Problems with Non-Linearity in Modified Gravity

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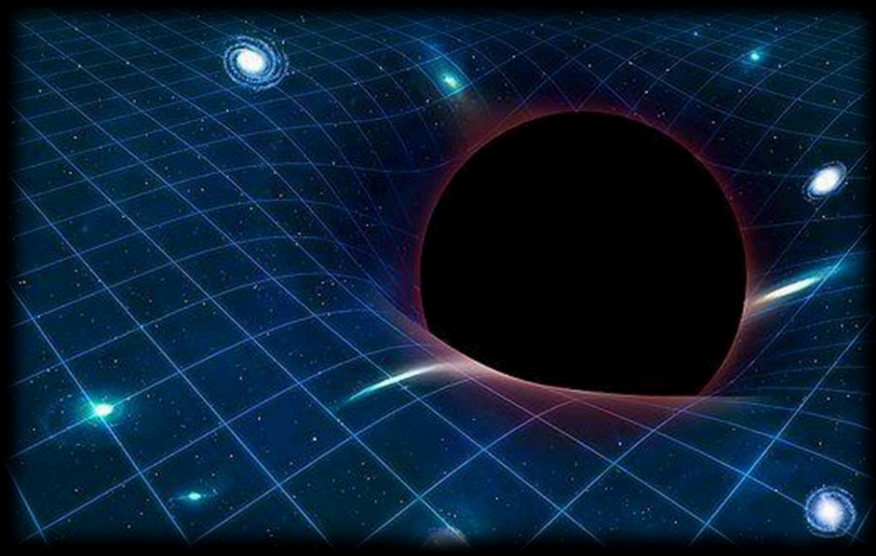
Talk Outline

- Introduction to Modified Gravity and Potential Problems due to Non-linearity.
- Example: Massive Gravity.
- Analogy: Nonlinear Proca Field.
- Example: f(T) Gravity [with introduction to teleparallel theories].



Talk Based On...

"Cosmological Perturbations of $f(T)$ Gravity Revisited", JCAP **06** (2013) 029, [1212.5774 [gr-qc]], joint work with Keisuke Izumi;

"Problems with Propagation and Time Evolution in $f(T)$ Gravity", To Appear on PRD, [1303.0993 [gr-qc]], joint work with Keisuke Izumi, James Nester, Pisin Chen;

"An Analysis of Characteristics in Non-Linear Massive Gravity", To Appear on CQG, [1304.0211 [hep-th]], joint work with Keisuke Izumi;

"Massive Gravity Acausality Redux", [1306.5457 [hep-th]], joint work with Stanley Deser, Andrew Waldron, Keisuke Izumi.

Modified Gravity or Missing Mass (Energy)?

One major motivation for considering modified gravity is the problem of dark matter [DM] and dark energy [DE].

History in perspective:

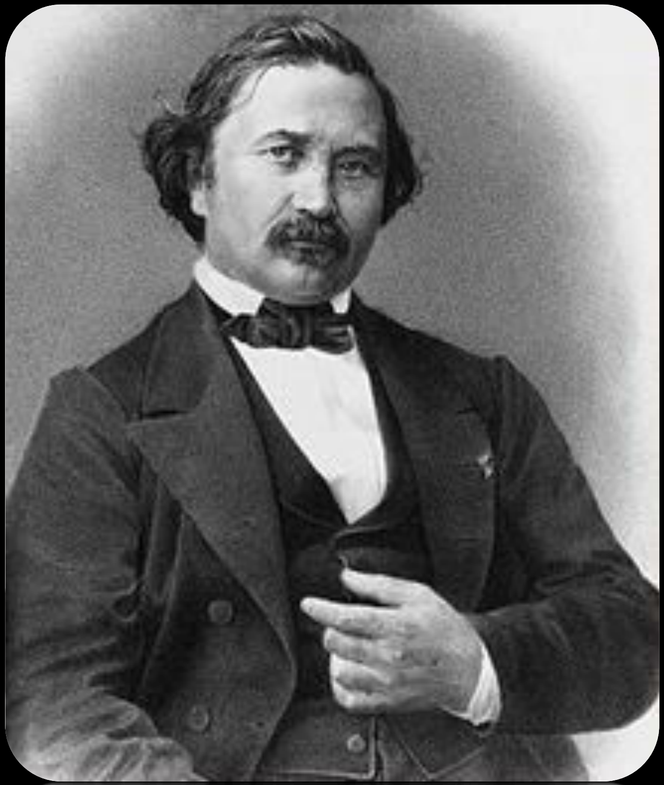
[1] Discovery of Neptune: Missing matter that perturbed orbit of Uranus. Airy believed it was due to breaking down of Newton's Law of Gravity. Victory of "dark matter".

[2] Precession of Mercury Perihelion: Search for missing planet "Vulcan" failed [despite several 'sightings']; explained by general relativity. Victory of "modified gravity".

The Risk of Modified Gravity

Bertrand's Theorem [1873]:

In 3-spatial dimension, there exist only two central force potentials that produce stable, closed orbits: inverse-square potential and radial harmonic potential.



Joseph Bertrand
[1822 - 1900]

Modified Gravity or Missing Mass (Energy)?

Is it possible to do away with dark matter using modified gravity?

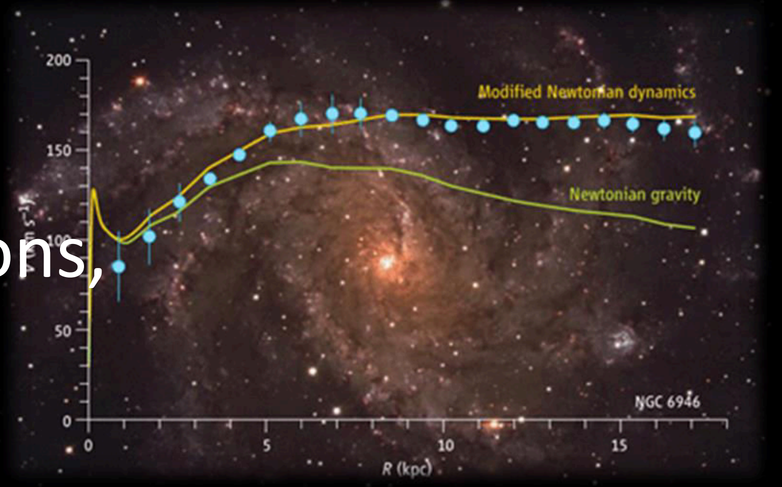
Not difficult to create a theory which could explain both the solar system and galactic observations.

