

Collectivity in ultrarelativistic collisions

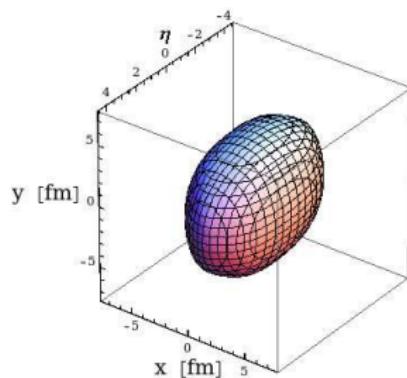
Piotr Bożek

AGH and Institute of Nuclear Physics, Kraków



hydrodynamic model

- ▶ expansion of dense matter
- ▶ close to local equilibrium
- ▶ initial conditions
- ▶ equation of state
- ▶ flow + thermal emission + decays + rescattering



ideal hydrodynamics

- ▶ energy momentum tensor

$$T_{id}^{\mu\nu} = (\epsilon + p)g^{\mu\nu} - P u^\mu u^\nu$$

- ▶ ideal hydro (5 functions in 3+1D),
energy density, pressure, flow velocity

$$\epsilon(x_\mu), \quad P(x_\mu), \quad \vec{v}(x_\mu)$$

- ▶ 4 hydrodynamic equations

$$\partial_\mu T^{\mu\nu} = 0$$

- ▶ equation of state (zero baryon density)

$$\epsilon = \epsilon(P)$$

massless gas $\epsilon = 3p$

hydro equations

- ▶ using $\epsilon + P = Ts$, $d\epsilon = Tds$, $dP = sdT$
- ▶ we calculate $u_\nu \partial_\mu T^{\mu\nu}$

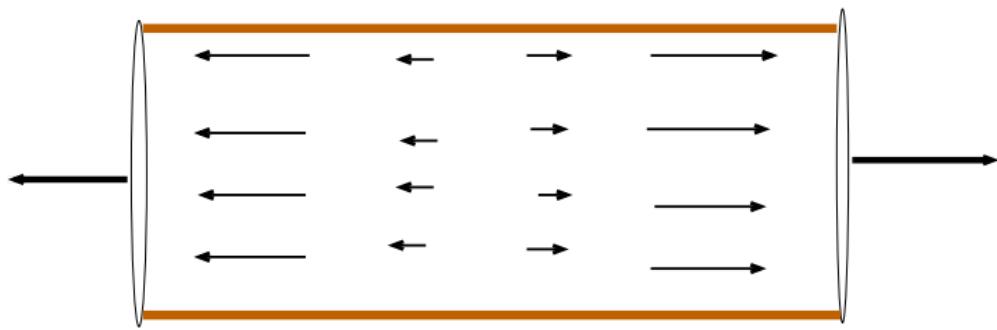
$$\partial_\mu(su^\mu) = 0$$

one equation = entropy conservation (ideal hydro)

- ▶ 3 other equations for flow velocity

$$su_\mu \partial^\mu u^\nu = (g^{\mu\nu} - u^\mu u^\nu) \partial_\mu P$$

longitudinal expansion



Bjorken flow

- ▶ the density depends on the proper time
 $\tau = \sqrt{t^2 - z^2}$ only,
flow velocity $u^\mu = (t/\tau, 0, 0, z/\tau)$

cooling in 2+1D and 3+1D

- ▶ one equation

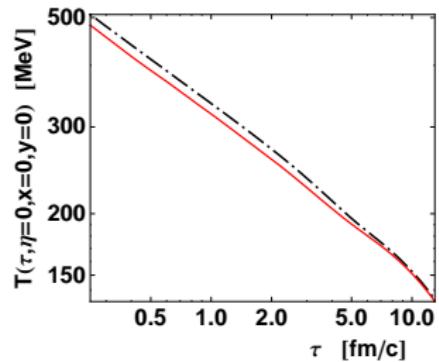
$$\frac{d\epsilon(\tau)}{d\tau} = -\frac{\epsilon(\tau) + P(\tau)}{\tau}$$

cooling from longitudinal expansion

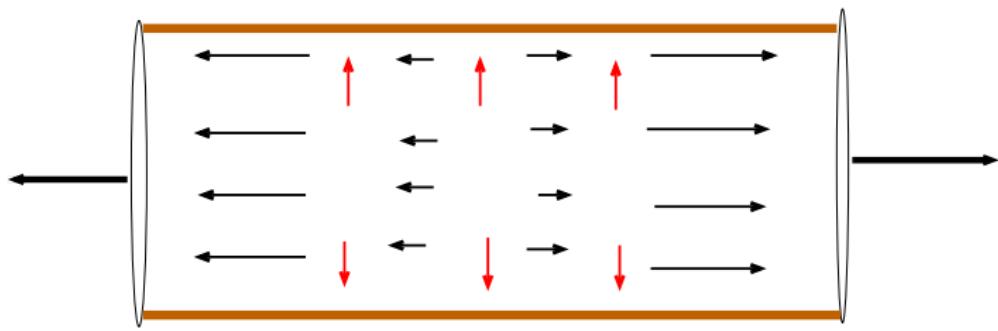
- ▶ if $\frac{1}{3}\epsilon \simeq P$

$$\epsilon(\tau) = \epsilon_0 \left(\frac{\tau_0}{\tau} \right)^{4/3}$$

- ▶ in 3+1D faster cooling when transverse expansion sets in



longitudinal+transverse expansion

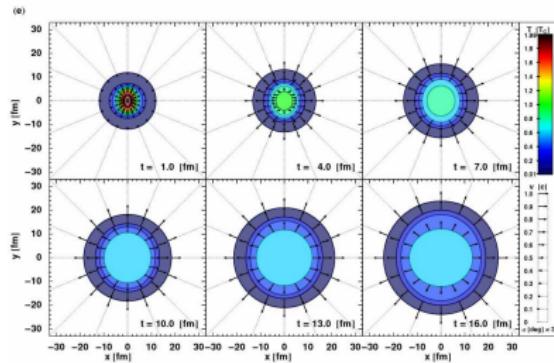


3+1D expansion

collective transverse expansion

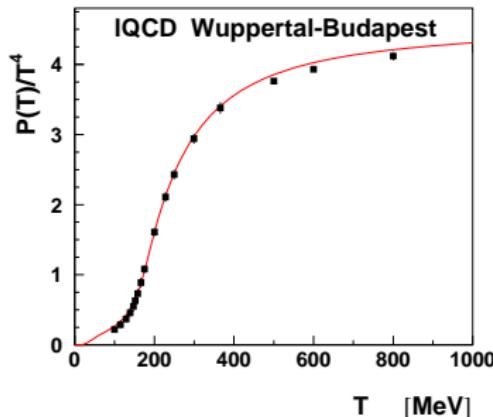
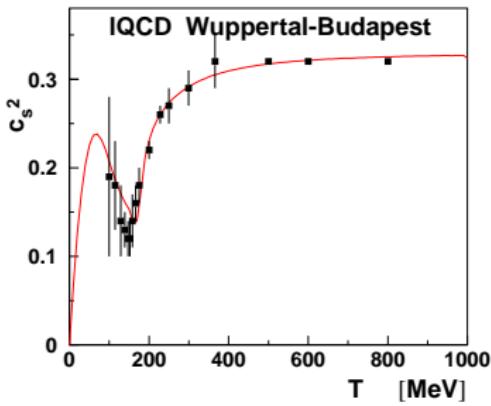
$$u^\mu \partial_\mu u_x = -\frac{1+u_x^2}{sT} \partial_x P - \frac{u_x u_y}{sT} \partial_y P + \dots$$
$$u^\mu \partial_\mu u_y = -\frac{1+u_y^2}{sT} \partial_y P - \frac{u_x u_y}{sT} \partial_x P + \dots$$

faster expansion in the direction of large pressure gradient



M. Chojnacki

equation of state - input



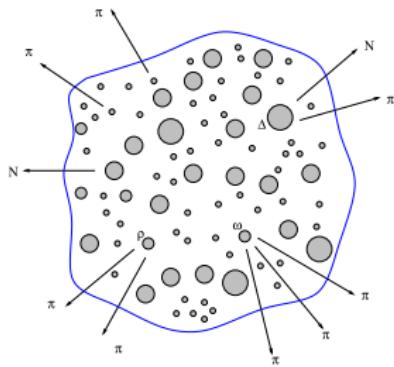
- ▶ lattice equation of state at high T
- ▶ hadron gas at low temperature (know resonances from PDG)
- ▶ matching at T_c Laine, Schroder 2006, Chojnacki, Florkowski 2007, Huovinen, Petreczky 2009
- ▶ crossover - not first order phase transition

freezeout

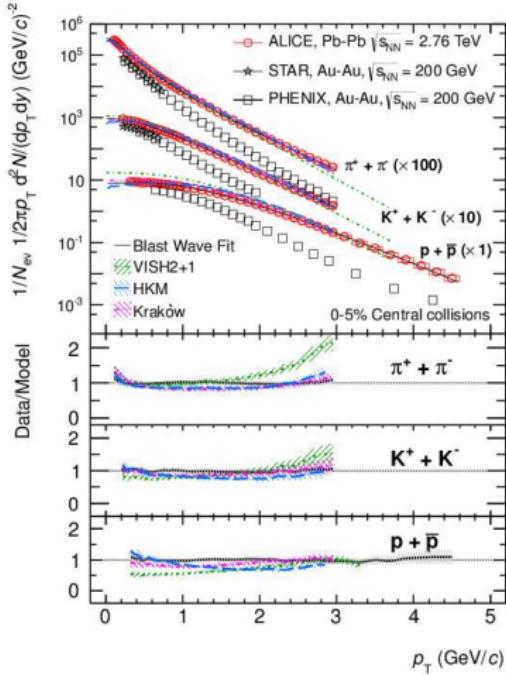
- ▶ density drops - hydrodynamic evolution stops
- ▶ usually surface of constant temperature
- ▶ particles emitted from fluid elements Cooper-Frye formula

$$E \frac{d^3 N}{dp^3} = \int d\Sigma_\mu p^\mu f[p_\mu u^\mu(x)]$$

- ▶ fluid velocity + thermal momenta
- ▶ latter - resonance decays and/or rescattering



