Higher Conformal Couplings from Weyl Gauge Theory

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LIII School, 29.06.2013, Zakopane

Outline

- Motivation
- Weyl gauge theory
- General method
- Applications
- Conclusions

Motivation

- On flat spacetime:
  no dimensionfull couplings ⇒ scale invariance
  Symmetry may be enhanced to full conformal (Iorio, O'Raifeartaigh, Sachs, Wiesendanger)

- On curved spacetime:
  Conformal (Weyl) rescalings, coupling to spacetime curvature

$$L=(\partial\phi)^2+\frac{d-2}{4(d-1)}R\phi^2$$

- How to obtain this non-minimal couplings preserving conformal covariance?

Motivation

Various methods have been used:

- BRST covariant algebra for Weyl-gravity (Boulanger)
- Unfolded Dynamics Approach (Shaynkman, Tipunin, Vasiliev)
- Tractor calculus (Gover, Shaukat , Waldron)
- Ambient space techniques (Fefferman, Graham)
- Ricci gauging (Iorio, O'Raifeartaigh, Sachs, Wiesendanger)
- Others (Manvelyan, Mkrtchyan, Erdmenger)

- In this talk method based on decoupling of Weyl gauge fields (related to Weyl geometry)

Weyl gauge theory

- Gauge transformation of lengths $ds'^2 = \Omega^2 ds^2$
- Gauge transformations on fields $g'_{\mu\nu} = \Omega^2 g_{\mu\nu}$ $\Phi' = \Omega^w \Phi$
- If $\Omega = \Omega(x)$ then need to introduce abelian Weyl gauge potential b_μ
  $b'_\mu = b_\mu + \partial_\mu \Omega$
- Weyl curvature (field strength) $B_{\mu\nu} = \frac{1}{2}\left(\partial_\mu b_\nu - \partial_\nu b_\mu\right)$
- Coordinates and derivatives do not transform $x'^\mu = x^\mu$ $\partial'_\mu = \partial_\mu$
- However Riemannian connection is not Weyl covariant!

$$\Gamma'^{\mu}_{LC\,\nu\rho} \neq \Gamma^{\mu}_{LC\,\nu\rho}$$

Weyl gauge theory

- Unique, symmetric, torsionfree linear connection ⇒ Weyl connection

$$\tilde{\Gamma}^{\mu}_{\nu\rho}=\Gamma^{\mu}_{LC\,\nu\rho}-\delta^{\mu}_{\nu}b_{\rho}-\delta^{\mu}_{\rho}b_{\nu}+g_{\nu\rho}b^{\mu}$$

- 1st consequence: Weyl connection is Weyl invariant $\tilde{\Gamma}'^{\mu}_{\nu\rho}=\tilde{\Gamma}^{\mu}_{\nu\rho}$
- 2nd consequence: cov. derivative $\tilde{\nabla}_{\mu}$ with $\tilde{\Gamma}^{\mu}_{\nu\rho}$ preserve Weyl symmetry
- Weyl covariant derivatives
  on scalar fields: $D_{\mu}\phi=(\partial_{\mu}-wb_{\mu})\phi$
  on tensor fields: $D_{\mu}T=(\tilde{\nabla}_{\mu}-wb_{\mu})T$
- Transformation law of cov. Weyl derivatives of any tensor *T* (indices suppressed) $\left(D_{\mu}T\right)'=\Omega^{w}D_{\mu}T$ $\qquad T'=\Omega^{w}T$

Weyl curvatures

- Riemann-Weyl curvature tensor $\left[D_\mu, D_\nu\right] v^\rho = r_{\mu\nu}{}^\rho{}_\sigma v^\sigma$
- Ricci-Weyl tensor and scalar $r_{\nu\sigma} = r_{\mu\nu}{}^\mu{}_\sigma$ $r = g^{\nu\sigma} r_{\nu\sigma}$
- Riemann-Weyl and Ricci-Weyl tensors are conf. inv.,
- Ricci-Weyl scalar is conf. cov. with *w*=-2
- Trivial Weyl gauge field $B_{\mu\nu} = 0$
  ⇒ Weyl curvatures in terms of ordinary spacetime curvatures and Weyl gauge potential b_μ

$$r_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + g_{\mu\rho}\left(\nabla_\nu b_\sigma + b_\nu b_\sigma\right) - g_{\mu\sigma}\left(\nabla_\nu b_\rho + b_\nu b_\rho\right) - g_{\nu\rho}\left(\nabla_\mu b_\sigma + b_\mu b_\sigma\right) + \\ + g_{\nu\sigma}\left(\nabla_\mu b_\rho + b_\mu b_\rho\right) - \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right) b^2$$