However, all such theories must violate some principles that we trust for a good theory of gravitation: [Nester & Zhytnikov, PRL 1994].

[1] Positivity of energy,

[2] 1st or 2nd order field equations,

[3] Linear for weak fields.



The Right Kind of Non-Linearity

General Relativity is non-linear, but surprisingly well-behaved: E.g. Minkowski spacetime is stable.

Theorem [Choquet-Bruhat, Deser (1972)]:
Minkowski spacetime is linearly stable.

Theorem [Christodoulou, Klainerman (1993)]:
Minkowski spacetime is non-linearly stable.

More about the *right kind* of non-linearity later...

Non-Linearity in Modified Gravity

When one attempts to modify gravity, it is almost always inevitable to introduce *extra degrees of freedoms* [DoF]. In principle this is a good thing: hope to model DM and DE.

However, the theory needs to recover GR at linear level, in the regime where GR is well-tested. This means new DoF must be *suppressed*. The theory is likely to be very non-linear.

Not easy to excite new DoF without giving rise to nasty problems like superluminal modes!

Lesson from Massive Gravity

QFT point of view: Gravity is a theory of spin-2 particles: It is natural to ask if graviton could have nonzero mass, after all, we thought neutrino is massless…

Also, might help to explain cosmic acceleration.

Massive Gravity: Phase 1

- Fierz-Pauli Theory [1939]: Construction of theory of massive spin-2 particle . DoF = 5 = 2s+1.
- van Dam-Veltman-Zakharov (vDVZ) Discontinuity [1970]: massless limit does not recover GR; light-bending prediction off by *whopping* 25%.
- Vainshtein Mechanism [1972]: Force the theory to recover the correct limit that matches linearized general relativity.
- Bolware-Deser ghost [1972]: Non-linearity introduced by Vainshtein mechanism excites a 6th DOF – a ghost mode arises.

Massive Gravity: Phase 2

- dGRT (de Rham, Gabadadze, Tolley) Non-Linear Massive Gravity [2010]: Exorcise BD ghost by an *even more non-linear* theory. Is everything ok?

$$S = \frac{{M_{pl}}^2}{2} \int d^4x \sqrt{-g} \left[R - 2\Lambda + m_g^2(\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4)\right]$$

$$\mathcal{L}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2],$$

$$\mathcal{L}_3 = \frac{1}{3}([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]),$$

$$\mathcal{L}_4 = \frac{1}{12}([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4])$$

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \left(\sqrt{g^{-1}f}\right)^\mu_\nu$$

$g_{\mu\nu}$ = physical metric

$f_{\mu\nu}$ = fiducial metric

Massive Gravity: Phase 2

Various hints of superluminal propagations:

Gruzinov [1106.3972 [hep-th]];

Burrage, de Rham, Heisenberg, Tolley, JCAP **1207** (2012) 004, [1111.5549 [hep-th]];

de Fromont, de Rham, Heisenberg, Matas, [1303.0274 [hep-th]];

Chiang, Izumi, Chen, JCAP **12** (2012) 025, [1208.1222v2 [hep-th]].

Instability in Cosmological Solutions:

De Felice, Gumrukcuoglu, Mukohyama, Phys. Rev. Lett. **109** (2012) 171101, [1206.2080 [hep-th]];
De Felice, Gumrukcuoglu, Lin, Mukohyama; [1303.4154 [hep-th]]; [1304.0484 [hep-th]];
Kuhnel, [1208.1764 [gr-qc]].

Digression: Characteristic Analysis

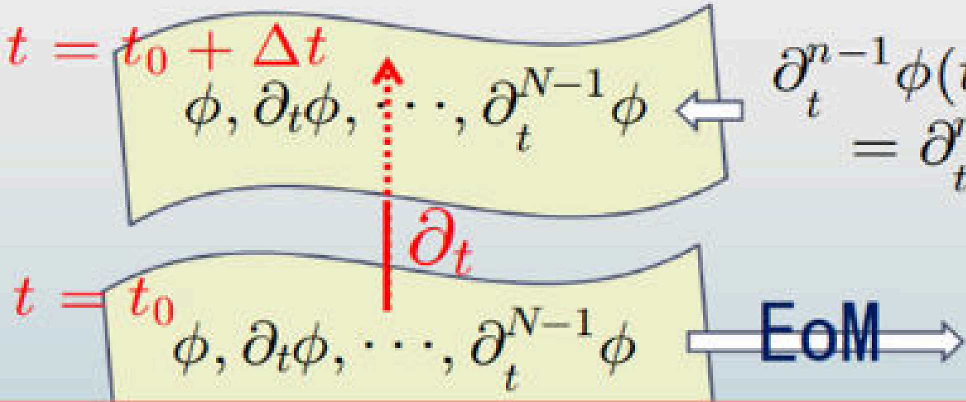
$$a_N^{\mu\nu\cdots\lambda}\partial_\mu\partial_\nu\cdots\partial_\lambda\phi + a_{N-1}^{\mu\nu\cdots\lambda}\partial_\mu\partial_\nu\cdots\partial_\lambda\phi + \cdots - V(\phi) = 0$$

is N

Time evolution from initial hypersurface

Decomposition of EoM

$$a^{tt\cdots t}\partial_t^N\phi + f(\partial_t^{N-1}\phi, \partial_t^{N-2}\phi, \cdots, \partial_t^{N-1}\partial_i\phi, \cdots, \phi) = 0$$



$$\partial_t^{n-1}\phi(t_0+\Delta t) = \partial_t^{n-1}\phi(t_0) + \partial_t^n\phi(t_0)\Delta t$$

$$\Uparrow$$

$$\partial_t^N\phi(t_0)$$

If $a^{tt\cdots t} = 0$, EoM becomes singular and $\partial_t^N\phi(t_0)$ can have any value.

