Gravity and Non-Commutative Geometry

An alternative bridge in two dimensions

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Outline

1. **Problem:** Geometric interpretation of spinors in gravity
2. **Strategy I:** Lattice gauge theory
3. **Strategy II:** Spinors and non-commutative geometry
4. **Epilogue:** Future directions

Euclidean gravity in two dimensions

The gravity action can be written in terms of constrained BF theory:

$$S[A,B,\lambda] = \int_{\Sigma} \mathrm{Tr}(BF) - \int_{\Sigma} \lambda\left(\mathrm{Tr}(B^2)+1\right),$$

invariant under the adjoint action of $SO(2)$ with Lie algebra $\mathfrak{so}(2)$.

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- Σ is (typically) a compact surface
- B is a scalar field; F the curvature of a connection A
- B and F are $\mathfrak{so}(2)$-valued
- $\mathrm{Tr}\colon \mathfrak{so}(2) \to \mathbb{R}$ is a Killing form
- λ is a volume form (Lagrange multiplier)

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We are interested in the quantum version.

Quantisation – 'Path integral' approach

The quantum theory is described by expectation values of operators:

$$\langle O \rangle_\Sigma = \int DADBD\lambda\, O[A, B, \lambda] e^{-iS[A,B,\lambda]}$$

Topological – all expectation values are **topological invariants**

Purpose of this talk

Problem

Can we describe operators encoding spinor information and find

$$\langle O\rangle_\Sigma = \int DADBD\lambda \int D\psi D\bar{\psi} O[A, B, \lambda, \psi, \bar{\psi}] e^{-iS[A,B,\lambda]}$$

without breaking topological invariance?

Expectation values in BF theory

Auxiliary problem

Calculate exactly the expectation values

$$\langle O \rangle_\Sigma = \int DADBD\lambda O[A, B, \lambda] e^{-i \int_\Sigma \mathrm{Tr}(BF) + i \int_\Sigma \lambda \left(\mathrm{Tr}(B^2) + 1\right)}$$

by giving them a **geometrical** interpretation.

Expectation values in BF theory

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Calculate exactly the expectation values

$$\langle O\rangle_\Sigma = \int DADB \int D\lambda O[A,B,\lambda] e^{i\int_\Sigma \lambda(\mathrm{Tr}(B^2)+1)} e^{-i\int_\Sigma \mathrm{Tr}(BF)}$$

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Starting point: calculating the vacuum expectation value

$$\langle 1 \rangle_\Sigma = \int DADB e^{-i\int_\Sigma \mathrm{Tr}(BF)} = \int DA \delta(F)$$

What is the importance of flat connections?

Quantisation – 'Canonical' approach

$$\begin{array}{ccc} T^{*}(\mathcal{A}) & \xrightarrow{\text{quantise}} & L^{2}(\mathcal{A}) \\ \downarrow{\scriptstyle \text{constraint}} & & \downarrow{\scriptstyle \text{constraint}} \\ T^{*}(\mathcal{A}_0/\mathcal{G}) & \xrightarrow{\text{quantise}} & L^{2}(\mathcal{A}_0/\mathcal{G}) \end{array}$$

- $\mathcal{A}$ is the space of connections A (configuration space)
- $\mathcal{A}_0/\mathcal{G}$ – the moduli space of flat connections
  - 'Moduli space' refers to the invariance under the action of $\mathcal{G}$
  - 'Flat connection' is the choice of 'gauge', $F = 0$

The vacuum expectation value

$$\langle 1 \rangle_\Sigma = \int DA\delta(F) = \mathrm{vol}\left(\mathcal{A}_0/\mathcal{G}\right)$$

Breakthrough of rigorous path integral approaches:

Using the geometry of Σ to probe the geometry of $\mathcal{A}_0/\mathcal{G}$

The vacuum expectation value

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Breakthrough of rigorous path integral approaches:

Using the geometry of Σ to probe the geometry of $\mathcal{A}_0/\mathcal{G}$

Key points

- Find some 'good' functions $f(A)$ – parallel transport variables
- Investigate them along some 'regions' of Σ – edges of a triangulation
- Impose gauge invariance – $\mathcal{G} = SO(2)$ – and gauge fixing

The vacuum expectation value

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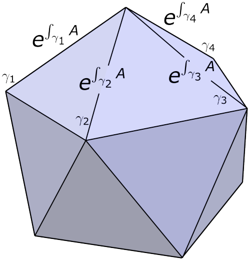
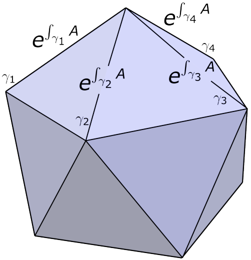
Breakthrough of rigorous path integral approaches:

Using the geometry of Σ to probe the geometry of $\mathcal{A}_0/\mathcal{G}$



The vacuum expectation value



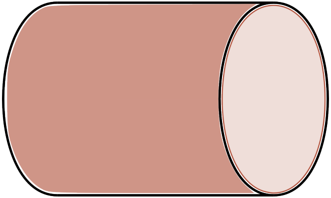
$$e^{\int_\gamma A} = g \in SO(2)$$

Basis for $L^2(\mathcal{A})$: $\rho^i(g)^a{}_b$

This basis is also dense in $\mathbb{R}[SO(2)]$, the group algebra.

$$L^2(\mathcal{A}) \simeq \mathbb{R}[SO(2)]$$

The vacuum expectation value



Loop variables: $\chi^i(g) = \mathrm{Tr}\rho^i(g)$

Invariance under $SO(2)$: $\chi^i(g) = \chi^i(hgh^{-1})$

$$L^2(\mathcal{A}/SO(2)) \simeq Z\left(\mathbb{R}[SO(2)]\right)$$

The vacuum expectation value

$$\langle 1 \rangle_\Sigma = \int DA\delta(F) = \mathrm{vol}\,(\mathcal{A}_0/\mathcal{G})$$

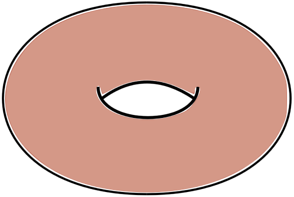
What do we conclude?