OBSERVATION 1 - transverse flow

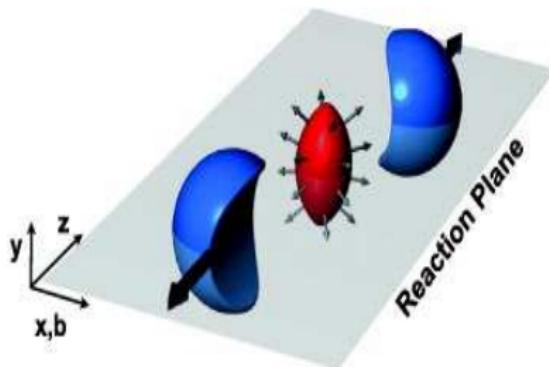


- ▶ spectra flatter due to flow
- ▶ stronger flow at the LHC
- ▶ stronger effect for heavy particles
- ▶ only soft spectra from hydro $p_\perp < 2 \text{ GeV}$
- ▶ chemical ratios - normalization
- ▶ excess of soft pions?

asymmetry in the transverse plane at finite impact parameter

Glauber model, KLN model, IP-Gasma

$$\text{eccentricity} - \epsilon_2 = -\frac{\int dx dy (x^2 - y^2) \rho(x, y)}{\int dx dy (x^2 + y^2) \rho(x, y)}$$



Snellings 2011

larger gradient and stronger flow in-plane - $v_2 > 0$ - **elliptic flow**

$$\frac{dN}{d\phi} \propto 1 + 2v_2 \cos(2\phi)$$

$\epsilon_2 + \text{HYDRO RESPONSE} \longrightarrow v_2$

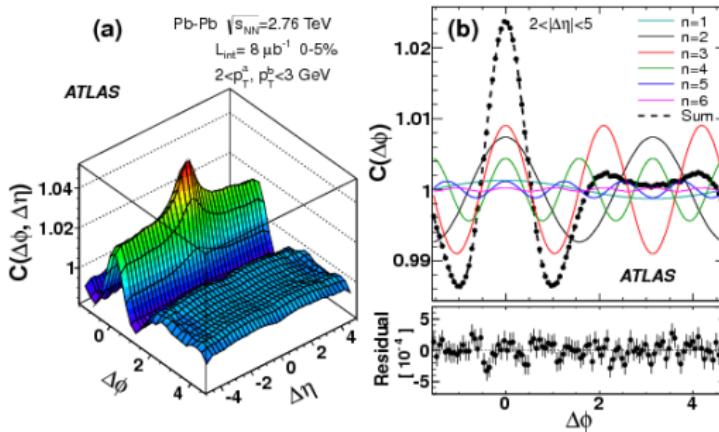
Event Plane (Reaction plane) must be reconstructed in each event

two-particle correlations in relative azimuthal angle

$$\begin{aligned} C(\Delta\phi) &\propto \int d\phi_1 d\phi_2 \delta(\phi_1 + \Delta\phi - \phi_2) \frac{dN}{d\phi_1 d\phi_2} \\ &\propto 1 + 2v_1^2 \cos(\Delta\phi) + 2v_2^2 \cos(2\Delta\phi) + 2v_3^2 \cos(3\Delta\phi) + \dots \end{aligned}$$

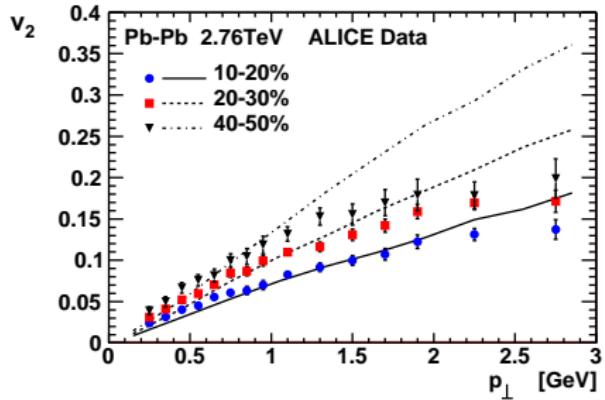
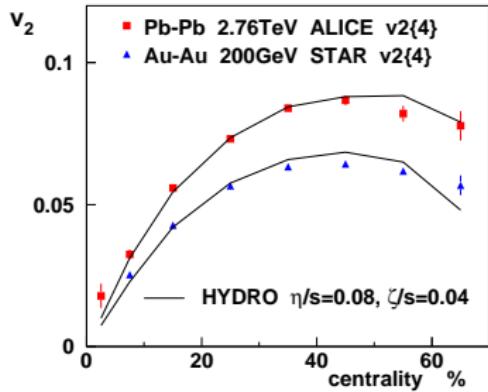
rms v_2 can be measured from two-particle correlations

$$\sqrt{\langle v_2^2 \rangle} = \left\langle \frac{1}{N_{pair}} \sum_{ij} \cos(\phi_i - \phi_j) \right\rangle \text{ (cummulant method)}$$



(v_1 - directed flow, v_3 - triangular flow) odd harmonics from fluctuations - see below

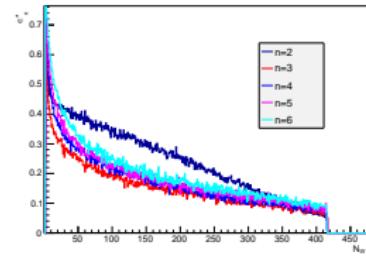
OBSERVATION 2 - Elliptic flow



GLISSANDO ver. 2.602

208+208, 50000 events
 $b=0.0 - 24.0$ fm
 mixed model: $\sigma_v = 67.7$ mb, $\sigma_{\text{vis}} = 67.7$ mb, $\alpha = 0.150$
 Gaussian wounding profile, $A = 0.92$

variable-axes eccentricities, $n=2,3,4,5,6$



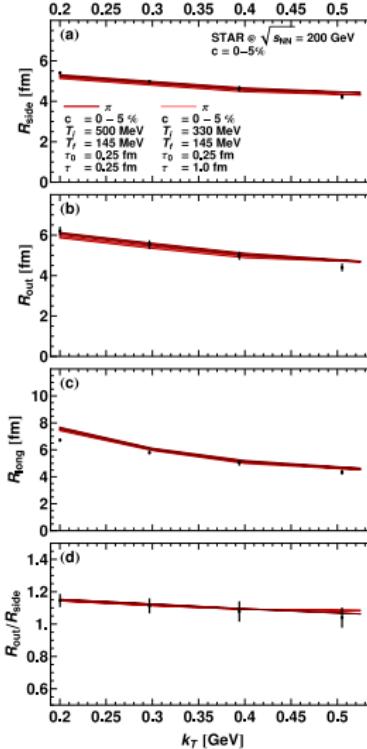
initial shape asymmetry transformed into flow asymmetry
 strong indication of collective behavior

Hanbury Brown-Twiss (HBT) correlations

correlation between identical particles

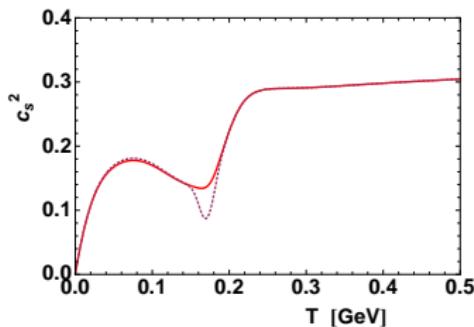
$$C(P, q) = \frac{N(p_1, p_2)}{N(p_1)N(p_2)}$$