K. Izumi's presentation slide

Digression: Characteristic Analysis

$$a_N^{\mu\nu\cdots\lambda}\partial_\mu\partial_\nu\cdots\partial_\lambda\phi + a_{N-1}^{\mu\nu\cdots\lambda}\partial_\mu\partial_\nu\cdots\partial_\lambda\phi + \cdots - V(\phi) = 0$$

$$b_M^{\mu\nu\cdots\lambda}\partial_\mu\partial_\nu\cdots\partial_\lambda\psi + b_{M-1}^{\mu\nu\cdots\lambda}\partial_\mu\partial_\nu\cdots\partial_\lambda\psi + \cdots - V(\psi) = 0$$

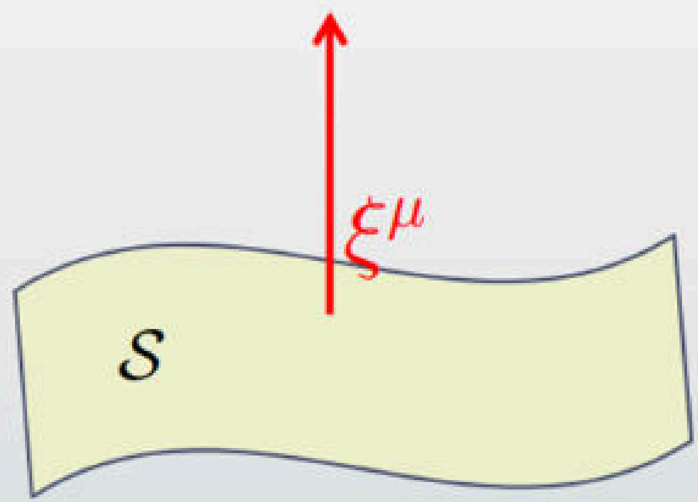
$$\vdots$$

If ξ^μ satisfies

$$a_N^{\mu\nu\cdots\lambda}\xi_\mu\xi_\nu\cdots\xi_\lambda\tilde{\phi} = 0$$

$$b_M^{\mu\nu\cdots\lambda}\xi_\mu\xi_\nu\cdots\xi_\lambda\tilde{\psi} = 0$$

$$\vdots$$



Hypersurface $\mathcal{S}$ is called characteristic hypersurface.

K. Izumi's presentation slide

Massive Gravity: Shock Analysis

Deser & Waldron [2012]: Found 2nd order superluminal shock waves, using eikonal approximation.

Due to *non-linearity that exorcizes BD Ghost!*

Izumi & Ong [2013]: Analyzed the structure of first order shocks, using full PDE analysis [Cauchy-Kovalevskaya Theorem]

Deser, Izumi, Ong, Waldron [2013]:

Proof of existence of first order superluminal shock, also improved analysis via spin connection. Existence for acausality established.

Note: There is no contradiction between ghostlessness and presence of superluminal mode.

How Superluminality Arises

Non-linear massive gravity has generically, *five* degrees of freedom, *but* in the second order action on open FLRW background we can see only *two* tensor degrees of freedom – where are the extra DoF’s? Naively, this is why we expect superluminality

$$\mathcal{L} = -f(\phi)\dot{\phi}^2 + g(\phi)(\partial_i \phi)^2$$

$$\Rightarrow \mathcal{L}_2 = -f(\phi)\dot{\delta\phi}^2 + g(\phi)(\partial_i \delta\phi)^2 + \cdots$$

No D.o.F. in linear analysis on $f(\phi) = 0$

Speed of sound : $c_s^2 = \frac{g(\phi)}{f(\phi)} \to \infty$

Example: Non-linear Proca Field

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m^2 A^\mu A_\mu - \frac{1}{4}\lambda\left(A^\mu A_\mu\right)^2$$

EOM: $\partial_\mu F^{\mu\nu} - m^2 A^\nu - \lambda A^\mu A_\mu A^\nu = 0$

Divergence yields $\left(m^2 + \lambda A^\mu A_\mu\right)\partial_\nu A^\nu + 2\lambda A^\mu A^\nu \partial_\nu A_\mu = 0$

Keep only the terms second order in derivatives and replace

$$\partial_\mu = k_\mu$$

We obtain the characteristic equation with k^μ being the normal to the characteristic hypersurface.

