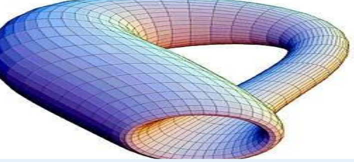
Matrix Product States for Lattice Gauge Theories

Krzysztof Cichy
NIC, DESY Zeuthen, Germany
Adam Mickiewicz University, Poznań, Poland

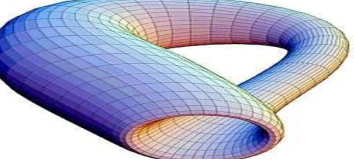
in collaboration with:
Mari Carmen Bañuls (MPQ Garching)
J. Ignacio Cirac (MPQ Garching)
Karl Jansen (DESY Zeuthen)
Agnieszka Kujawa-Cichy (Goethe-Universität Frankfurt am Main)
Hana Saito (Humboldt Universität, DESY Zeuthen)
Marcin Szyniszewski (Lancaster University, University of Manchester)



Interdisciplinary project

The project involves people from very different branches of theoretical physics:

- Lattice QCD at zero temperature
  K.C., Karl Jansen (DESY Zeuthen)
- Lattice QCD thermodynamics
  Hana Saito (DESY Zeuthen, Univ. of Tsukuba)
- Quantum Information, Quantum Many-Body Physics
  Mari Carmen Bañuls, J. Ignacio Cirac
  (Max-Planck Insitute for Quantum Optics, Garching)
- Theoretical Condensed Matter Physics, Ultracold Atomic Gases
  Agnieszka Kujawa-Cichy (Goethe-Universität Frankfurt am Main)
- Theoretical Nanophysics
  Marcin Szyniszewski (Lancaster University, Univ. of Manchester)



Seminar outline

1. Introduction
   - Motivation – Lattice QCD
   - Schwinger model
   - Hamiltonian approach
   - Tensor Network States
2. Results
   - Ground state energy
   - Vector and scalar mass gap
   - Chiral condensate $T = 0$
   - Chiral condensate $T > 0$
3. Prospects

Based on:

- K. Cichy, A. Kujawa-Cichy and M. Szyniszewski, "Lattice Hamiltonian approach to the massless Schwinger model: Precise extraction of the mass gap," Comput. Phys. Commun. **184** (2013) 1666, [arXiv:1211.6393 [hep-lat]]
- M. C. Bañuls, K. Cichy, K. Jansen and J. I. Cirac, "The mass spectrum of the Schwinger model with Matrix Product States," JHEP **1311** (2013) 158, [arXiv:1305.3765 [hep-lat]]
- M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen and H. Saito, "Matrix Product States for Lattice Field Theories," PoS(LATTICE 2013)332, [arXiv:1310.4118 [hep-lat]]

Lattice QCD

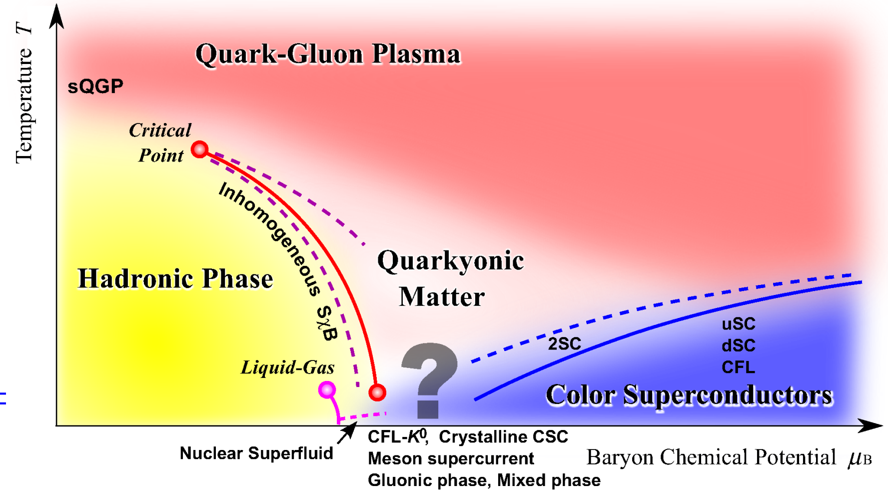
- The most common approach to Lattice QCD simulations consists in sampling the QCD path integral numerically via the Monte Carlo method.
- The QCD path integral: $Z = \int D\bar{\psi} D\psi DU \; e^{-S_{gauge}[U] - S_{ferm}[\psi,\bar{\psi},U]}$.
- The fermionic degrees of freedom can be integrated out:
  $Z = \int DU \; e^{-S_{gauge}[U]} \prod_{f=1}^{N_f} \det(\hat{D}_f[U])$,
  where $\det(\hat{D}_f[U])$ is the determinant of the Dirac operator matrix for fermion flavour f.
- The fermionic determinant $\det(\hat{D}_f[U])$ is by far the highest cost in a MC simulation. But, due to γ_5-Hermiticity ($\gamma_5 \hat{D}_f \gamma_5 = \hat{D}_f^\dagger$) it is real, so MC simulations are possible:

$$\det\left(\gamma_5(\hat{D}_f + m)\gamma_5\right) = \det\left(\hat{D}_f^\dagger + m\right) = \det\left(\hat{D}_f + m\right)^\dagger.$$

- First approximation ⇒ neglect the determinant ("quenched approximation") – commonly used until early 2000s.
- Dynamical simulations ⇒ take the determinant into account.

Problems of Lattice QCD

LQCD simulations led to spectacular successes. However, there are some areas where progress is hard to achieve:

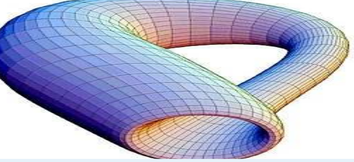


[K. Fukushima, T. Hatsuda, Rep. Prog. Phys. 74 (2011) 14001]

- non-vanishing chemical potential μ – if $\mu \neq 0$, the determinant becomes complex:
  $$\det\left(\gamma_5(\hat{D}_f + m + \mu\gamma_0)\gamma_5\right) = \det\left(\hat{D}_f^\dagger + m - \mu\gamma_0\right) =$$
  $$= \det\left(\hat{D}_f + m - \mu^*\gamma_0\right)^\dagger,$$
  determinant real only if μ taken to be purely imaginary.
  Ways to tackle the problem: reweighting, Taylor expansion, analytic continuation from imaginary μ.
- LQCD works in Euclidean space, related to Minkowski space by analytic continuation – hence time is imaginary. Hence, it is not possible to simulate real-time phenomena, i.e. non-equilibrium dynamics.

Alternative approaches wanted for these classes of problems!

Tensor Networks?



Road map to QCD with Tensor Networks

The way to apply TNS to QCD is a long one.

- START: Schwinger model, i.e. an Abelian gauge theory with U(1) gauge group, 1+1 dimensions

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\,\partial\!\!\!/ - g\,A\!\!\!/ - m)\psi$$

- NATURAL NEXT STEP: non-Abelian gauge theories (SU(2), SU(3)) in 1+1 dimensions
- AND ALSO: go to 2+1 dimensions
- FINALLY: go to 3+1 dimensions, non-Abelian gauge group SU(3) for QCD

All these next steps non-trivial and challenging.

The Schwinger model

The 1-flavour Schwinger model is QED in 1+1 dimensions:
[J. S. Schwinger, Phys. Rev. **128** (1962) 2425]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\,\partial\!\!\!/ - g\,A\!\!\!/ - m)\psi$$

where ψ is a 2-component spinor field. The field strength term is:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The coupling g has dimensions of mass (theory super-renormalizable). Using g as the scale of energy, the physical properties of the model are then functions of the dimensionless ratio m/g.

- simplest gauge theory
- but physics still surprisingly rich
- in several aspects resembles much more complex theories (QCD)
- standard toy model for testing lattice techniques

The Hamiltonian of the Schwinger model in the staggered discretization:

$$H = \frac{x}{2}\sum_{n=0}^{N-1}\Big(\sigma^+(n)e^{i\theta(n)}\sigma^-(n+1) + \sigma^+(n+1)e^{-i\theta(n)}\sigma^-(n)\Big) +$$

$$+\frac{m}{ag^2}\sum_{n=0}^{N-1}\left(1+(-1)^n\sigma^3(n)\right) + \sum_{n=0}^{N-1}L^2(n),$$

where: $x = 1/a^2g^2$.

- Natural choice of basis: direct product of Ising basis $\{|i\rangle\}$, acted upon by Pauli spin operators, and the ladder space of states $\{|l\rangle\}$:

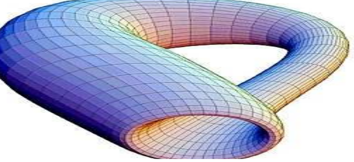
$$|i_0 i_1 \ldots i_{N-2} i_{N-1}\rangle \otimes |l_{0,1} l_{1,2} \ldots l_{N-2,N-1} l_{N-1,0}\rangle,$$

- The gauge degrees of freedom $l_{i,i+1}$ can be eliminated using the Gauss law:

$$L_n - L_{n-1} = \frac{1}{2}\left(\sigma^z_n + (-1)^n\right),$$

leaving the basis states as: $|i_0 i_1 \ldots i_{M-2} i_{M-1}\rangle \otimes |l\rangle,$

- An arbitrary state from a Hilbert space of an N-body interacting system needs in general an exponential number of coefficients – thus computational complexity increases very fast and prohibits exact diagonalization of systems larger than e.g.:
  - ⋆ $\mathcal{O}(20)$ Heisenberg spins (with a naive approach) or
  - ⋆ $\mathcal{O}(40)$ Heisenberg spins (using symmetries etc.).
- However, physical states (ground states, thermal states) of most systems are far from arbitrary.
- In many cases, they can be described by Tensor Network states that have only a polynomial number of parameters.
- In other words, only a small "corner" of the Hilbert space is physically relevant.



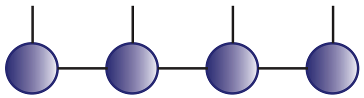
Matrix Product States

- A particularly successful and efficient family of Tensor Network states is called Matrix Product States (MPS).
- The MPS ansatz for some state $|\Psi\rangle$ has the following form:

$$|\Psi\rangle = \sum_{i_0 \dots i_{N-1}=1}^{d} \mathrm{tr}\left(A_0^{i_0} \dots A_{N-1}^{i_{N-1}}\right) |i_0 \dots i_{N-1}\rangle,$$

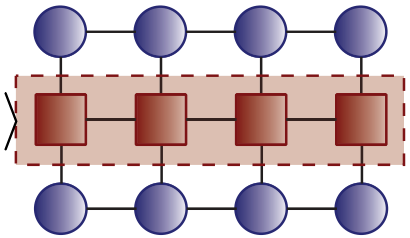
where:
$|i_k\rangle$ are individual basis states for each site ($k = 0, \dots, d-1$),
d – dimension of one-site Hilbert space,
each A_j^i is a D-dimensional matrix
and D is called the bond dimension.

$|\Psi\rangle$

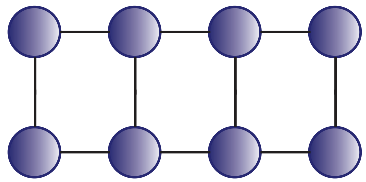


A_k

$\langle\Psi|H|\Psi\rangle$



$\langle\Psi|\Psi\rangle$



- The ground state can be found variationally by successively minimizing the energy $\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}$ with respect to each tensor A_j until convergence is achieved.
- Having the ground state, one can find ground state expectation values of any operator of interest.

