{from hadrons to X}\end{matrix} \equiv \sum_{\substack{\text{partons}\\ \{q,g\}}} \begin{matrix}\text{probability to find}\\ \{q,g\} \text{ in } \{h_1,h_2\}\end{matrix} \otimes \left| \sum \begin{matrix}\text{Amplitudes}\\ \{q,g\} \rightarrow X\end{matrix} \right|^2$$

- This property is known as factorization. It can be justified in QCD, and it is a consequence of the separation between the timescale of confinement and the collision timescale

Parton distributions – and possible complications at small x



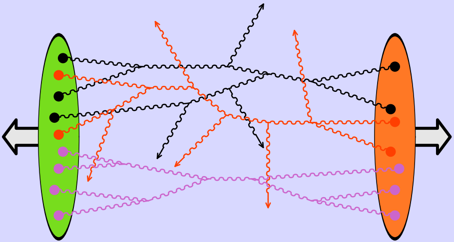
Parton distributions – and possible complications at small x

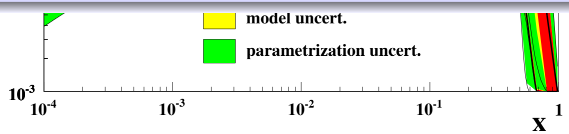


Large x : dilute, dominated by single parton scattering

Parton distributions – and possible complications at small x

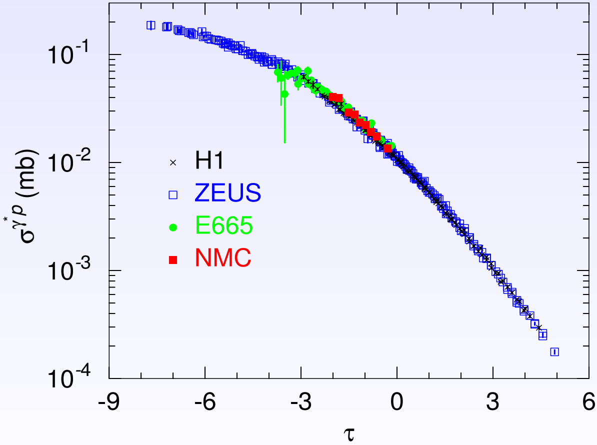
Small x : dense, multi-parton interactions become likely





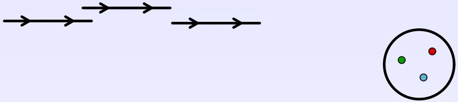
Small x data displayed differently... (Geometrical scaling)

- Small x data ($x \le 10^{-2}$) displayed against $\tau \equiv \log(x^{0.32}\, Q^2)$:



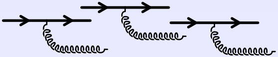
Gluon Saturation

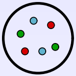
Parton evolution under boosts



▷ at low energy, only valence quarks are present in the hadron wave function

Parton evolution under boosts



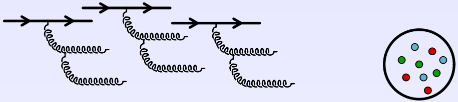


▷ when energy increases, new partons are emitted

▷ the emission probability is $\alpha_s \int \frac{dx}{x} \sim \alpha_s \ln(\frac{1}{x})$, with x the longitudinal momentum fraction of the gluon

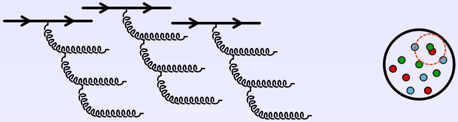
▷ at small-x (i.e. high energy), these logs need to be resummed

Parton evolution under boosts



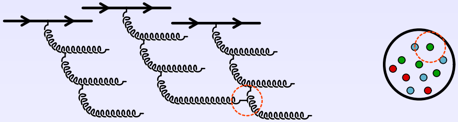
▷ as long as the density of constituents remains small, the evolution is linear: the number of partons produced at a given step is proportional to the number of partons at the previous step

Parton evolution under boosts



▷ eventually, the partons start overlapping in phase-space

Parton evolution under boosts



▷ parton recombination becomes favorable

▷ after this point, the evolution is non-linear:

the number of partons created at a given step depends non-linearly on the number of partons present previously

Balitsky (1996), Kovchegov (1996,2000)

Jalilian-Marian, Kovner, Leonidov, Weigert (1997,1999)

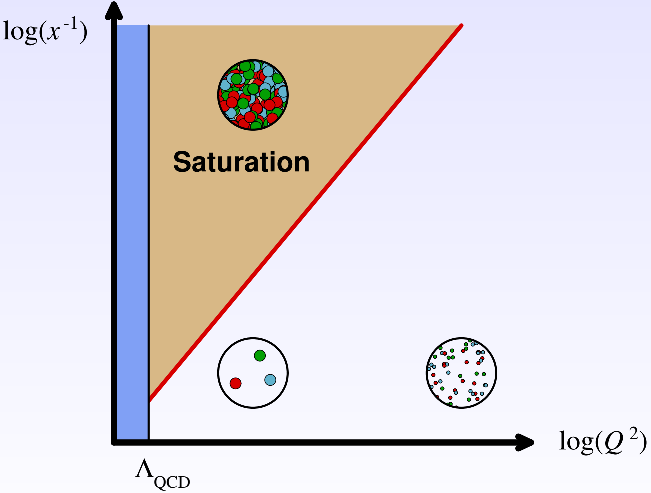
Iancu, Leonidov, McLerran (2001)

Saturation criterion [Gribov, Levin, Ryskin (1983)]

$$\underbrace{\alpha_s Q^{-2}}_{\sigma_{gg\to g}} \times \underbrace{A^{-2/3} xG(x,Q^2)}_{\text{surface density}} \geq 1$$

$$Q^2 \leq \underbrace{Q_s^2 \equiv \frac{\alpha_s xG(x,Q_s^2)}{A^{2/3}}}_{\text{saturation momentum}} \sim A^{1/3} x^{-0.3}$$

Saturation domain



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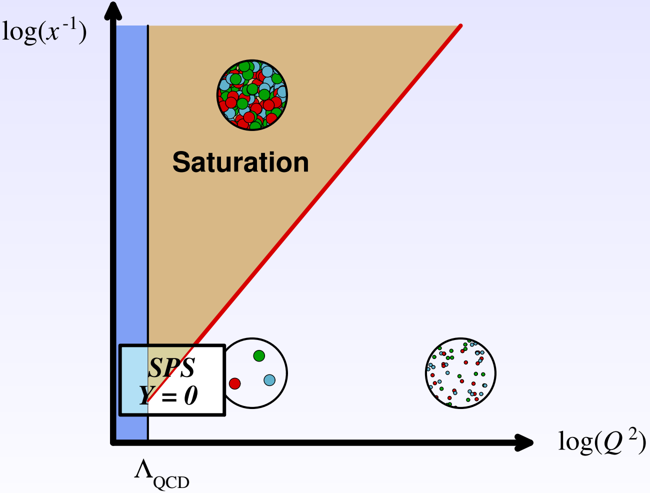

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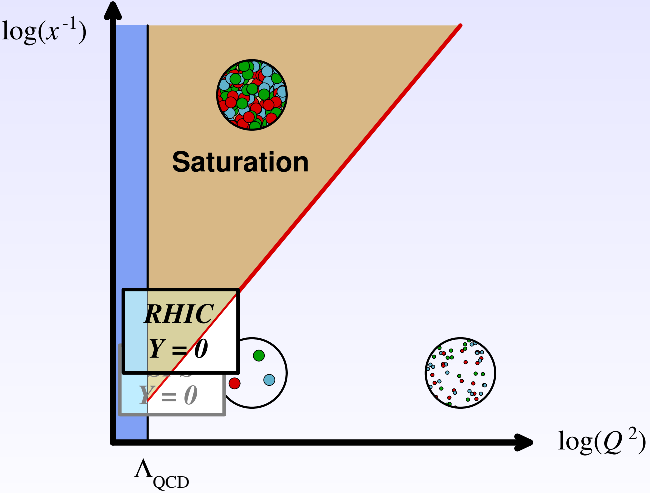

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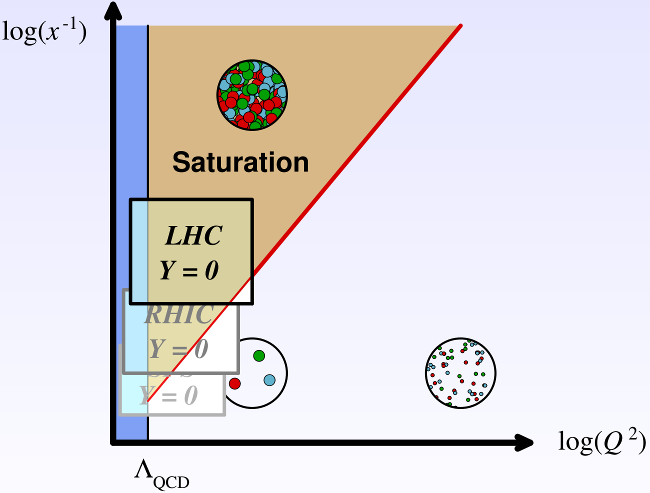

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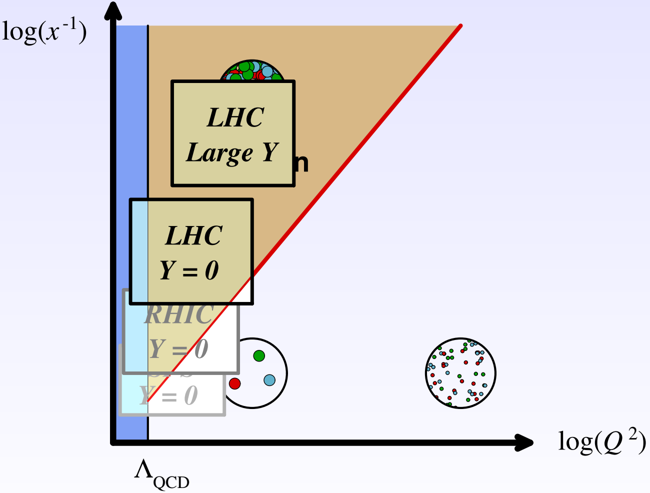

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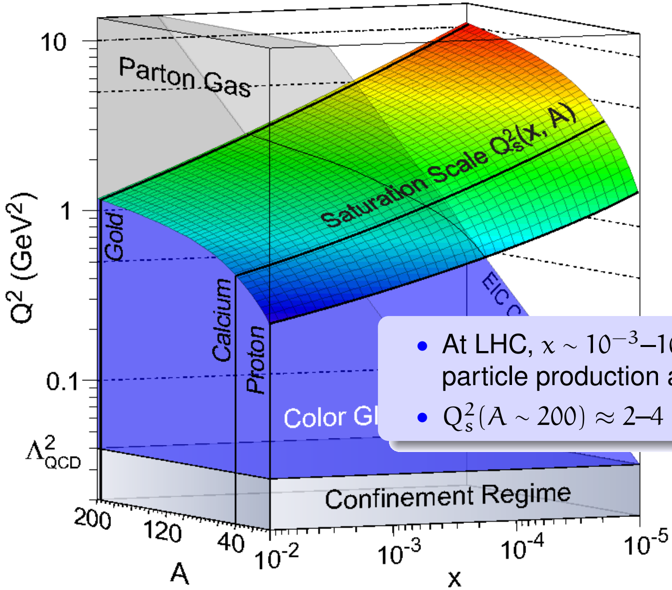
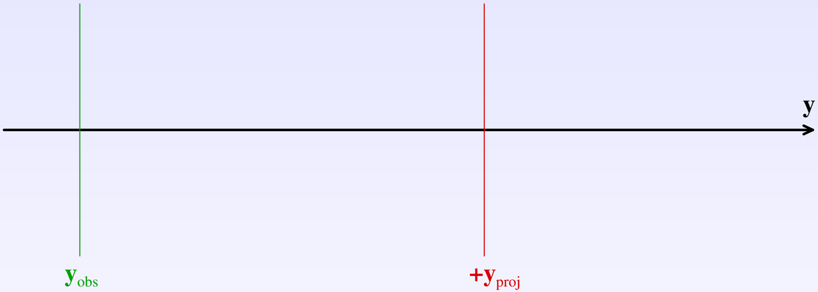