$$(m^2 + \lambda A^\nu A_\nu)k^\mu k_\mu - 2\lambda(k_\nu A^\nu)^2 = 0$$

Suppose $\lambda > 0$, and $k_\mu = (1,0,0,0)$. Write $A_\mu = (A_0, A_i)$, then

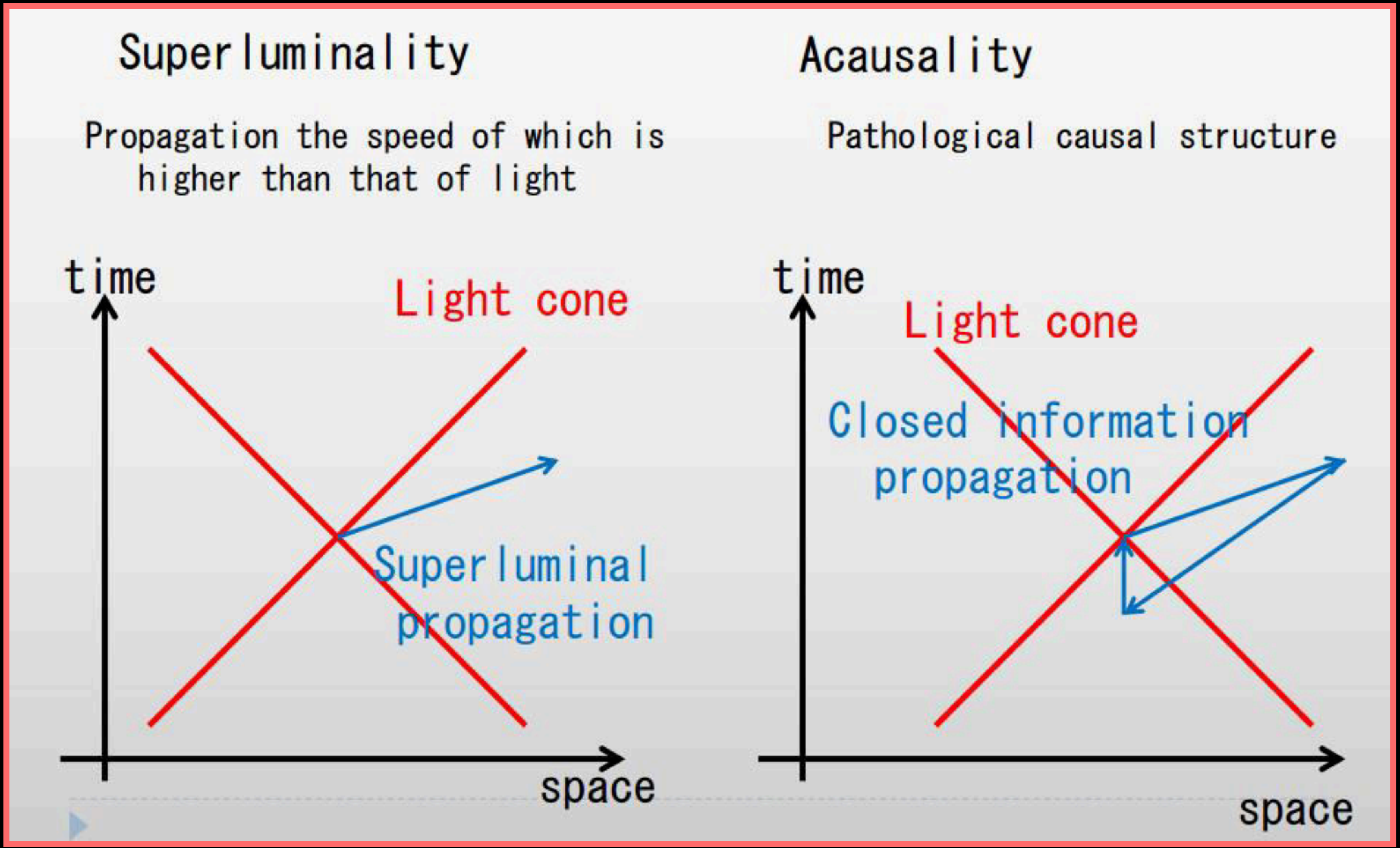
Example: Non-linear Proca Field

$$m^2 + \lambda\left(A^\mu A_\mu - 3A_0^2\right) = 0$$

$$\Rightarrow -3A_0^2 + A^i A_i = -\frac{m^2}{\lambda} < 0.$$

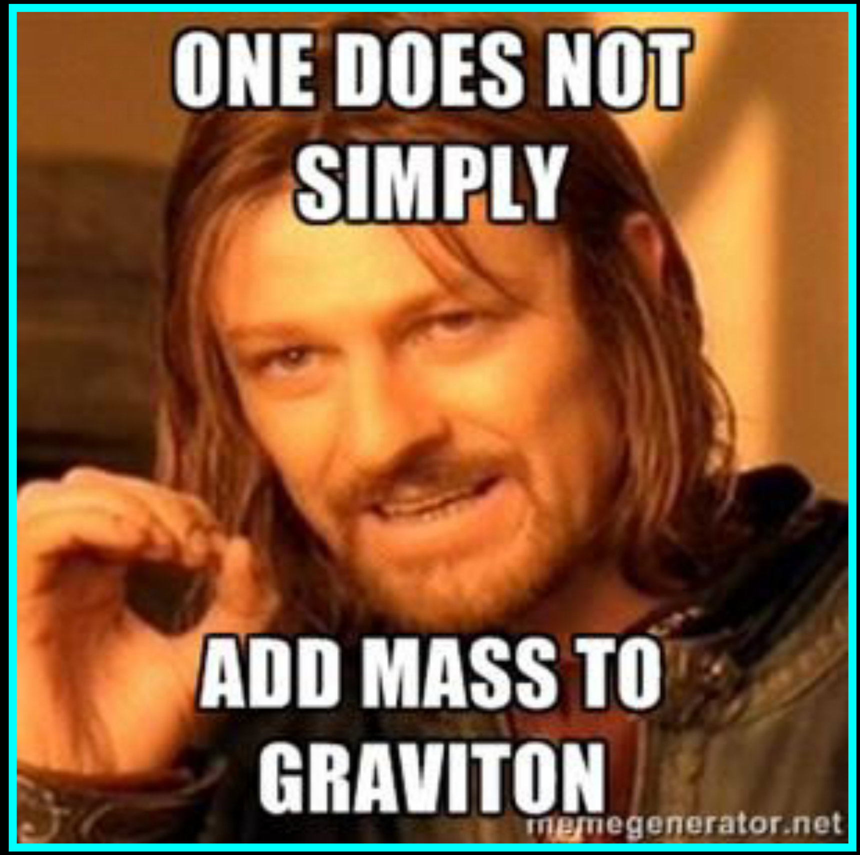
This is satisfied by *timelike* vector, i.e. the characteristic hypersurface is *spacelike*.

Superluminal vs. Acausal



Acausality in massive gravity is *local*, c.f. CTC in GR.

Conclusion on Massive Gravity



"The addition of a mass term is a brutality upon the beautiful structure of GR, and does not go unpunished."

– K. Hinterbichler, Theoretical Aspects of Massive Gravity [1105.3735v2 [hep-th]]

Non-Linearity in f(T) Gravity

We will now discuss an a priori very different theory of gravity, and see that the same sort of problem arises…

Teleparallel gravity: Originated from Einstein's attempt to unify general relativity and electromagnetism.

