On the Hydrodynamic Gradient Expansion of Holographic Fluid

Based on work with R. Janik and M. Heller

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Outline

- Introduction and motivation
- Higher order hydrodynamics
- Hydrodynamic expansion from holography
- Properties of hydrodynamic series
- Conclusions

Motivation

There are several reasons to study the hydrodynamics within AdS/CFT framework:

- Ongoing programme of strongly coupled plasma investigation at LHC, RHIC
- Strongly coupled dynamics of non-abelian gauge theory at finite temperature
- Possible insight into non-equilibrium physics beyond hydrodynamics

What is hydrodynamics?

- It is an effective theory of low energy dynamics of conserved charges, which remain after integrating high energy d.o.f.:

$$\nabla_\mu T^{\mu\nu}(x) = 0$$

- It assumes local thermal equilibrium, which introduces effective collective degrees of freedom: local temperature $T(x)$, velocity field $u^\mu(x)$ and other conserved quantities, varying only on large scales: $\epsilon = \frac{l_{mfp}}{L} << 1$
- Thermodynamic variables in $T_{\mu\nu}(T(x), u^\mu(x), ...)$ can be expanded in gradients, $\frac{\nabla T}{T^2} \sim \epsilon$, to obtain viscous contributions to perfect fluid
- Usually we take this series "for granted" and use it as the hydrodynamic equation, but is it a convergent expansion?

Equations of hydrodynamics

- Recent progress in fluid/gravity duality allowed for derivation of 2^{nd} order viscous conformal hydrodynamics in $d = 4$ (up to 3^{rd} ord. in boost-invariant case)

$$\begin{aligned} T^{\mu\nu} &= (\pi T)^4(\eta^{\mu\nu} + 4u^\mu u^\nu) - 2(\pi T)^3\sigma^{\mu\nu} \\ &+ (\pi T)^2(ln(2)T^{\mu\nu}_{2a} + 2T^{\mu\nu}_{2b} + (2 - ln(2))(\frac{1}{3}T^{\mu\nu}_{2c} + T^{\mu\nu}_{2d} + T^{\mu\nu}_{2e})) \end{aligned}$$

- Where transport coefficients are: η, τ_Π, κ, λ_1, λ_2, λ_3, and

$$\begin{aligned} \sigma^{\mu\nu} &= P^{\mu\alpha}P^{\nu\beta}\partial_{(\alpha}u_{\beta)} - \frac{1}{3}P^{\mu\nu}\partial_\alpha u^\alpha \\ T^{\mu\nu}_{2a} &= \epsilon^{\alpha\beta\gamma(\mu}\sigma^{\nu)}_{\gamma}u_\alpha l_\beta \\ T^{\mu\nu}_{2b} &= \sigma^{\mu\alpha}\sigma^{\nu}_{\alpha} - \frac{1}{3}P^{\mu\nu}\sigma^{\alpha\beta}\sigma_{\alpha\beta} \\ T^{\mu\nu}_{2c} &= \partial_\alpha u^\alpha \sigma^{\mu\nu} \\ T^{\mu\nu}_{2d} &= Du^\mu Du^\nu - \frac{1}{3}P^{\mu\nu}Du^\alpha Du_\alpha \\ T^{\mu\nu}_{2e} &= P^{\mu\alpha}P^{\nu\beta}D(\partial_{(\alpha}u_{\beta)}) - \frac{1}{3}P^{\mu\nu}P^{\alpha\beta}D(\partial_\alpha u_\beta) \\ l_\mu &= \epsilon_{\alpha\beta\gamma\mu}u^\alpha\partial^\beta u^\gamma,\ D = u^\alpha\partial_\alpha,\ P^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu}. \end{aligned}$$

Particular dual hydrodynamics from AdS/CFT
MH, RJ, PW, PRL 108, 201602 (2012)

- Consider boost-invariant $d = 4$ conformal fluid, the Bjorken model for RHIC
- Boundary coordinates are such that: $ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2$
- The stress tensor obeying symmetries has just one unknown function $\epsilon(\tau)$, to be specified by AdS dual evolution:

$$\begin{aligned} T^\mu_\nu &= Diag(-\epsilon(\tau), p_L(\tau), p_T(\tau), p_T(\tau)), \;\; T^\mu_\mu = 0 \\ p_L &= -\epsilon - \tau\epsilon', \;\; p_T = \epsilon + \frac{1}{2}\tau\epsilon', \;\; u = \partial_\tau, \;\; u^2 = -1 \end{aligned}$$