$$C(q) = 1 + \lambda \exp[-q_S^2 R_s^2 - q_I^2 R_I^2 - q_o^2 R_o^2]$$

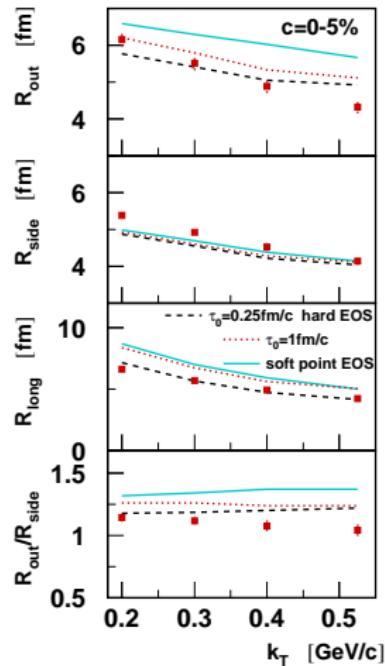


HBT requires hard equation of state

OBSERVATION 3 - hard equation of state

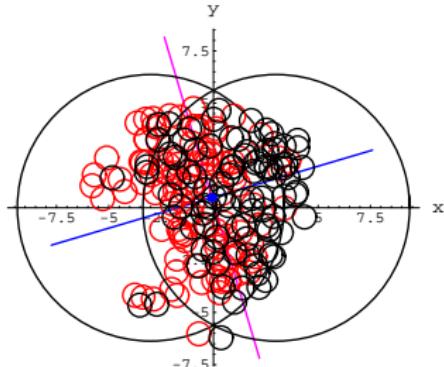


soft EOS - wrong HBT

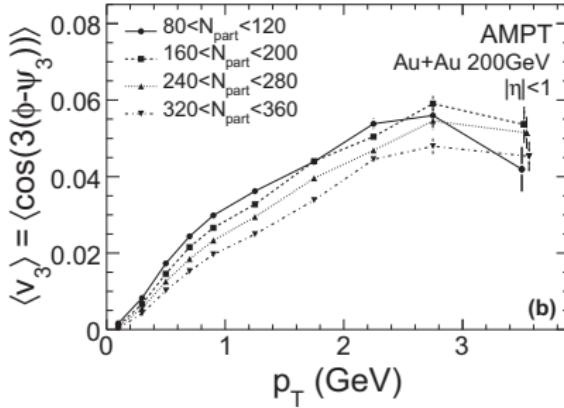


Heavy-ion experiments consistent with lattice QCD

fluctuating initial conditions



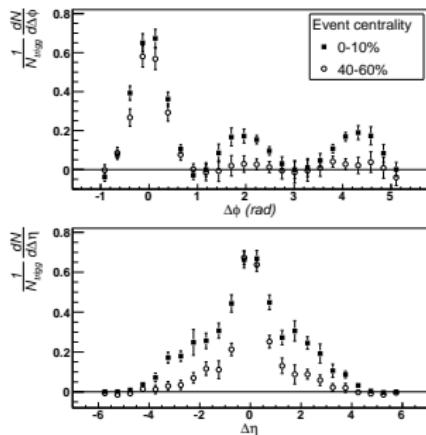
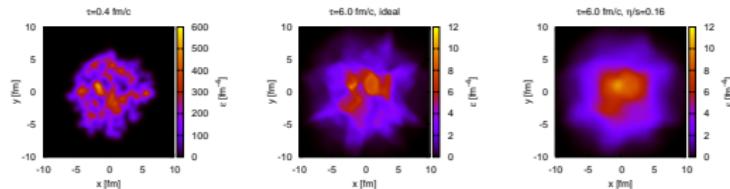
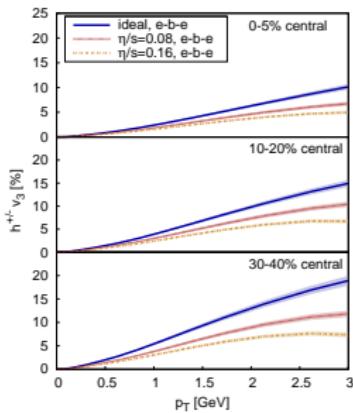
- fluctuating initial density
- larger eccentricity
- fluctuating eccentricity
- **triangular deformation** ϵ_3
- dipolar (directed) flow v_1
- $\langle p_\perp \rangle$ fluctuations
- RP orientation, torqued fireball



B. Alver, G. Roland

$$\frac{dN}{d\phi} \propto 1 + 2v_1 \cos(\phi - \Psi_1) + 2v_2 \cos(2(\phi - \Psi_2)) + 2v_3 \cos(3(\phi - \Psi_3)) + \dots$$

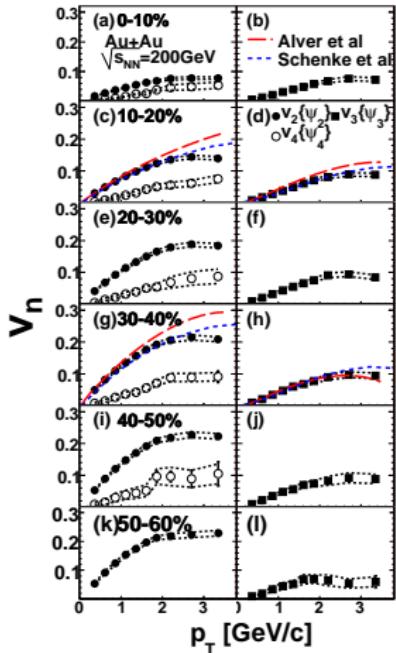
Event by event hydro



Schenke, Jeon, Gale, Phys. Rev. 106,
042301 (2011)

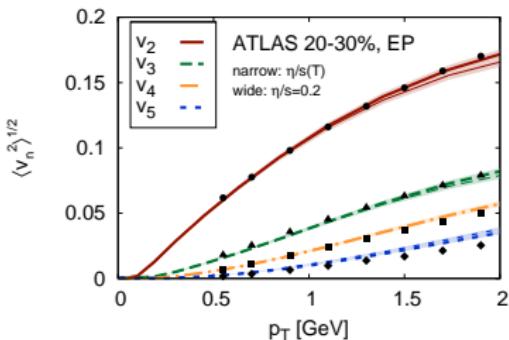
Takahashi et al., Phys. Rev. 103, 242301 (2009)
event by event ideal hydro - Andrade, Grassi, Hama, Kodama, Socolowski 2006

OBSERVATION 4 - triangular flow



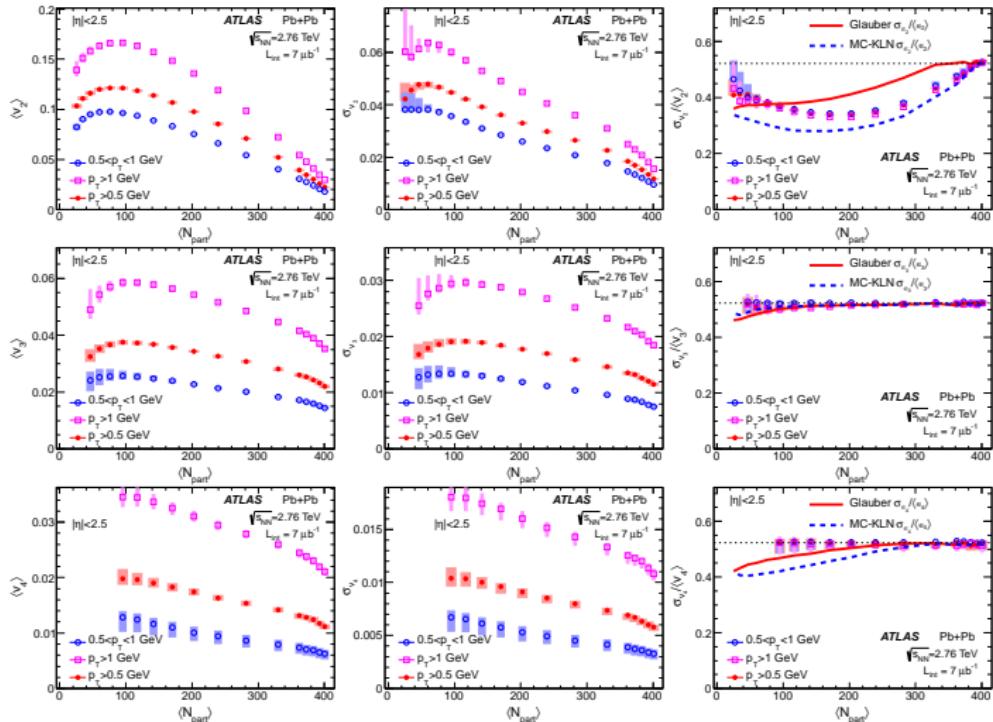
PHENIX

(viscous) event by event hydrodynamics



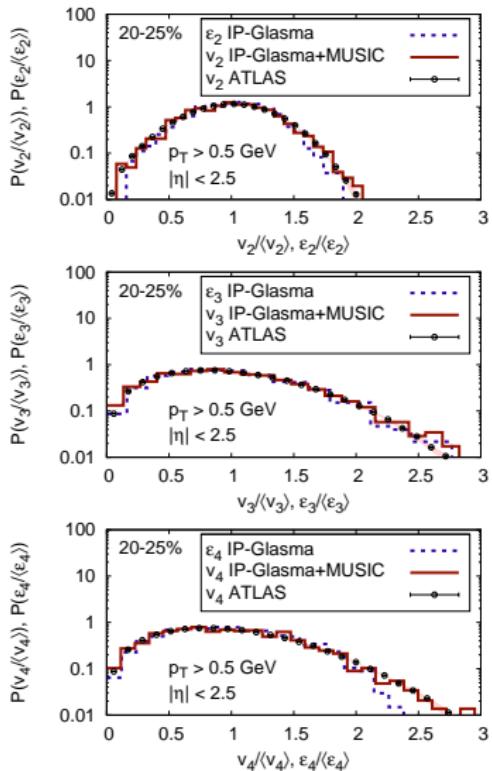
C.Gale et al. arXiv:1209.6330

OBSERVATION 5 -flow fluctuations

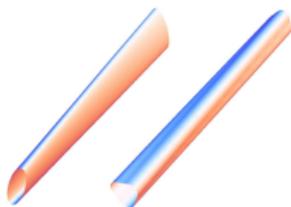


$\langle v_n \rangle$, $\langle v_n^2 \rangle$ and the whole distribution of v_n can be reconstructed.

