Uniqueness of extreme horizons in 4-dimensional Einstein-Yang-Mills theory

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Introduction

1. Black hole entropy proportional to area of the horizon
2. Extreme black holes have zero surface gravity hence zero Hawking temperature
3. Easier to establish a microscopic description to extreme BH entropy
4. All SUSY black holes are extremal
5. All known extreme black holes have an AdS_2 factor in their near-horizon (NH) geometries
6. Existence of AdS_2 near-horizon symmetry enhancement proved for certain extreme black holes in various dimensions
7. It holds for extreme BH in any Einstein gravity theory coupled to arbitrary number of Maxwell fields and uncharged scalars in D$= 4, 5$ (Kunduri, Lucietti, Reall '07)

Introduction

1. Can we extend this to Einstein gravity coupled to non-abelian gauge fields?
2. No uniqueness theorem for extreme black holes
3. Black hole uniqueness does not apply to Einstein-Yang-Mills (EYM) black holes (Smoller, Wasserman, Yau '93)
4. 4-D $SU(2)$ EYM theory with $\Lambda < 0$ is a consistent truncation of 11-D SUGRA on S^7 (Pope '85)
5. Consider the simplest set up: D=4 Einstein-Yang-Mills with a compact semi-simple gauge group and a cosmological constant
6. Focus on stationary and $\Lambda \leq 0$, but local results remain valid for $\Lambda > 0$

Near-horizon geometry

1. Use Gaussian null coordinates (v, r, x^a) which are regular on the Killing horizon $\mathcal{N}$
2. Horizon at $r = 0$, x^a are coordinates on 2-d compact spatial cross-section H, $K = \frac{\partial}{\partial v}$ is Killing vector
3. In neighbourhood of $\mathcal{N}$ spacetime metric takes the form

$$g = rf(r,x)dv^2 + 2dvdr + 2rh_a(r,x)dvdx^a + \gamma_{ab}(r,x)dx^a dx^b$$

4. Extreme horizon $\kappa = 0$ implies $f(r,x) = rF(r,x)$
5. Consider diffeomorphism $v \to v/\epsilon$ and $r \to \epsilon r$ where $\epsilon > 0$
6. Near-horizon limit: take $\epsilon \to 0$ (Reall '02)

Near-horizon symmetry

1. So in NH limit extreme black hole metric is

$$g_{NH} = r^2 F(x) dv^2 + 2 dv dr + 2 r h_a(x) dv dx^a + \gamma_{ab}(x) dx^a dx^b$$

2. NH metric symmetries: $r \to \epsilon r$, $v \to v/\epsilon$ and $v \to v + c$ form a 2-d non-abelian isometry group

Non-abelian gauge fields near extreme horizon

1. Compact Lie group whose Lie algebra $\mathfrak{g}$ is semisimple
2. Thus $\mathfrak{g}$ admits a positive definite invariant metric given by $(A, B) \equiv \mathrm{Tr}(AB)$ for $A, B \in \mathfrak{g}$
3. Yang-Mills gauge field $\mathcal{A}$, field strength $\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]$
4. Gauge-covariant derivative $\mathcal{D}X = dX + [\mathcal{A}, X]$
5. Einstein-Yang-Mills equations are

$$\begin{aligned} R_{\mu\nu} &= 2\ \mathrm{Tr}\left(\mathcal{F}_{\mu}{}^{\delta}\mathcal{F}_{\nu\delta} - \frac{1}{4}g_{\mu\nu}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}\right) + \Lambda g_{\mu\nu} \\ \mathcal{D}\star\mathcal{F} &= 0 \end{aligned}$$

Non-abelian gauge fields near extreme horizon

1. Choose the gauge such that $\mathcal{L}_K \mathcal{A} = 0$ and $\mathcal{L}_K \mathcal{F} = 0$
2. Use residual gauge freedom to fix $\mathcal{A}_r = 0$, thus most general gauge field is
$$\mathcal{A} = \mathcal{W}(r,x)dv + \mathcal{A}_a(r,x)dx^a$$
3. $R_{\mu\nu}K^\mu K^\nu|_{\mathcal{N}} = 0$ allows us to recast EYM field equations as equations on H
4. "hat"denotes restriction of any quantity to H e.g. $\hat{\mathcal{A}} = \hat{\mathcal{A}}_a(x)dx^a$, $\hat{\mathcal{D}} = \hat{d} + [\hat{\mathcal{A}}, \cdot]$