- This system models the QGP expansion at mid-rapidity region ("∞" collision energy) and is motivated by the search for the rapid themalization mechanism
- To such a fluid one can construct a gravity dual
- In order to trace the whole evolution and thermalization, full nonlinear spectral numerical simulation was developed, employing ADM-like formulation
- $\epsilon(\tau)$ was then obtained from the numerical solution

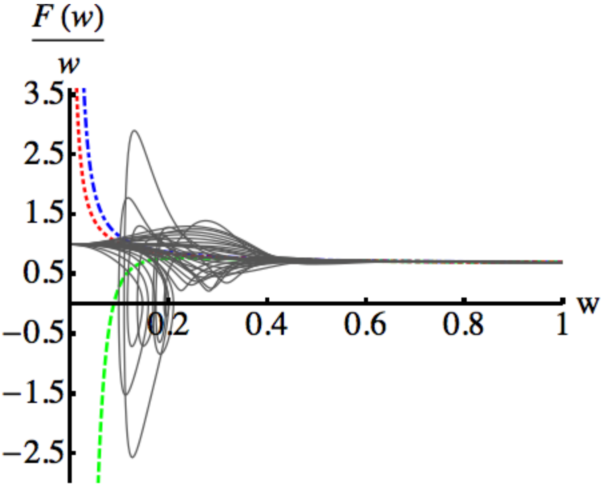
"Infinite order" hydrodynamics

- In this much constrained system one can rewrite the hydrodyamics equations $\nabla_\mu T^{\mu\nu} = 0$ in a very interesting fashon
- Those equations are first order in time, and we only have proper time τ as an independent variable
- By introducing dimensionless variable $w = T_{eff}(\tau)\tau \sim \epsilon^{1/4}(\tau)\tau$ we can write:

$$\frac{\tau}{w}\frac{d}{d\tau}w = \frac{F_{hydro}(w)}{w}$$

- $F_{hydro}(w)/w$ is in hydrodynamic regime completely determined by transport coefficients and universal
- On every hydrodynamic solution it evaluates to unity (it is the definition of the hydrodynamic equation)
- From numerical simulations we can independently read-off $\epsilon'(\tau)$ and $\epsilon(\tau)$, so we can parametrically plot the function $F_{hydro}(w)/w$
- But that function from full nonlinear evolution contains the whole information on the plasma dynamics, even beyond equlibrium and hydrodynamics
- Thus one can observe the transition to 'all-order' hydrodynamics, and also what happens before it

"Infinite order" hydrodynamics



Tempting possibility

- The late time expansion of the function $F(w)/w$ is known explicitly up to 3^{rd} order in the boost invariant case:

$$\frac{F(w)}{w} = \frac{2}{3} + \frac{1}{9\pi w} + \frac{1 - ln2}{27\pi^2 w^2} + \frac{15 - 2\pi^2 - 45 ln2 + 24 ln^2 2}{972\pi^3 w^3} + \dots$$

- Is it possible to obtain expression for $F(w)/w$ to some very high order in $1/w$ and resume it?
- Could it give some insight into the deep non-equilibrium region of Fig 1?
- Extension of hydrodynamic $F(w)/w$ could incorporate somehow the genuine non-equilibrium D.O.F.
- Result of simulation shows that the plot is regular (althoug very diverse) before the hydrodynamic regime
- Similarily, how would the energy density $\epsilon(\tau)$ look like after such a resummation?
- Is it possible to extend the plots to $\tau = 0$?
- We address these questions next

Two approaches to the energy density series

- There are in principle two ways to obtain perturbative contributions to energy density from gravity
- One is to consider fluid/gravity duality in the long wavelength regime and employ gradients expansion
- This permits only slow metric variations but of arbitrarily large scale
- Thus it includes black hole formation, horizon dynamics and viscous processes, like entropy production
- The other way is to consider linearized evolution on a given background
- This way leads to quasinormal modes and arbitrarily fast evolution, however without dissipation
- One can even consider "QNM" of dynamic geometries (R. Janik, R. Peschanski, 2006)
- Is there some link between these two approaches?