- At LHC, $x \sim 10^{-3}$–10^{-4} (for bulk particle production at mid-rapidity)
- $Q_s^2(A \sim 200) \approx 2$–$4$ GeV^2

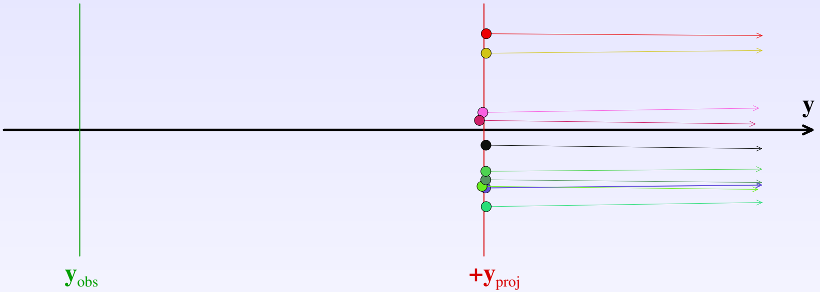
Color Glass Condensate

Degrees of freedom and their interplay



- $p_\perp^2 \sim Q_s^2 \sim \Lambda_{QCD}\, e^{\lambda(y_{proj}-y)} \quad , \quad p_z \sim Q_s\, e^{y-y_{obs}}$
- Fast partons : frozen dynamics, negligible $p_\perp \;\Rightarrow\;$ classical current
- Slow partons : evolve with time $\;\Rightarrow\;$ gauge fields

Degrees of freedom and their interplay



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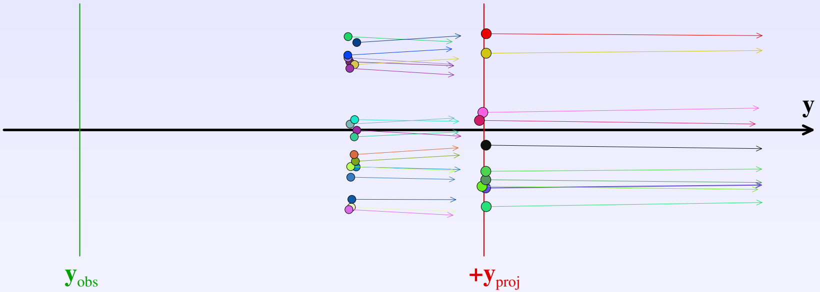

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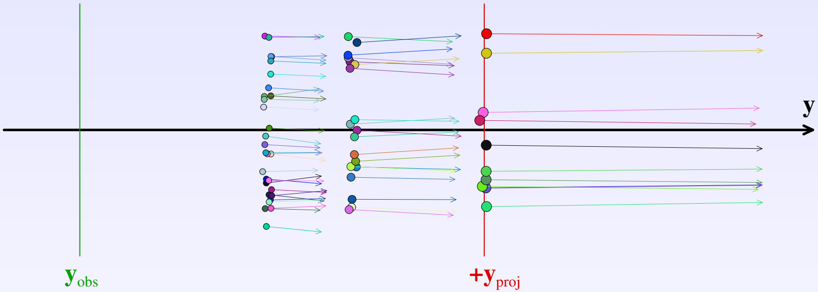

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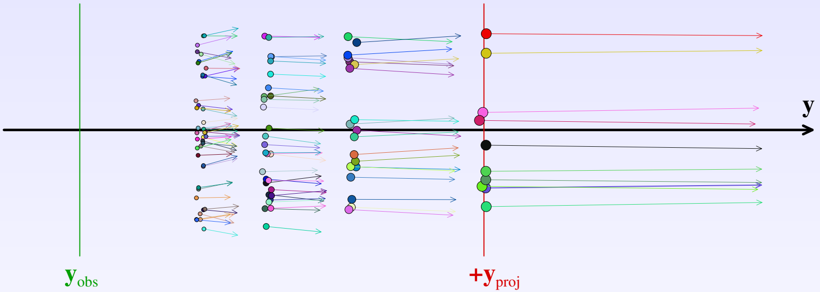

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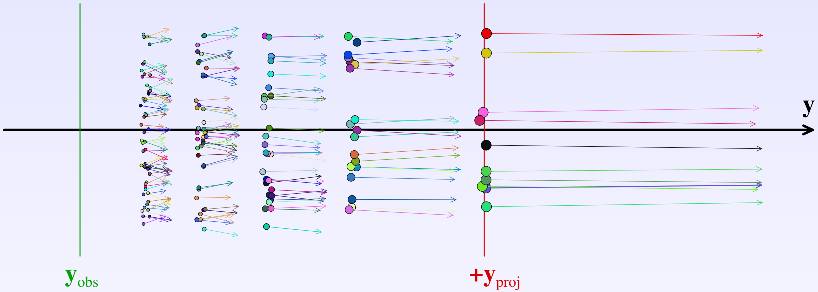

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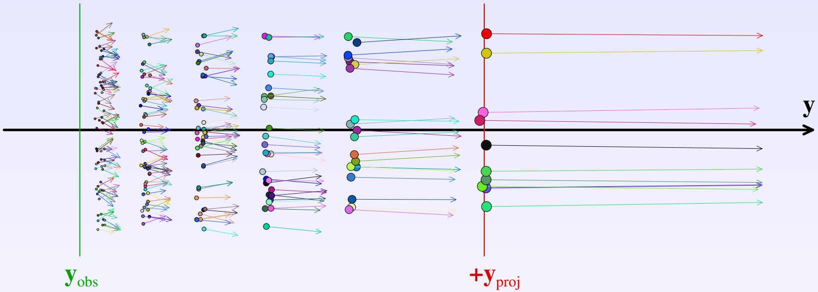

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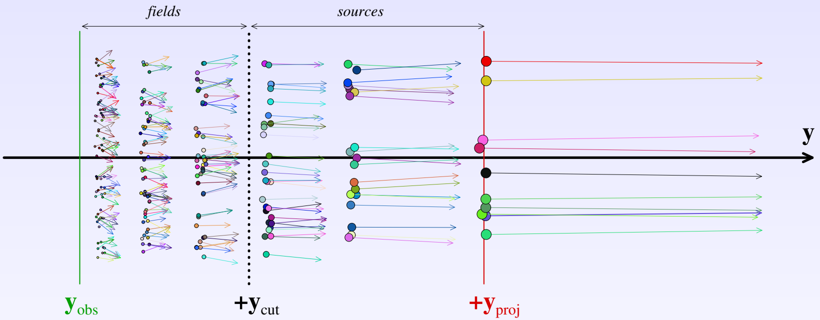

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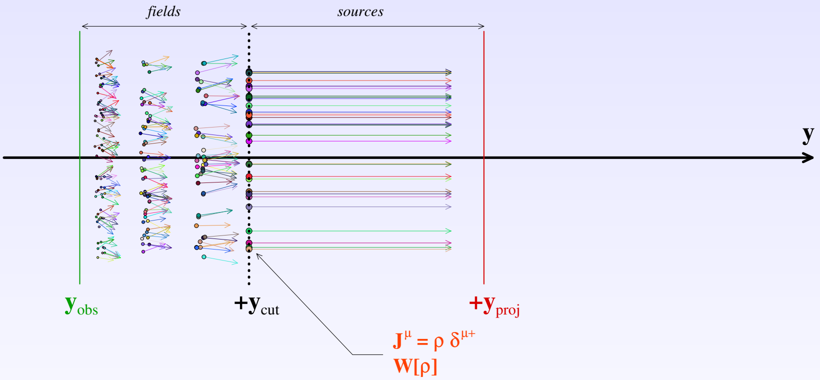

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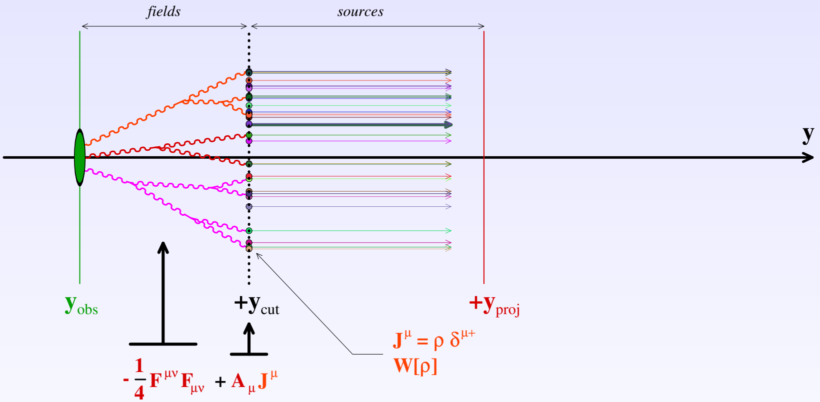

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Semantics

- Color : quarks and gluons are colored
- Glass : the system has degrees of freedom whose timescale is much larger than the typical timescales for interaction processes. Moreover, these degrees of freedom are stochastic variables, like in “spin glasses” for instance
- Condensate : the soft degrees of freedom are as densely packed as they can (the density remains finite, of order α_s^{-1}, due to the interactions between gluons)

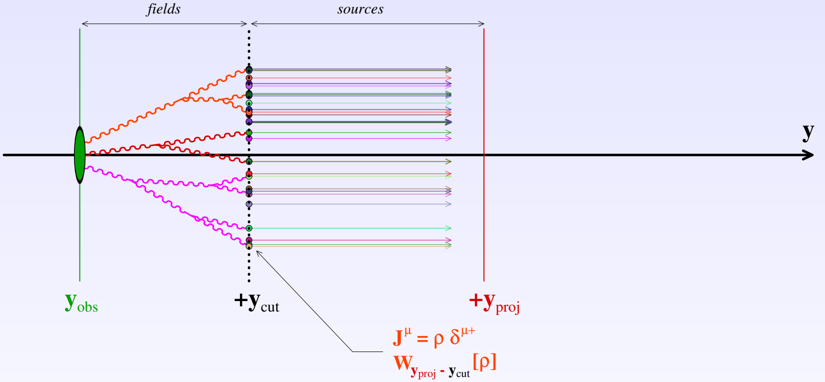
Target average

- Expectation values can be written as :

$$\langle \mathcal{O} \rangle = \int [D\rho] \; W[\rho] \; \mathcal{O}[\rho]$$

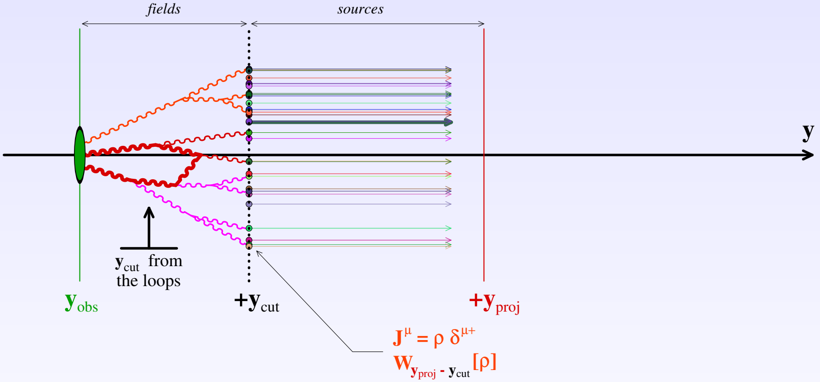
- In this formalism, the Y dependence of the expectation value $\langle \mathcal{O} \rangle$ must come from the probability density $W[\rho]$

