Time-periodic solutions in Einstein AdS – massless scalar field system

joint work with Andrzej Rostworowski [arXiv:1303.3186]

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Outline

Perturbative and **numerical** construction of time-periodic solutions within the system of self-gravitating massless scalar field in $d+1$ dimensions at spherical symmetry with $\Lambda < 0$.

$$G_{\alpha\beta}+\Lambda\, g_{\alpha\beta} = 8\pi G \left(\nabla_\alpha\phi\,\nabla_\beta\phi - \frac{1}{2} g_{\alpha\beta}\nabla_\mu\phi\nabla^\mu\phi\right),\ \Lambda = -d(d-1)/(2\ell^2)\,,$$

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0\,.$$

Actual and potential outcomes:

- More complete picture of AdS instability
- Efficient method for numerical integration of Einstein's equations
- AdS/CFT correspondence interpretation

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Motivation

Main motivation by the conjectures [Bizoń&Rostworowski, 2011]

- *Anti-de Sitter space is unstable against the formation of a black hole under arbitrarily small generic perturbations* (also in higher dimensions [Jałmużna,Rostworowski&Bizoń, 2011], [Buchel,Lehner&Liebling, 2012])

- *There are non-generic initial data which may stay close to AdS solution; Einstein-scalar-AdS equations may admit time-quasiperiodic solutions*

Analogous conjecture for vacuum Einstein's equations – existence of geons [Dias,Horowitz&Santos, 2011], [Dias,Horowitz,Marolf&Santos, 2012].

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Model

- Parametrization of asymptotically AdS spacetimes

$$ds^2 = \frac{\ell^2}{\cos^2 x}\left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \sin^2 x\, d\Omega^2_{S^{d-1}}\right) ,$$

$(t, x) \in \mathbb{R} \times [0, \pi/2)$.

- Field equations (units $8\pi G = d - 1$)

$$A' = 2(1-A)\frac{d-1-\cos 2x}{\sin 2x} - A\delta' , \qquad \delta' = -\frac{\sin 2x}{2}\left(\Phi^2 + \Pi^2\right) ,$$

$$\dot{\Phi} = \left(Ae^{-\delta}\Pi\right)' , \qquad \dot{\Pi} = \frac{1}{\tan^{d-1} x}\left(\tan^{d-1} x A e^{-\delta}\Phi\right)' .$$

- Auxiliary variables ($' = \partial_x$, $\dot{} = \partial_t$): $\Phi = \phi'$ and $\Pi = A^{-1}e^{\delta}\dot{\phi}$

- AdS space: $\phi \equiv 0$, $A \equiv 1$, $\delta \equiv \text{const.}$

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Boundary conditions

- Smoothness at the center implies symmetry of the fields.
- There is no freedom in prescribing boundary data at $x = \pi/2$ if we require smooth evolution and finiteness of the total mass.
- Mass function and asymptotic mass:

$$m(t,x) = \frac{\sin^{d-2} x}{\cos^d x} \left(1 - A(t,x)\right),$$

$$M = \lim_{x \to \pi/2} m(t,x) = \int_0^{\pi/2} A \left(\Phi^2 + \Pi^2\right) \tan^{d-1} x \, dx \,.$$

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Linear perturbations of AdS

- Linearized equation [Ishibashi&Wald, 2004]

$$\ddot{\phi} + L\phi = 0\,, \quad L = -\frac{1}{\tan^{d-1}x}\,\partial_x\left(\tan^{d-1}x\,\partial_x\right)\,,$$

- Eigenvalues and eigenvectors of L are ($j = 0, 1, \ldots$)

$$\omega_j^2 = (d+2j)^2, \quad e_j(x) = N_j\,\cos^d x\,P_j^{(d/2-1,d/2)}(\cos 2x)\,,$$

- AdS is linearly stable, linear solution

$$\phi(t,x) = \sum_{j\geq 0} \alpha_j \cos(\omega_j t + \beta_j)\,e_j(x)\,,$$

with amplitudes α_j and phases β_j determined by the initial data.

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Perturbative construction

- We search for solutions of the form

$$\phi = \varepsilon \cos(\omega_\gamma t) e_\gamma(x) + \mathcal{O}(\varepsilon^3)\,,$$

  with one *dominant* mode, ε is a small parameter.