"In the tranquility of my sickness, I have laid a wonderful egg in the area of general relativity. Whether the bird that will hatch from it will be vital and long-lived only the gods know. So far I am blessing my sickness that has endowed me with it."

- Einstein, 1928.

On the Fame of Einstein

"You may be amused to hear that one of our great Department Stores [Selfridges] has pasted up in its window your paper [the six pages pasted up side by side] so that passers by can read it all through. Large crowds gather round to read it!"

[Eddington to Einstein, 11 February, 1929]

"Field Equations in Teleparallel Spacetime: Einstein's *Fernparallelismus* Approach Towards Unified Field Theory", [0405142v1 [physics.hist-ph]

The Mathematical Structure of Teleparallel Theories

Coordinate frame

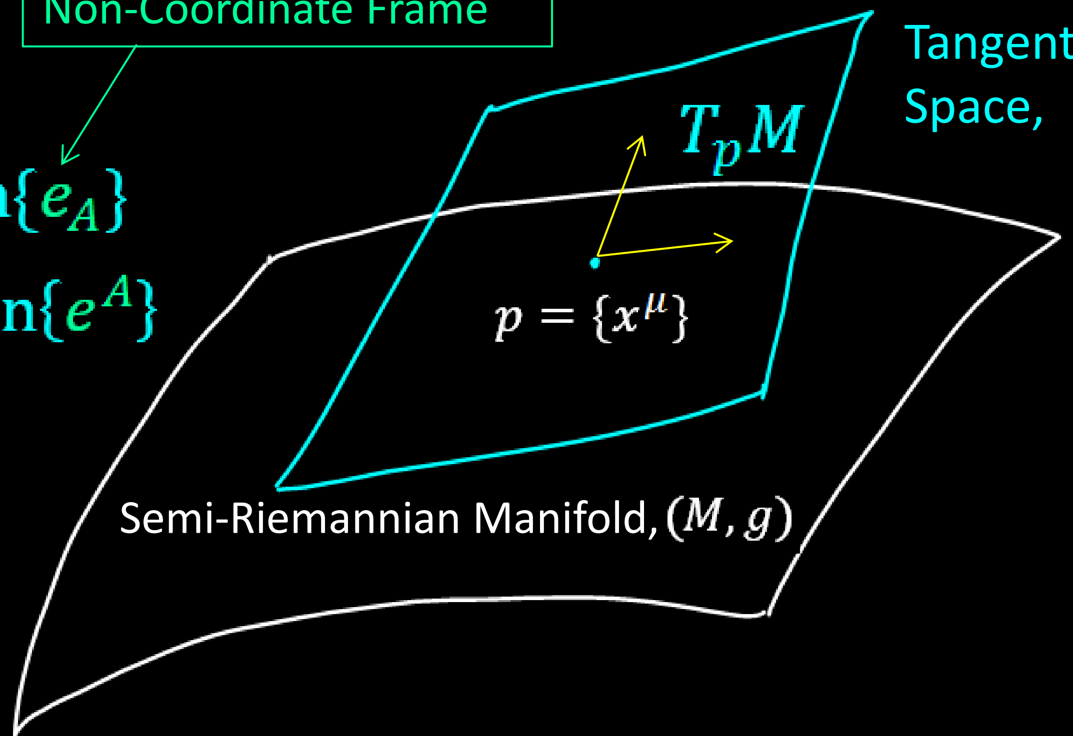
Non-Coordinate Frame

$$T_pM = \text{span}\{\partial_\mu\} = \text{span}\{e_A\}$$

$$T_p^*M = \text{span}\{dx^\mu\} = \text{span}\{e^A\}$$

$$\partial_\mu = e_A{}^\lambda \partial_\lambda$$

$$dx^\mu = e_\lambda{}^A dx^\lambda$$



Assuming that spacetime is parallelizable (i.e. there exist n vector fields $\{v_1, ..., v_n\}$ such that at *any* point $p \in M$ the tangent vectors $v_i|_p$'s provide a basis of the tangent space at p), we can view the mapping between the bases in coordinate frame $\{\partial_\mu\}$ to that of non-coordinate frame $\{e_A\}$ as an isomorphism $TM \to M \times \mathbb{R}^4$.

$$\eta(e_A, e_B) = \eta_{AB} = \text{diag}(-1,1,1,1)$$

$$g_{\mu\nu} = \eta_{AB} e_\mu{}^A e_\nu{}^B$$

$$T_pM \cong \mathbb{R}^{3,1} = (\mathbb{R}^4, \eta)$$

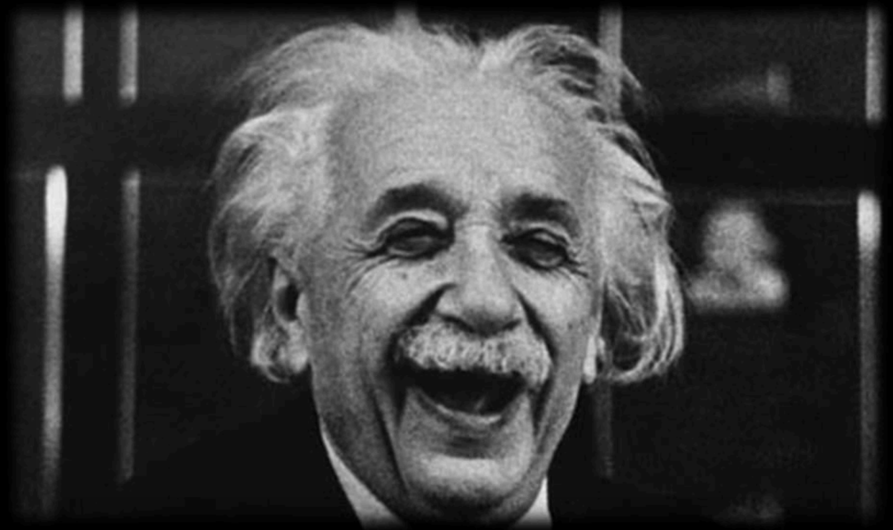
Notation:

Greek indices $[\mu, \nu, \ldots]$: Manifold coordinates

Small 'Middle' Latin indices $[i, j, \ldots]$: Manifold spatial coordinates

Capital 'Beginning' Latin indices $[A, B, \ldots]$: Fiber coordinates

Small 'Beginning' Latin indices $[a, b, \ldots]$: Fiber spatial coordinates



Weitzenböck connection:

$$\overset{w}{\nabla}_X Y \equiv (XY^A)e_A, \quad \text{where } Y = Y^A e_A.$$

Connection Coefficient

$$\overset{w}{\Gamma}{}^{\lambda}{}_{\mu\nu} = e_A^{\ \lambda}\partial_\nu e^A_{\ \mu} = -e^A_{\ \mu}\partial_\nu e_A^{\ \lambda}.$$

Roland Weitzenböck
(26 May 1885 – 24 July 1955)

Torsion:

$$\overset{w}{T}(X,Y) = \overset{w}{\nabla}_X Y - \overset{w}{\nabla}_Y X - [X,Y]$$
$$= X^A Y^B [e_A, e_B].$$

$$\overset{w}{T}{}^{\lambda}{}_{\mu\nu} \equiv \overset{w}{\Gamma}{}^{\lambda}{}_{\nu\mu} - \overset{w}{\Gamma}{}^{\lambda}{}_{\mu\nu} = e_A^{\ \lambda}(\partial_\mu e^A_{\ \nu} - \partial_\nu e^A_{\ \mu}) \neq 0.$$

Riemann Curvature Endomorphism:

$$\overset{w}{R}(X,Y)Z = \left(\overset{w}{\nabla}_X \overset{w}{\nabla}_Y - \overset{w}{\nabla}_Y \overset{w}{\nabla}_X - \overset{w}{\nabla}_{[X,Y]}\right) Z.$$

$$\Rightarrow \overset{w}{R}(e_A, e_B)e_C = \overset{w}{\nabla}_{e_A}(\overset{w}{\nabla}_{e_B} e_C) - \overset{w}{\nabla}_{e_B}(\overset{w}{\nabla}_{e_A} e_C) - \overset{w}{\nabla}_{[e_A,e_B]} e_C = 0$$

since

$$\overset{w}{\nabla}_X e_A = (X\delta^C_A)e_C = 0$$

That is, the geometry is *flat*.

Contortion:

The difference between the Weitzenböck connection coefficient and Christoffel symbol of Levi-Civita connection is a tensor:

$$\overset{w}{K}{}^{\mu\nu}{}_{\rho} = -\frac{1}{2}\left(\overset{w}{T}{}^{\mu\nu}{}_{\rho} - \overset{w}{T}{}^{\nu\mu}{}_{\rho} - \overset{w}{T}{}_{\rho}{}^{\mu\nu}\right).$$

Physics is Where the Action Is

Hilbert-Einstein action:

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R, \quad \kappa = 8\pi G,$$

can be re-written as TEGR action:

$$S = -\frac{1}{2\kappa}\int d^4x\ e\overset{w}{T},$$

since $R = -\overset{w}{T} - 2\nabla^{\mu}\overset{w}{T}{}^{\nu}{}_{\mu\nu}$

where $e = |\det(e^{A}{}_{\mu})|$, which is equal to $\sqrt{-g}$ in GR, and

$$\overset{w}{T} \equiv \overset{w}{S}{}_{\rho}{}^{\mu\nu}\overset{w}{T}{}^{\rho}{}_{\mu\nu},$$

$$\overset{w}{S}{}_{\rho}{}^{\mu\nu} \equiv \frac{1}{2}\left(\overset{w}{K}{}^{\mu\nu}{}_{\rho} + \delta^{\mu}_{\rho}\overset{w}{T}{}^{\alpha\nu}{}_{\alpha} - \delta^{\nu}_{\rho}\overset{w}{T}{}^{\alpha\mu}{}_{\alpha}\right).$$

Covariant derivative of Levi-Civita connection.

Note local Lorentz invariance is broken!

The "Torsion Scalar"

$$\overset{w}{T} = \boxed{\frac{1}{4}}\overset{w}{T}{}^{\rho}{}_{\eta\mu}\overset{w}{T}{}_{\rho}{}^{\eta\mu} + \boxed{\frac{1}{2}}\overset{w}{T}{}^{\rho}{}_{\mu\eta}\overset{w}{T}{}^{\eta\mu}{}_{\rho}\boxed{-\ 1}\overset{w}{T}{}_{\rho\mu}{}^{\rho}\overset{w}{T}{}^{\nu\mu}{}_{\nu}.$$

a b c

TEGR: $(a, b, c) = \left(\frac{1}{4}, \frac{1}{2}, -1\right)$

Solar system constraints:

K. Hayashi, T. Shirafuji, *New General Relativity*, Phys. Rev. D 19, 3524-3553 (1979).

$$|2a + b + c| \leq O(10^{-3}), |c + 1| \leq O(10^{-3})$$

$$\dashrightarrow 2a + b = 1, c = -1$$

OPTP: $(a, b, c) = \left(\frac{1 + 2\lambda}{4}, \frac{1 - 2\lambda}{2}, -1\right)$

↑ One-Parameter Teleparallel Theory

Teleparallel Theories and f(T)

- TEGR, OPTP

$$S = -\frac{1}{2\kappa}\int d^4x\; e\overset{w}{T},$$

- $f(T)$

$$S = -\frac{1}{2\kappa}\int d^4x\, e f(T)$$

We will suppress overscript w from now on.

Teleparallel theories have rich literature: Moller, Hayashi and Shirafuji, Hehl, Nester, Kawai, Itin, Obukhov, Maluf etc.

Is Parallelizability a Strong Demand?

The Following are Equivalent:

- A manifold *M* is parallelizable.
- The tangent bundle *TM* is trivial.
- The frame bundle *FM* has a global section.
- [In 4-dimension] Vanishing of the second Stiefel-Whitney characteristic class.