Cancellation of the cutoff dependence



- The cutoff y_{cut} is arbitrary and should not affect the result
- The probability distribution $W[\rho]$ changes with the cutoff

Cancellation of the cutoff dependence



- The cutoff y_{cut} is arbitrary and should not affect the result
- The probability distribution $W[\rho]$ changes with the cutoff
- Loop corrections are also y_{cut}-dependent and cancel the cutoff dependence coming from $W[\rho]$, to all orders $(\alpha_s y_{cut})^n$ (Leading Log)

JIMWLK evolution equation

Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

$$\frac{\partial W_Y[\rho]}{\partial Y} = \underbrace{\frac{1}{2}\int_{\vec{x}_\perp,\vec{y}_\perp} \frac{\delta}{\delta\rho_a(\vec{x}_\perp)}\chi_{ab}(\vec{x}_\perp,\vec{y}_\perp)\frac{\delta}{\delta\rho_b(\vec{y}_\perp)}}_{\mathcal{H}\quad\text{(JIMWLK Hamiltonian)}} \; W_Y[\rho]$$

with

$$\chi_{ab}(\vec{x}_\perp,\vec{y}_\perp) \equiv \frac{\alpha_s}{4\pi^3}\int d^2\vec{z}_\perp \frac{(\vec{x}_\perp-\vec{z}_\perp)\cdot(\vec{y}_\perp-\vec{z}_\perp)}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\left[\left(1-\widetilde{U}^\dagger(\vec{x}_\perp)\widetilde{U}(\vec{z}_\perp)\right)\left(1-\widetilde{U}^\dagger(\vec{z}_\perp)\widetilde{U}(\vec{y}_\perp)\right)\right]_{ab}$$

- $\widetilde{U}$ is a Wilson line in the adjoint representation (exponential of the gauge field A^+ such that $\nabla_\perp^2 A^+ = -\rho$)

JIMWLK evolution equation

- Sketch of a derivation : exploit the frame independence in order to write :

$$\langle \mathcal{O} \rangle_{Y} = \underbrace{\int [D\rho]\; W_0[\rho]\; \mathcal{O}_{Y}[\rho]}_{\text{Balitsky-Kovchegov description}} = \underbrace{\int [D\rho]\; W_{Y}[\rho]\; \mathcal{O}_0[\rho]}_{\text{CGC description}}$$

- Calculate the 1-loop correction to some generic observable, and extract the terms in $\alpha_s Y$

- Universality : the evolution of $W_{Y}[\rho]$ does not depend on the observable one is considering

Mean-field approximation : Balitsky-Kovchegov equation

$$\frac{\partial\, T(\vec{x}_\perp,\vec{y}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_\perp\;\frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$

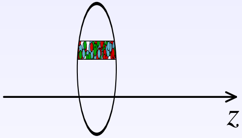
$$\times\Big\{T(\vec{x}_\perp,\vec{z}_\perp)+T(\vec{z}_\perp,\vec{y}_\perp)-T(\vec{x}_\perp,\vec{y}_\perp)-T(\vec{x}_\perp,\vec{z}_\perp)T(\vec{z}_\perp,\vec{y}_\perp)\Big\}$$

$$T(\vec{x}_\perp,\vec{y}_\perp)\equiv 1-\frac{1}{N_c}\,\mathrm{Tr}\,U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)$$

- T is small in the dilute regime, and grows when x decreases
- The r.h.s. vanishes when T reaches 1, and the growth stops
- Both $T=0$ and $T=1$ are fixed points of this equation
  - $T=\epsilon$: r.h.s.>0 $\Rightarrow$ $T=0$ is unstable
  - $T=1-\epsilon$: r.h.s.>0 $\Rightarrow$ $T=1$ is stable

Initial condition : McLerran–Venugopalan model

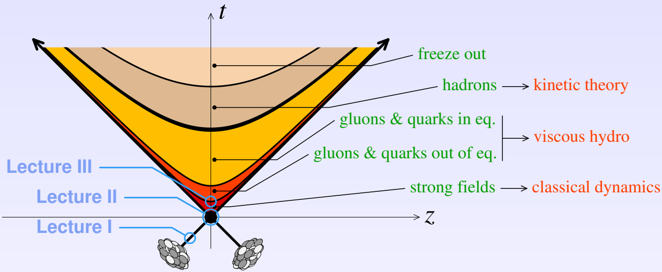
- The JIMWLK equation must be completed by an initial condition, given at some moderate x_0
- As with DGLAP, the initial condition is non-perturbative
- The McLerran-Venugopalan model is often used as an initial condition at moderate x_0 for a large nucleus :



- partons distributed randomly
- many partons in a small tube
- no correlations at different $\vec{x}_\perp$

- The MV model assumes that the density of color charges $\rho(\vec{x}_\perp)$ has a Gaussian distribution :

$$W_{x_0}[\rho] = \exp\left[-\int d^2\vec{x}_\perp \frac{\rho_a(\vec{x}_\perp)\rho_a(\vec{x}_\perp)}{2\mu^2(\vec{x}_\perp)}\right]$$



- Lecture I : Nucleon at high energy, Color Glass Condensate
- Lecture II : Collision, Factorization at high energy
- Lecture III : Evolution after the collision

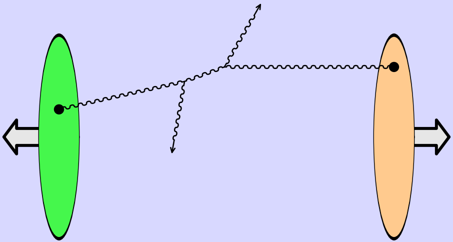
CGC at LO

CGC and Nucleus-Nucleus collisions

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \underbrace{(J_1^\mu + J_2^\mu)}_{J^\mu} A_\mu$$

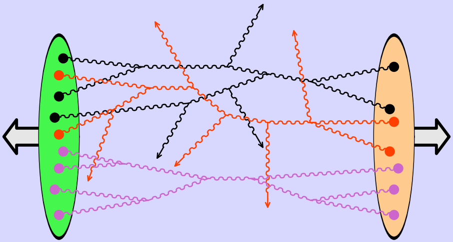
- Given the sources $\rho_{1,2}$ in each projectile, how do we calculate observables? Is there some kind of perturbative expansion?
- Loop corrections and factorization?

Saturation : strong sources and strong fields



- Dilute regime : one parton in each projectile interact

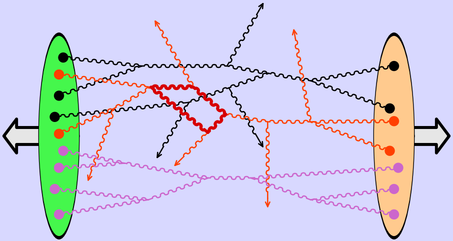
Saturation : strong sources and strong fields



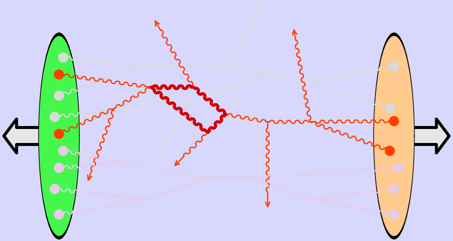
- Dilute regime : one parton in each projectile interact
- Dense regime : multiparton processes become crucial

  (+ pileup of many partonic scatterings in each AA collision)

Power counting



Power counting



- In the saturated regime, the sources are of order $1/g$ (because $\langle\rho\rho\rangle \sim$ occupation number $\sim 1/\alpha_s$)

The order of each connected subdiagram is

$$\frac{1}{g^2} \quad g^{\#\text{ produced gluons}} \quad g^{2(\#\text{ loops})}$$

Power counting

- Example : gluon spectrum :

$$\frac{dN_1}{d^3\vec{p}} = \frac{1}{g^2}\left[c_0 + c_1\,g^2 + c_2\,g^4 + \cdots\right]$$

- The coefficients $c_0, c_1, \cdots$ are themselves series that resum all orders in $(g\rho_{1,2})^n$. For instance,

$$c_0 = \sum_{n=0}^{\infty} c_{0,n}\,(g\rho_{1,2})^n$$

- At Leading Order, we want to calculate the full c_0/g^2 contribution

Inclusive observables

- Average gluon multiplicity $\sim 1/g^2 \gg 1$
- Probability of a given final state $\sim \exp(-1/g^2) \ll 1$

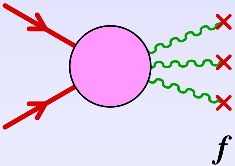
  $\Longrightarrow$ not very useful

- Inclusive observables :
  average of some quantity over all possible final states

$$\big\langle \mathcal{O} \big\rangle \equiv \sum_{\substack{\text{all final}\\ \text{states } \mathbf{f}}} \mathcal{P}(AA \to \mathbf{f})\; \mathcal{O}[\mathbf{f}]$$

Schwinger-Keldysh formalism : technique to perform the sum over final states without computing the individual transition probabilities $\mathcal{P}(AA \to \mathbf{f})$

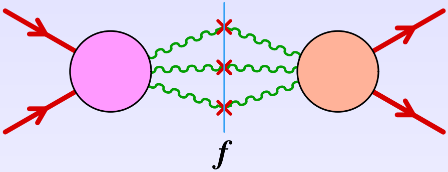
Schwinger-Keldysh formalism



Time-ordered perturbation theory :

$$G_{++}(p) = \frac{i}{p^2 + i\epsilon}$$

Schwinger-Keldysh formalism



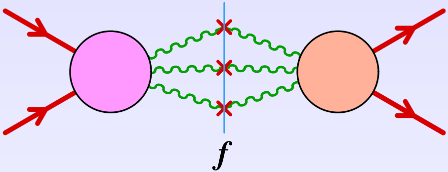
Time-ordered perturbation theory :

$$G_{++}(p) = \frac{i}{p^2 + i\epsilon}$$

Anti time-ordered perturbation theory :

$$G_{--}(p) = \frac{-i}{p^2 - i\epsilon}$$

Schwinger-Keldysh formalism



Time-ordered perturbation theory :

$$G_{++}(p) = \frac{i}{p^2 + i\epsilon}$$

Anti time-ordered perturbation theory :

$$G_{--}(p) = \frac{-i}{p^2 - i\epsilon}$$

Schwinger-Keldysh formalism :

- Accross the cut : $G_{+-}(p) \equiv 2\pi\,\theta(-p^0)\,\delta(p^2)$
- Draw all the graphs $AA \to AA$ that have a given order in g^2
- Sum over all the possibilities of assigning the labels $+$ and $-$ to the internal vertices

Schwinger-Keldysh formalism

- Schwinger-Keldysh formalism $\Longleftrightarrow$ Cutkosky's cutting rules
- The generating functional $Z[j_+, j_-]$ of the Schwinger-Keldysh formalism can be obtained from the generating functional $Z[j]$ of time-ordered perturbation theory :

$$Z[j_+, j_-] = \exp\left[\iint d^4x d^4y\ G^0_{+-}(x,y)\,\square_x \square_y\ \frac{\delta^2}{\delta j_+(x)\delta j_-(y)}\right] Z[j_+]\, Z^*[j_-]$$

- Physical sources : $j_+ = j_-$
- $G_{++} + G_{--} = G_{+-} + G_{-+}$
- $G_{++} - G_{+-} = G_{-+} - G_{--} = G_R$ (retarded propagator)

Inclusive observables at Leading Order

- The Leading Order is the sum of all the tree diagrams
  The sum over the $\pm$ labels turns all propagators into retarded
  Expressible in terms of classical solutions of Yang-Mills equations :
  $$\mathcal{D}_\mu \mathcal{F}^{\mu\nu} = J_1^\nu + J_2^\nu$$
- Boundary conditions : $\lim_{x^0 \to -\infty} \mathcal{A}^\mu(x) = 0$
  (WARNING : this is not true for exclusive observables!)