Excited states

- After having found the ground state of the system, $|\Psi_0\rangle$, we can construct the projector onto the orthogonal subspace, $\Pi_0 = 1 - |\Psi_0\rangle\langle\Psi_0|$.
- The projected Hamiltonian, $\Pi_0 H \Pi_0$, has $|\Psi_0\rangle$ as eigenstate with zero eigenvalue, and the first excited state as eigenstate with energy E_1.
- Given that $E_1 < 0$, what we can always ensure by adding an appropriate constant to H, the first excitation corresponds then to the state that minimizes the energy of the projected Hamiltonian:

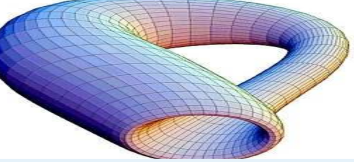
$$E_1 = \min_{|\Psi\rangle} \frac{\langle\Psi|\Pi_0 H \Pi_0|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\langle\Psi|\left(H - E_0|\Psi_0\rangle\langle\Psi_0|\right)|\Psi\rangle}{\langle\Psi|\Psi\rangle}.$$

- This minimization corresponds to finding the ground state of the effective Hamiltonian $H_{\text{eff}}[1] = \Pi_0 H \Pi_0$.
- The procedure can be concatenated to find subsequent energy levels, so that, to find the M-th excited state, we will search for the ground state of the Hamiltonian:

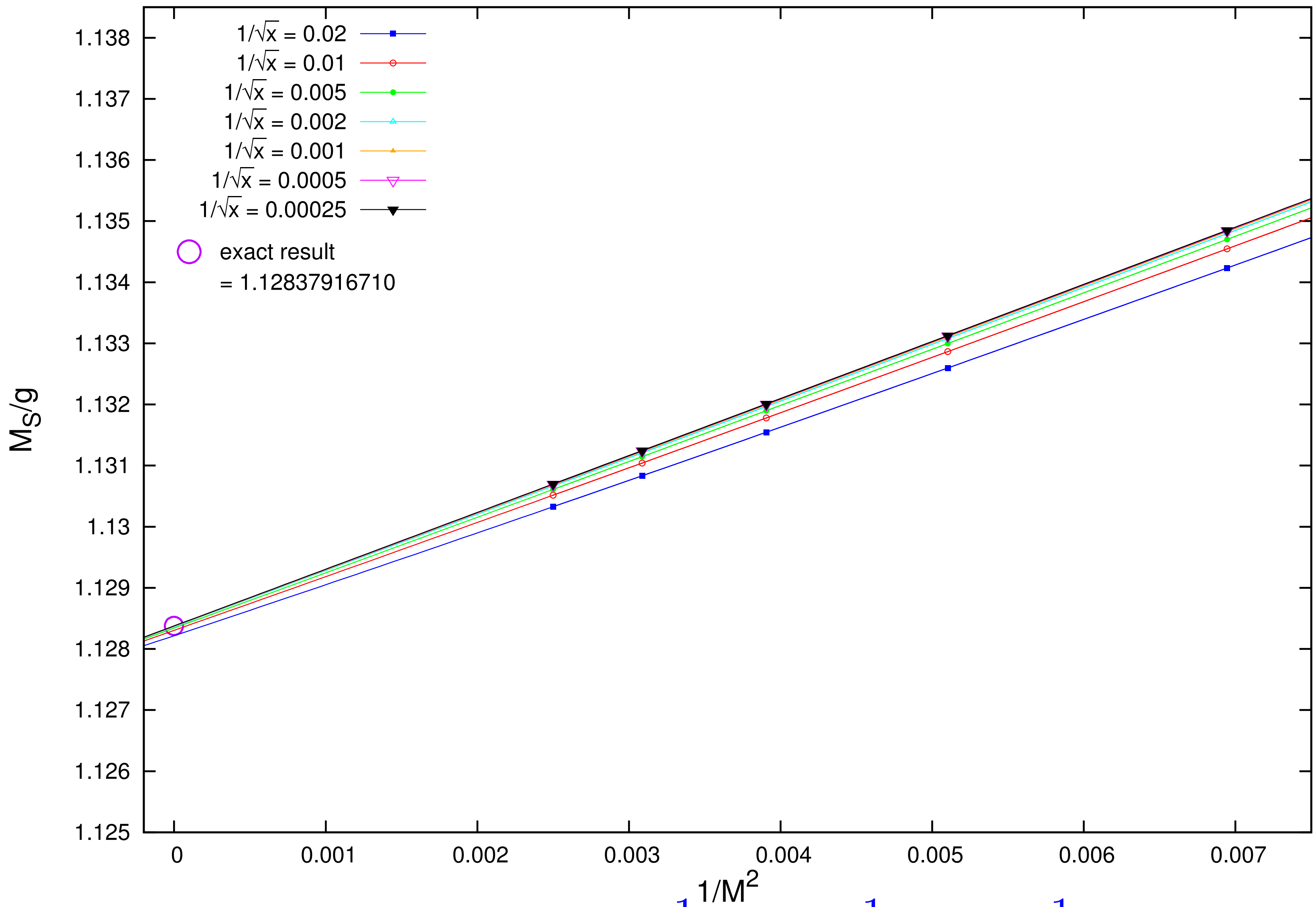
$$H_{\text{eff}}[M] = \Pi_{M-1} \ldots \Pi_0 H \Pi_0 \ldots \Pi_{M-1} = H - \sum_{k=0}^{M-1} E_k |\Psi_k\rangle\langle\Psi_k|.$$

Active area of research

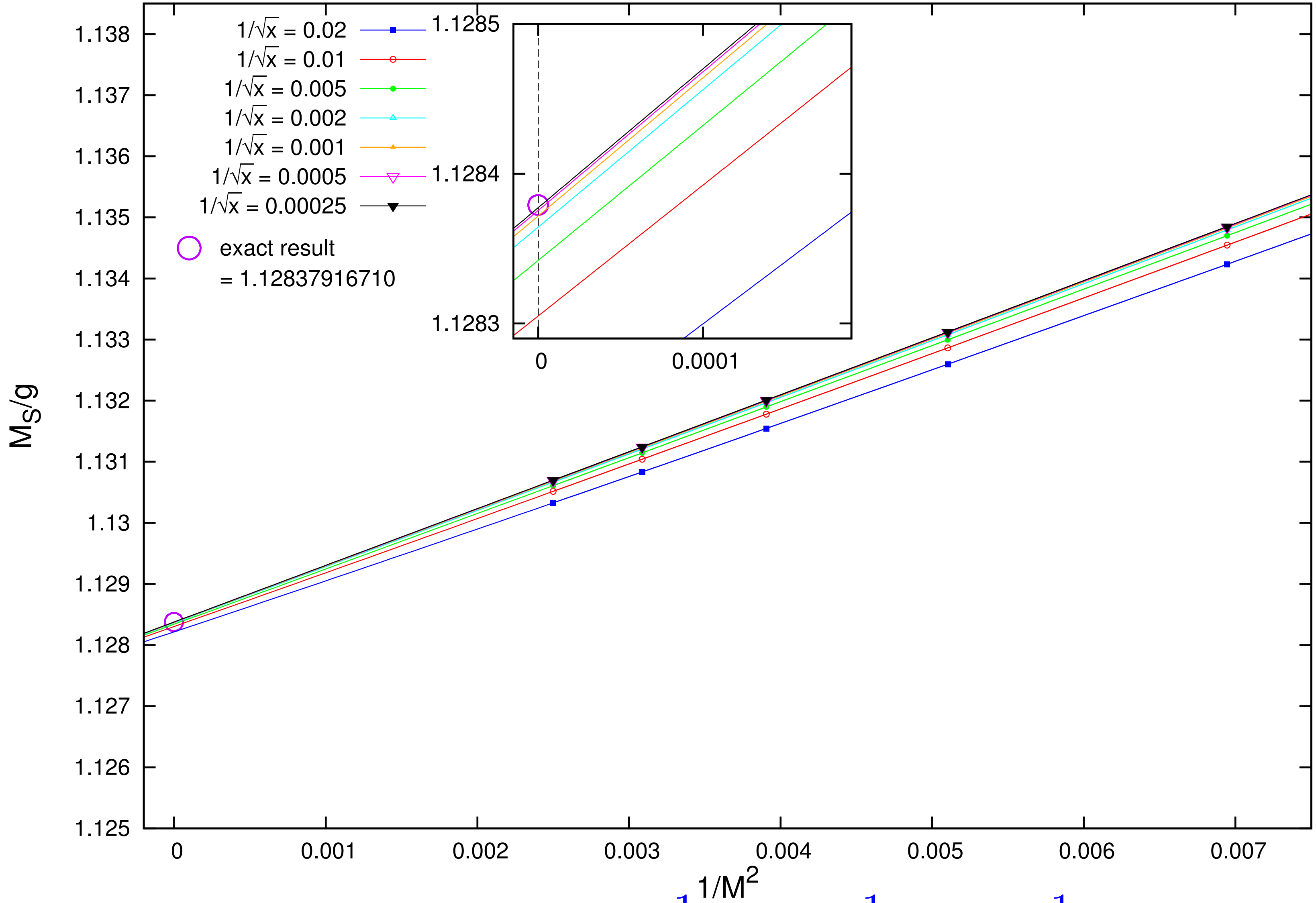
- DMRG approach, Schwinger model – T. Byrnes et al.
  [Phys.Rev. **D66** (2002) 013002 [hep-lat/0202014]]
- DMRG, 2d $\lambda\phi^4$ – T. Sugihara [JHEP **0405** (2004) 007, [hep-lat/0403008]]
- MPS, Z(2) LGT – T. Sugihara [JHEP **0507** (2005) 022, [hep-lat/0506009]]
- Atomic Quantum Simulation of LGTs – D. Banerjee et al.
  [Phys.Rev.Lett. 109 (2012) 175302, arXiv:1205.6366 [cond-mat.quant-gas]]
  [Phys.Rev.Lett. 110 (2013) 125303, arXiv:1211.2242 [cond-mat.quant-gas]]
- MPS, 2d $\lambda\phi^4$ – A. Milsted et al. [Phys.Rev. D**88**, 085030 (2013), arXiv:1302.5582 [hep-lat]]
- TN, quantum link models – E. Rico et al.
  [Phys.Rev.Lett. 112 (2014) 201601, arXiv:1312.3127 [cond-mat.quant-gas]]
- MPS, Schwinger model – B. Buyens et al. [arXiv:1312.6654 [hep-lat]]
- Grassmann TRG, Schwinger model – Y. Shimizu et al. [arXiv:1403.0642 [hep-lat]]
- Lattice Gauge Tensor Networks – P. Silvi et al. [arXiv:1404.7439 [quant-ph]]
- Tensor Networks for Lattice Gauge Theories with continuous groups – L. Tagliacozzo et al.
  [arXiv:1405.4811 [cond-mat.str-el]]