Gradient expansion for numerics
MH, RJ, PW, PRL 110, 211602 (2013)

- We can construct semi-analytic series for $\epsilon(\tau)$ and $F(w)/w$ from a dual metric
- Utilizing fluid/gravity duality, we start by choosing the metric (in proper-time E-F coordinates) as:

$$ds^2 = 2d\tau dr - Ad\tau^2 + \Sigma^2 e^{-2B} dy^2 + \Sigma^2 e^B (dx_1^2 + dx_2^2)$$

- Functions A, B, Σ depend on τ and r, and are systematically corrected in powers of $\tau^{-2/3}$
- Analytically known late time energy density,

$$\epsilon(\tau) = \frac{3}{8} N^2 \pi^2 \frac{1}{\tau^{4/3}} (\epsilon_2 + \epsilon_3 \frac{1}{\tau^{2/3}} + \epsilon_4 \frac{1}{\tau^{4/3}} + \ldots),$$

  reflects gradient expansion of velocity u^μ in units of temperature T: $T^{-1} \nabla_\mu u_\nu \sim \epsilon^{-1/4} \tau^{-1} \sim \tau^{1/3} \tau^{-1} = \tau^{-2/3}$
- Terms ϵ_i are the first few transport coefficients
- We want to compute this series to very high order

High order energy density

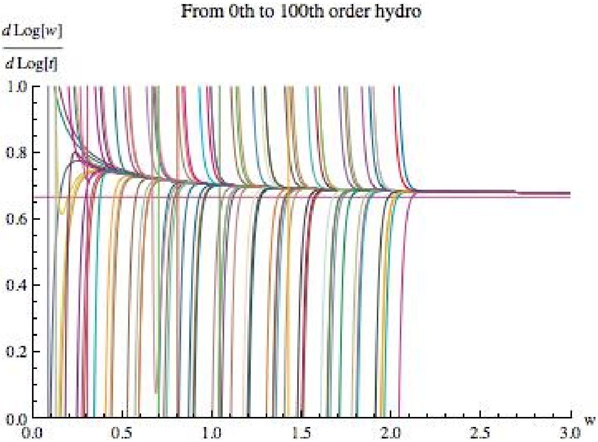
- Using gradient expansion in proper time, expression for energy density $\epsilon(\tau)$ (and thus $F(w)/w$) was obtained, up to order 240, à la M. Heller et al., 2009
- Einstein equations were analytically expanded in time τ and numerically integrated in the bulk variable r
- Resulting semi-analytic expression for the energy density reads:

$$\epsilon(\tau) = \frac{1}{\tau^{4/3}} \sum_{i=0}^{N} \epsilon_i \tau^{-2/3i}, N \sim 240$$

- By construction it should describe only hydrodynamic information, as it is performed in the late time/long wavelength regime
- One could hope that including higher and higher terms would improve the quality of the energy approximation
- It turns out, not quite so..

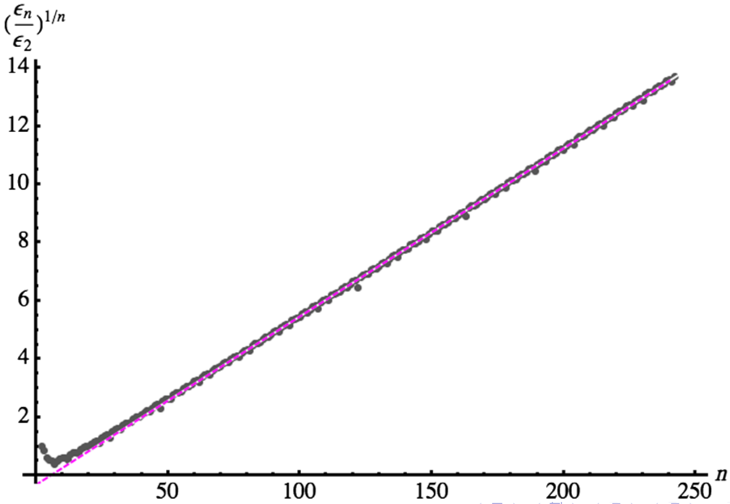
Perturbative-numeric $F(w)/w$

- The expected improvment in 'universal hydrodynamic' function is not there:



Divergence

- A closer look at the character of the series coefficients (basic step when handling series..) reveals the anticipated truth:



The nature of the series

- This type of behavior is characteristic of asymptotic series with factorial grownth of coefficients,

$$\epsilon_n \sim C a^n n!$$

- The Cauchy criterion and Stirling's formula indicate linear grownth of such a series:

$$(\epsilon_n/\epsilon_2)^{\frac{1}{n}} \sim (|a^n n!|)^{\frac{1}{n}} \sim (|a^n n^n e^{-n}|)^{\frac{1}{n}} \sim an/e$$

- It is thus natural to employ the Borel resummation technique to resum the series and recover possible nonperturbative information limiting its convergence radius to zero