Components of the energy-momentum tensor at LO :

$$T^{00}_{\rm LO} = \frac{1}{2}\big[\underbrace{\mathbf{E}^2 + \mathbf{B}^2}_{\text{class. fields}}\big] \qquad T^{0i}_{\rm LO} = \big[\mathbf{E} \times \mathbf{B}\big]^i$$

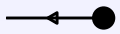
$$T^{ij}_{\rm LO} = \frac{\delta^{ij}}{2}\big[\mathbf{E}^2 + \mathbf{B}^2\big] - \big[\mathbf{E}^i\mathbf{E}^j + \mathbf{B}^i\mathbf{B}^j\big]$$

Retarded classical fields

This sum of trees obeys :

$$\square\mathcal{A} + U'(\mathcal{A}) = J \qquad , \qquad \lim_{x_0\to-\infty} \mathcal{A}(x) = 0$$

- Perturbative expansion (illustrated here for $U(\mathcal{A}) \propto \mathcal{A}^3$) :



- Built with retarded propagators

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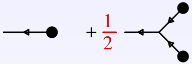

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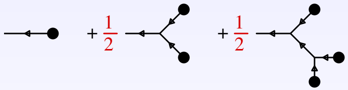

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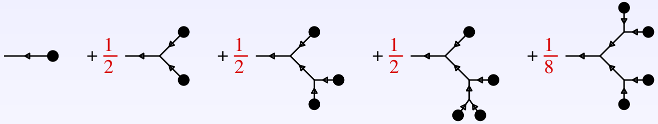


- Built with retarded propagators

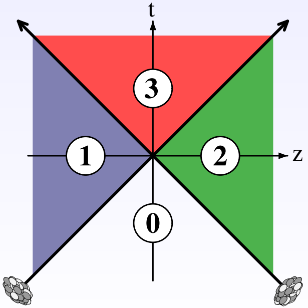
- Classical fields resum the full series of tree diagrams

Space-time evolution of the classical field

[Kovner, McLerran, Weigert (1995)]
[Krasnitz, Venugopalan (1999)] [Lappi (2003)]

- Sources located on the light-cone :

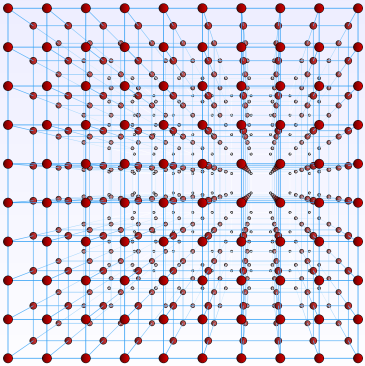
$$J^\mu = \delta^{\mu +} \underbrace{\rho_1(x^-, \boldsymbol{x}_\perp)}_{\sim\delta(x^-)} + \delta^{\mu -} \underbrace{\rho_2(x^+, \boldsymbol{x}_\perp)}_{\sim\delta(x^+)}$$



- Region 0 : $\mathcal{A}^\mu = 0$
- Regions 1,2 : $\mathcal{A}^\mu$ depends only on ρ_1 or ρ_2 (known analytically)
- Region 3 : $\mathcal{A}^\mu$ = radiated field known analytically at $\tau = 0^+$ numerical solution for $\tau > 0$

Hamiltonian Lattice QCD

- Choose a variable that you call "time"
  ($\tau = \sqrt{t^2 - z^2}$ in a high energy collision)
- Conjuguate momenta : $E \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\tau A)}$. Hamiltonian : $\mathcal{H} = EA - \mathcal{L}$
- Discretize space on a 3-dim cubic lattice :



Hamiltonian Lattice QCD

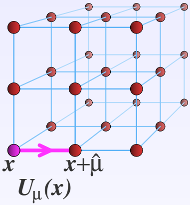
- Naively, one may think of putting the gauge potential A^i on the nodes of the lattice. Problem : this breaks the gauge invariance by terms proportional to the lattice spacing

- Wilson formulation : introduce a link variable

$$U_i(x) \equiv P \exp i\, g \int_x^{x+\hat{\imath}} ds\; A^i(s)$$

that lives on the edge between the nodes x and $x+\hat{\imath}$. Under a gauge transformation, it transforms as

$$U_i(x) \quad \to \quad \Omega(x)\, U_i(x)\, \Omega^\dagger(x+\hat{\imath})$$



- The A^τ potential should live on the nodes (but in practice, one ignores it altogether by choosing the $A^\tau = 0$ gauge)

- The electrical fields E^i live on the nodes of the lattice

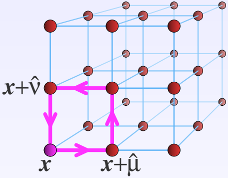
Hamiltonian Lattice QCD

- Hamiltonian in $A^\tau = 0$ gauge :

$$\mathcal{H} = \sum_{\vec{x};i} \frac{E^i(\boldsymbol{x})E^i(\boldsymbol{x})}{2} - \frac{6}{g^2} \sum_{\vec{x};ij} 1 - \frac{1}{3}\mathrm{Re}\ \mathrm{Tr}\underbrace{\left(U_i(x)U_j(x+\hat{\imath})U_i^\dagger(x+\hat{\jmath})U_j^\dagger(x)\right)}_{\text{plaquette at the point } \vec{x} \text{ in the } ij \text{ plane}}$$

Properties :

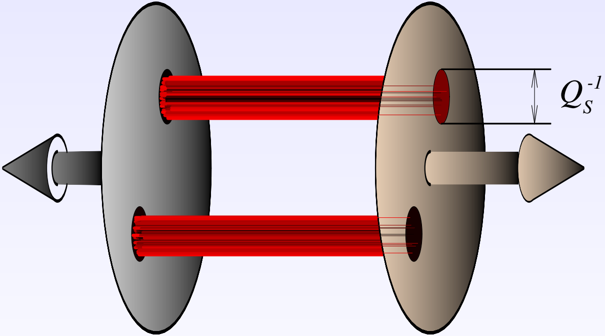
- Invariant under the residual gauge transformations that preserve $A^\tau = 0$ (i.e. time independent gauge transformations)
- Hamilton equations $\Leftrightarrow$ lattice classical Yang-Mills equations



- The Hamilton equations on the lattice form a (large) set of ordinary differential equations, that can be solved with the leapfrog algorithm

Field lines at $\tau \ll Q_s^{-1}$: Glasma flux tubes

[McLerran, Lappi (2006)]

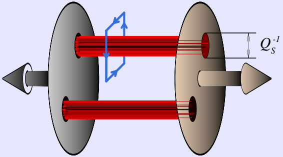


- Seed for the long range rapidity correlations (ridge)
  [Dumitru, FG, McLerran, Venugopalan (2008)]
  [Dusling, FG, Lappi, Venugopalan (2009)]

Field lines at $\tau \ll Q_s^{-1}$: Glasma flux tubes

[McLerran, Lappi (2006)]

[Dumitru, Nara, Petreska (2013)]
[Dumitru, Lappi, Nara (2014)]



$$W \equiv \left\langle \mathrm{P} \exp ig \int_\gamma dx^i \mathcal{A}^i \right\rangle$$

$W \sim \exp(-\text{Area})$ for Area $\times\, Q_s^2 \gtrsim 1$

$\Rightarrow$ magnetic flux bundled in domains of area $\sim Q_s^{-2}$

- Seed for the long range rapidity correlations (ridge)
  [Dumitru, FG, McLerran, Venugopalan (2008)]
  [Dusling, FG, Lappi, Venugopalan (2009)]

Inclusive gluon spectrum at LO

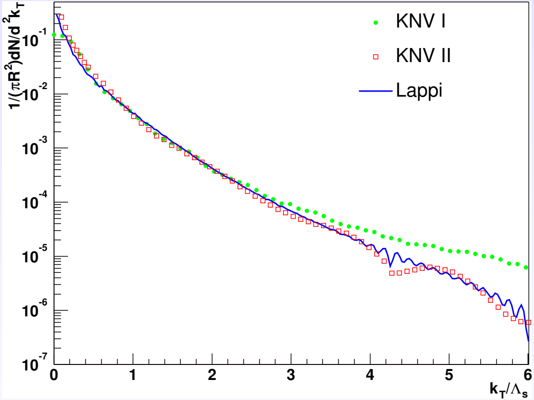
- The gluon spectrum at LO is given by :

$$\left.\frac{dN_1}{dYd^2\vec{\mathbf{p}}_\perp}\right|_{LO} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot(x-y)}\,\square_x\square_y \sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu \;\mathcal{A}_\mu(x)\mathcal{A}_\nu(y)$$

Inclusive multigluon spectra at Leading Order

$$\left.\frac{dN_n}{d^3\mathbf{p}_1\cdots d^3\mathbf{p}_n}\right|_{LO} = \left.\frac{dN_1}{d^3\mathbf{p}_1}\right|_{LO}\times\cdots\times\left.\frac{dN_1}{d^3\mathbf{p}_n}\right|_{LO}$$

Single gluon spectrum at LO



- Lattice artifacts at large momentum
  (they do not affect much the overall number of gluons)
- Important softening at small $k_\perp$ compared to pQCD (saturation)

Gluon yield

Gluon multiplicity (in a symmetric collision)

$$\frac{dN_{\text{gluons}}}{dy} \sim \int d^2\mathbf{b}\ \frac{Q_s^2(\mathbf{b})}{\alpha_s}$$

- The energy dependence of the multiplicity is inherited from that of the saturation momentum :

$$Q_s^2 \sim x^{-0.3} \sim s^{0.15}$$

Next–to–
Leading Order

Why is the LO insufficient ?