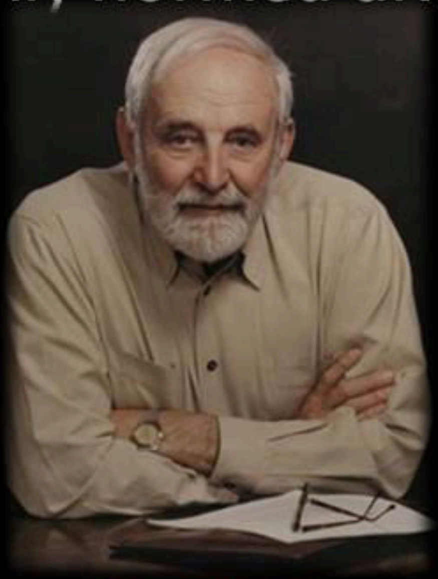
[In the general case the vanishing of the second characteristic classes of Stiefel- Whitney, Chern and Pontryagin are necessary but not sufficient conditions for a manifold to be parallelizable.]

Is Parallelizability a Strong Demand?

Fact: Many manifolds are *not* parallelizable. E.g. among the *n*-sphere, only S^0, S^1, S^3, and S^7 are parallelizable.

[R. Bott, J. Milnor, *On the Parallelizability of Spheres*, Bull. Amer. Math. Soc. 64 (1958)]

[The only normed division algebras are $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{O}$.]



Raoul Bott
(September 24, 1923 –
December 20, 2005)

John Milnor
(February 20, 1931 -)

Is Parallelizability a Strong Demand?

- Fact: All orientable 3-manifolds are parallelizable [Steenrod's theorem], consequently all 4-dimensional spacetimes with orientable spatial section are parallelizable.
- Fact: A non-compact 4-dimensional spacetime admits a spin structure if and only if it is parallelizable [Geroch's theorem].

Norman Steenrod
(April 22, 1910 – October 14, 1971)

Robert Geroch
(1 June 1942 -)

Why Teleparallel Theories?

- TEGR *was* considered to have advantages with regard to the identification of the energy-momentum of gravitating systems.

  [TEGR gauge current of gravitational energy is *tensorial* instead of *pseudo-tensorial* -- but it depends on choice of tetrads. Consequently, energy determined this way is this *quasi-local*, as in GR. But: Has nice property!]

- TEGR can be regarded as a gauge theory of local translations.

Remark on TEGR Gauge Current

Since the notion of quasi-local energy corresponds to non-isolated system where gravity could be very strong, there are some criteria for good definition [M.-T. Wang, Quasilocal Mass and Surface Hamiltonian in Spacetime, 1211.1407 [gr-qc]]:

[1] Good asymptotics: the limit should recover the ADM mass in the asymptotically flat case and the Bondi mass in the asymptotically null case. Most pseudotensorial quantities satisfy this -- asymptotics are weak field, only need to agree on linear level.

[2] Good behavior for *small* sphere limit in vacuum -- should recover the Bel-Robinson tensor up to 2nd order [Bonus: This gives a proof for positive energy theorem]. *None* of GR pseudotensors satisfy this! [Although artificial combinations do] However TEGR gauge current satisfies this!

Non-Linearity In f(T) Gravity

- Cosmological linear perturbation [on flat FLRW background] up to second order, does *not* see any extra degrees of freedom [Izumi & Ong]. The theory has generically 5 DoF's. [Li et. al, JHEP **07** (2011) 108, [1105.5934 [hep-th]]]
- Note: Expected to *not* see extra DoF's on background, but one really does not know what happens at perturbation level before calculation.
- This means that f(T) gravity is *very* non-linear.

$$\left[f_T M_A{}^{\mu\nu}{}_B{}^{\alpha\beta} + 2 f_{TT} S_A{}^{\mu\nu} S_B{}^{\alpha\beta}\right] k_\mu k_\alpha \tilde{e}^B{}_\beta = 0,$$

$$M_A{}^{\mu\nu}{}_B{}^{\alpha\beta} = \frac{\partial S_A{}^{\mu\nu}}{\partial T^B{}_{\alpha\beta}}$$

Non-Uniqueness of Evolution

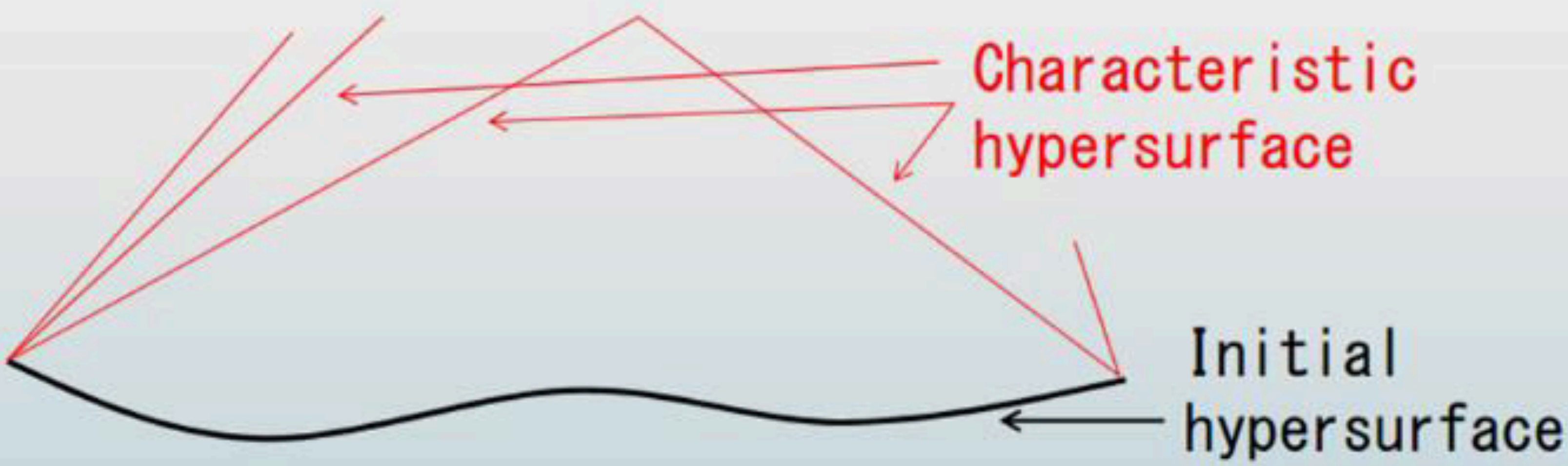
On characteristic hypersurface

$\partial_t^N \phi(t_0)$ can be any

➡ $\partial_t^{N-1} \phi(t_0 + \Delta t)$ can be any

Time evolution is not unique!!