Geometrical properties have been **translated into algebraic** ones

- The Hilbert spaces have extra structure – they are also algebras
- Triangulation independence $\Leftrightarrow$ axioms of the algebra $\mathbb{R}[SO(2)]$

Ansatz

$$\langle 1 \rangle_{\Sigma} \quad \longleftrightarrow$$



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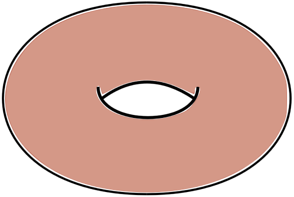
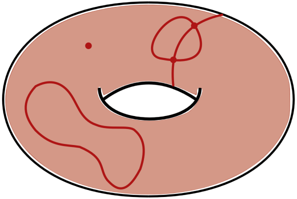
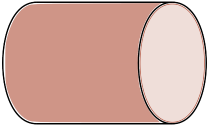


$$\langle O \rangle_\Sigma \quad \longleftrightarrow$$



Ansatz

The linear map



$$L^2(\mathcal{A}/SO(2)) \to L^2(\mathcal{A}/SO(2))$$

is an operator in the Hilbert space.

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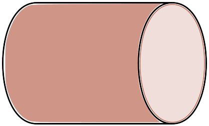

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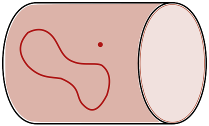
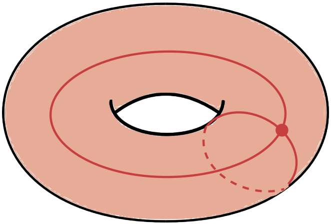
is an operator in the Hilbert space. It follows



$$L^2(\mathcal{A}/SO(2)) \to L^2(\mathcal{A}/SO(2))$$

must also be one.

What is the information encoded in the defects?



- Defect lines: vector spaces with inner product.
- The vector space carries left and right actions from $\mathbb{R}[SO(2)]$
- Vertices: linear maps

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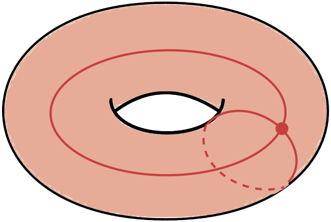

Some results:

- Expectation values constructed from defects are homotopy invariant
- Operator implementing gravity constraint has been constructed
- Simply connected cover data can be recovered

What about the spinors?

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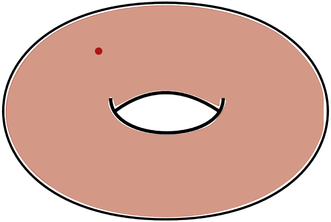
The role of non-commutative geometry



Non-commutative geometry

$\star$ algebra $\mathcal{A}$

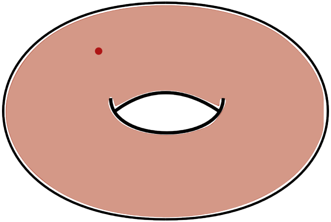
Hilbert space $\mathcal{H}$

$D: \mathcal{H} \to \mathcal{H}$

$J: \mathcal{H} \to \mathcal{H}, J^2 = \pm 1$

$\Gamma: \mathcal{H} \to \mathcal{H}, \Gamma^2 = \pm 1$

The role of non-commutative geometry



Classical case	**Non-commutative geometry**
$C^\infty(\Sigma)$	$\star$ algebra $\mathcal{A}$
Spinor fields on Σ	Hilbert space $\mathcal{H}$
Dirac operator	$D : \mathcal{H} \to \mathcal{H}$
Charge conjugation	$J : \mathcal{H} \to \mathcal{H}, J^2 = \pm 1$
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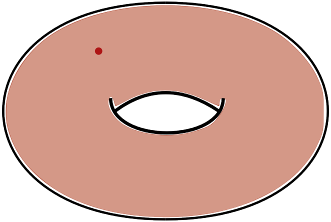

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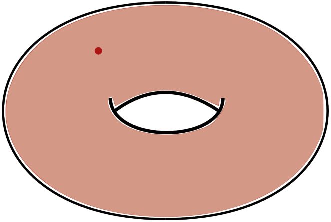

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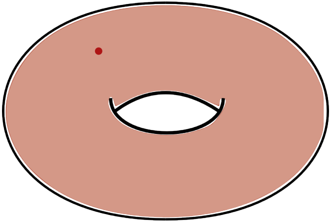

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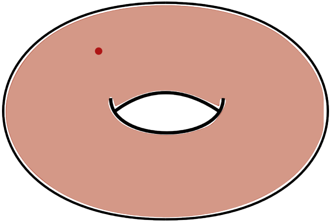

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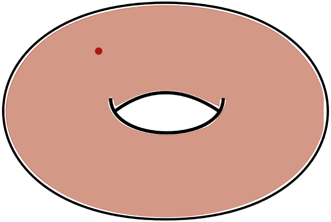

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Work in progress:

- Find all defect vector spaces V satisfying the NC axioms
- Use non-orientable case to suppress false candidates
- Relate expectation values with classical fermionic fields

Thank you!

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