- Naive perturbative expansion :

$$\frac{dN}{d^3\vec{p}} = \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$

Note : so far, we have seen how to compute c_0

- Problem : $c_{1,2,\ldots}$ contain powers of the cutoff y_{cut} :

$$\begin{aligned} c_1 &= \qquad\qquad c_{10} + c_{11}\, y_{cut} \\ c_2 &= c_{20} + c_{21}\, y_{cut} + \underbrace{c_{22}\, y_{cut}^2}_{\text{Leading Log terms}} \end{aligned}$$

- These terms are unphysical. However, they are universal and can be absorbed into the distributions $W[\rho_{1,2}]$

Inclusive observables at Next to Leading Order

[FG, Lappi, Venugopalan (2007–2008)]

- Observables at NLO can be obtained from the LO by "fiddling" with the initial condition of the classical field :

$$\mathcal{O}_{\text{NLO}} = \left[\frac{1}{2} \int_{\mathbf{u},\mathbf{v}} \boldsymbol{\Gamma}_2(\mathbf{u},\mathbf{v}) \frac{\partial}{\partial \mathcal{A}_{\text{init}}(\mathbf{u})} \frac{\partial}{\partial \mathcal{A}_{\text{init}}(\mathbf{v})} + \int_{\mathbf{u}} \boldsymbol{\alpha}(\mathbf{u}) \frac{\partial}{\partial \mathcal{A}_{\text{init}}(\mathbf{u})} \right] \mathcal{O}_{\text{LO}}$$

- NLO : the time evolution remains classical;
  $\hbar$ only enters in the initial condition
- NNLO : $\hbar$ starts appearing in the time evolution itself
- NOT true for exclusive observables
- This formula is the basis for proving the factorization of the $W[\rho]$ and their universality (at Leading Log)

Leading Log corrections to the gluon spectrum

- By keeping only the terms that contain the cutoff :

$$\frac{1}{2}\int_{u,v} \Gamma_2(u,v)\,\frac{\partial}{\partial\mathcal{A}_{\rm init}(u)}\,\frac{\partial}{\partial\mathcal{A}_{\rm init}(v)}+\int_{u} \alpha(u)\,\frac{\partial}{\partial\mathcal{A}_{\rm init}(u)}=y^+_{\rm cut}\;\mathcal{H}_1+y^-_{\rm cut}\;\mathcal{H}_2$$

$\mathcal{H}_{1,2}$: JIMWLK Hamiltonians for the two nuclei

- Notes :
  - the $y_{\rm cut}$ terms do not mix the two nuclei $\Rightarrow$ Factorization
  - same operator in all inclusive observables $\Rightarrow$ Universality

Factorization

- By integrating over $\rho_{1,2}$'s, one can absorb the y_{cut}-dependent terms into universal distributions $W_{1,2}[\rho_{1,2}]$
- $\mathcal{H}$ is a self-adjoint operator :

$$\int [D\rho]\; W\, \big(\mathcal{H}\, \mathcal{O}\big) = \int [D\rho]\; \big(\mathcal{H}\, W\big)\; \mathcal{O}$$

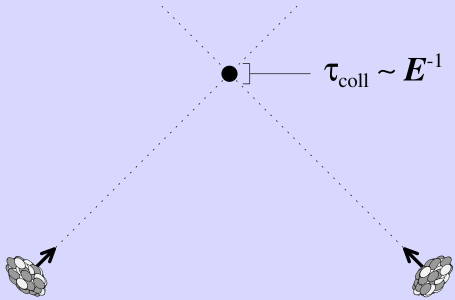
Single inclusive gluon spectrum at Leading Log accuracy

$$\frac{dN_1}{d^3\vec{\mathbf{p}}} \underset{\text{Leading Log}}{=} \int \left[D\rho_1\; D\rho_2\right]\; W_1\left[\rho_1\right]\; W_2\left[\rho_2\right]\; \underbrace{\left.\frac{dN_1}{d^3\vec{\mathbf{p}}}\right|_{\rm LO}}_{\text{fixed }\;\rho_{1,2}}$$

- Cutoff absorbed into the evolution of $W_{1,2}$ with rapidity

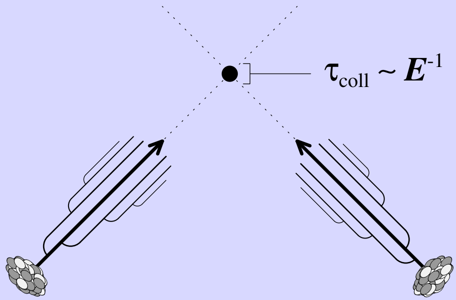
$$\frac{\partial W}{\partial y} = \mathcal{H}\, W \qquad \text{(JIMWLK equation)}$$

Handwaving argument for factorization



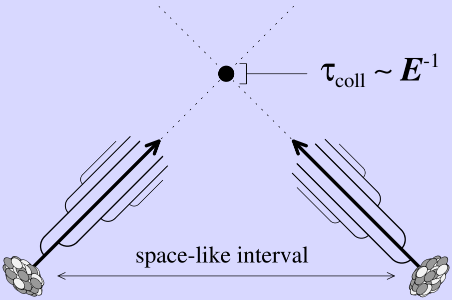
- The duration of the collision is very short: $\tau_{\rm coll} \sim E^{-1}$

Handwaving argument for factorization



- The duration of the collision is very short: $\tau_{coll} \sim E^{-1}$
- The terms we want to resum are due to the radiation of soft gluons, which takes a long time
  ▷ it must happen (long) before the collision

Handwaving argument for factorization



- The duration of the collision is very short: $\tau_{coll} \sim E^{-1}$
- The terms we want to resum are due to the radiation of soft gluons, which takes a long time
  ▷ it must happen (long) before the collision
- The projectiles are not in causal contact before the impact
  ▷ the y_{cut}-dependent terms are intrinsic properties of the projectiles, independent of the measured observable

Multi-gluon correlations at Leading Log

- The previous factorization can be extended to multi-particle inclusive spectra :

$$\frac{dN_n}{d^3\vec{p}_1 \cdots d^3\vec{p}_n} \underset{\text{Leading Log}}{=} \int \left[D\rho_1\, D\rho_2\right] W_1\left[\rho_1\right] W_2\left[\rho_2\right] \; \frac{dN_1}{d^3\vec{p}_1} \cdots \frac{dN_1}{d^3\vec{p}_n}\bigg|_{LO}$$

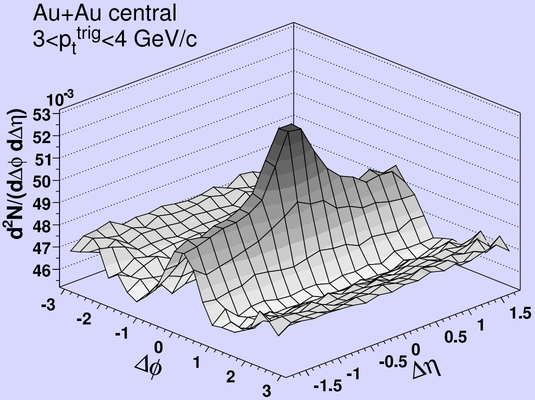
- At Leading Log accuracy, all the rapidity correlations come from the evolution of the distributions $W[\rho_{1,2}]$
  ▷ they are a property of the pre-collision initial state

- Predicts long range ($\Delta y \sim \alpha_s^{-1}$) correlations in rapidity

Ridge correlations

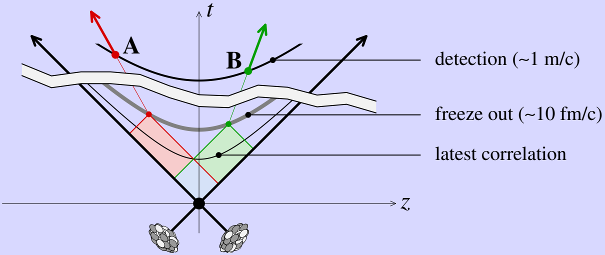
2-particle correlations in AA collisions

[STAR Collaboration, RHIC]



- Long range rapidity correlation
- Narrow correlation in azimuthal angle
- Narrow jet-like correlation near $\Delta y = \Delta\varphi = 0$

Probing early times with rapidity correlations

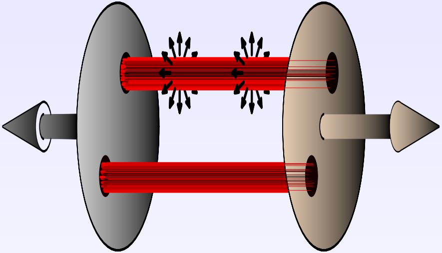


- By causality, long range rapidity correlations are sensitive to the dynamics of the system at early times :

$$\tau_{\text{correlation}} \leq \tau_{\text{freeze out}} \; e^{-|\Delta y|/2}$$

2-hadron correlations from color flux tubes

- η-independent fields lead to long range correlations :



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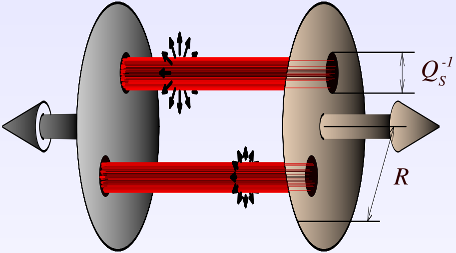

- Particles emitted by different flux tubes are not correlated
  $\triangleright$ $(RQ_s)^{-2}$ sets the strength of the correlation

2-hadron correlations from color flux tubes

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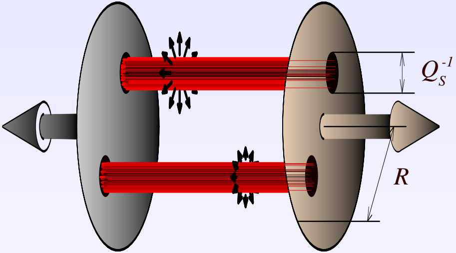

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- At early times, the correlation is flat in $\Delta\varphi$

2-hadron correlations from color flux tubes

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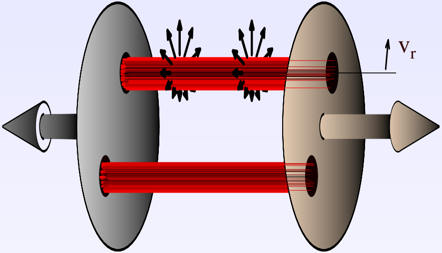
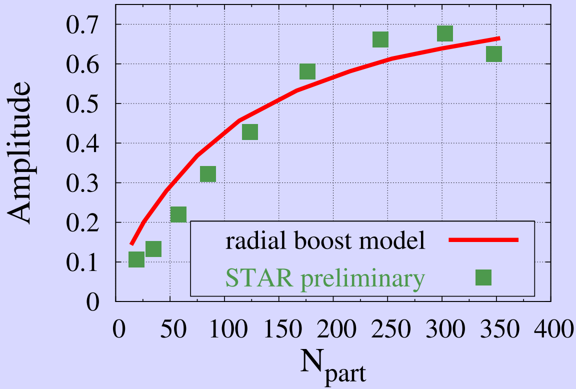


- Particles emitted by different flux tubes are not correlated
  ▷ $(RQ_s)^{-2}$ sets the strength of the correlation
- At early times, the correlation is flat in $\Delta\varphi$