- We rescale the time variable

$$\tau = \Omega_\gamma t, \quad \Omega_\gamma = \omega_\gamma + \sum_{\text{even } \lambda \geq 2} \varepsilon^\lambda \omega_{\gamma,\lambda}$$

  and we make an ansatz for the expansion in ε

$$\phi = \varepsilon \cos(\tau) e_\gamma(x) + \sum_{\text{odd } \lambda \geq 3} \varepsilon^\lambda \phi_\lambda(\tau, x),$$

$$\delta = \sum_{\text{even } \lambda \geq 2} \varepsilon^\lambda \delta_\lambda(\tau, x), \qquad 1 - A = \sum_{\text{even } \lambda \geq 2} \varepsilon^\lambda A_\lambda(\tau, x),$$

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Perturbative construction – expansion

- We expand functions ϕ_λ, δ_λ, A_λ into the eigenbasis

$$\phi_\lambda = \sum_j f_{\lambda,j}(\tau) e_j(x),$$

$$\delta_\lambda = d_{\lambda,-1}(\tau) + \sum_j d_{\lambda,j}(\tau) e_j(x), \quad A_\lambda = \sum_j a_{\lambda,j}(\tau) e_j(x)\,,$$

with $f_{\lambda,j}(\tau)$, $a_{\lambda,j}(\tau)$, $d_{\lambda,j}(\tau)$ being **periodic** in τ.

This works well for d even – the sums are finite at each order λ (the boundary conditions).

Notation:

- Inner product

$$(f\,|g) := \int_0^{\pi/2} f(x) g(x) \tan^{d-1} x dx\,,$$

- Coefficient at ε^λ in the power series expansion of $f = \sum_\lambda \varepsilon^\lambda f_\lambda$

$$\left[\varepsilon^\lambda\right] f = f_\lambda\,,$$

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Perturbative construction – constraint equations

- Metric function δ

$$d_{\lambda,k} = -\frac{1}{2\omega_k^2}\left(e'_k \,\middle|\, \left[\varepsilon^\lambda\right] \sin 2x \left(\Phi^2 + \Pi^2\right)\right),$$

gauge fixing condition: $\left[\varepsilon^\lambda\right] \delta\big|_{x=0} = 0 = d_{\lambda,-1} + \sum_j d_{\lambda,j} e_j(0)$

- Metric function A

$$\sum_j \left[(d-1)\delta_{kj} + \left(e_k \,\middle|\, \frac{1}{2}\sin 2x\, e'_j - \cos 2x\, e_j\right)\right] a_{\lambda,j} =$$

$$\frac{1}{4}\left(e_k \,\middle|\, \left[\varepsilon^\lambda\right] (\sin 2x)^2 A \left(\Phi^2 + \Pi^2\right)\right),$$

boundary condition: $\left[\varepsilon^\lambda\right] (1 - A)\big|_{x=0} = 0 = \sum_j a_{\lambda,j} e_j(0)$

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Perturbative construction – wave equation I

- Solve inhomogeneous wave equation

$$\left(\omega_\gamma^2 \partial_{\tau\tau} + L\right) \phi_\lambda = S_\lambda \,,$$

plugging $\phi_\lambda = \sum_j f_{\lambda,j}(\tau) e_j(x)$, gives

$$\left(\omega_\gamma^2 \partial_{\tau\tau} + \omega_k^2\right) f_{\lambda,k} = \left(e_k \,|S_\lambda\,\right) \,,$$

- How do we get secular terms?

$$\ddot{g}(t) + \omega_0^2 g(t) = a\cos(\omega t), \quad g(0) = c, \quad \dot{g}(0) = \tilde{c}\,,$$

$$g(t) = \frac{\tilde{c}}{\omega_0}\sin\left(\omega_0 t\right) + c\cos\left(\omega_0 t\right) + \begin{cases} \frac{a(\cos(\omega t) - \cos(\omega_0 t))}{\omega_0^2 - \omega^2}\,, & \omega_0 \neq \omega\,, \\ \frac{a}{2\omega_0} t \sin\left(\omega_0 t\right)\,, & \omega_0 = \omega\,, \end{cases}$$

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Perturbative construction – wave equation II

- Use the integration constants $\{c_{\lambda,k}, \tilde{c}_{\lambda,k}\}$ to remove resonant terms $\cos(\omega_k/\omega_\gamma)\tau$ or $\sin(\omega_k/\omega_\gamma)\tau$.

- *Dominant* mode condition fixes two constants in $f_{\lambda,\gamma}$

  $\left(f_{\lambda,\gamma},\, \partial_\tau f_{\lambda,\gamma}\right)\Big|_{\tau=0} = (0,\, 0) \iff \left(\left(e_\gamma\,|\phi\right),\, \left(e_\gamma\,|\partial_\tau\phi\right)\right)\Big|_{\tau=0} = (\varepsilon,\, 0)$

  ($\Rightarrow \tilde{c}_{\lambda,k} = 0$).