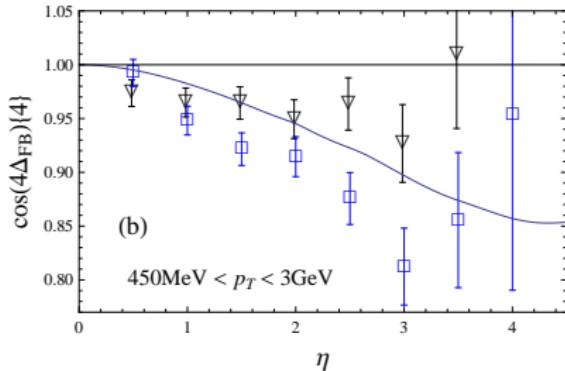
good description of flow fluctuations with IP-plasma + hydro



event by event twist of the reaction plane



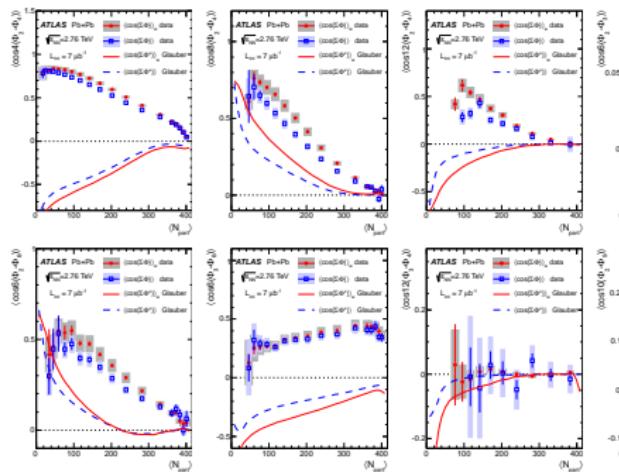
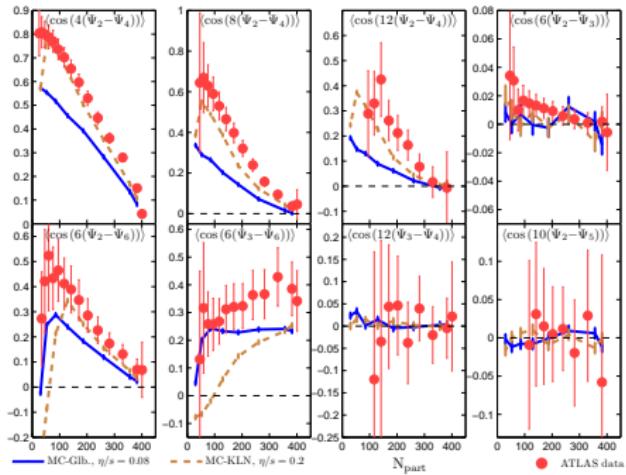
Fluctuating RP twist
can be measured



PB, Broniowski, Moreira, Phys. Rev. C83, 034911 (2011)

$$\begin{aligned}\cos(2k\Delta_{FB})\{4\} &\equiv \frac{\langle e^{ik[(\phi_{F,1}+\phi_{F,2})-(\phi_{B,1}+\phi_{B,2})]}\rangle}{\langle e^{ik[(\phi_{F,1}-\phi_{F,2})-(\phi_{B,1}-\phi_{B,2})]}\rangle} = \\ &\langle \cos(2k\Delta_{FB}) \rangle_{\text{events}} + \text{nonflow}\end{aligned}$$

Event planes correlations

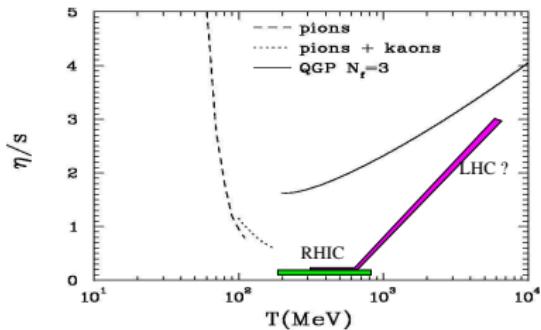
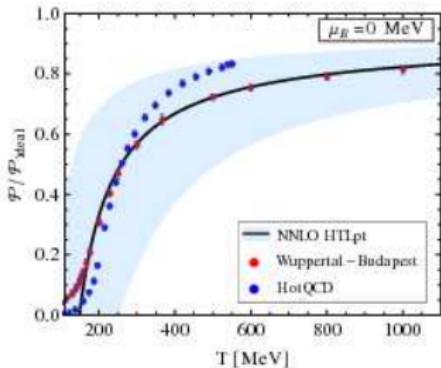


- ▶ correlations between EP
 $\langle \cos(nm(\Phi_n - \Phi_m)) \rangle$,
- ▶ initial correlations
- ▶ nonlinearities of viscous hydrodynamics

Bhalerao, Luzum, Ollitrault 2011, Qiu, Heinz 2012

sQGP or wQGP ?

How to see the difference?



Csernai, Kapusta, McLerran, 2006

Haque, Andersen, Mustafa, Strickland, Su, 2013

- ▶ strongly coupled theory $\frac{\eta}{s} = \frac{1}{4\pi} \simeq 0.08$
- ▶ short mean-free path, large cross section

$$\eta = \frac{1}{3} np L_{mfp}$$

- ▶ for $\eta/s = 0.08$

$$L_{mfp} = \frac{3s}{4\pi np} \simeq 0.15 - 0.3 fm$$

- ▶ mean-free path comparable to wavelength

$$L_{mfp} \simeq 0.9\lambda$$

- ▶ mean-free path comparable to inter-particle distance

$$L_{mfp} \simeq 0.5n^{-1/3}$$

all of the above means no quasi-particles, no kinetic description

energy-momentum tensor

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p + \Pi & 0 & 0 \\ 0 & 0 & p + \Pi & 0 \\ 0 & 0 & 0 & p + \Pi \end{pmatrix} + \pi^{\mu\nu}$$

- ▶ shear viscosity

$$\Delta^{\mu\alpha} \Delta^{\nu\beta} u^\gamma \partial_\gamma \pi_{\alpha\beta} = \frac{2\eta\sigma^{\mu\nu} - \pi^{\mu\nu}}{\tau_\pi} - \frac{1}{2}\pi^{\mu\nu} \frac{\eta T}{\tau_\pi} \partial_\alpha \left(\frac{\tau_\pi u^\alpha}{\eta T} \right)$$

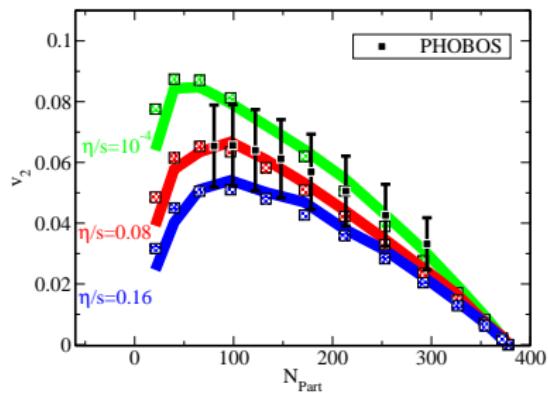
- ▶ bulk viscosity

$$u^\gamma \partial_\gamma \Pi = \frac{-\zeta \partial_\gamma u^\gamma - \Pi}{\tau_\Pi} - \frac{1}{2}\Pi \frac{\zeta T}{\tau_\Pi} \partial_\alpha \left(\frac{\tau_\Pi u^\alpha}{\zeta T} \right)$$

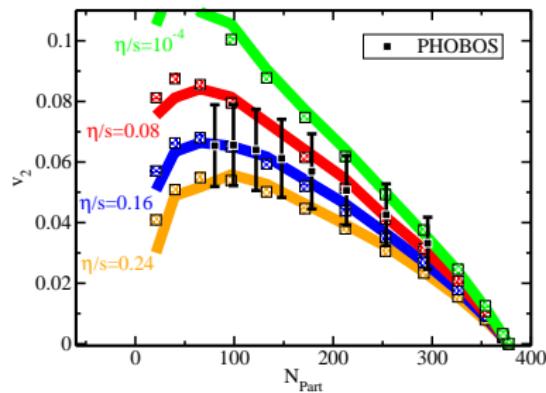
viscosity corrections from velocity gradients

OBSERVATION 6 - small viscosity

Glauber



CGC

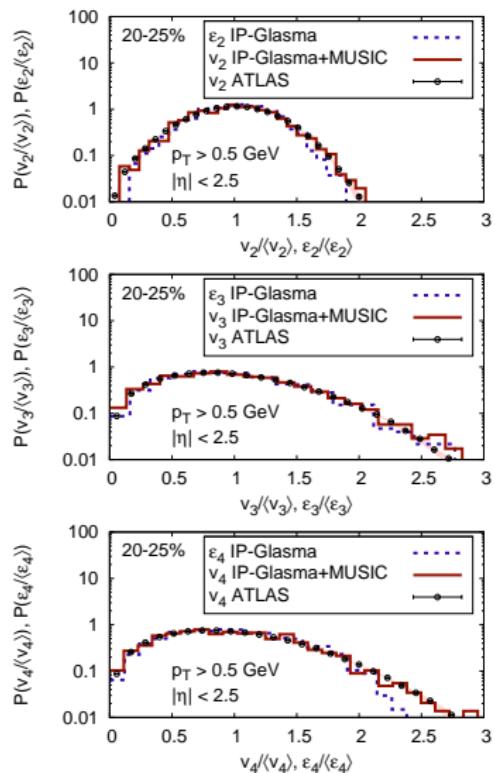


Luzum, Romatschke, Phys. Rev. C78, 034915 (2009)