Borel resummation

- Borel transform and sum are commonly used tools in the context of perturbation theory in QFT
- Here however we deal with expansion in time, not coupling constant
- It is then interesting to see how the method will work
- Borel tranform for a 'divergent function' $\epsilon(u) = \sum_k \epsilon_k u^k, \;\; u = \tau^{-2/3}$, is defined as:

$$\epsilon_B(\zeta) = \sum_{k=0}^{\infty} \tilde{\epsilon}_k \zeta^k = \sum_{k=0}^{\infty} \frac{\epsilon_k}{k!} \zeta^k$$

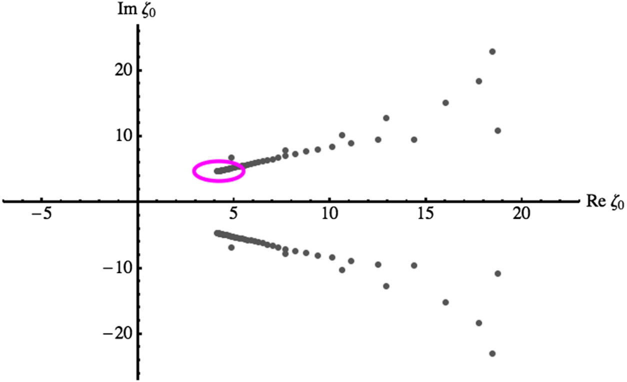
- This auxiliary series should possess non-zero radius of convergence, and define an analytic function around $\zeta = 0$
- Inverse operation defining resummation (and undoing prohbited change of sums) is:

$$\epsilon_R(u) = \int_0^{\infty} d\zeta e^{-\zeta} \epsilon_B(\zeta u)$$

- Unique Laplace transform should exist, provided integrand is regular in $Re\zeta > 0$

Borel transformed energy

- To perform the transform, function $\epsilon_B(\zeta)$ should be integrable up to infinity, at least in some neighborhood of R_+
- In practice, this means that we must analytically continue the transform, if it does not reach to infinity
- For finite expressions obtained from numerics, one often employs Padé approximants (like in e.g. Z. Ambrozinski, J. Wosiek, 2013)
- Our energy density treated like that reveals interesting poles structure:



Borel transformed energy

- Pade continuation reveals something that may be complicated cut structure
- The radius of convergence is now estimated from Cauchy's criterion to be $|\tilde{\omega}_0| \sim 6.37$ (and agrees with a fit of subleading power grownth of the series, a^n)
- Absence of Pade poles on the R_+ is a good sign and suggests possibility of energy and $F(w)/w$ resummation
- However presence of poles on the right halfplane introduces certain (interesting!) ambiguity
- Now complex integral may reach infinity in many nonequivalent ways!
- What are these numbers: poles and $|\tilde{\omega}_0|$?

A different approach

- Let us now consider the other way, linearized evolution on the dynamic backgroud
- The dynamics of metric perturbation can be investigated numerically on the background of perfect fluid with first viscous correction:

$$B(\tau, z) = B_{hydro}(\tau, z) + \delta B(\tau, z), \;\; \Box_g \delta B(\tau, z) = 0$$

- From the latter, the contribution to energy density $\epsilon(\tau)$ is computed following AdS/CFT dictionary
- The result, very interestingly, confirms and generalizes previous analytic dynamic background calculations of “Viscous QNM” by R. Janik, R. Peschanski in 2006,

$$\delta\epsilon(\tau) \sim \tau^{\alpha_{qnm}} e^{-i\frac{3}{2}\omega_{qnm}\tau^{2/3}},$$

with:

$$\alpha_{qnm} = -1.5422 + 0.5199i, \;\; \omega_{qnm} = 3.1195 - 2.7467i$$

- Now due to viscous correction to geometry, subleading power contribution α_{qnm} is present

Back to Borel transformed energy

- It is interesting to compare this with the Borel resummed energy density $\epsilon_R(\tau)$
- Due to ambiguity in computing Borel-Laplace integral, we can include some poles in the contour
- Encircling some of the poles introduce exponental factors to the resummed energy density!
- Contribution from the very tip of the structure is:

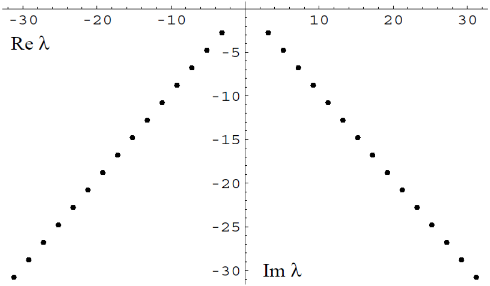
$$\delta\epsilon(\tau) \sim \tau^{\alpha_{Borel}} e^{-i\frac{3}{2}\omega_{Borel}\tau^{2/3}},$$

  and $\omega_{Borel} = 3.1193 - 2.7471i = \omega_{qnm}$!!!
- Moreover, $\alpha_{Borel} = -1.5426 + 0.5192i = \alpha_{qnm}$!!!
- Non-equilibrium leading QNM is the obstruction controlling the radius of convergence of Borel transformed energy $\epsilon_B(\zeta)$
- The subleading viscous power term comes from the 'cut' nature of contribution, $\tau^{\alpha+i\beta}e^{\cdot\cdot}$ and can be computed by a fit to a certain deformation of $\epsilon_R(\tau)$
- This effect closely resembles instantonic phenomena in QFT, but here the exponent is non-integer and in time, not coupling!

We have seen some of these numbers before

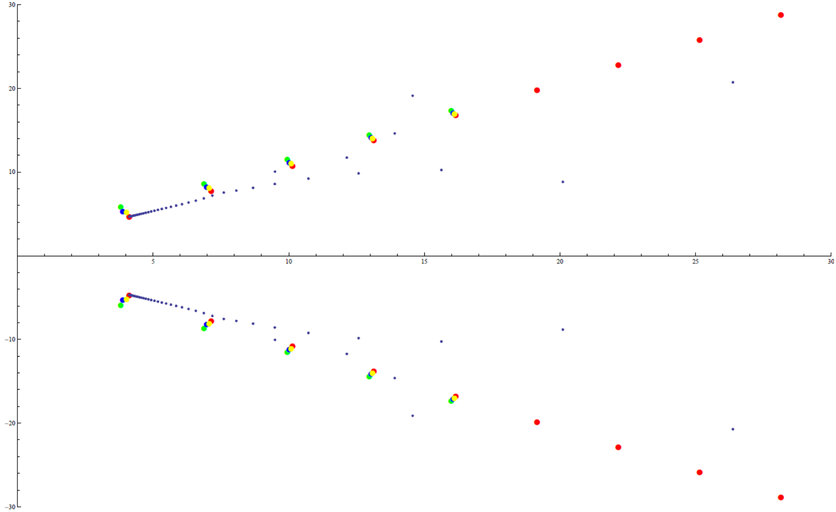
- Set of zero-momentum complex quasinormal modes of static black brain, $q_n/\pi T$, A. Starinets 2002:

n	$\mathrm{Re}\,\lambda_n$	$\mathrm{Im}\,\lambda_n$
1	± 3.119452	-2.746676
2	± 5.169521	-4.763570
3	± 7.187931	-6.769565
4	± 9.197199	-8.772481
5	± 11.202676	-10.774162
6	± 13.206247	-12.775239
7	± 15.208736	-14.775979
8	± 17.210558	-16.776515
9	± 19.211943	-18.776919
10	± 21.213025	-20.777232
11	± 23.213896	-22.777489
12	± 25.213896	-24.777489
13	± 27.213896	-26.777489
14	± 29.213896	-28.777489
15	± 31.213896	-30.777489



QNM in Borel transformed energy $\epsilon_B(\zeta)$

- Now several previous QNM can be found on the Borel-Pade plane!



QNM in Borel transformed energy $\epsilon_B(\zeta)$

- We see now, that the radius of convergence of Borel transformed energy density $\epsilon_B(\zeta)$ is controlled by the lowest non-hydrodynamic quasinormal mode ω_0,

$$\omega_{Borel} = \omega_{qnm} = \omega_0$$

and radius of convergence is in fact:

$$|\tilde{\omega}_0| \sim 6.37 \sim \frac{3}{2}|\omega_0|$$

- It is purely non-equlibrium fast degree of freedom, nevertheless it is contained in the late time hydrodynamic equilibrium expansion
- QNM are exponentially small (non-perturbative) in the late time limit $\tau \to \infty$
- It seems that the energy is an example on a resurgent series, in small parameters $\tau^{-\#}$ and $e^{-\#\tau^{+\#}}$

Conclusions

- Hydrodynamic gradient expansion is asymptotic
- Genuine non-equilibrium d.o.f. are the master villains standing behind this threat
- The non-perturbative ambiguity is in agreement with explicit QNM calculation, even up to subleading "viscous" prefactor
- Lack of poles on R_+ may suggest the possibility of resummed hydrodynamics
- It then may allow for a refined criterion of QGP hydrodynamization time
- Maybe non-equilibrium d.o.f. could be explicitly included in the dynamics of $T_{\mu\nu}$?

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