Splitting thin shells and distributional solutions of Einstein equations

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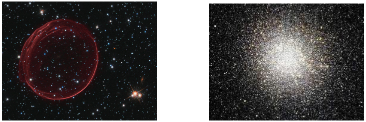
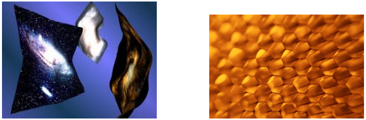
Overview

Why thin shells?

- Simplifying assumption to obtain concrete solutions of the coupled matter-field equations (obtaining realistic exact solutions of Einstein equations with matter is hard)
- There exist some astrophysical systems that can be approximately modelled by thin shells



- There could exist (or have existed) hypothetical identities described by thin shells



- They have been used as toy models to gain insight into theoretical issues, like the cosmic censorship conjecture and the hoop conjecture

Thin shells

- Timelike embedded orientable hypersurface Σ
- Smooth intrinsic metric (unambiguous induced metric)
- Discontinuous normal derivatives of the metric (discontinuous extrinsic curvature)
- $T = T_{smooth} + \delta_\Sigma S$

Israel-Darmois junction conditions

$$\begin{aligned} \kappa S_{ij} &= -([K_{ij}] - \gamma_{ij}[tr(K)]) && (1)\\ S^{ij}[K_{ij}]_+ &= \kappa[T_{\mu\nu} n^\mu n^\nu] && (2)\\ S^j_{i;j} &= [T_{\mu\nu} n^\mu e^\nu_i] && (3) \end{aligned}$$

Technical remarks

- They are well-defined solutions of Einstein equations in the sense of distributions (Riemann tensor and contractions between Riemann and the metric are well-defined)
- And the only ones among concentrated sources on submanifolds (Geroch, Traschen 88')
- Nevertheless, Bianchi identities do not make sense as distributional equations

Spherically symmetric thin shells in $n+2$ dimensions

$$\begin{aligned} ds_\Sigma^2 &= -d\tau^2 + R(\tau)^2 d\Omega_n^2 && (4) \\ S^i_j &= \mathrm{diag}[-\rho(\tau), p(\tau), ..., p(\tau)] && (5) \\ S^{ij} &= p h^{ij} + (\rho + p) u^i u^j && (6) \end{aligned}$$

If there is no hysteresis

$$\frac{d\rho}{dR} + \frac{n(\rho + p)}{R} = 0 \tag{7}$$

If we give explicitly $\rho(R)$ or a barotropic equation of state, then we obtain an equation of state

$$\frac{1}{2}\dot{R}^2 + V(R) = 0 \tag{8}$$

Collisionless particles

- Non-interacting but self-gravitating particles (Vlasov matter)
- Each particle follows a geodesic trajectory
- But, as the geodesics are not well-defined on Σ, we suppose that they follow trajectories of constant angular momentum (geodesics of the induced metric)

$$S^{ij} = -\mu \int f(x,p)\sqrt{-\gamma}u^i u^j \frac{dp^1..dp^n}{p_0} \tag{9}$$

- It turns out that the particle distribution can be completely defined by giving a distribution on the angular momenta space $n(L)$
- They have been previously analysed in the literature (Evans 77'), but always considering a single possible value for the angular momentum modulus
- In principle $n(L)$ is arbitrary. Newtonian intuition suggests that the particles move within a range of velocities given by the depth of the potential well that the shell generates

$$\rho(R) = \frac{\mu}{S_n R^{n+1}} \int n(L)\sqrt{R^2+L^2}dL \quad , \quad p(R) = \frac{\mu}{nS_n R^{n+1}} \int \frac{n(L)L^2}{\sqrt{R^2+L^2}}dL \tag{10}$$

Stability of solutions

Different kinds thereof...

- Are thin shells solutions "realistic"? (Are they stable? Do they approximate smooth solutions?)
- Perturbations around static solutions (Newtonian shells made of barotropic fluids are unstable; Bicak, Schmidt 99')
- Perturbations of an initial data set (continuity of the map between initial data and solutions)
- If a mode expansion is possible, they typically mix metric and matter degrees of freedom
- Let us perturb the matter variables ...

Heuristic analysis for stability against fragmentation

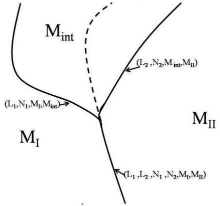
Particle evaporation analysis

- Suppose an infinitesimal separation of an individual particle
- Solve geodesic equation for both sides of the shell
- Compute the relative sign of the normal acceleration with respect to displacement
- For a given shell solutions, this criteria puts constraints on the support of $n(L)$. There is a dynamic range of stability $(L_{min}(R), L_{max}(R))$.
- The shell is stable along its entire evolution if $supp\{n\} \subset (\min_R L_{min}, \max_R L_{max})$
- There are solutions that are always stable, always unstable, and solutions are initially stable but become unstable at a certain point of the evolution