Model uncertainties

Small viscosity : $\eta/s = 0.08 - 0.2$ - sQGP

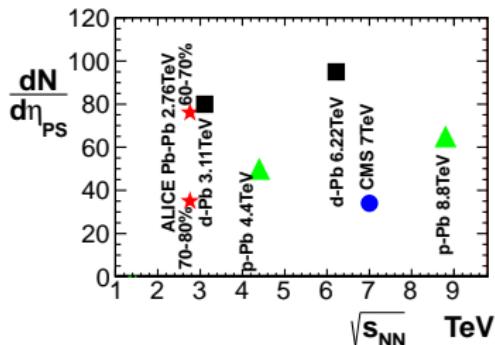
IP-glasma $\eta/s \simeq 0.2$



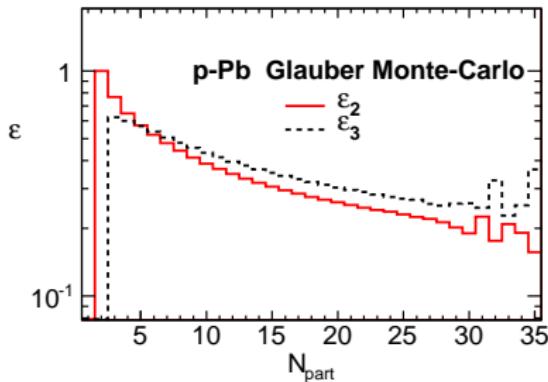
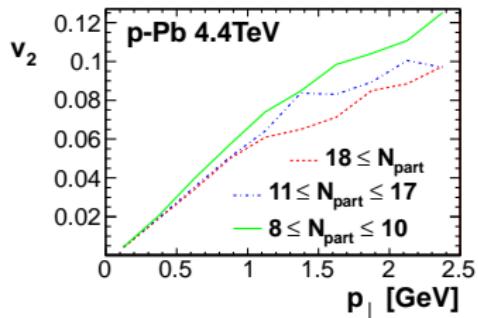
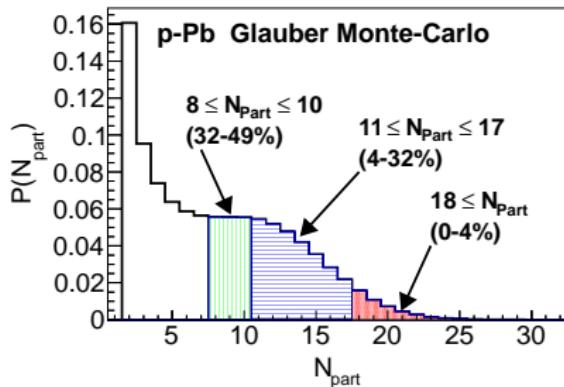


pA collision

p-Pb at 5.02TeV
small dense fireball formed !



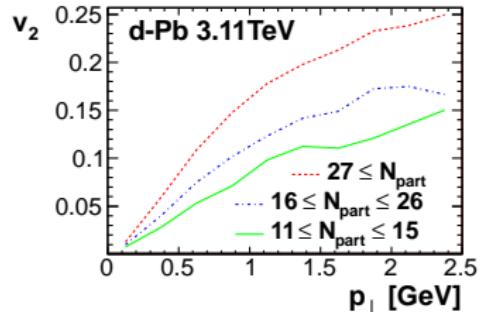
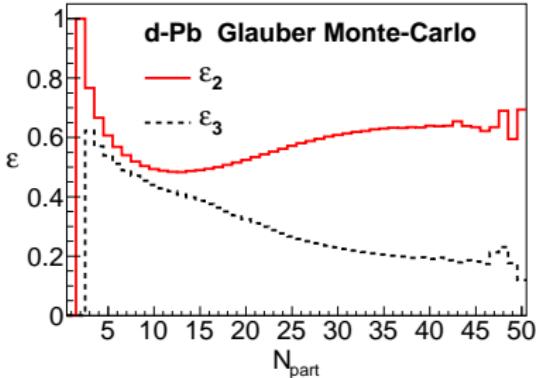
Fireball in p-Pb



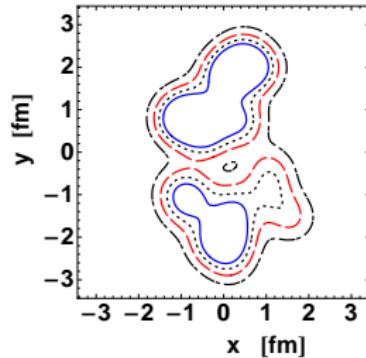
PB, arXiv:1112.0912

- ▶ collective flow effects \simeq peripheral Pb-Pb
- ▶ can be observed
- ▶ p-A (d-A) is not p-p superposition

d-Pb



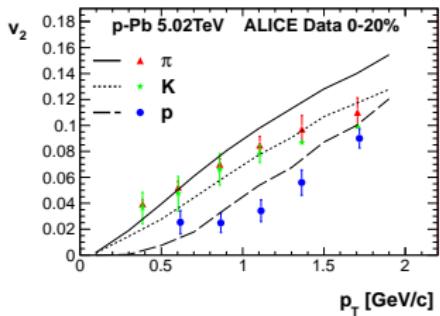
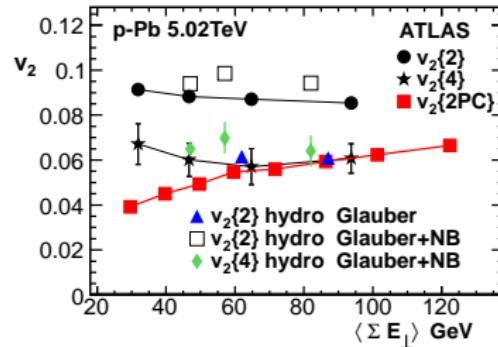
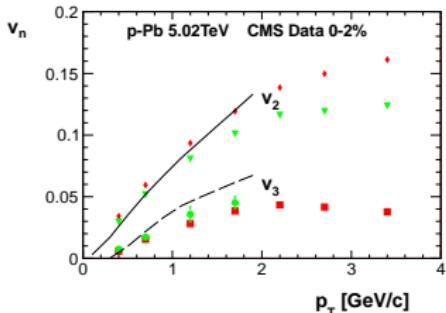
large elliptic flow



PB, arXiv:1112.0912

small-on-large system
with controlled eccentricity

OBSERVATION 7 - flow in pA collisions

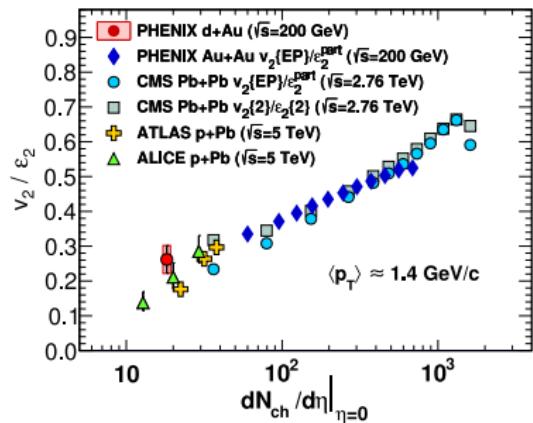
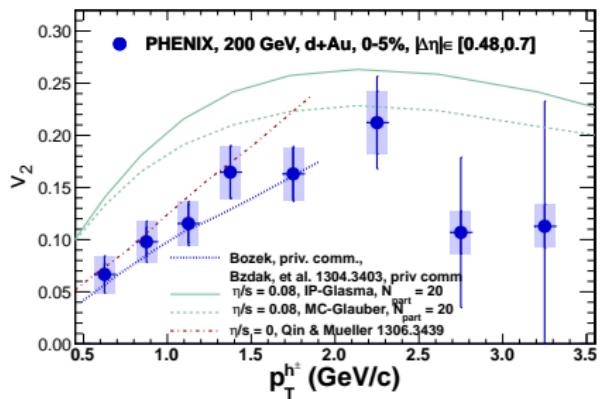


PB, Broniowski, arXiv:1304.3044

PB, Broniowski, Torrieri arXiv:1306.5442

- ▶ elliptic and triangular flow
- ▶ mass hierarchy of v_2

d-Au at 200GeV



PHENIX, arXiv:1303.1794

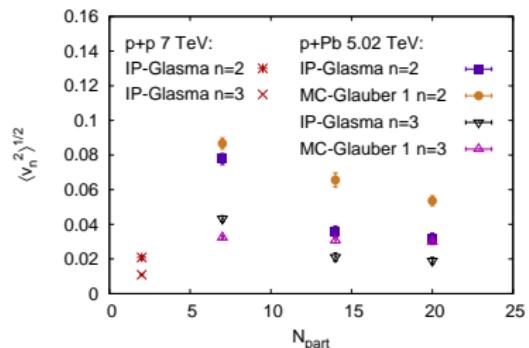
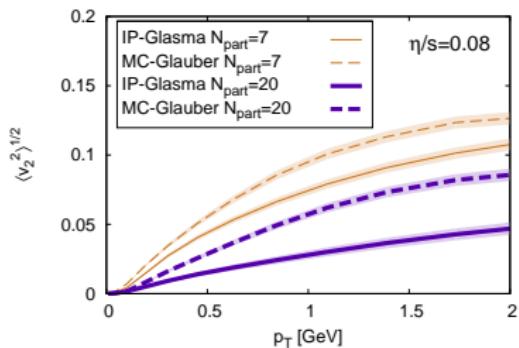
large eccentricity - large elliptic flow

Summary

Relativistic viscous hydrodynamics describes
the evolution of the fireball

1. significant transverse flow generated
2. elliptic flow reflecting initial asymmetry
3. hard equation of state
4. triangular flow - fluctuating initial conditions
5. flow fluctuations
6. small viscosity fluid - sQGP
7. flow in $p(d)A$?