  The collimation in $\Delta\varphi$ is produced later by radial flow

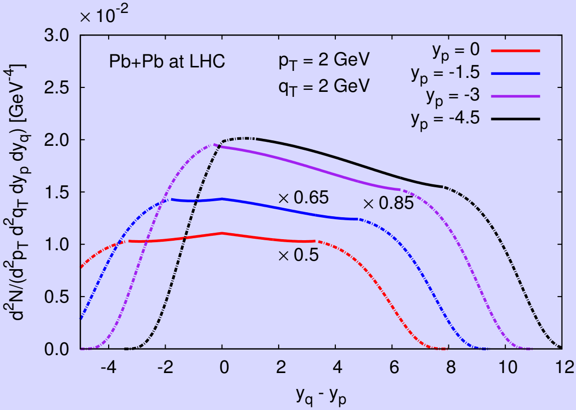
Centrality dependence

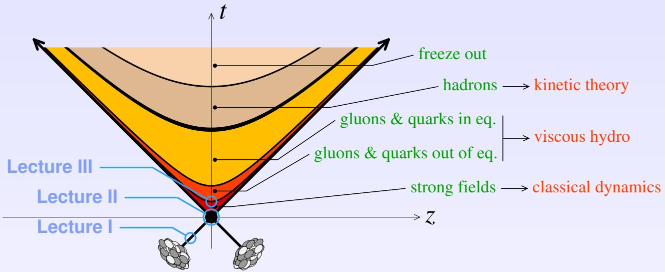


- Main effect : increase of the radial flow velocity with the centrality of the collision

Rapidity dependence

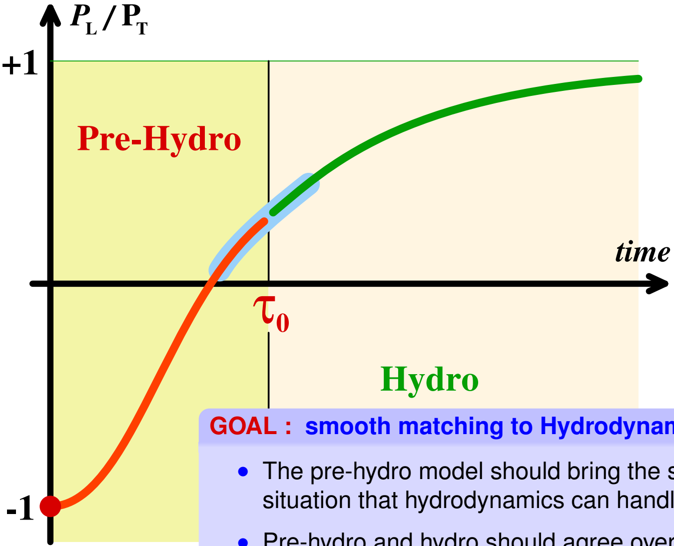
Estimate at LHC energy





- Lecture I : Nucleon at high energy, Color Glass Condensate
- Lecture II : Collision, Factorization at high energy
- Lecture III : Evolution after the collision

Post collision evolution



GOAL : smooth matching to Hydrodynamics

- The pre-hydro model should bring the system to a situation that hydrodynamics can handle
- Pre-hydro and hydro should agree over some range of time $\Rightarrow$ no τ_0 dependence
- NOTE : the anisotropic hydro model of Ryblewski, Strickland et al. may help for a matching at small τ_0

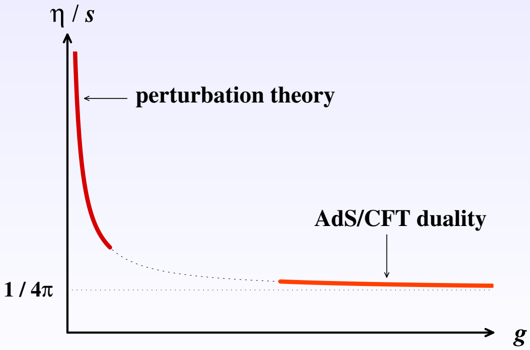
Conditions for hydrodynamics

- The initial P_L/P_T should not be too small (for the stability of hydro codes)
- The ratio η/s should be small enough (for an efficient transfer from spatial to momentum anisotropy)

Shear viscosity at weak and strong coupling (in equilibrium)

Weak coupling QCD result [Arnold, Moore, Yaffe (2000)]

$$\frac{\eta}{s} \approx \frac{5.1}{g^4 \ln\left(\frac{2.4}{g}\right)}$$



Is there another possibility?

From kinetic theory :

$$\frac{\eta}{s} \sim \frac{\text{mean free path}}{\text{de Broglie wavelength}}$$

- (**de Broglie wavelength**)$^{-1} \sim Q$
- (**mean free path**)$^{-1} \sim \underbrace{g^4 Q^{-2}}_{\text{cross section}} \times \underbrace{\int_k f_k}_{\text{density}} \underbrace{(1+f_k)}_{\text{Bose enhancement}}$

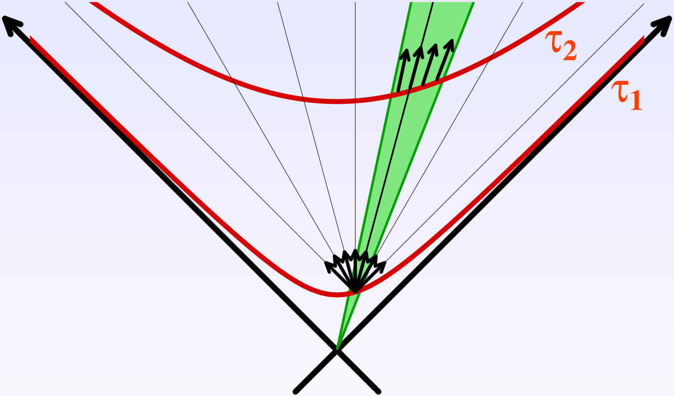
If $g \ll 1$ but $f_k \sim g^{-2}$ (weakly coupled, but strongly interacting)

$$\frac{\eta}{s} \sim g^0$$

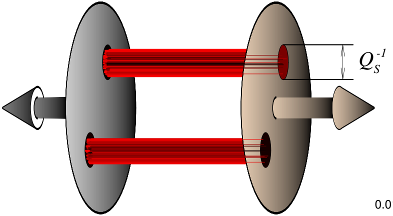
(even w/o quasiparticles, $\mathbf{B} \sim \frac{Q^2}{g}$ has the same effect)

[Asakawa, Bass, Mueller (2006)]

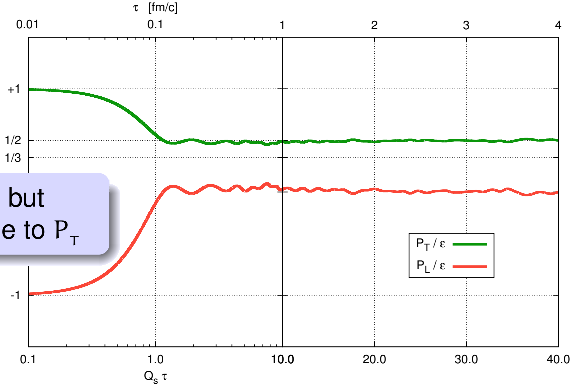
Competition between Expansion and Isotropization



CGC at LO : strong pressure anisotropy at all times

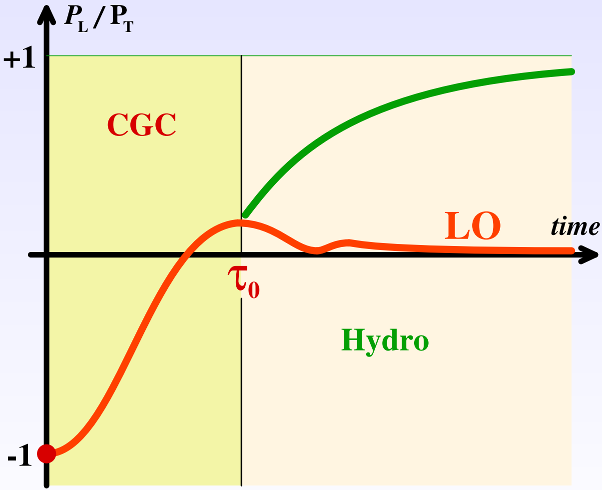


When $\mathbf{E} \parallel \mathbf{B}$:
$P_T = \epsilon, P_L = -\epsilon$



P_L rises to positive values, but never becomes comparable to P_T

CGC at LO : unsatisfactory matching to hydrodynamics



P_L / P_T

Matching to hydro :

- Compute $T^{\mu\nu}$ from CGC
- Find time-like eigenvector : $u_\mu T^{\mu\nu} = \epsilon\, u^\nu$
- Get pressure from some equation of state $P = f(\epsilon)$
- Get viscous stress as difference between full and ideal $T^{\mu\nu}$

"CGC initial conditions" very often means :

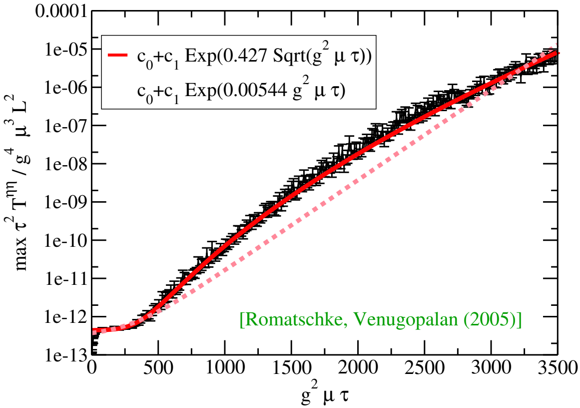
- $\epsilon = T^{00}$ from CGC (or a CGC-inspired model)
- Initial flow neglected, Viscous stress $= 0$

NOTE : glasma fields start to flow at $\tau \sim Q_s^{-1}$:

[Krasnitz, Nara, Venugopalan (2002)] [Chen, Fries (2013)]

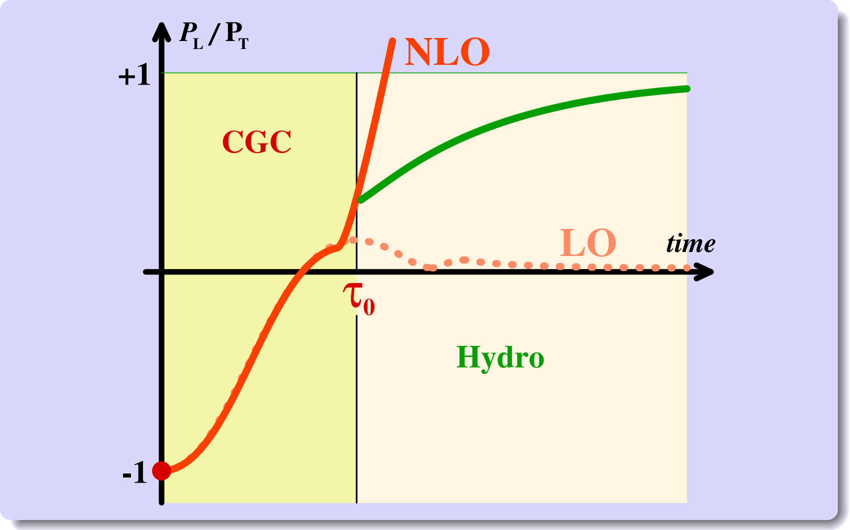
CGC at NLO : instabilities

[Mrowczynski (1988), Romatschke, Strickland (2003), Arnold, Lenaghan, Moore (2003), Rebhan, Romatschke, Strickland (2005), Arnold, Lenaghan, Moore, Yaffe (2005), Romatschke, Rebhan (2006), Bodeker, Rummukainen (2007), Fujii, Itakura (2008),...,**Attems, Rebhan, Strickland (2012)**, **Fukushima (2013)**]