Results

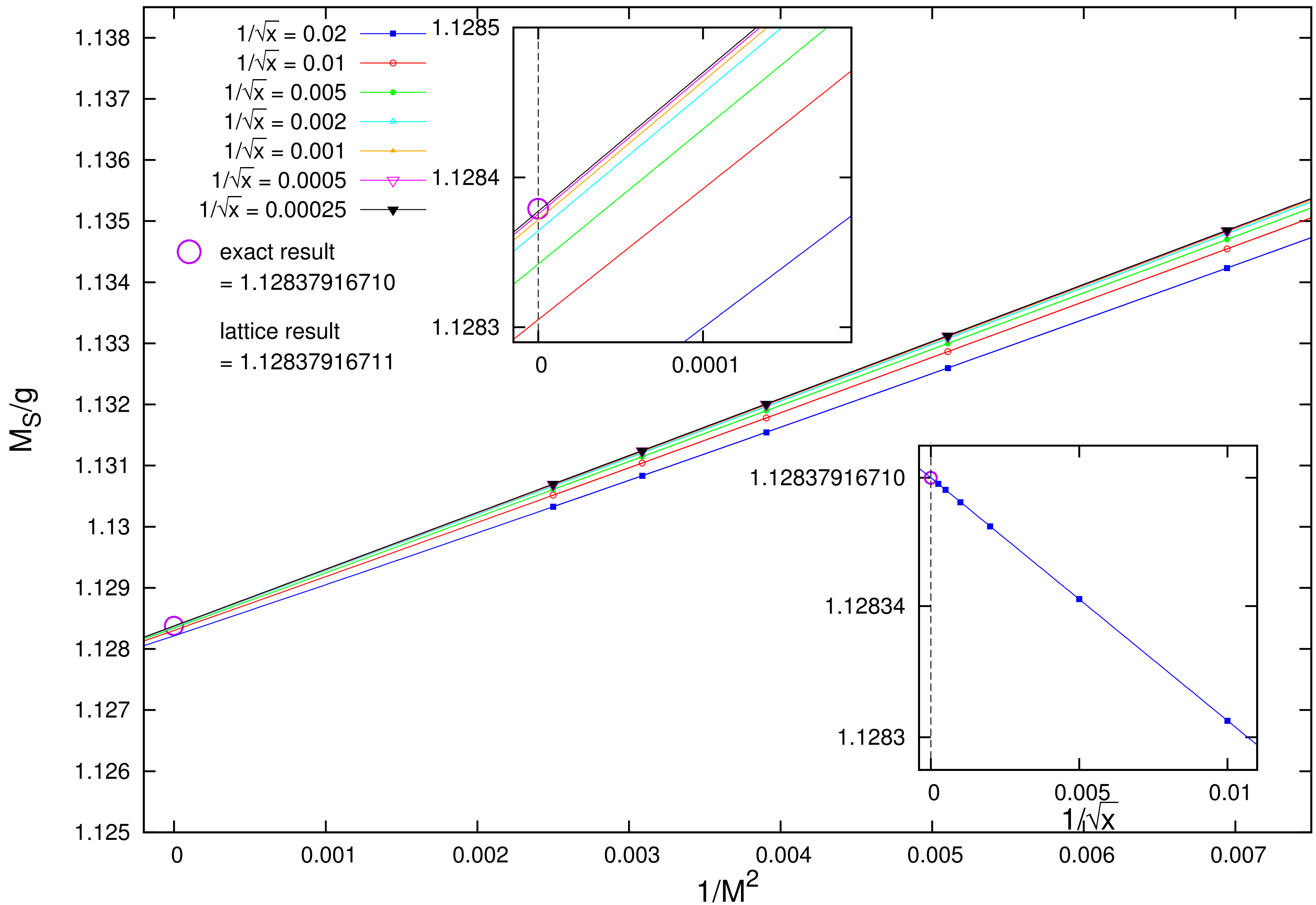


$$F(1/M^2) = F_0 + F_2\frac{1}{M^2} + F_4\frac{1}{M^4} + F_6\frac{1}{M^6}$$

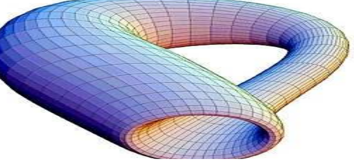


$$F(1/M^2) = F_0 + F_2\frac{1}{M^2} + F_4\frac{1}{M^4} + F_6\frac{1}{M^6}$$

SCE+ED, continuum extrapolation



$$F_0(ag) = F_{00} + F_{01} \cdot ag + F_{02} \cdot (ag)^2$$



SCE+ED, comparison with literature

	M_S/g		M_V/g	
	result	error	result	error
exact	1.12837916710	–	0.5641895836	–
this work	1.12837916711	$1.3 \cdot 10^{-9}\%$	0.5641895845	$1.8 \cdot 10^{-7}\%$
[Crewther, Hamer 1980]	1.120	0.7%	0.560	0.7%
[Irving, Thomas 1982]	1.128	0.03%	0.565	0.1%
[Hamer et al. 1997] (I)	1.25	11%	0.56	0.7%
[Hamer et al. 1997] (II)	1.14	1%	0.57	1%
[Sriganesh et al. 1999] (I)	1.11	1.6%	0.563	0.2%
[Sriganesh et al. 1999] (II)	1.1284	0.002%	0.56417	0.003%
[Byrnes et al. 2002]	–	–	0.56419	$7 \cdot 10^{-5}\%$

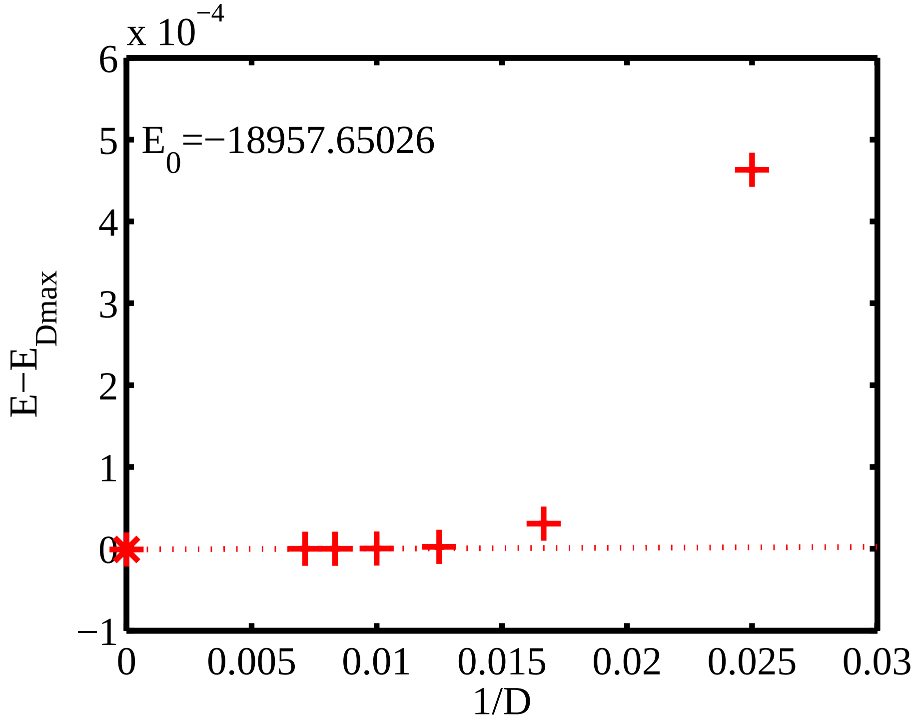
We want to find:

- ground state energy
- vector mass gap
- scalar mass gap

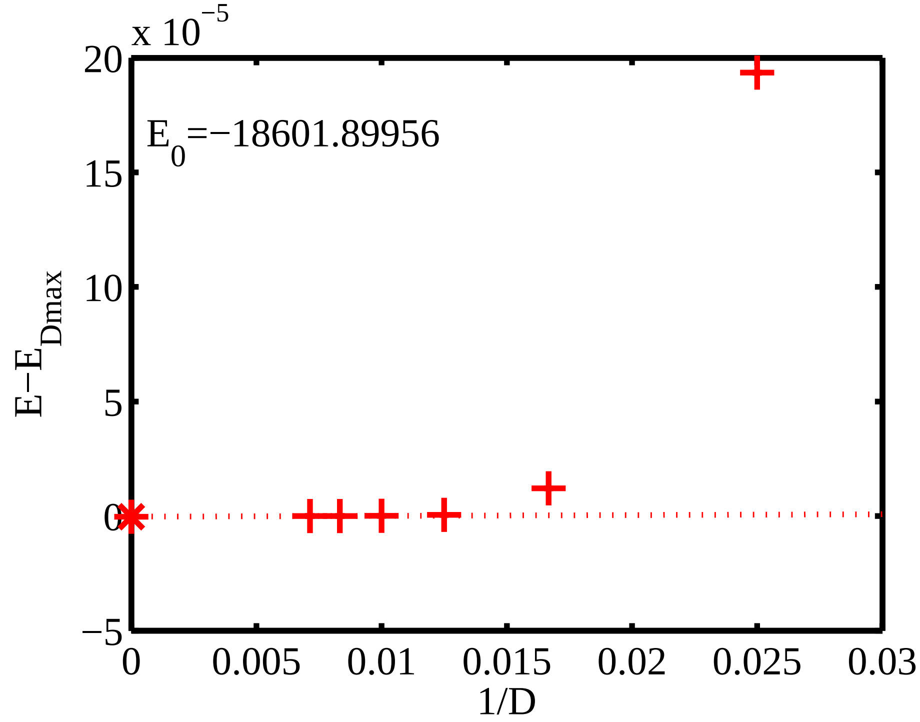
for selected values of the fermion mass $m/g = 0,\ 0.125,\ 0.25,\ 0.5$.

Simulate with finite D (bond dimension), N (system size), x (inverse lattice spacing). We want:

- large enough D – check $D \in [20, 140]$,
- $N \to \infty$ – choose $N \in [100, 850]$ (note that $N \propto x$),
- $x \to \infty$ – choose $x \in [5, 600]$.