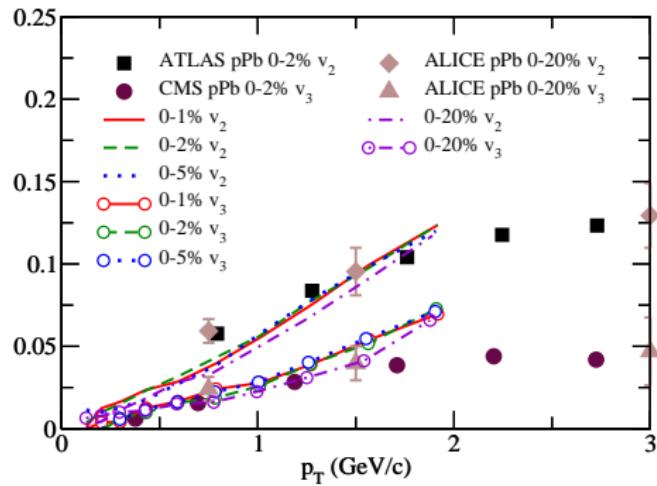
3+1D visc. hydro



dependence on initial model, v_n small for IP-Glasma i.c.

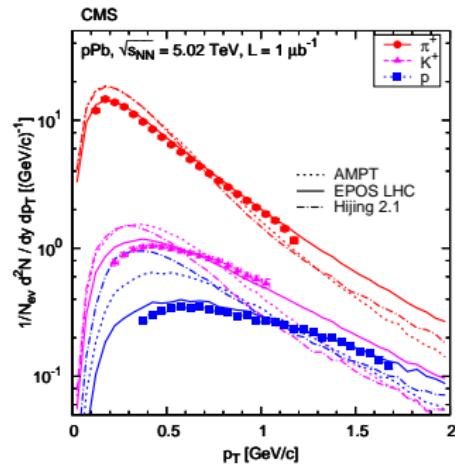
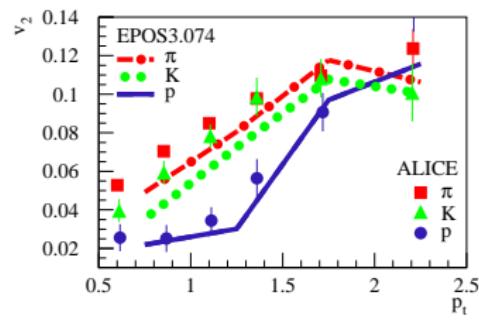
A.Bzdak, B.Schenke, P.Tribedy, R.Venugopalan - arXiv: 1304.3403

3+1D hydro



G-Y.Qin, B. Müller arXiv: 1306.3439

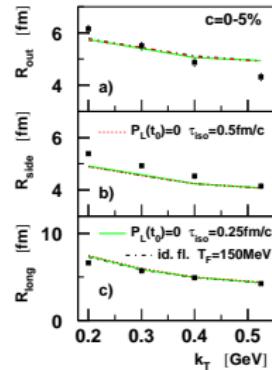
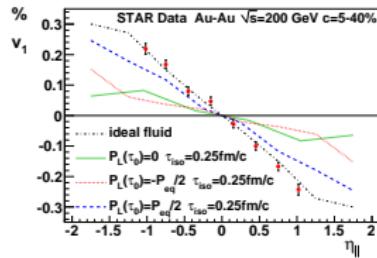
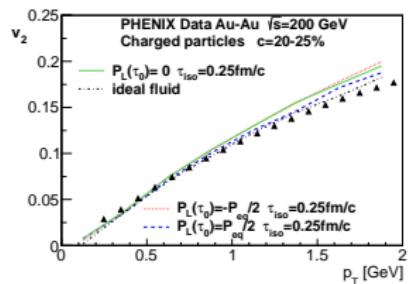
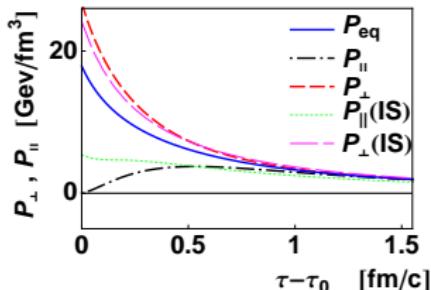
EPOS



excellent description of spectra

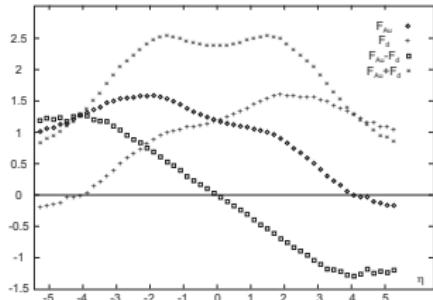
K. Werner, M. Bleicher, B. Guiot, Iu. Karpenko, T. Pierog - arXiv:1307.4379

pressure anisotropy



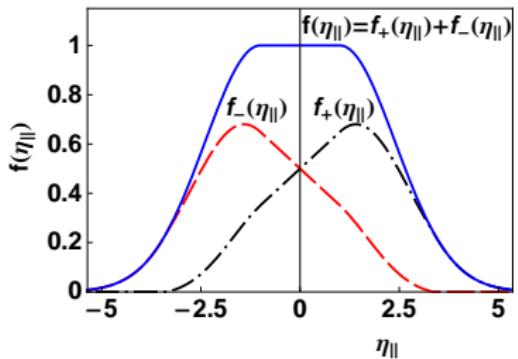
PB, I. Wyskiel - arXiv:1009.0701

- early pressure anisotropy irrelevant!

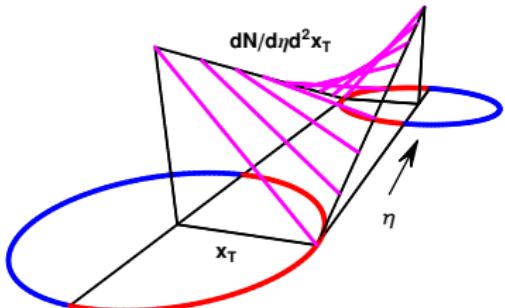


Asymmetric emission

(Białas, Czyż, Acta Phys. Polon. B36, 905 (2005))

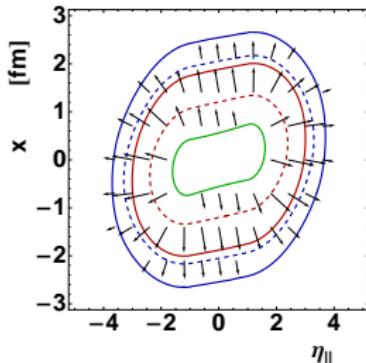
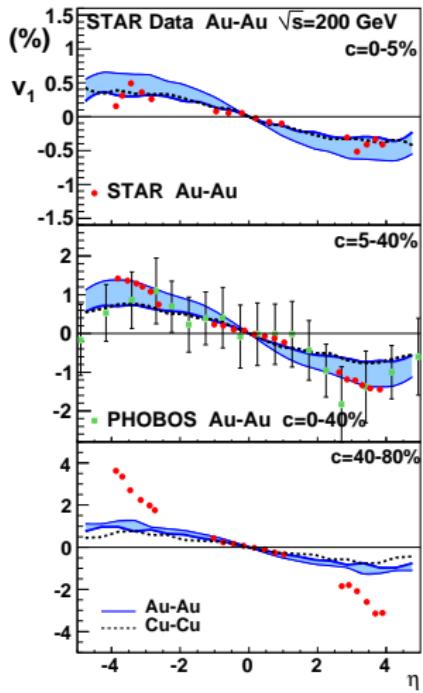


$$\begin{aligned}\rho(\eta, x, y) \propto & f_+(\eta)N_+(x, y) \\ & + f_-(\eta)N_-(x, y)\end{aligned}$$



bremsstrahlung (Adil Gyulassy, Phys. Rev. C72, 034907 (2005))

Directed flow- tilted source



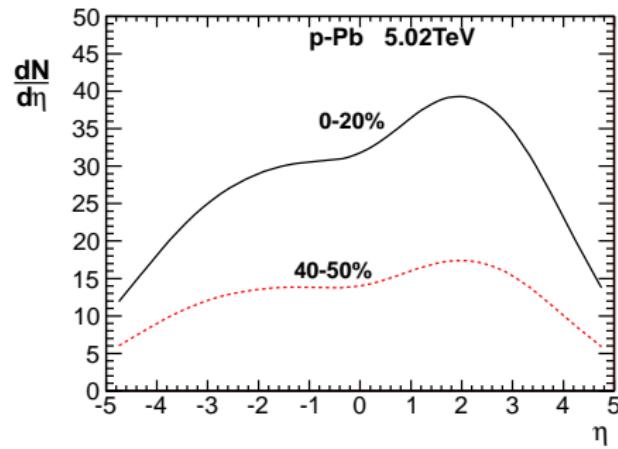
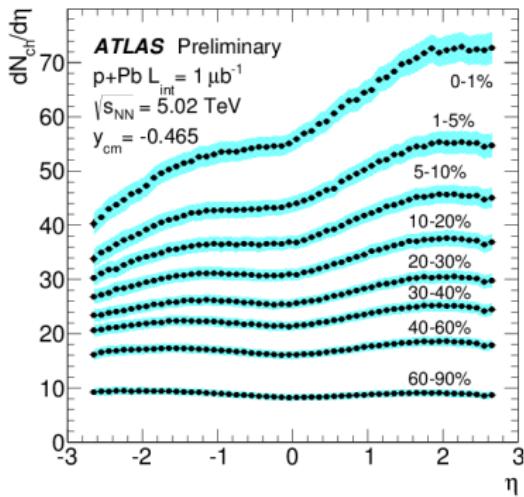
Bozek, Wyskiel, Phys. Rev. C81, 054902 (2010)

$$\partial_\tau u_x = - \frac{\partial_x p_\perp}{p + \epsilon}$$

$$\partial_\tau Y = - \frac{\partial_\eta p_{||}}{\tau(p + \epsilon)}$$

tilted source \rightarrow transverse pressure + longitudinal pressure
Glauber model