- At any odd $\lambda \geq 3$

  $$\left(e_k\,|S_\lambda\right) \equiv 0 \quad \text{for} \quad k > \gamma + (d+1+2\gamma)\frac{\lambda-1}{2},$$

  we are left with $(\lambda-1)/2 + \lfloor(\lambda-1)/(2(d+2\gamma))\rfloor$ undetermined integration constants $\{c_{\lambda,k}\}$ and frequency shift $\omega_{\gamma,\lambda-1}$.

- Use $\{c_{\lambda,k}\}$ together with $\omega_{\gamma,\lambda+1}$ to remove $(\lambda+1)/2 + \lfloor(\lambda-1)/(2(d+2\gamma))\rfloor$ secular terms in $\phi_{\lambda+2}$.

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Numerical construction

We make an ansatz ($\tau = \Omega t$)

$$\phi = \sum_{0\le i<N} \sum_{0\le j<K} f_{i,j} \cos((2i+1)\tau) e_j(x)\,,$$
$$\Pi = \sum_{0\le i<N} \sum_{0\le j<K} p_{i,j} \sin((2i+1)\tau) e_j(x)\,.$$

- Find the solution by determining $2 \times K \times N + 1$ numbers
- Set the equations on a numerical grid of $K \times N$ collocation points
- Add one equation for *dominant* mode condition

$$\sum_{0\le i<N} f_{i,\gamma} = \varepsilon$$

Highly nonlinear system solved with the Newton-Raphson algorithm.

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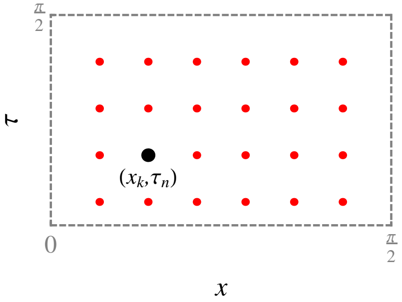
$$\sum_{0\leq i<N} f_{i,\gamma} = \varepsilon$$



Highly nonlinear system solved with the Newton-Raphson algorithm.

Numerical construction

We make an ansatz ($\tau = \Omega t$)

$$\phi = \sum_{0\leq i<N} \sum_{0\leq j<K} f_{i,j} \cos((2i+1)\tau) e_j(x) \,,$$
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- Find the solution by determining $2 \times K \times N + 1$ numbers
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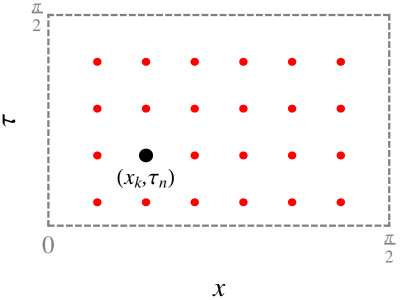

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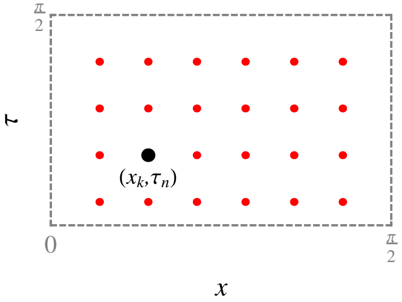

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Mathematica notebook

Series summation

Improve convergence with the Padé resummation of Ω_γ for $d = 4$, $\gamma = 0$

ε	direct sum	Padé	numerics
0.005	4.0016596666501	4.0016596666501	4.0016596666501
0.015	4.0151220741462	4.0151220741462	4.0151220741462
0.025	4.0430867838460	4.0430867838521	4.0430867838521
0.035	4.0879197007	4.0879197035435	4.0879197035448
0.045	4.15407139	4.15407167953	4.1540716797440
0.055	4.249920	4.249932516	4.2499325336279
0.065	4.39267	4.3929928	4.3929938556099
0.075	4.6230	4.629225	4.6292962269712
0.085	5.05	5.184	5.2017714694183

Estimate for the radius of convergence – threshold for the black-hole formation

$$\left([n/n]_{\Omega_\gamma}(\varepsilon^*)\right)^{-1} = 0\,,$$

n	2	4	6	8	...
ε^*	0.128	0.102	0.095	0.092	...

Results

- High order expansion for **time-periodic solution** — lenghty formulas in ε (solution for $d = 4$, $\gamma = 0$ up to 17th order consists of: 1257 terms in ϕ, 1137 in A and 1180 in δ expansion)

- Numerical solutions for descrete values of ε — extended floating-point arithmetic for highly **accurate solution**

- **Consistency** of the results — verification by two independent methods

- Indication on the **stability** of the obtained solutions

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