Globally regular instability of AdS_3

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Introduction

- There is strong evidence that AdS is unstable for $D \geq 4$
- The endstate of the evolution: AdS-Schwarzschild black hole:

$$g = -A\,dt^2 + A^{-1}dr^2 + r^2\,d\varphi^2\,, \qquad A = 1 - \frac{M}{r^{D-3}} + \frac{r^2}{\ell^2}$$

- Spectral properties and nonlinear perturbation analysis are qualitatively the same in all dimensions $D \geq 3$

Introduction

What is different in $D = 3$?

- Dimensionless measure of gravity's strength is GM/L^{D-3}
  so in $D = 3$ the *total* mass matters (not its concentration)
- AdS-Schwarzschild family in $D = 3$

$$g = -A\,dt^2 + A^{-1}dr^2 + r^2\,d\varphi^2\,, \qquad A = 1 - M + \frac{r^2}{\ell^2}$$

  There is a mass gap between AdS_3 and the lightest black hole:
  - $M = 0$ AdS
  - $0 < M < 1$ naked (conical) singularities
  - $M > 1$ BTZ black holes
- Small perturbations of AdS_3 cannot evolve into black holes. What is the endstate of the evolution?

- Convenient parametrization of 3D asymptotically AdS spacetimes

$$ds^2 = \frac{\ell^2}{A\cos^2 x}\left(-e^{2\beta}dt^2 + dx^2 + A\sin^2 x\, d\varphi^2\right)$$

where A and β are functions of $(t,x) \in (-\infty,\infty) \times [0,\pi/2)$

- Define mass function $m(t,x)$ by $A = 1 - m\cos^2 x$
- Field equations (using $' = \partial_x$, $\dot{} = \partial_t$ and $8\pi G = 1$)

$$\left(e^{-\beta}\dot{\phi}\right)^{\cdot} = \frac{1}{\tan x}(\tan x\, e^{\beta}\phi')'$$

$$m' = \tan x\, A\,(e^{-2\beta}\dot{\phi}^2 + \phi'^2)$$

$$\beta' = 2\sin x\cos x\,\frac{m}{A}$$

- We want to solve the initial-boundary value problem for this system for small perturbations of the AdS_3 space $\phi = 0, m = 0, \beta = 0$.

Initial and boundary conditions

- We assume that initial data $(\phi, \dot{\phi})_{|t=0}$ are smooth
- Smoothness implies that near $x = 0$

$$\phi(t,x) = f_0(t) + \mathcal{O}(x^2), \quad \beta(t,x) = \mathcal{O}(x^4), \quad m(t,x) = \mathcal{O}(x^2)$$

- Smoothness at spatial infinity and finiteness of the total mass M imply that near $x = \pi/2$ (using $\rho = \pi/2 - x$)

$$\phi(t,x) = f_\infty(t)\,\rho^2 + \mathcal{O}\left(\rho^4\right), \quad \beta(t,x) = \beta_\infty(t) + \mathcal{O}\left(\rho^4\right),$$
$$m(t,x) = M + \mathcal{O}\left(\rho^2\right)$$

  Remark: There is no freedom in prescribing the boundary conditions.
- Local well-posedness follows from (Holzegel-Smulevici 2011)

Spectral properties

- Linearized equation (Breitenlohner-Freedman 1982, Ishibashi-Wald 2004)

$$\ddot{\phi} + L\phi = 0, \quad L = -\frac{1}{\tan x}\,\partial_x\left(\tan x\,\partial_x\right)$$

  L is essentially self-adjoint on $L^2([0,\pi/2], \tan x\,dx)$.
- Eigenvalues and eigenvectors of L are $(k = 0, 1, \dots)$

$$\omega_k^2 = (2+2k)^2, \quad e_k(x) = 2\sqrt{k+1}\cos^2 x\,P_k^{0,1}(\cos 2x)$$

- Inner product: $(f,g) = \int\limits_0^{\pi/2} f(x)g(x)\tan x dx$
- Spectral representation: we introduce momentum $\Pi = e^{-\beta}\dot{\phi}$ and define:

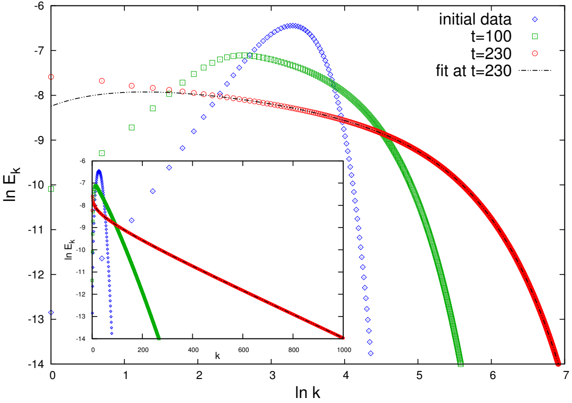
$$\Phi_k := (\sqrt{A}\phi', e_k'), \qquad \Pi_k := (\sqrt{A}\Pi, e_k)$$

  Then

$$M = \int\limits_0^{\pi/2} A\left(e^{-2\beta}\dot{\phi}^2 + \phi'^2\right)\tan x\,dx = \sum_{k=0}^{\infty} E_k(t)$$

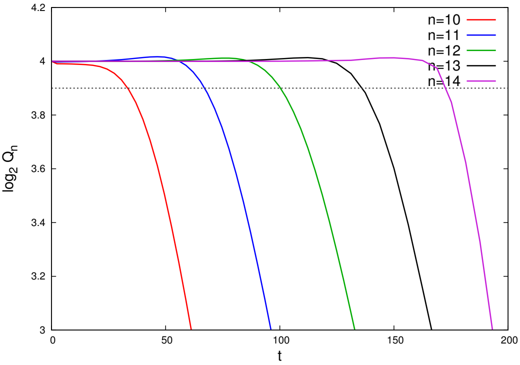
  where $E_k = \Pi_k^2 + \omega_k^{-2}\Phi_k^2$ is the energy of the k-th mode.