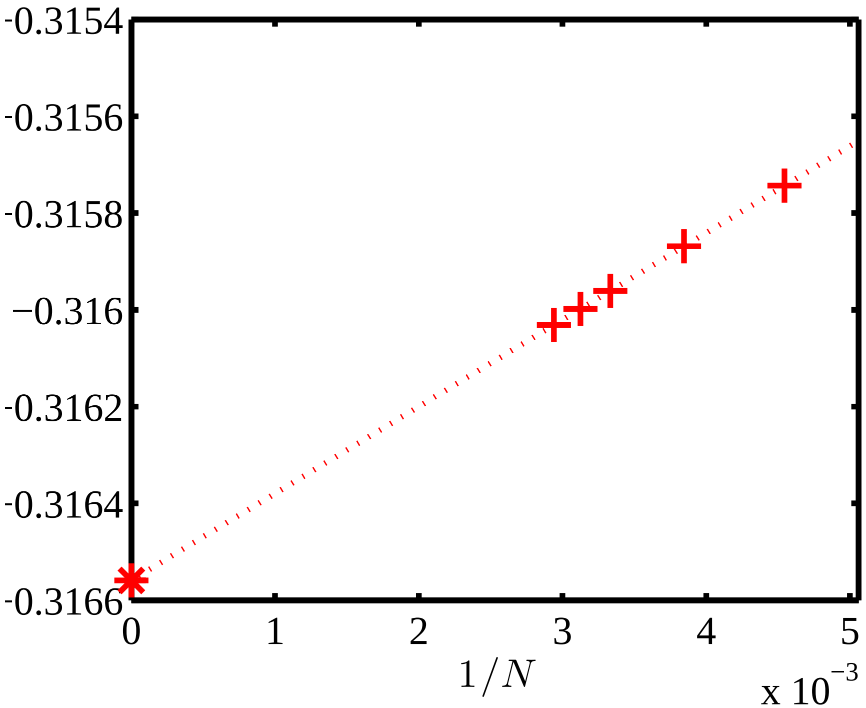


$m/g = 0,\ x = 100,\ N = 300$

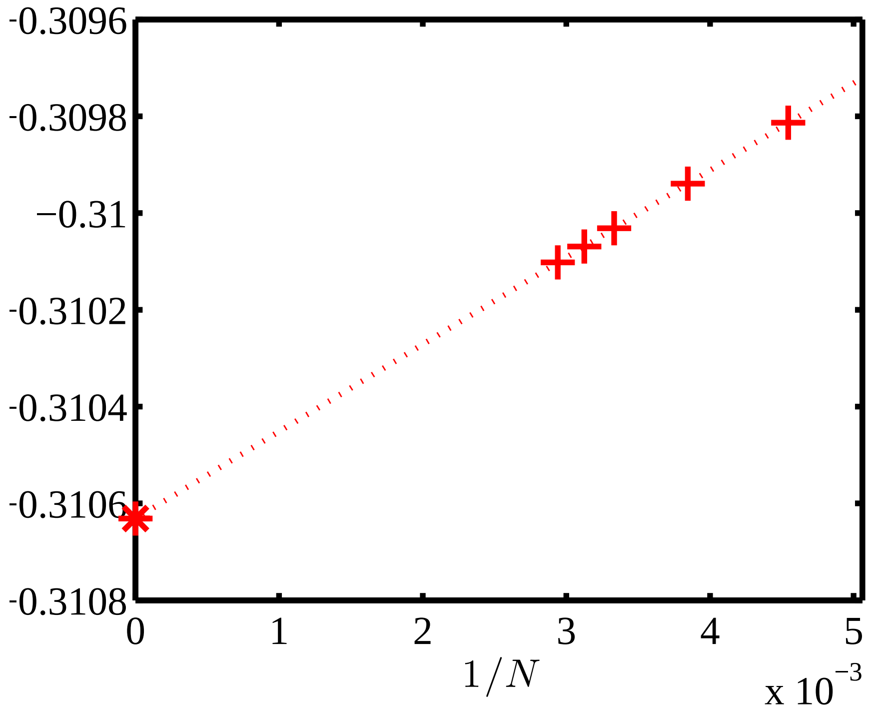


$m/g = 0.125,\ x = 100,\ N = 300$

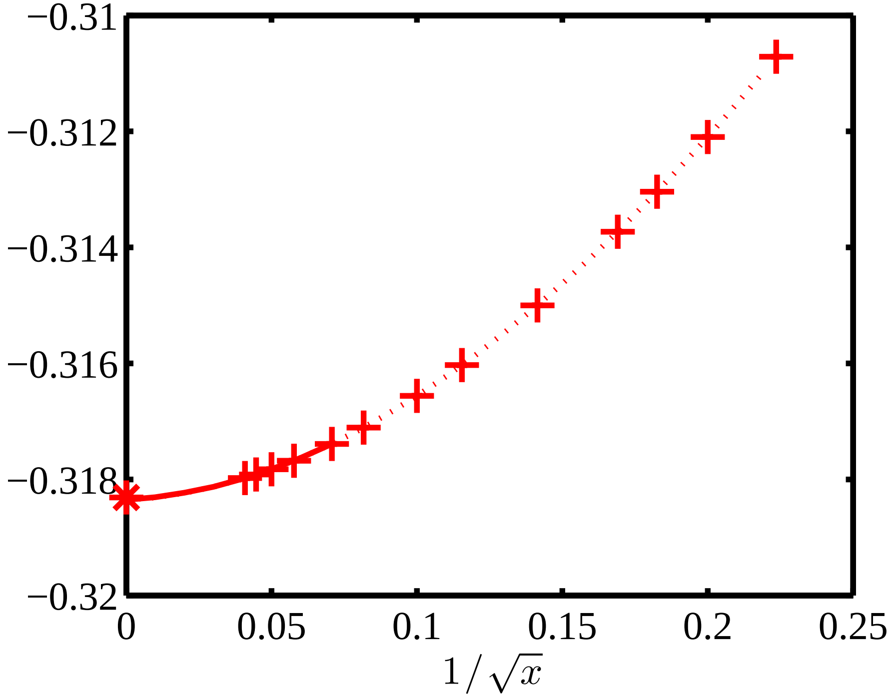
GS energy. Finite size scaling



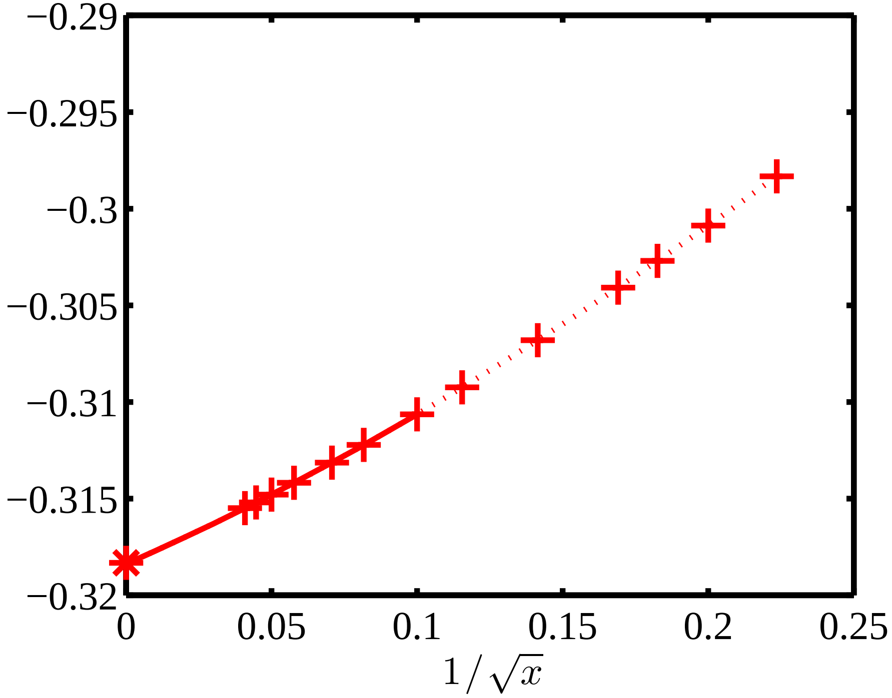
$m/g = 0,\ x = 100$



$m/g = 0.125,\ x = 100$



$m/g = 0$



$m/g = 0.125$

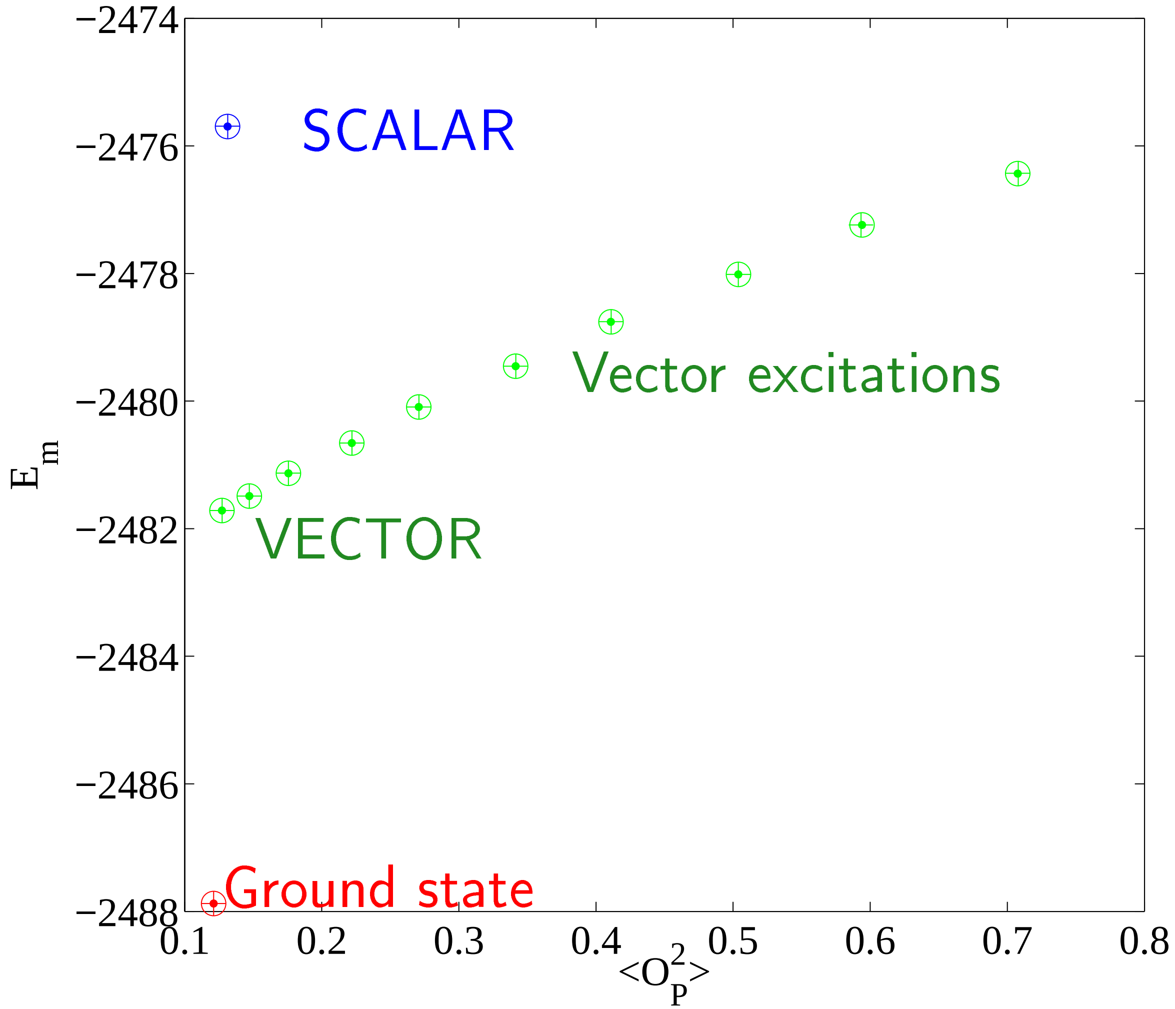
continuum result:

$-0.318338(22)_{D,N,x\,\mathrm{extrapol.}}(24)_{\mathrm{fitansatz}}$ $\quad$ $-0.318343(96)_{D,N,x\,\mathrm{extrapol.}}(25)_{\mathrm{fit\ ansatz}}$

exact result: $1/\pi \approx -0.318310$

Computing the mass gap

- After having computed the GS energy, we want to compute the masses of the two lightest bound states ("mesons") of the theory:
  - ⋆ vector meson,
  - ⋆ scalar meson.
- Important: we have to recognize the vector and scalar states – use the charge conjugation transformation:
  - ⋆ PBC – $C = -1$ ⇒ vector state, $C = +1$ ⇒ scalar state,
  - ⋆ OBC – C no longer an exact symmetry, but "enough" to differentiate vector vs. scalar.
- Note: with OBC translational symmetry is lost – hence we also have momentum excitations of the vector meson *before* we reach the scalar.