$$r_{\mu\nu} = R_{\mu\nu} + (d-2) b_\mu b_\nu + (d-2) \nabla_\mu b_\nu - (d-2) b^2 g_{\mu\nu} + \left(\nabla \cdot b\right) g_{\mu\nu}$$

$$r = R + 2(d-1)\left(\nabla \cdot b\right) - (d-1)(d-2) b^2$$

General method

1) Start with generally covariant expression *X*
2) Weyl covariantize all derivatives in your expression $\nabla_\mu \rightarrow D_\mu$
3) Impose condition of vanishing Weyl gauge field strength $B_{\mu\nu}=0$
4) Add all possible Weyl covariant terms T_i with the same weight *w* constructed out of Weyl curvatures, Weyl derivatives and original fields in *X* $X+a_i T_i$ $w(T_i)=w(X)$
   2 Weyl derivatives *D* ⇒ one Weyl curvature
5) Expand resulting sum in Weyl gauge potentials – b_μ fields
6) Search for coefficients a_i and *w*, s.t. Weyl potentials decouple and terms with b_μ vanish
7) If a solution exists, then *X* conformally coupled to spacetime is given by $X+a_i T_i$

General method

- Linear algebra problem
- Applicable to any expression w/o dimensionful constants
- Works for fields with any spin and in any spacetime dimension
- Found conformal weights w agree with canonical dimensions of fields as assigned on flat spacetime
- Shows obvious links with more general gravitation-dilaton-conformal theory (Weyl gauge theory of gravitation)

Applications

- Scalar field with 4 derivatives $L=\phi\Box^2\phi$ $\Box^2\phi=0$
- Old result from *d=4*, when *w(ϕ)=0*

$$\Box^2\phi+\frac{1}{3}(\nabla_\mu R)(\nabla^\mu\phi)+2\,\mathrm{Ric}^{\mu\nu}\nabla_\mu\nabla_\nu\phi-\frac{2}{3}R\Box\phi=0$$

- In general dimension 3 new terms must be added

$$r^2\phi \qquad \mathrm{ric}^2\phi \qquad (\Box r)\phi \qquad \Delta_{(4)}\phi=0$$

- General result

$$\Delta_{(4)}\phi=\Box^2\phi+A\,R^2\phi+B\,\mathrm{Ric}^2\phi+C(\Box R)\phi+D(\nabla_\mu R)(\nabla^\mu\phi)+E\,\mathrm{Ric}^{\mu\nu}\nabla_\mu\nabla_\nu\phi+F\,R\Box\phi$$

- Dimension-dependent coefficients

$$A=\frac{(d-4)(d^3-4d^2+16d-16)}{(4(d-1)(d-2))^2}\qquad B=\frac{4-d}{(d-2)^2}\qquad C=\frac{4-d}{4(d-1)}\qquad D=\frac{6-d}{2(d-1)}$$

$$E=\frac{4}{d-2}\qquad F=-\frac{(d-2)^2+4}{2(d-1)(d-2)}\qquad w=\frac{4-d}{2}$$

Applications 2

- Abelian gauge field in *d*=*6* with Minkowski spacetime action $F\nabla^2F$
  Gauge fields have no conformal weights $w(A_\mu)=w(F_{\mu\nu})=0$
  However ordinary cov. derivatives are not conf. cov.
  $$D_\mu A_\rho=\tilde{\nabla}_\mu A_\rho\neq\nabla_\mu A_\rho$$
- Conformally covariant lagrangian exists
  Terms to combine $F^{\rho\sigma}\Box F_{\rho\sigma}$ $F^{\rho\sigma}\nabla_\rho\nabla_\mu F^\mu{}_\sigma$
- In our method we add $r\,F^{\rho\sigma}F_{\rho\sigma}$ $\mathrm{ric}_{\rho\mu}F^{\rho\sigma}F^\mu{}_\sigma$
- Final conf. coupled lagrangian with weight *w*=-6 on curved spacetime
$$F^{\rho\sigma}\left(\Box-\frac{R}{10}\right)F_{\rho\sigma}-2F^{\rho\sigma}\left(\nabla_\rho\nabla_\mu+\frac{R_{\rho\mu}}{2}\right)F^\mu{}_\sigma$$

Summary

- Conformal couplings to curved spacetimes
- Weyl gravitation as a gauge theory with Weyl rescalings
- Weyl curvatures and Weyl geometry
- Decoupling of Weyl gauge potential as an useful method for finding conformally coupled lagrangians and equations
- Applications to old and new results

Thank you
for attention!