Energy spectrum



Initial data $\phi(0,x) = \varepsilon \exp(-\tan^2 x/\sigma^2)\,, \quad \dot{\phi}(0,x) = 0$

Convergence tests



Convergence factor for the solution ϕ_n computed on the 2^n-grid is defined by $Q_n = \frac{||\phi_n - \phi_{n+1}||}{||\phi_{n+1} - \phi_{n+2}||}$, where $||\cdot||$ is the spatial ℓ_2-norm.

Analyticity strip method (Sulem-Sulem-Frisch 1983)

- Let $u(t,x)$ be a solution of an evolution equation starting from real-analytic initial data and let $u(t,z)$ be its analytic extension to the complex z-plane.
- Typically $u(t,z)$ will have complex singularities. Let $z = x + i\rho$ be the location of the singularity closest to the real axis (hence ρ measures the width of the analyticity strip around the real axis).
- If $\rho(t)$ vanishes at some $t = T < \infty$, then the solution "blows up"; otherwise it is globally regular in time.
- Fourier coefficients of $u(t,x)$ behave for large k as

$$\hat{u}_k(t) \sim k^{-\alpha} \exp(-\rho k)$$

- Method: compute $\rho(t)$ by fitting an exponential decay to the tail of the numerically computed Fourier spectrum

Example

Consider an equation:

$$\boxed{u_t = x u_x \,, \quad u(0,x) = \frac{\varepsilon}{1+x^2}}$$

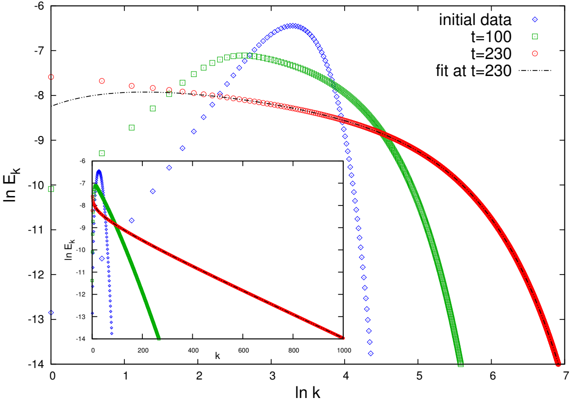
$$\begin{aligned} u(t,x) &= \frac{\varepsilon}{1+e^{2t}x^2} \\ \hat{u}(t,k) &= \varepsilon \pi e^{-t} \, H(k) \exp(-k \underbrace{e^{-t}}_{\rho(t)}) + (k \leftrightarrow -k) \end{aligned}$$

The solution is globally regular but

$$||u||^2_{\dot{H}^s} := \int_{-\infty}^{\infty} (\partial_x^s u)^2 dx = c_s \, e^{(2s-1)t}$$

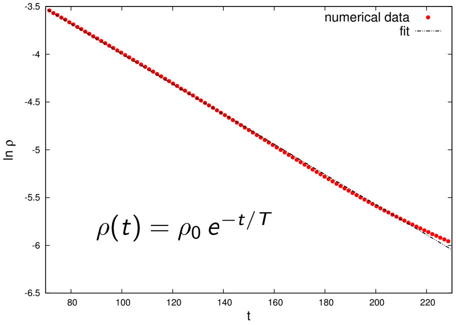
L^2-asymptotic stability ($s = 0$) and instability for $s > 1/2$.

Computation of $\rho(t)$ from the energy spectrum



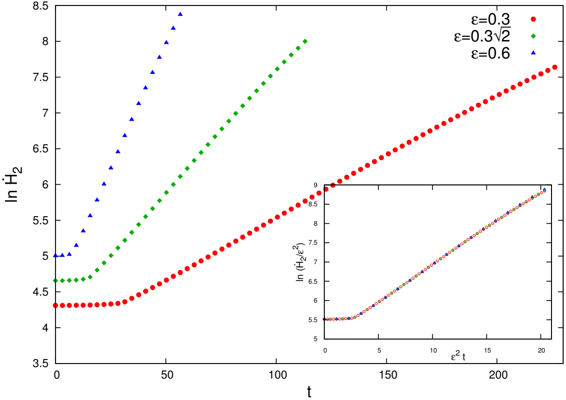
$$E_k(t) = C(t)\, k^{-\alpha(t)} e^{-2\rho(t)k}$$

Evidence for global regularity



Solutions develop progressively finer spatial scales as $t \to \infty$ without ever losing smoothness (weak turbulence). Characteristic decay time $T \sim \varepsilon^{-2}$.

$\dot{H}^s$-instability for $s > 1$



Time evolution of the L^2-norm of the second spatial derivative $\dot{H}^2 = ||\phi''(t,x)||_2$ (only the upper envelope of oscillations is plotted)

Final remarks

- Gradual loss of regularity due to weak turbulence has been well known in fluid dynamics (example: Euler equation in two spatial dimensions, Yudovich 1974). Weak turbulence is expected to be common for nonlinear wave equations in bounded domains.
- In the case of Einstein's equations, the weakly turbulent dynamics can proceed forever *only* in 3D, whereas in higher dimensions it is unavoidably cut off in finite time by the black hole formation.
- Here we considered only small mass solutions but we conjecture that all solutions with $M < 1$ are globally regular in time.
- Threshold at $M = 1$ is not well understood (numerical studies by Pretorius-Choptuik 2000). Does every solution with $M > 1$ evolve into a black hole?