CGC at NLO : instabilities

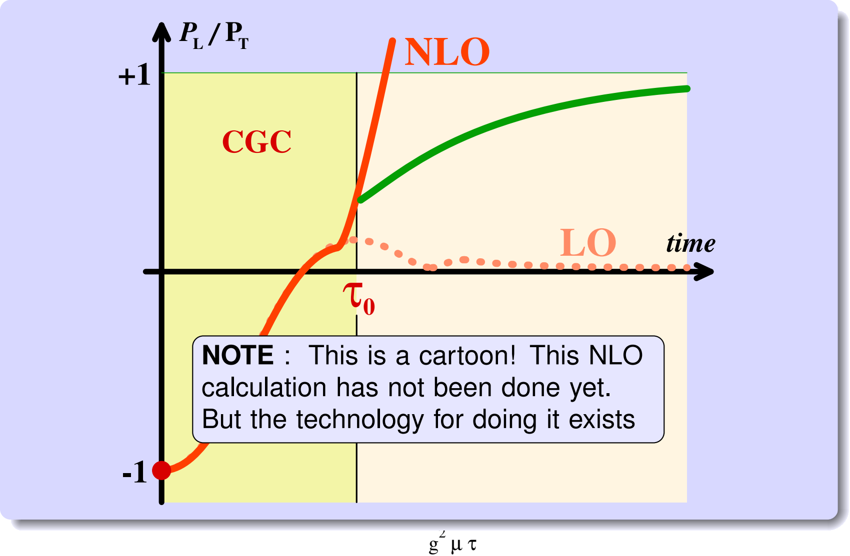
[Mrowczynski (1988), Romatschke, Strickland (2003), Arnold, Lenaghan,



$g^2 \mu \tau$

CGC at NLO : instabilities

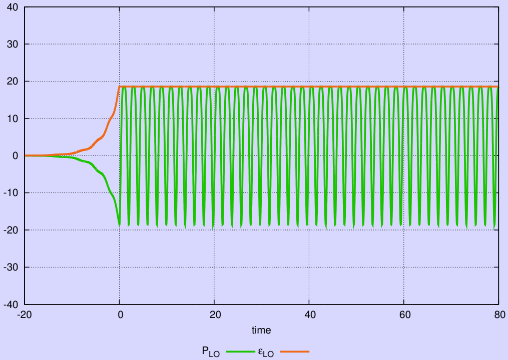
[Mrowczynski (1988), Romatschke, Strickland (2003), Arnold, Lenaghan,



NOTE : This is a cartoon! This NLO calculation has not been done yet. But the technology for doing it exists

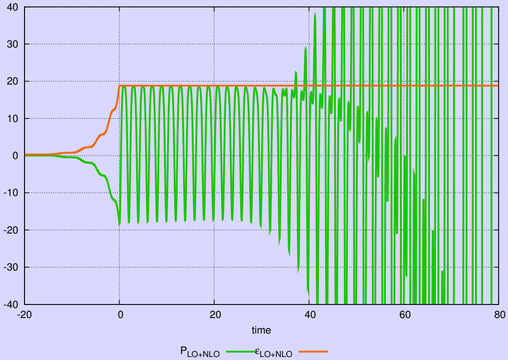
Example of pathologies in fixed order calculations (scalar theory)

LO



Example of pathologies in fixed order calculations (scalar theory)

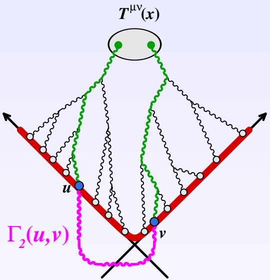
LO + NLO



- Small correction to the energy density
  (protected by energy conservation)
- Secular divergence in the pressure

Improved CGC power counting

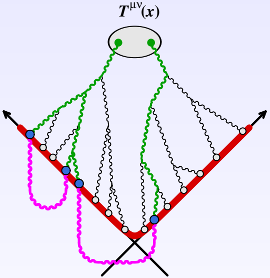
Loop $\sim g^2$, $e^{\sqrt{\mu\tau}}$ for each field perturbation



- 1 loop : $(ge^{\sqrt{\mu\tau}})^2$

Improved CGC power counting

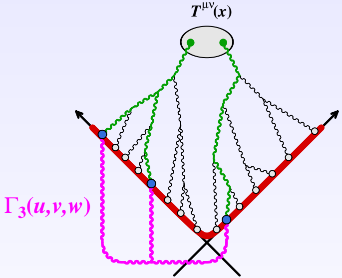
Loop $\sim g^2$, $e^{\sqrt{\mu\tau}}$ for each field perturbation



- 1 loop : $(g e^{\sqrt{\mu\tau}})^2$
- 2 disconnected loops : $(g e^{\sqrt{\mu\tau}})^4$

Improved CGC power counting

Loop $\sim g^2$, $e^{\sqrt{\mu\tau}}$ for each field perturbation



- 1 loop : $(ge^{\sqrt{\mu\tau}})^2$
- 2 disconnected loops : $(ge^{\sqrt{\mu\tau}})^4$
- 2 nested loops : $g(ge^{\sqrt{\mu\tau}})^3$
  ▷ subleading

Leading terms at τ_{max}

- All disconnected loops to all orders
  ▷ exponentiation of the 1-loop result

- We need the operator :

$$\exp\left[\frac{1}{2}\int_{u,v}\Gamma_2(u,v)\,\frac{\partial}{\partial\mathcal{A}_{\rm init}(u)}\,\frac{\partial}{\partial\mathcal{A}_{\rm init}(v)}+\int_{u}\alpha(u)\,\frac{\partial}{\partial\mathcal{A}_{\rm init}(u)}\right]$$

One can prove that

$$e^{\frac{\alpha}{2}\partial_x^2}\;f(x)=\int_{-\infty}^{+\infty}dz\;\frac{e^{-z^2/2\alpha}}{\sqrt{2\pi\alpha}}\;f(x+z)$$

Resummation of the leading secular terms

$$T^{\mu\nu}_{\text{resummed}} = \int [D\mathfrak{a}] \, \exp\left[-\frac{1}{2} \int_{u,v} \mathfrak{a}(u) \Gamma_2^{-1}(u,v) \mathfrak{a}(v) \right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}} + \mathfrak{a}]$$

- There is a unique choice of the variance Γ_2 such that

$$T^{\mu\nu}_{\text{resummed}} = T^{\mu\nu}_{\text{LO}} + T^{\mu\nu}_{\text{NLO}} + \cdots$$

- This resummation collects all the terms with the worst time behavior

Resummation of the leading secular terms

$$T^{\mu\nu}_{\rm resummed} = \int [{\rm D}a]\; \exp\left[-\frac{1}{2}\int_{u,v} a(u)\Gamma_2^{-1}(u,v)a(v)\right]\; T^{\mu\nu}_{\rm LO}[\mathcal{A}_{\rm init}+a]$$

- There is a unique choice of the variance Γ_2 such that

$$T^{\mu\nu}_{\rm resummed} = T^{\mu\nu}_{\rm LO} + T^{\mu\nu}_{\rm NLO} + \cdots$$

- This resummation collects all the terms with the worst time behavior
- Equivalent to Gaussian fluctuations of the initial field + classical time evolution
- At $Q_s\tau_0 \ll 1$: $\mathcal{A}_{\rm init} \sim Q_s/g \quad , \quad a \sim Q_s$

Main steps

1. Determine the 2-point function $\Gamma_2(u, v)$ that defines the Gaussian fluctuations, for the initial time $Q_s\tau_0$ of interest

   Note : this is an initial value problem, whose outcome is uniquely determined by the state of the system at $x^0 = -\infty$, and depends on the history of the system from $x^0 = -\infty$ to $\tau = \tau_0$

   Problem solvable only if the fluctuations are weak, $a^\mu \ll Q_s/g$

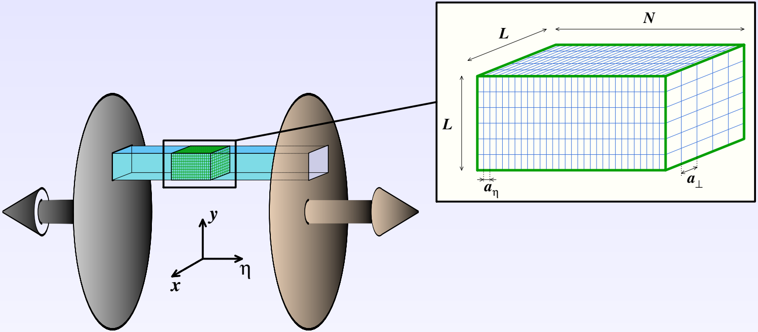
   $Q_s\tau_0 \ll 1$ necessary for the fluctuations to be Gaussian

2. Solve the classical Yang-Mills equations from τ_0 to τ_f

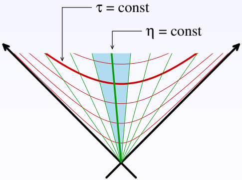
   Note : the problem as a whole is boost invariant, but individual field configurations are not $\implies$ 3+1 dimensions necessary

3. Do a Monte-Carlo sampling of the fluctuating initial conditions

Discretization of the expanding volume



- Comoving coordinates : $\tau, \eta, \boldsymbol{x}_\perp$
- Only a sub-volume is simulated
  + periodic boundary conditions
- $\mathrm{L}^2 \times \mathrm{N}$ lattice

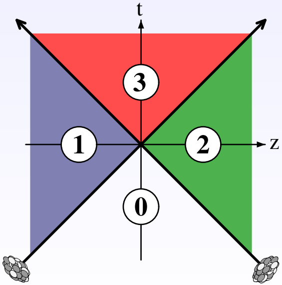


Gaussian spectrum of fluctuations [Epelbaum, FG (2013)]

Expression of the variance (from 1-loop considerations)

$$\Gamma_2(u,v) = \int_{\text{modes } \mathbf{k}} a_{\mathbf{k}}(u)\, a^*_{\mathbf{k}}(v)$$

$$\left[\mathcal{D}_\rho \mathcal{D}^\rho \delta^\nu_\mu - \mathcal{D}_\mu \mathcal{D}^\nu + ig\,\mathcal{F}_\mu{}^\nu\right] a^\mu_{\mathbf{k}} = 0 \quad , \quad \lim_{x^0 \to -\infty} a_{\mathbf{k}}(x) \sim e^{ik\cdot x}$$



- **0.** $\mathcal{A}^\mu = 0$, trivial
- **1,2.** $\mathcal{A}^\mu =$ pure gauge, analytical solution
- **3.** $\mathcal{A}^\mu$ non-perturbative
  $\Rightarrow$ expansion in $Q_s\tau$
- $a^\mu_{\mathbf{k}}(\tau, \eta, \boldsymbol{x}_\perp)$ known analytically at $Q_s\tau \ll 1$, in the gauge $a^\tau = 0$

- At the moment, two implementations of this method, but discrepancy in the results regarding the behavior of P_L/P_T...