Continuum momentum operator

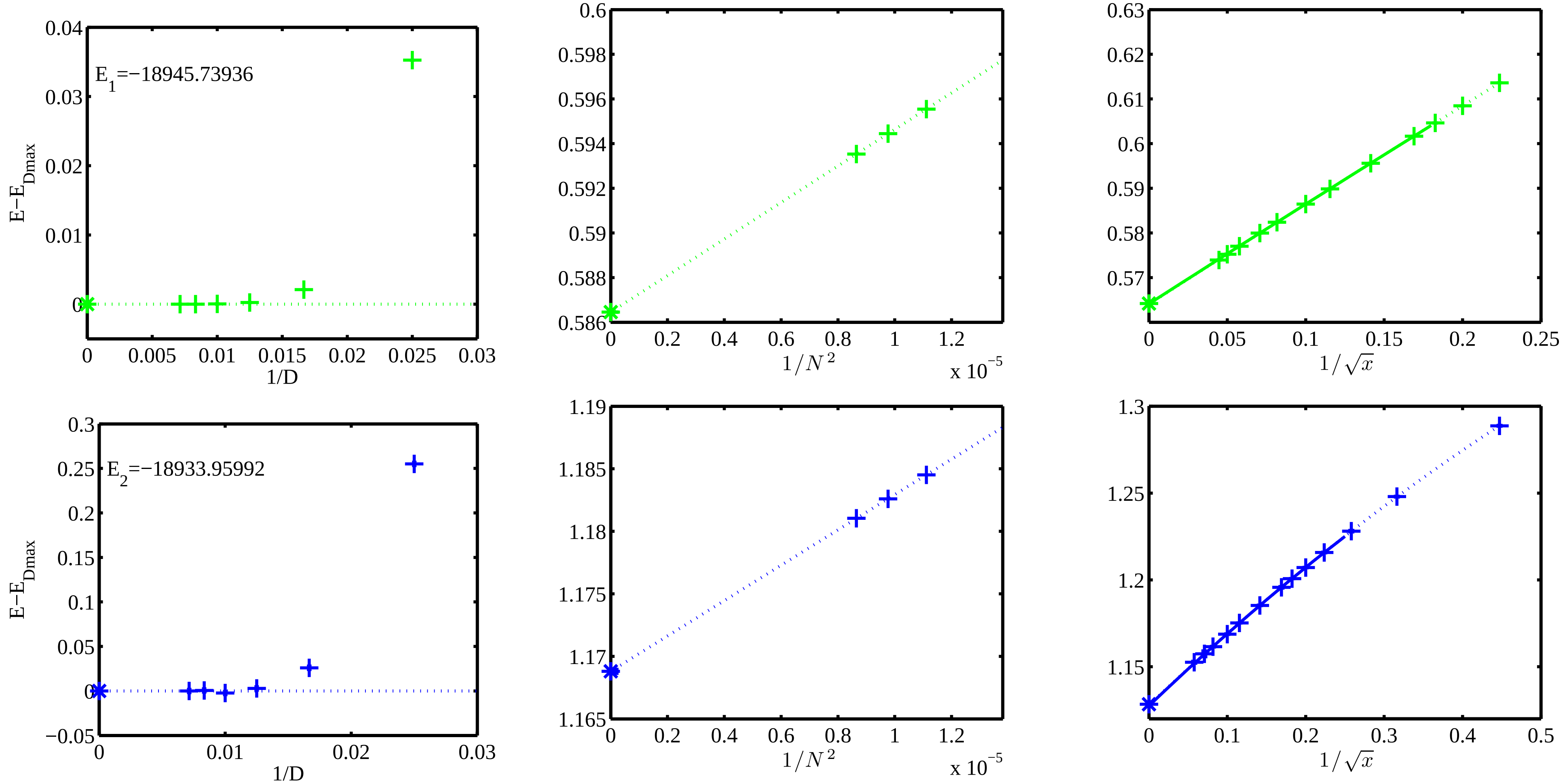
$$\hat{P} = \int dx \Psi^\dagger(x) i\partial_x \Psi(x)$$

Lattice spin representation

$$\hat{O}_P = -ix \sum_n \left(\sigma_n^- \sigma_{n+1}^z \sigma_{n+2}^+ - H.c.\right)$$

VECTOR

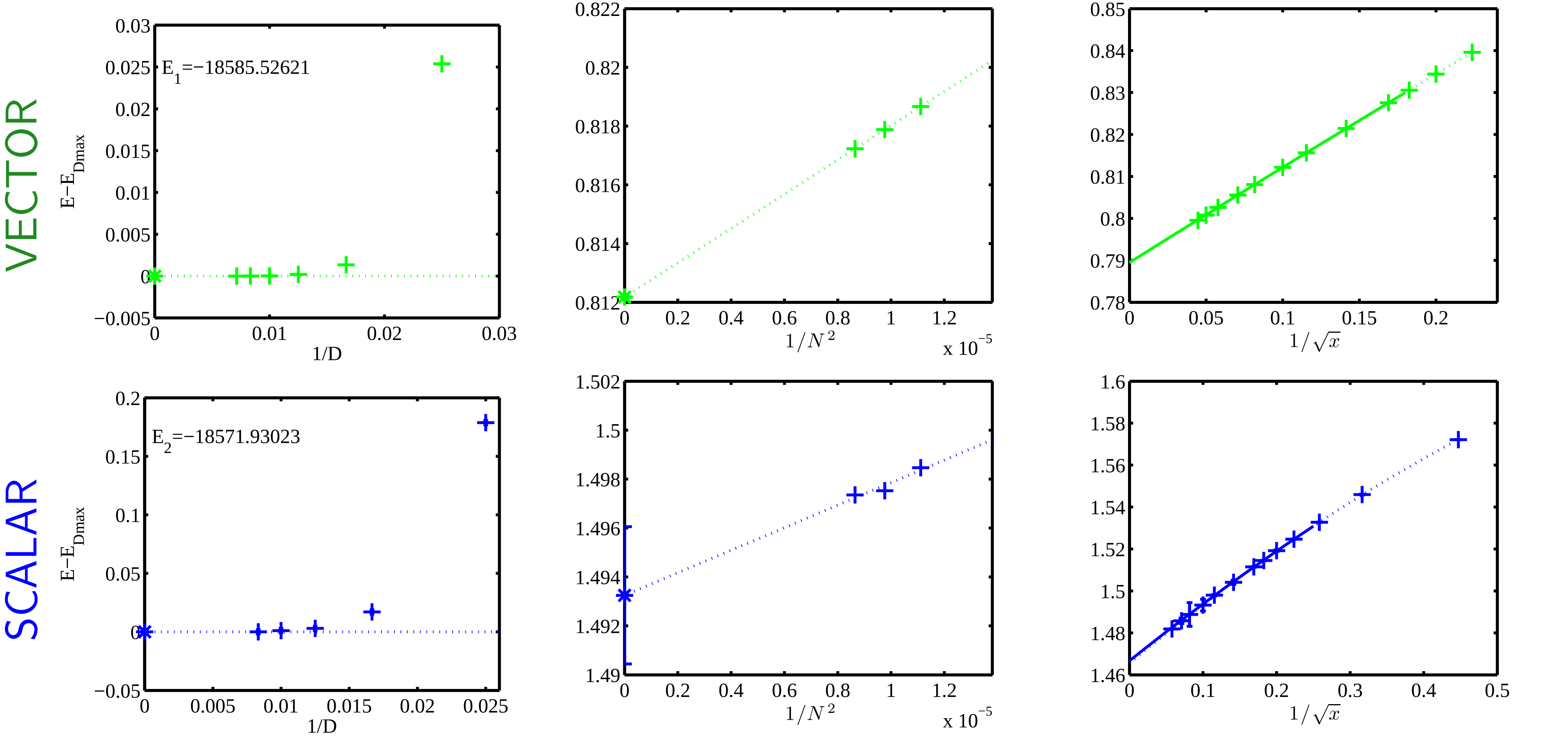
SCALAR



Truncation
$x = 100,\ N = 300$

Finite size scaling
$x = 100$

Continuum extrapolation



Truncation	Finite size scaling	Continuum extrapolation
$x = 100,\ N = 300$	$x = 100$	

Results for the mass gaps

	Vector binding energy exact 0.5641895	
m/g	**MPS with OBC**	DMRG result
0	**0.56421(9)**	0.56419(4)
0.125	**0.53953(5)**	0.53950(7)
0.25	**0.51922(5)**	0.51918(5)
0.5	**0.48749(3)**	0.48747(2)

	Scalar binding energy exact 1.12838	
m/g	**MPS with OBC**	SCE result
0	**1.1279(12)**	1.11(3)
0.125	**1.2155(28)**	1.22(2)
0.25	**1.2239(22)**	1.24(3)
0.5	**1.1998(17)**	1.20(3)

DMRG result:

[T. Byrnes, P. Sriganesh, R. J. Bursill and C. J. Hamer, Phys. Rev. D **66** (2002) 013002]

SCE result:

[P. Sriganesh, R. Bursill and C. J. Hamer, Phys. Rev. D **62** (2000) 034508]

Chiral condensate

- The Schwinger model posesses a $U(1)_A$ chiral symmetry, which is broken by the chiral anomaly.
- This symmetry breaking is signaled by a non-zero value of the chiral condensate:

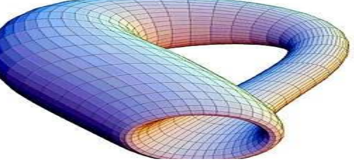
$$\Sigma = \frac{\sqrt{x}}{N} \sum_n (-1)^n \frac{1+\sigma_n^z}{2}$$

$\longrightarrow$ compute GS expectation value of Σ.