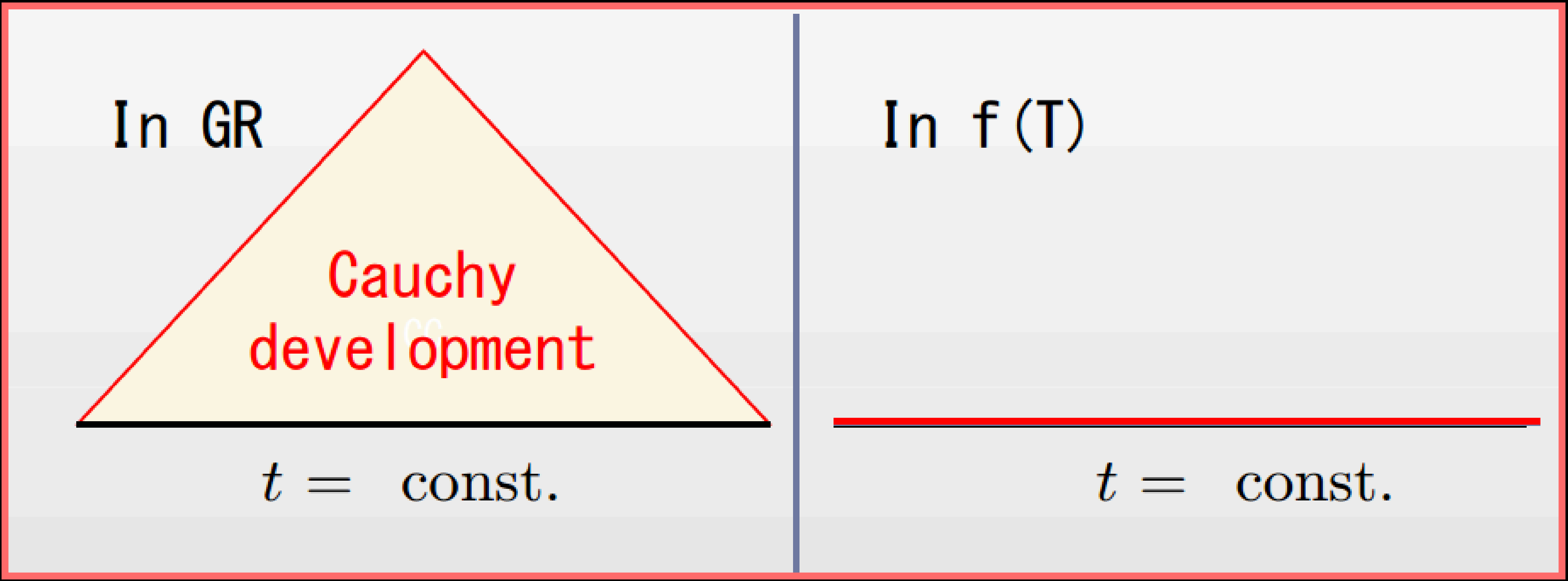
Cauchy horizon



K. Izumi's presentation slide

Non-Uniqueness of Evolution

In fact f(T) gravity allows superluminal propagation, and furthermore, Cauchy evolution is not well-posed, *even on Minkowski background!* [Ong, Izumi, Nester, Chen]

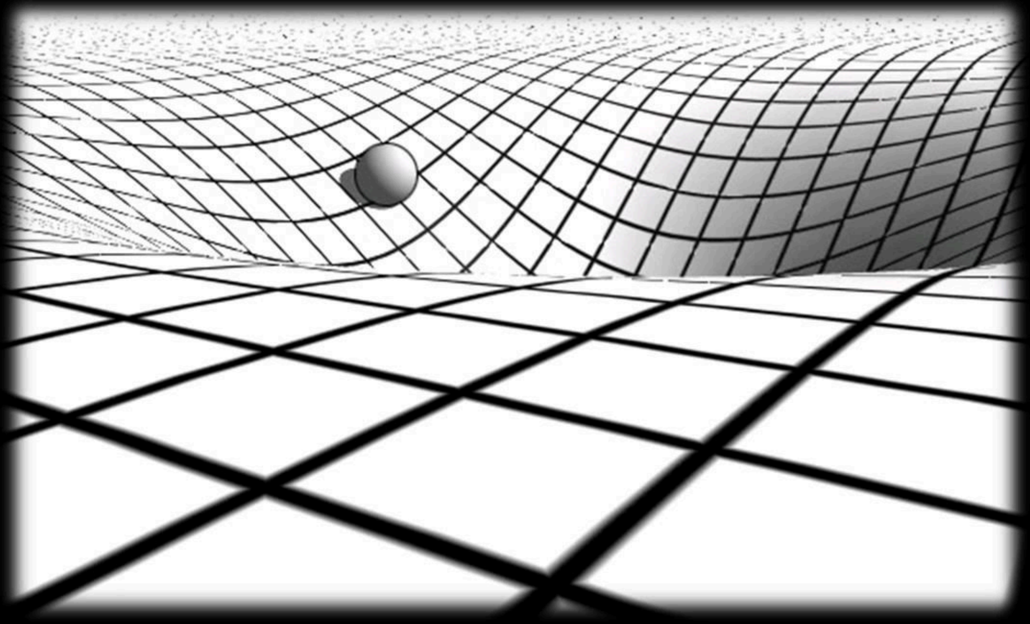


Conclusion

"Don't modify gravity, understand it!"

- Nima Arkani-Hamed

Or perhaps some *drastic* modification [new framework]?



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