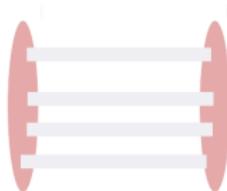
Asymmetric distributions



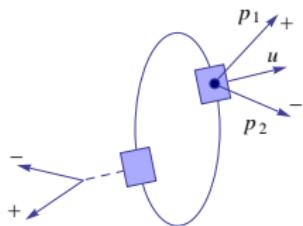
FSI scenarios

fields+thermalization

color fields

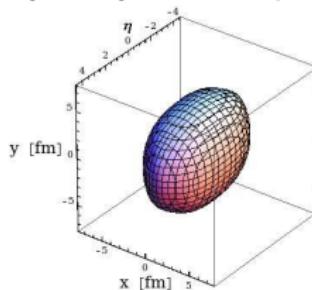


local thermalization \rightarrow hadronization

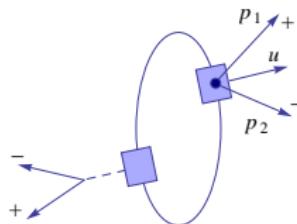


hydrodynamics

hydrodynamic expansion

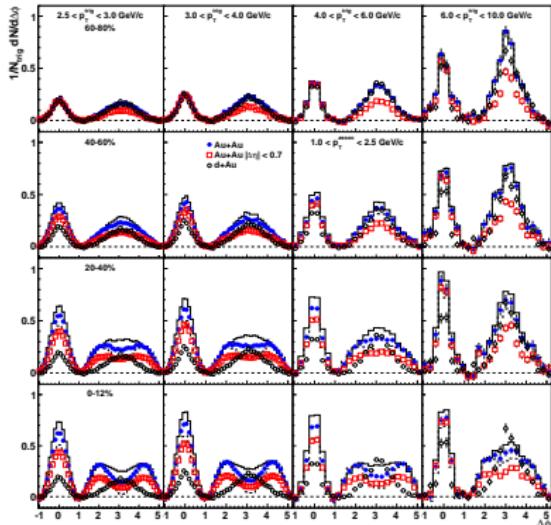


hadronization, statistical emission

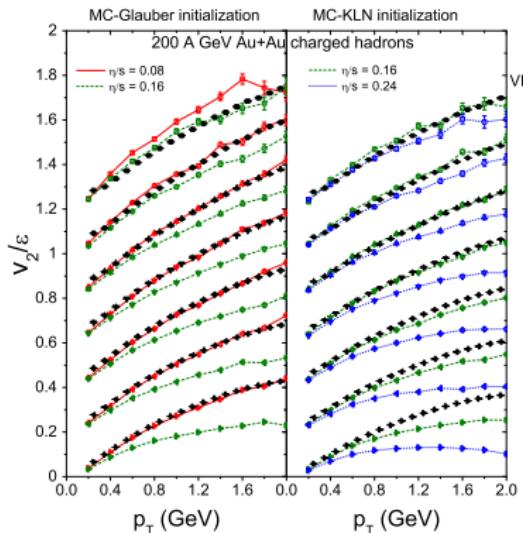


Give similar flow

Can we reduce uncertainties? go back to very peripheral A-A



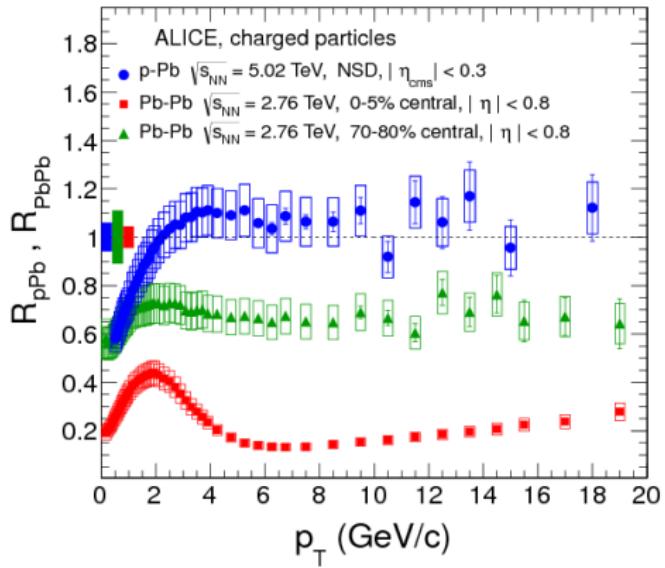
STAR-arXiv:1004.2377



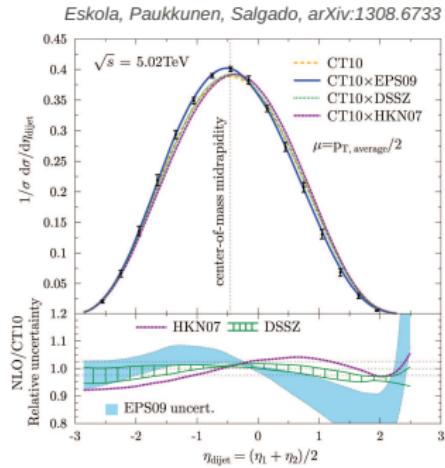
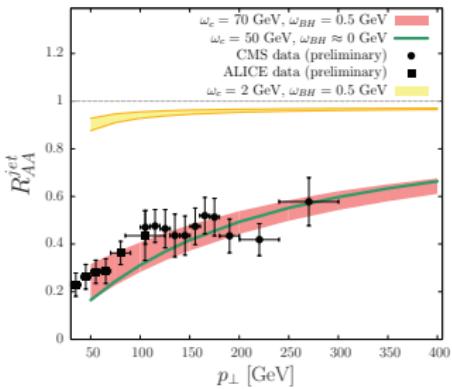
Song, Bass, Heinz, Hirano, Shen-arXiv:1101.4638

also jet modification, dijet asymmetry, PID flow, HBT

Flow without jet quenching?



No final-state effects



- Excellent situation to extract initial-state effects

also see H. Paukkunen's and J. Qiu's talks yesterday

energy-momentum tensor

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p + \Pi & 0 & 0 \\ 0 & 0 & p + \Pi & 0 \\ 0 & 0 & 0 & p + \Pi \end{pmatrix} + \pi^{\mu\nu}$$

- ▶ shear viscosity

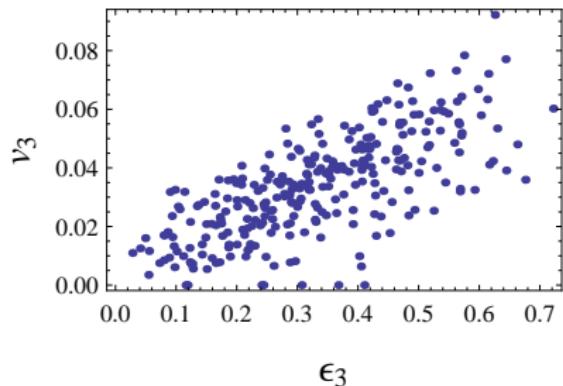
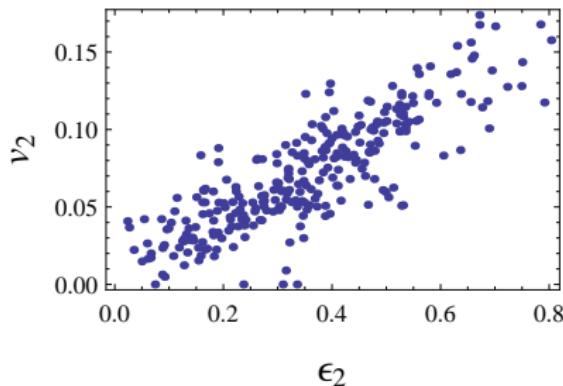
$$\Delta^{\mu\alpha} \Delta^{\nu\beta} u^\gamma \partial_\gamma \pi_{\alpha\beta} = \frac{2\eta \sigma^{\mu\nu} - \pi^{\mu\nu}}{\tau_\pi} - \frac{1}{2} \pi^{\mu\nu} \frac{\eta T}{\tau_\pi} \partial_\alpha \left(\frac{\tau_\pi u^\alpha}{\eta T} \right)$$

- ▶ bulk viscosity

$$u^\gamma \partial_\gamma \Pi = \frac{-\zeta \partial_\gamma u^\gamma - \Pi}{\tau_\Pi} - \frac{1}{2} \Pi \frac{\zeta T}{\tau_\Pi} \partial_\alpha \left(\frac{\tau_\Pi u^\alpha}{\zeta T} \right)$$

- ▶ viscosity corrections from velocity gradients
- ▶ **initial** stress tensor - pressure anisotropy
- ▶ equation of state

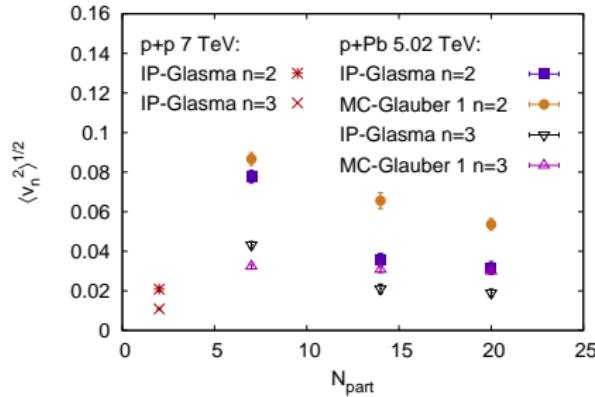
fireball asymmetry - flow asymmetry



- Ev-by-Ev hydro response to geometry valid
- response strength depends on details

Hydrodynamic flow in p-p?