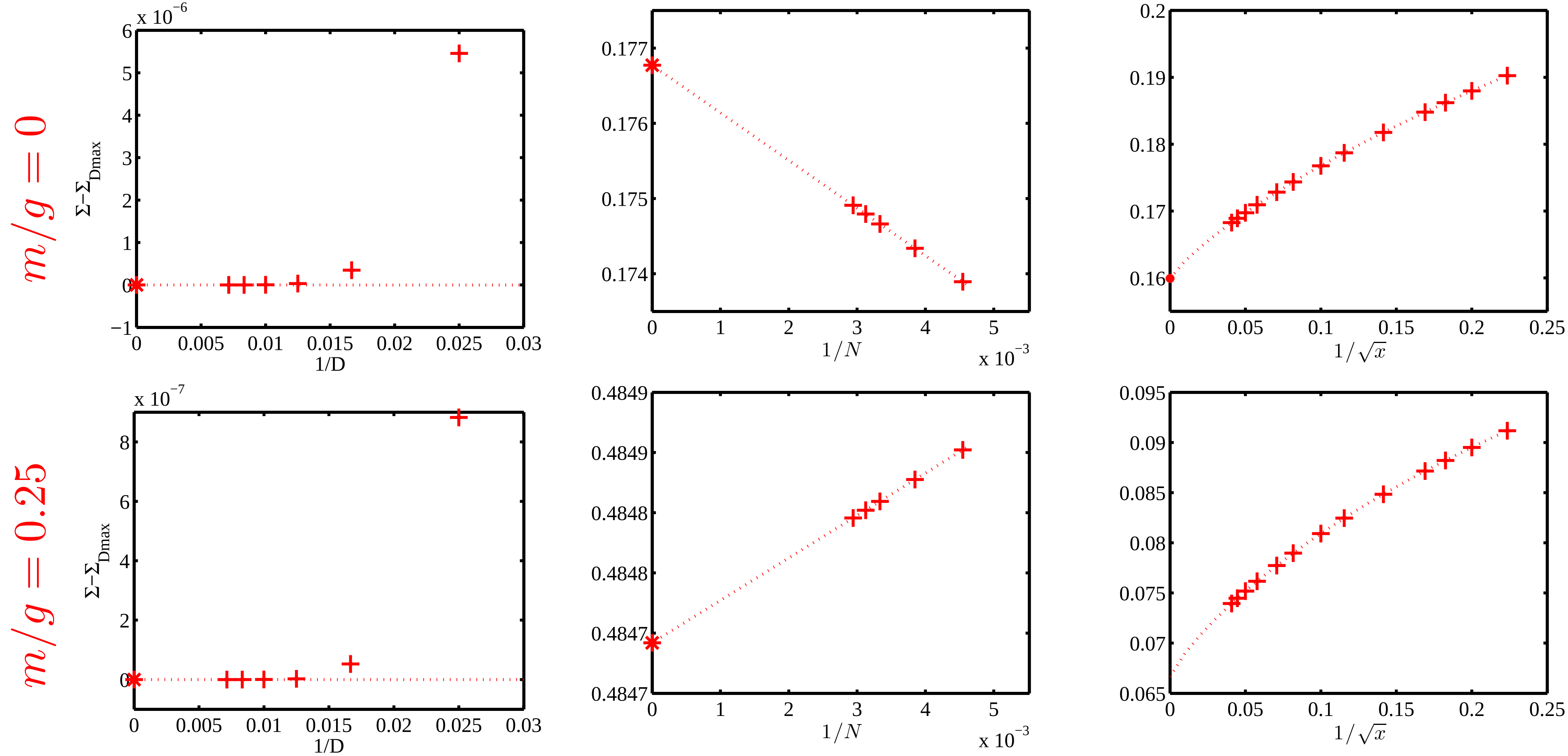
- The naively computed condensate has a logarithmic divergence $\propto \frac{m}{g} \log ag$. This divergence can be subtracted off by subtracting the free theory contribution (in the infinite volume limit):

$$\Sigma_{\text{free}}^{(\text{bulk})}(m/g, x) = \frac{m}{\pi g} \frac{1}{\sqrt{1+\frac{m^2}{g^2 x}}} \mathrm{K}\left(\frac{1}{1+\frac{m^2}{g^2 x}}\right),$$

where $\mathrm{K}(u)$ is the complete elliptic integral of the first kind.



Results for the chiral condensate



Truncation
$x = 100,\ N = 300$

Finite size scaling
$x = 100$

Continuum extrapolation

	Substracted condensate		
m/g	**MPS with OBC**	exact	Hosotani
0	**0.159930(8)**	0.159929	-
0.125	**0.092023(4)**	-	0.0918
0.25	**0.066660(11)**	-	-
0.5	**0.042383(22)**	-	-

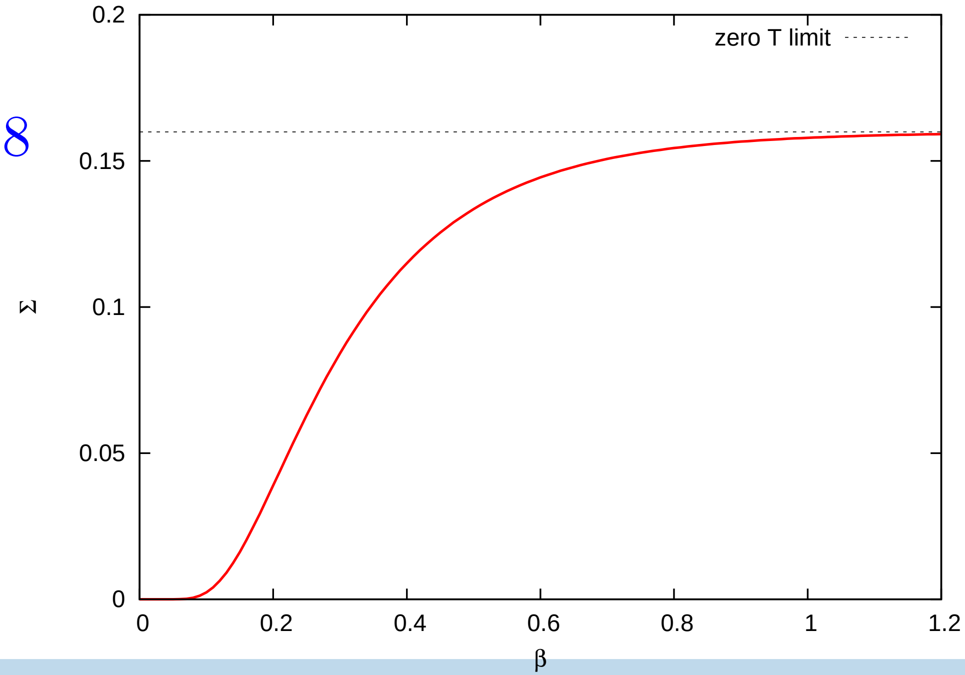
Exact result: $\frac{\Sigma}{g} = \frac{1}{2\pi^{3/2}} e^{\gamma_E} \approx 0.1599288$.
Hosotani (reduction to a quantum mechanics problem and numerical solution of the resulting Schrödinger equation):
[Y. Hosotani, "Chiral dynamics in weak, intermediate, and strong coupling QED in two-dimensions," In: Nagoya 1996, Perspectives of strong coupling gauge theories, 390-397 [hep-th/9703153].]

Analytic prediction for the behaviour of the condensate at finite T:
[I. Sachs, A. Wipf, "Finite Temperature Schwinger Model," Helv. Phys. Acta 65, 652 (1992), arXiv:1005.1822 [hep-th]]

$$\frac{\Sigma}{g} = \frac{1}{2\pi^{3/2}} e^{\gamma_E} e^{2I\left(\frac{g}{\sqrt{\pi}T}\right)},$$

where: $I(x) = \int_0^\infty dt \left(1 - e^{x\cosh(t)}\right)^{-1}\right)$.

- Low-temperature limit:
  $\frac{\Sigma}{g} = \frac{1}{2\pi^{3/2}} e^{\gamma_E} \approx 0.1599288$
- High-temperature limit:
  $\frac{\Sigma}{g} \approx \frac{2T}{g} e^{-\pi^{3/2}T/g}$



Idea of the computation

Given some operator $\mathcal{O}$, we want to calculate its thermal expectation value:

$$\langle \mathcal{O} \rangle_\beta = \frac{\mathrm{Tr}\left(\mathcal{O}\rho(\beta)\right)}{\mathrm{Tr}\left(\rho(\beta)\right)},$$

where $\beta = 1/T$, $\rho(\beta)$ is the thermal density operator.

- $\rho(\beta) = \rho(\beta/2)^\dagger \rho(\beta/2)$ to ensure positivity
- divide the interval $\beta/2$ into $N = \beta/\delta$ steps of length $\delta/2$:

$$\rho(\beta/2) = \underbrace{e^{-\frac{\delta}{2}H} \ldots e^{-\frac{\delta}{2}H}}_{N=\beta/\delta \text{ times}}$$

- 2nd order Trotter expansion:

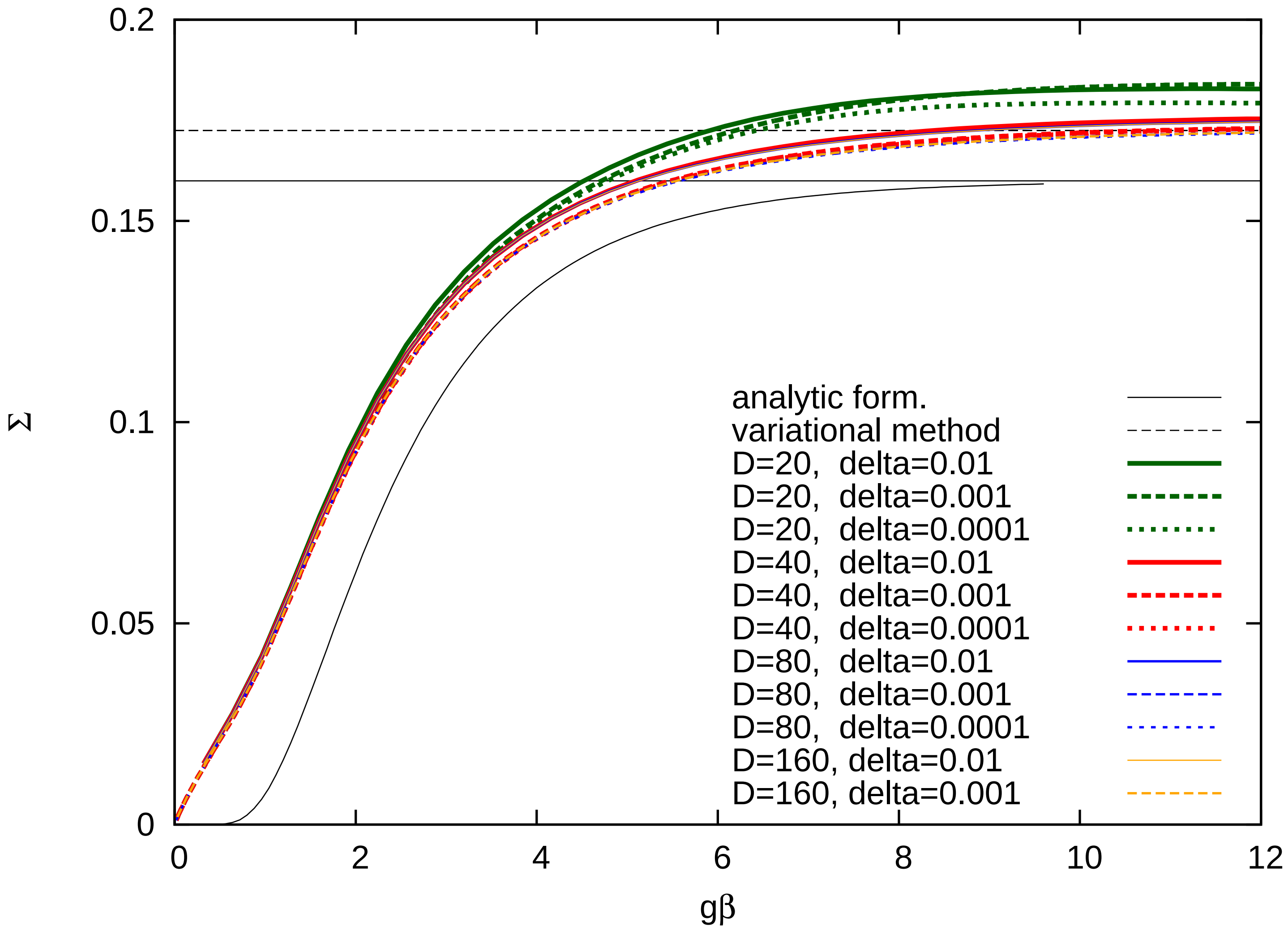
$$e^{-\frac{\delta}{2}H} \approx e^{-\frac{\delta}{4}H_g} \underbrace{e^{-\frac{\delta}{2}(H_{hop}+H_{mass})}}_{\approx e^{-\frac{\delta}{4}H_e} e^{-\frac{\delta}{2}H_o} e^{-\frac{\delta}{4}H_e}} e^{-\frac{\delta}{4}H_g},$$