Classical Statistical Approximation

Classical Statistical Approximation (CSA)

- Classical time evolution
- Quantum fluctuations in the initial conditions

- Dynamics fully non-linear $\Rightarrow$ no unbounded growth
- Individual classical trajectories may be chaotic $\Rightarrow$ a small initial ensemble can span a large phase space volume

CSA in Quantum Mechanics

- Consider the von Neumann equation for the density operator :

$$\frac{\partial \widehat{\rho}_\tau}{\partial \tau} = i\hbar \left[\widehat{H}, \widehat{\rho}_\tau\right] \qquad (1)$$

- Introduce the Wigner transforms :

$$\begin{aligned} W_\tau(x,p) &\equiv \int ds\; e^{ip\cdot s}\, \left\langle x+\frac{s}{2}\middle|\widehat{\rho}_\tau\middle|x-\frac{s}{2}\right\rangle \\ \mathcal{H}(x,p) &\equiv \int ds\; e^{ip\cdot s}\, \left\langle x+\frac{s}{2}\middle|\widehat{H}\middle|x-\frac{s}{2}\right\rangle \quad \text{(classical Hamiltonian)} \end{aligned}$$

- (1) is equivalent to

$$\begin{aligned} \frac{\partial W_\tau}{\partial \tau} &= \mathcal{H}(x,p)\,\frac{2}{i\hbar}\,\sin\left(\frac{i\hbar}{2}\left(\overleftarrow{\partial}_p\overrightarrow{\partial}_x - \overleftarrow{\partial}_x\overrightarrow{\partial}_p\right)\right) W_\tau(x,p) \\ &= \underbrace{\left\{\mathcal{H}, W_\tau\right\}}_{\text{Poisson bracket}} + \mathcal{O}(\hbar^2) \end{aligned}$$

CSA in Quantum Mechanics

- Approximating the right hand side by the Poisson bracket
  $\Longleftrightarrow$ classical time evolution instead of quantum
  $\Longrightarrow$ $\mathcal{O}(\hbar^2)$ error

- In addition : $\hbar$ dependence in the initial state
  Uncertainty principle, $\Delta x \cdot \Delta p \geq \hbar$
  $\Longrightarrow$ the Wigner distribution $W_{\tau=0}(x, p)$ must have a width $\gtrsim \hbar$

- All the $\mathcal{O}(\hbar)$ effects can be accounted for by a Gaussian initial distribution $W_{\tau=0}(x, p)$

CSA from the path integral

$$\langle \mathcal{O} \rangle = \int [D\phi_+ D\phi_-] \; \mathcal{O}[\phi_\pm] \; e^{i(S[\phi_+]-S[\phi_-])}$$

- ϕ_+ = field in the amplitude
- ϕ_- = field in the conjugate amplitude
- $\phi_+ - \phi_-$ = quantum interference

- Introduce : $\phi_1 \equiv \phi_+ - \phi_-$, $\phi_2 \equiv \frac{1}{2}(\phi_+ + \phi_-)$

$$S[\phi_+] - S[\phi_-] = \phi_1 \cdot \frac{\delta S[\phi_2]}{\delta \phi_2} + \text{terms cubic in } \phi_1$$

- Strong field regime : $\phi_\pm$ large, but $\phi_+ - \phi_-$ small
  Neglect the terms cubic in ϕ_1
  $D\phi_1 \;\; \rightarrow$ classical Euler-Lagrange equation for ϕ_2
- The only remaining fluctuations are in the initial condition for ϕ_2

CSA from perturbation theory

- Start from Schwinger-Keldysh perturbation theory
- Rotate from the basis $\phi_\pm$ to the basis $\phi_{1,2}$
- New perturbative rules :
  - Propagators G_{12}, G_{21} and G_{22} $\quad (G_{11} = 0)$
  - Vertices 1222 and 1112
- CSA : drop the 1112 vertex

Non renormalizability of the CSA

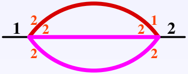
- CSA $\neq$ underlying theory at 2-loops and beyond

- Sources of fluctuations of the initial fields :

$$G_{22}(p) \sim \left(f_0(\mathbf{p}) + \frac{1}{2} \right) \delta(p^2)$$

quasiparticles $\hookleftarrow$ $\hookrightarrow$ vacuum fluctuations

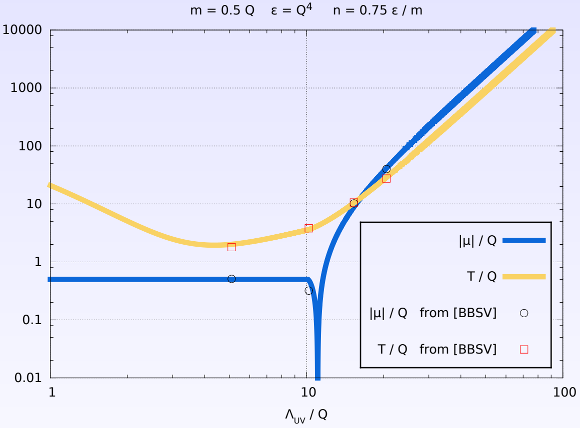
- **Vacuum fluctuations** make the CSA **non-renormalizable**. Example of problematic graph :

Im 

$$= -\frac{g^4}{1024\pi^3} \left(\Lambda_{_{UV}}^2 - \frac{2}{3} p^2 \right)$$

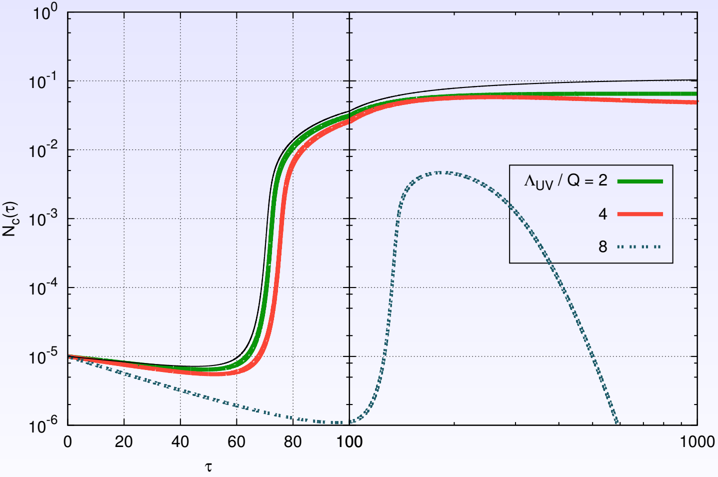
- With only **quasiparticle**-induced fluctuations :
  - Finite if $f_0(p)$ falls faster than p^{-1}
  - Super-renormalizable if $f_0(p) \sim p^{-1}$ [Aarts, Smit (1997)]

Cutoff dependence at late time



- Sweet range : $\Lambda_{\rm UV} \sim (3-6) \times Q$
- But no continuum limit

Occupation in the zero mode for various UV cutoffs



Bose-Einstein condensation

Overpopulated CGC initial conditions

CGC initial conditions

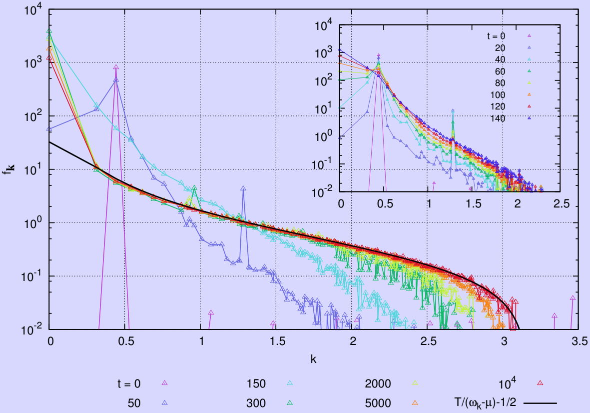
$$\epsilon_0 \sim \frac{Q_s^4}{\alpha_s} \qquad n_0 \sim \frac{Q_s^3}{\alpha_s} \qquad (n\epsilon^{-3/4})_0 \sim \alpha_s^{-1/4}$$

Equilibrium state

$$\epsilon \sim T^4 \qquad n \sim T^3 \qquad n\epsilon^{-3/4} \sim 1$$

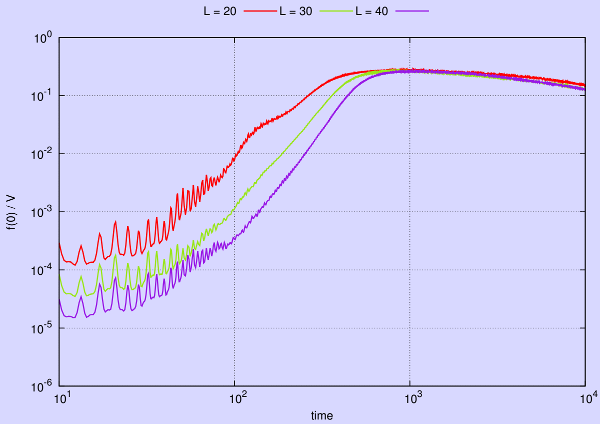
- The excess of gluons can be eliminated in two ways :
  - via inelastic processes $3 \to 2$
  - by condensation on the zero mode

Bose-Einstein condensation (in a scalar field theory)



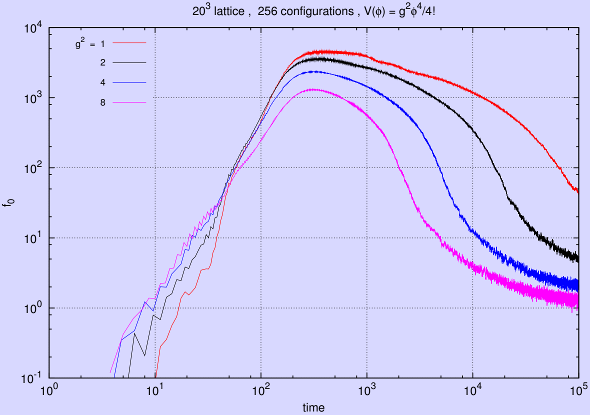
- Start with an overpopulated initial condition, with an empty zero mode
- Very quickly, the zero mode becomes highly occupied

Volume dependence



$$f(k) = \frac{1}{e^{\beta(\omega_k - \mu)} - 1} + n_0 \delta(k) \quad \Longrightarrow f(0) \propto V = L^3$$

Evolution of the condensate



- Formation time almost independent of the coupling
- Condensate lifetime much longer than its formation time
- Smaller amplitude and faster decay at large coupling

Summary
and Outlook

Summary

- Gluon saturation and recombination
  - prevents the gluon occupation number to go above $1/\alpha_s$
  - prevents violations of unitarity in scattering amplitudes
- Two equivalent descriptions
  - Balitsky-Kovchegov :

    Non-linear evolution equation for specific matrix elements

    The non-linear terms lead to the dynamical generation of geometrical scaling

    Applicable to collisions between a saturated and a dilute projectile
  - Color Glass Condensate :

    The color fields of the target evolve with rapidity

    More suitable to collisions of two saturated projectiles
- Isotropization, Thermalization
  - Instabilities require the resummation of additional contributions
  - Possibility of the formation of a Bose-Einstein condensate