- ▶ Humanic-nucl-th/0612098 (pythia, cascade)
- ▶ Romatschke, Luzum-arXiv:0901.4588 (overlap)
- ▶ Prasad, Roy, Chattopadhyay, Chaudhuri -arXiv: 0910.4844 (overlap)
- ▶ Bozek-arXiv: 0911.2393 (flux-tubes)
- ▶ Yan, Dong, Zhou, Li, Ma, Sa- arXiv: 0912.3342 (transport)
- ▶ Werner, Karpenko, Pierog, Bleicher, Mikhailov- arXiv: 1010.0400 (EPOS)
- ▶ Deng, Xu, Greiner-arXiv: 1112.0470 (hot-spots, transport model)
- ▶ Shuryak, Zahed-arXiv:1301.4470 (symmetric)
- ▶ Bzdak, Schenke, Tribedy, Venugopalan-arXiv: 1304.3403 (IP-Glasma)



Bzdak et al. arXiv: 1304.3403

- Is hydrodynamics valid?
- What is the initial eccentricity?

p-Pb reference system - No FSI expected

arXiv:1105.3919v1 [hep-ph] 19 May 2011

CERN-PH-TH/2011-119
LHC-Project-Report-1181

Proton-Nucleus Collisions at the LHC: Scientific Opportunities and Requirements

Editor: C. A. Salgado¹

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Abstract

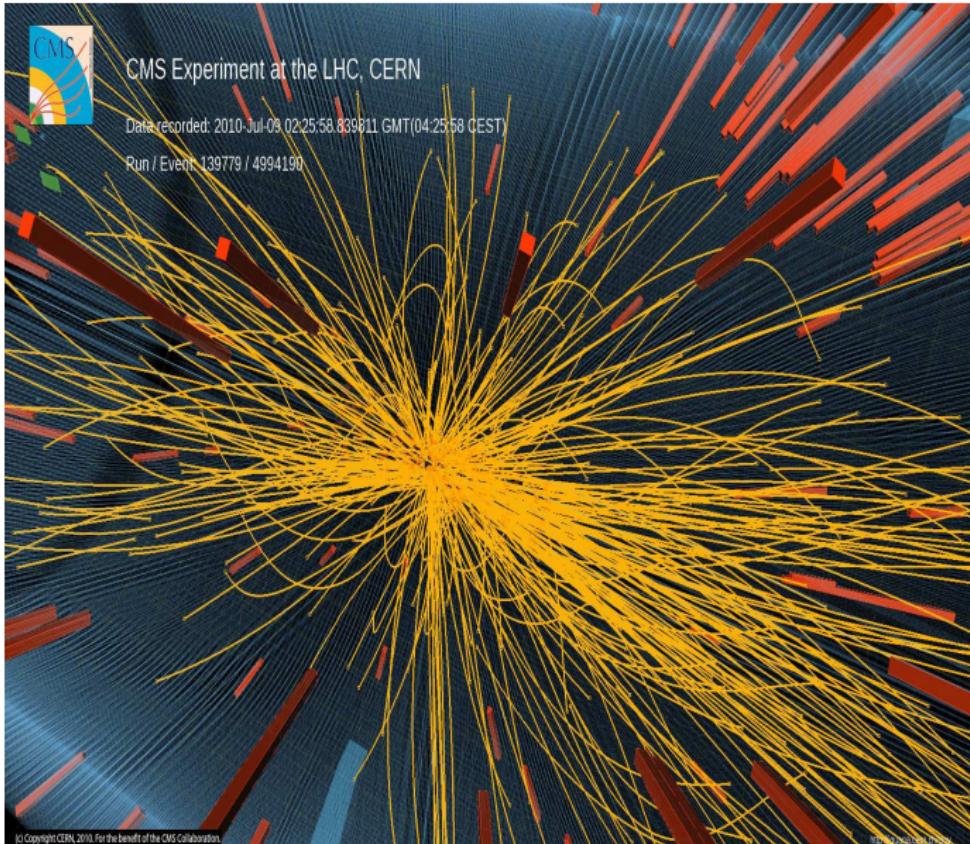
Proton-nucleus (p+A) collisions have long been recognized as a crucial component of the physics programme with nuclear beams at high energies, in particular for their reference role to interpret and understand nucleus-nucleus data as well as for their potential to elucidate the partonic structure of matter at low parton fractional momenta (small- x). Here, we summarize the main motivations that make a proton-nucleus run a decisive ingredient for a successful heavy-ion programme at the Large Hadron Collider (LHC) and we present unique scientific opportunities arising from these collisions. We also review the status of ongoing discussions about operation plans for the p+A mode at the LHC.

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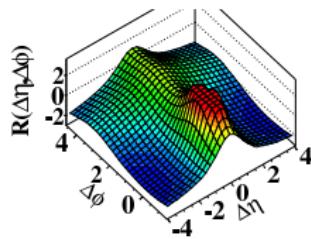
*Current address: ABB Switzerland Ltd., Corporate Research, Baden-Dättwil, Switzerland

High multiplicity events in pp

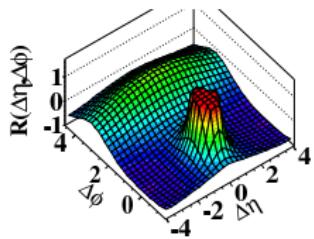


Ridge in pp

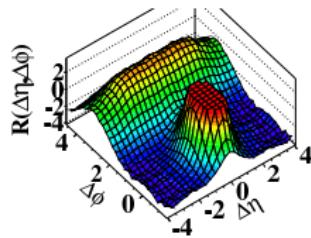
(a) CMS MinBias, $p_T > 0.1 \text{ GeV}/c$



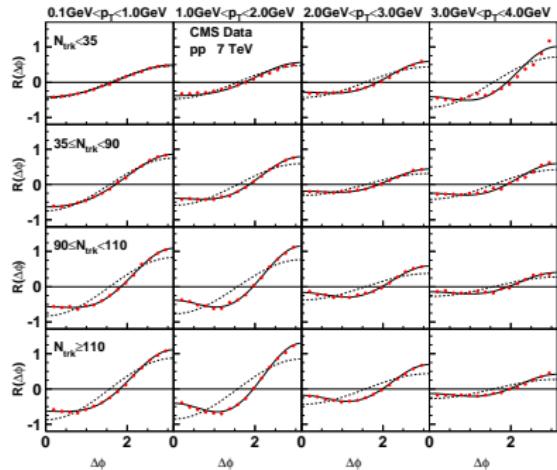
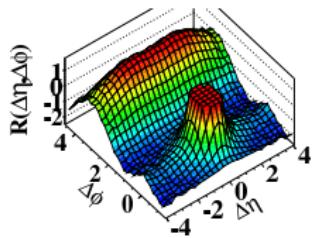
(b) CMS MinBias, $1.0 \text{ GeV}/c < p_T < 3.0 \text{ GeV}/c$



(c) CMS $N \geq 110$, $p_T > 0.1 \text{ GeV}/c$



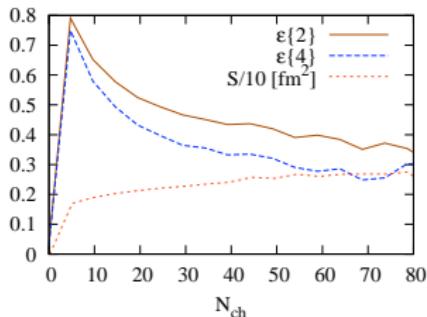
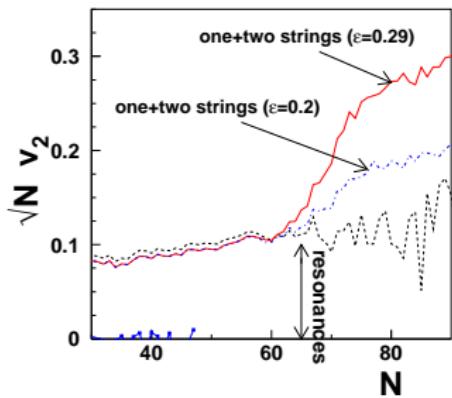
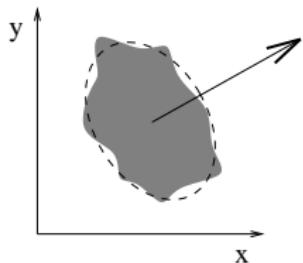
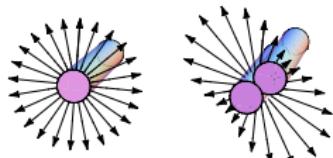
(d) CMS $N \geq 110$, $1.0 \text{ GeV}/c < p_T < 3.0 \text{ GeV}/c$



PB arXiv:1010.0405

can we measure (calculate) v_2

Fireball shape in pp



E. Asar et al., 1009.5643

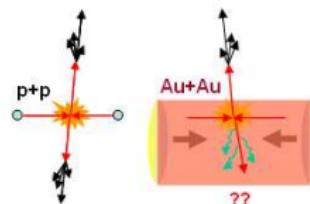
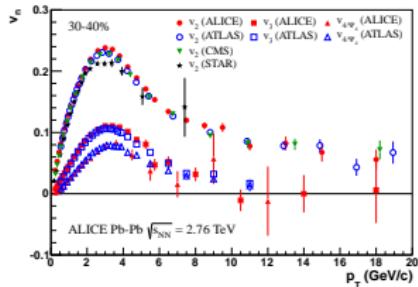
Casalderrey-Solana, Wiedemann, 0911.4400

Bozek, 0911.2397

sQGP formed in A-A collisions

Jet quenching

sQGP



$$\frac{dN}{d\phi} \propto 1 + 2v_2 \cos(2(\phi - \Psi_2)) + 2v_3 \cos(3(\phi - \Psi_3)) + \dots$$