H_e/H_o – on even/odd sites

where:

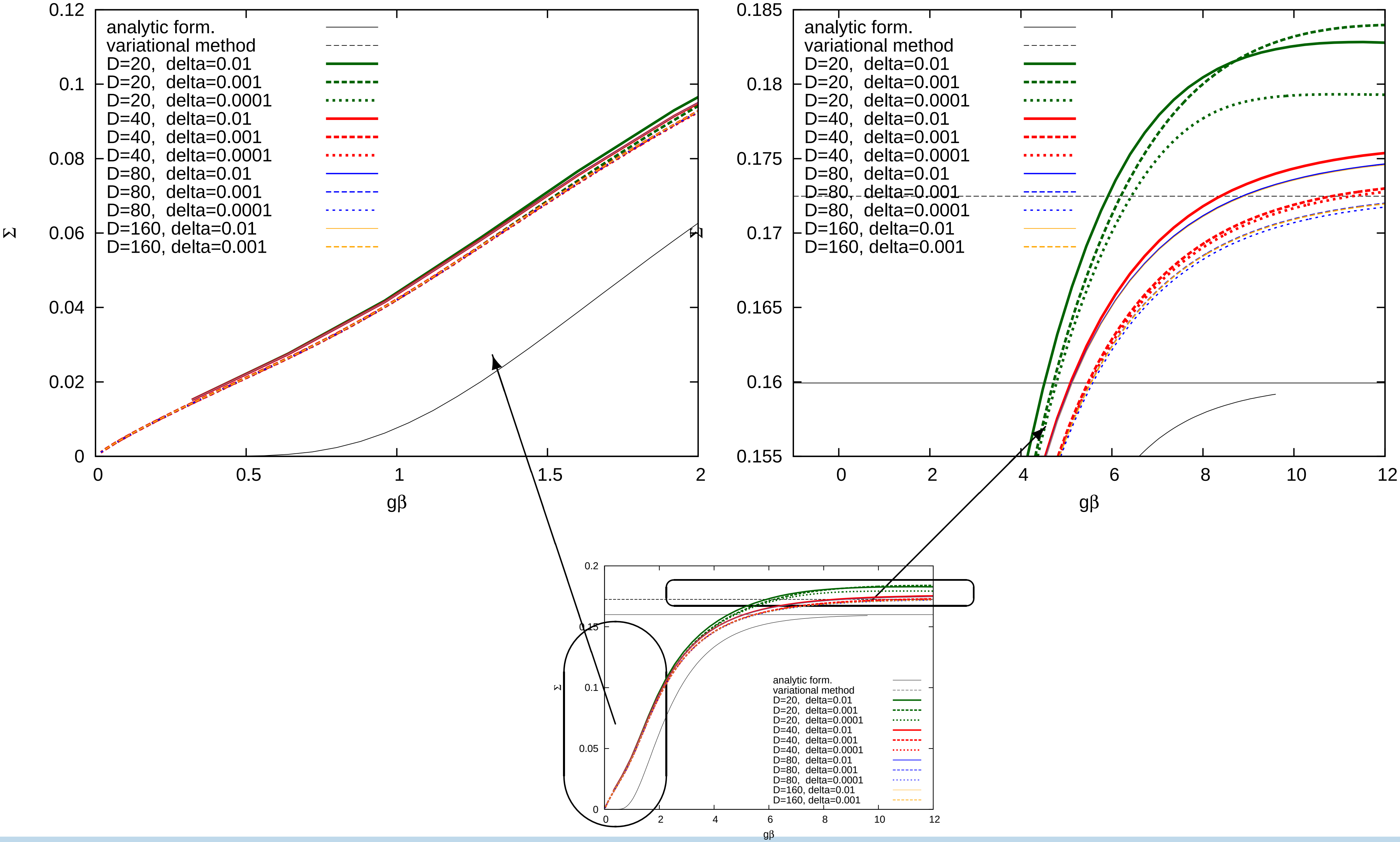
$$H = \underbrace{x \sum_{n=0}^{N-2} \left(\sigma_n^+ e^{i\theta_n} \sigma_{n+1}^- + H.c.\right)}_{H_{hop}} + \underbrace{\mu \sum_{n=0}^{N-1} \left(1 + (-1)^n \sigma_n^3\right)}_{H_{mass}} + \underbrace{\sum_{n=0}^{N-2} \left(l + \frac{1}{2} \sum_{k=0}^{n} \left((-1)^k + \sigma_k^3\right)\right)^2}_{H_g}$$