Heuristic analysis for stability against fragmentation

Split the particle ensemble into two parts

- Separate the particle ensemble into groups and compute the relative acceleration of the resulting shells
- We impose continuity of the radius and velocities of the shell at the moment of separation
- This prescription completely determines the resulting spacetime, and makes it possible to compute the relative accelerations



Heuristic analysis for stability against fragmentation

Split the particle ensemble into two parts

- We compute the stability of the simplest non-trivial case: two possible values for the angular momentum of the particles, and propose the separation of these two groups
- Fixing all parameters except one of the possible values of the angular momentum, we get a dynamic stability range for it, which is included in the stability range for the single particle analysis
- This criteria is in this sense more restrictive than the previous one
- As in the previous case, there are solutions that are either always stable, always unstable or initially stable but become unstable at some point of the evolution
- In the last case, a solution that represents the splitting of a shell into two parts can be constructed by solving the equations of motion of the shells separately
- These resulting shells can either collapse to a black hole, collide, or expand forever

Heuristic analysis for stability against fragmentation

Split the particle ensemble into two parts

- Nevertheless, one can solve the equations of motion at all times without considering any fragmentation
- If we consider a Cauchy surface before the splitting, the initial data for the splitting and for the non-splitting solutions are identical
- This compromises the uniqueness in the evolution of thin shells made of Vlasov matter
- But the Einstein-Vlasov system is well-posed, and there are global results for the spherically symmetric case (Dafermos, Rendall 2006')
- The loss of uniqueness should have something to do with the thin-shell-limit of Einstein-Vlasov shells
- We constructed families of thick static solutions of the Einstein-Vlasov system that tend to a thin shell, and all the limiting configurations turned out to be stable in the above sense
- It could be that unstable shells can not be thin-shell-limits of thick configurations, and therefore unphysical

Thin shell composed of two non-interacting matter fields

- More general matter content for the shell: two spherically symmetric but arbitrary non-interacting matter fields

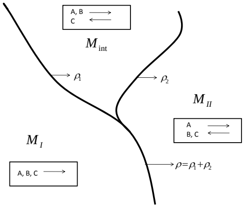
$$S^{ij} = (p_1 + p_2)h^{ij} + (\rho_1 + \rho_2 + p_1 + p_2)u^i u^j \tag{11}$$

Conservation of the source holds separately for each component

$$\frac{d\rho_i}{dR} + \frac{n(\rho_i + p_i)}{R} = 0 \tag{12}$$

- In Newtonian terms, two arbitrary fluids may move together as a shell because of the potential well they generate

Thin shell composed of two non-interacting matter fields



- The same procedure as before: infinitesimal separation of the non-interacting fluids and computation of the relative acceleration
- Define $\omega_i \equiv p_i/\rho_i$
- For $\alpha_i > -1/3$, the stability criteria gives a dynamical range for α $\alpha_i \in (\min_R \alpha_{min}, \max_R \alpha_{max})$

A brane-world cosmology setting

- Consider a five dimensional spacetime with a negative cosmological constant that admits a codimension 2 foliaton of isotropic submanifolds
- An arbitrary solution on a vacuum region with this properties can be expressed in the following form (Bowcock, Charmousis, Gregory 2002'):

$$ds^2 = -F(r)dt^2 + F(r)^{-1}dr^2 + r^2 d\Omega_n^2 \tag{13}$$

where $F(r) = k - 2M/r^2 + 6r^2/\ell^2$ (k is the curvature constant of the leaves of the foliation, and $\ell^2 = -6/\Lambda_5$)

- These are the so-called "generalized Schwarzschild - AdS" solutions
- They have been used in brane-world cosmology as a way to incorporate matter to the Randall-Sundrum model with one brane (the so-called SMS brane-world model)
- In this context the "observable" universe is a 4-d thin shell made of the standard cosmological ingredients (except dark energy) embedded in a 5-d spacetime with Z_2 symmetry centred on the shell
- To have standard cosmology as a low-energy limit a brane-tension must be imposed: a "cosmological constant fluid" within the shell (ideal fluid with $\omega = -1$), and fine tuning...

Stability of a brane-world composed of non-interacting matter fields

- If there is no Z_2 symmetry, the stability in this setting displays the same qualitative features than the spherically symmetric w/o cosmological constant case
- We may also propose a Z_2-symmetric version of this, where the shell splits into three parts
- In this case, also the same qualitative features appear
- These families of brane-world models turn out to be extremely unstable in this sense

Final remarks

- We developed a new kind of stability analysis for thin shells solutions
- There are thin shells solutions that are initially stable but become unstable later in the evolution
- When this kind of instability appears, a splitting solution can be constructed
- This compromises the uniqueness in the evolution of initial data associated with thin shells
- A possible interpretation of this is to consider unstable shells as not being thin-shell-limits of thick configurations, and therefore unphysical
- As in the case of shock-waves, there could be a way to define an entropy for specific matter models and in this way identify the physical solution

Thank you