Influence of D and δ

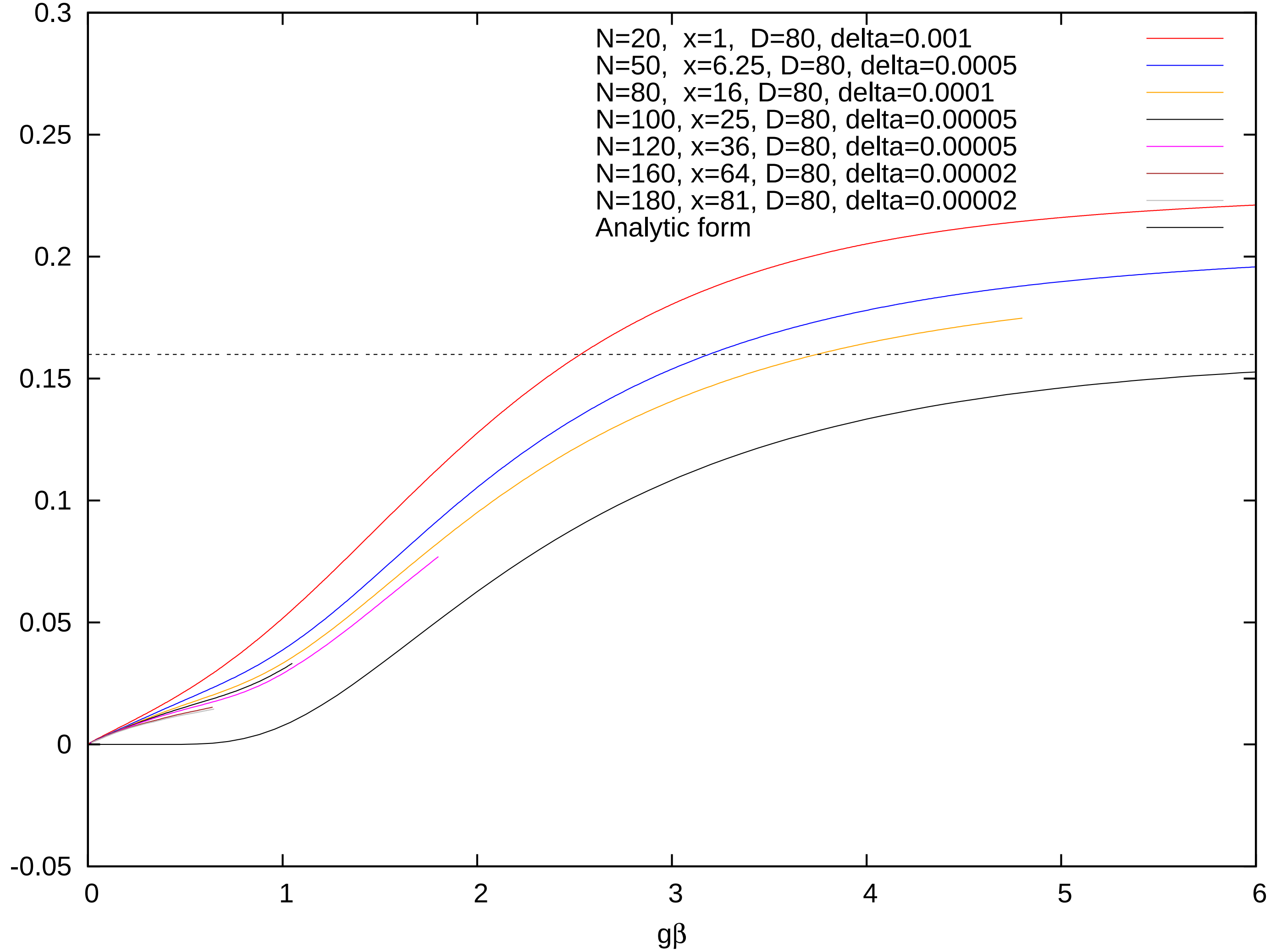


$N = 20$, $x = 16$, $D = 20$—160, $\delta = 0.0001$—0.01

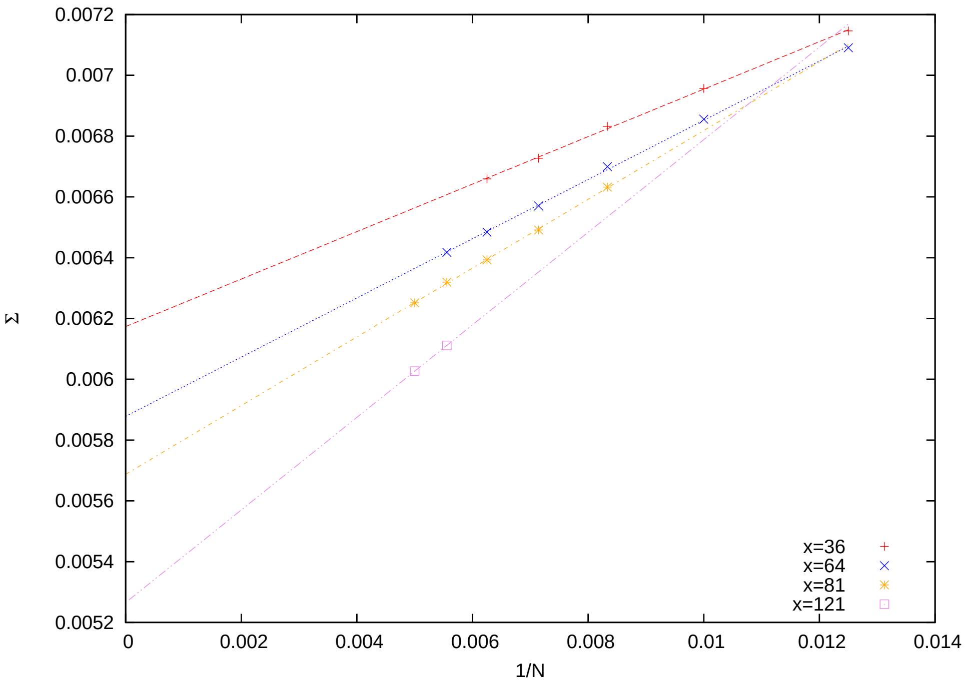
Zoom into high and low T

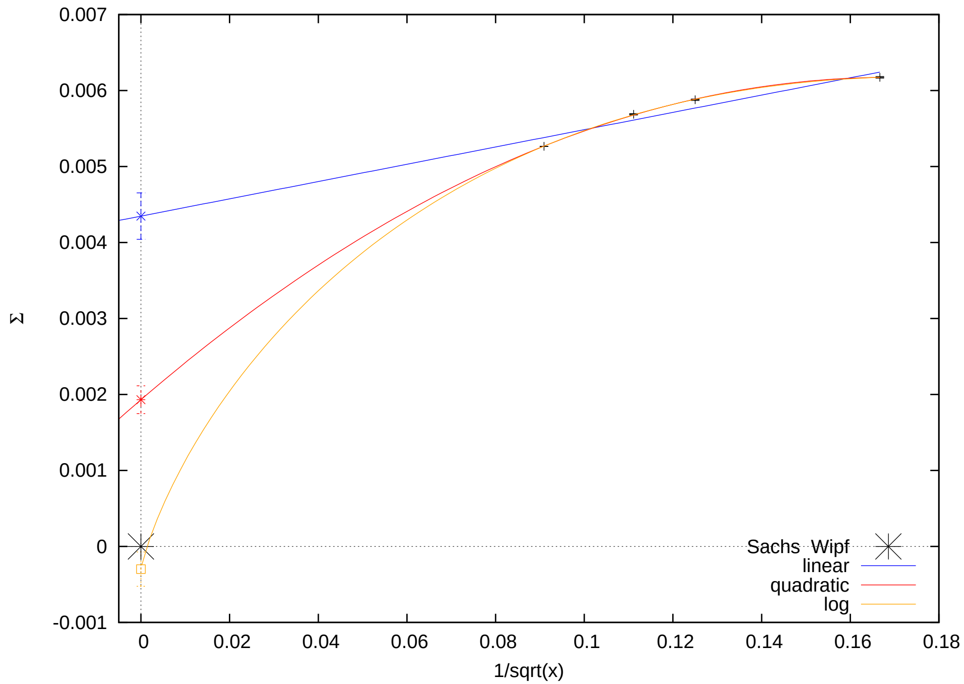


Towards the continuum limit

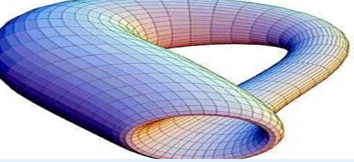


fixed $N/\sqrt{x} = 20$, i.e. pretty large volume





analytic result at $g\beta = 0.2$ [I. Sachs, A. Wipf, 1992]: $\frac{\Sigma}{g} \approx 8.1 \cdot 10^{-12}$

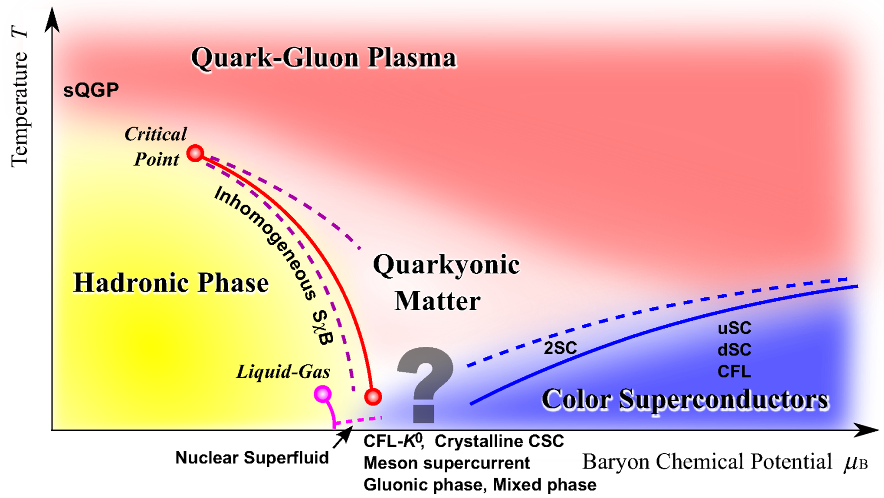


Conclusions

- Proof of concept – the MPS approach can be used to extract:
  - ⋆ mass spectrum (GS energy, masses of lightest particles of a theory),
  - ⋆ ground state expectation values (chiral condensate).
- Precision better or comparable to best results in the literature, in some cases better than 0.01%.
- **The success of our work so far encourages to look in more detail into the use of Tensor Network methods in lattice gauge theories.**

Prospects

- We would like to look into aspects of lattice gauge theories where the standard methods have problems:
  - ⋆ thermodynamics at non-zero chemical potential,
  - ⋆ non-equilibrium properties.



[K. Fukushima, T. Hatsuda, Rep. Prog. Phys. 74 (2011) 14001]

- First attempts already underway: computation of the chiral condensate at finite temperature.
- Ultimate aim: full QCD, i.e.:
  - ⋆ a non-Abelian theory (with SU(3) gauge group),
  - ⋆ in **3+1** dimensions.
- **Needs a lot of work of the Tensor Network + lattice gauge theory community...**

Thank you for your attention!

$\mathcal{L} \rightarrow \mathcal{H}$

The Hamiltonian $\mathcal{H}$ is the Legendre transform of the Lagrangian $\mathcal{L}$:

$$\mathcal{H} = \pi^\mu \dot{A}_\mu - \mathcal{L},$$

where:

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu}.$$

We choose the time like axial gauge $A_0 = 0$:

$$H = \int dx \left(-i\bar{\psi}\gamma^1(\partial_1 + igA_1)\psi + m\bar{\psi}\psi + \frac{1}{2}E^2 \right).$$

The γ matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Going to the lattice:

$$U(n, n+1) = e^{i\theta(n)} = e^{-iagA^1(n)}$$

fermionic fields are associated with lattice sites and gauge fields with lattice links

Staggered discretization

The Hamiltonian becomes:

$$H = -\frac{i}{2a}\sum_{n=0}^{M-1}\Big(\phi^\dagger(n)e^{i\theta(n)}\phi(n+1) - \phi^\dagger(n+1)e^{-i\theta(n)}\phi(n)\Big) +$$

$$+m\sum_{n=0}^{M-1}(-1)^n\phi^\dagger(n)\phi(n) + \frac{ag^2}{2}\sum_{n=0}^{M-1}L^2(n),$$

in the Kogut-Susskind discretization:

[T. Banks, L. Susskind and J. B. Kogut, Phys. Rev. D **13** (1976) 1043]

[J. B. Kogut and L. Susskind, Phys. Rev. D **11** (1975) 395.]

$$\phi(n)/\sqrt{a} \rightarrow \begin{cases} \psi_{\text{upper}}(x) & n \text{ even} \\ \psi_{\text{lower}}(x) & n \text{ odd} \end{cases}$$

The correspondence between lattice and continuum fields is:

$$\frac{1}{ag}\theta(n) \rightarrow -A^1(x)$$

$$gL(n) \rightarrow E(x).$$

Basic ingredients

- $\phi(n)$ is a single-component fermion field, defined on each site of a M-site lattice with periodic b.c. and obeying the anticomm. relations: $\{\phi^\dagger(n), \phi(m)\} = \delta_{nm}$, $\{\phi(n), \phi(m)\} = 0$, $\{\phi^\dagger(n), \phi^\dagger(m)\} = 0$
- The gauge field variable $\theta(n)$ is defined on the link between sites n and $n+1$ and is related to the spatial component of the Abelian vector potential by $\theta(n) = agA(n)$
- The angular momentum variable $L(n)$ is related to the electric field $E(n)$ by the relation $L(n) = E(n)/g$ and to the gauge field by the commutation relations: $[\theta(n), L(m)] = i\delta_{nm}$. The possible values of $L(n)$ are quantized: $L(n)|l\rangle = l|l\rangle$, $l = 0, \pm 1, \pm 2, \ldots$. This implies: $e^{\pm i\theta(n)}|l\rangle = |l \pm 1\rangle$
- m – fermion mass
- g – gauge coupling
- a – lattice spacing
- M – lattice size

Jordan-Wigner transformation

$$H = -\frac{i}{2a}\sum_{n=0}^{M-1}\Big(\phi^\dagger(n)e^{i\theta(n)}\phi(n+1) - \phi^\dagger(n+1)e^{-i\theta(n)}\phi(n)\Big) +$$

$$+m\sum_{n=0}^{M-1}(-1)^n\phi^\dagger(n)\phi(n) + \frac{ag^2}{2}\sum_{n=0}^{M-1}L^2(n),$$

For numerics, it is convenient to perform the Jordan-Wigner transformation: [P. Jordan, E. Wigner, Z. Phys. **47** (1928) 631.]

$$\phi(n) = \prod_{p<n}(i\sigma^3(p))\sigma^-(n),$$

where $\sigma^i(n)$ are Pauli matrices ($\sigma^\pm = \sigma^1 \pm i\sigma^2$). This gives:

$$H = -\frac{1}{2a}\sum_{n=0}^{M-1}\Big(\sigma^+(n)e^{i\theta(n)}\sigma^-(n+1) + \sigma^+(n+1)e^{-i\theta(n)}\sigma^-(n)\Big) +$$

$$+\frac{m}{2}\sum_{n=0}^{M-1}\big(1 + (-1)^n\sigma^3(n)\big) + \frac{ag^2}{2}\sum_{n=0}^{M-1}L^2(n).$$

Choice of basis

Rewrite Hamiltonian in a dimensionless form: $W = \frac{2}{ag^2} H_{\mathrm{JW}} = W_0 - xV$, with:

$$W_0 = \frac{m}{ag^2} \sum_{n=0}^{M-1} \left(1 + (-1)^n \sigma^3(n)\right) + \sum_{n=0}^{M-1} L^2(n),$$

$$V = \sum_{n=0}^{M-1} \left(\sigma^+(n) e^{i\theta(n)} \sigma^-(n+1) + \sigma^+(n+1) e^{-i\theta(n)} \sigma^-(n)\right)$$

$$x \equiv \beta = 1/a^2 g^2.$$

- Natural choice of basis: direct product of Ising basis $\{|i\rangle\}$, acted upon by Pauli spin operators, and the ladder space of states $\{|l\rangle\}$:

  $$|i_0 i_1 \dots i_{M-2} i_{M-1}\rangle \otimes |l_{0,1} l_{1,2} \dots l_{M-2,M-1} (l_{M-1,0})\rangle,$$

  where $(l_{M-1,0})$ is present if PBC are considered and absent for OBC.
- Formally, the operator W_0 can be treated as an unperturbed part and V as a perturbation. Ground state of W_0: $|0\rangle = |\downarrow\uparrow\downarrow\uparrow \dots \downarrow\uparrow\rangle \otimes |0000 \dots 00\rangle$,
- The perturbation operator V flips two neighbouring spins and couples them via a gauge field excitation (flux line): $V\,|\bullet \quad \bullet\rangle = |\uparrow\rightsquigarrow\downarrow\rangle$

- The gauge degrees of freedom $l_{i,i+1}$ can be eliminated using the Gauss law:

$$L_n - L_{n-1} = \frac{1}{2}\left(\sigma_n^z + (-1)^n\right),$$

leaving the basis states as: $|i_0 i_1 \ldots i_{M-2} i_{M-1}\rangle \otimes |l\rangle$,

with:

  ⋆ $l = 0, \pm 1, \pm 2, \ldots$ for PBC,

  ⋆ $l = 0$ (or other constant) for OBC.

- With M-site lattice, dim(spin part)=2^M, while for the gauge part the basis is

  ⋆ infinite-dimensional for PBC ⇒ truncation needed,

  ⋆ one-dimensional for OBC.

- Truncation for PBC:

  ⋆ at some finite $\pm l_{\max}$, thus reducing the basis to dimension $(2l_{\max}+1)2^M$,

  ⋆ or use strong coupling expansion (SCE):

  [T. Banks, L. Susskind and J. B. Kogut, Phys. Rev. D **13** (1976) 1043]

  [J. B. Kogut and L. Susskind, Phys. Rev. D **11** (1975) 395.]