Non-abelian gauge fields near extreme horizon

1. Define $\hat{E} = \partial_r \mathcal{W}|_{r=0}$, $\hat{G} = \star_2 \hat{\mathcal{F}}$ and $\hat{\mathcal{W}} = \mathcal{W}|_{r=0}$
2. Find $\hat{\mathcal{A}}_a, \hat{G}, \hat{E} \in Z_{\hat{\mathcal{W}}}$
3. $\mathcal{F}$ always admits NH limit; $\mathcal{A}$ only admits NH limit if $\hat{\mathcal{W}} = 0$

$$\begin{aligned} \mathcal{A}_{NH} &= \hat{E}(x) r dv + \hat{\mathcal{A}}_a(x) dx^a \\ \mathcal{F}_{NH} &= \hat{E}(x) dr \wedge dv - r \hat{\mathcal{D}}_a \hat{E} dv \wedge dx^a + \frac{1}{2} \hat{G}(x) \hat{\epsilon}_{ab} dx^a \wedge dx^b \end{aligned}$$

4. Regardless of the value of $\hat{\mathcal{W}}$, NH limit exists for the YM equation even though it contains $\mathcal{A}$ explicitly
5. Can use semi-simplicity of the algebra to show the problematic term $[\hat{\mathcal{W}}, \hat{\mathcal{A}}_{a,r}]$ always vanishes

Einstein-Yang-Mills Equations

1. Near-horizon EYM equations equivalent to the following set of equations defined purely on H for near-horizon data $(\hat{\gamma}_{ab}, \hat{h}_a, \hat{F}, \hat{E}, \hat{G})$:

$$\begin{aligned} \hat{R}_{ab} &= \frac{1}{2}\hat{h}_a\hat{h}_b - \hat{\nabla}_{(a}\hat{h}_{b)} + \Lambda\hat{\gamma}_{ab} + \mathrm{Tr}\left(\hat{E}^2 + \hat{G}^2\right)\hat{\gamma}_{ab} \\ \hat{F} &= \frac{1}{2}\hat{h}_a\hat{h}^a - \frac{1}{2}\hat{\nabla}_a\hat{h}^a + \Lambda - \mathrm{Tr}\left(\hat{E}^2 + \hat{G}^2\right) \\ \hat{\mathcal{D}}\hat{G} - \hat{h}\hat{G} &= \hat{\star}_2\left(\hat{\mathcal{D}}\hat{E} - \hat{h}\hat{E}\right) \end{aligned}$$

2. Also contracted Bianchi identity can be written as

$$\hat{\nabla}_a\hat{F} = \hat{F}\hat{h}_a + 2\hat{h}_b\hat{\nabla}_{[a}\hat{h}_{b]} - 2\mathrm{Tr}\left[\left(\hat{G}\hat{\epsilon}_{ab} + \hat{E}\hat{\gamma}_{ab}\right)\left(\hat{\mathcal{D}}^b\hat{E} - \hat{h}^b\hat{E}\right)\right]$$

3. Drop hats from now on

Stationary Near-Horizon Geometries

1. Rigidity: all stationary rotating black holes are axisymmetric
2. Stationary NH metric admits in addition $U(1)$ isometry, generated by Killing field m which commutes with K
3. m tangent to H so H can be S^2 or T^2; focus on S^2 for now
4. Can introduce coordinates (x, ϕ) on H for $x_1 < x < x_2$ such that $m = \partial/\partial\phi$ and parametrize NH metric as

$$
\begin{aligned}
\gamma_{ab} dx^a dx^b &= \frac{dx^2}{B(x)} + B(x) d\phi^2 \\
h_a dx^a &= \Gamma(x)^{-1} (Bk(x) d\phi - \Gamma'(x) dx)
\end{aligned}
$$

- where $B(x) > 0$ with $B(x_1) = B(x_2) = 0$ and $\Gamma(x) > 0$ everywhere
- smoothness requires $\phi \sim \phi + 2\pi$ and $B'(x_1) = -B'(x_2) = 2$

Stationary Near-Horizon Geometries

1. Can fix the gauge such that horizon gauge field is simply

$$\mathcal{A}_a dx^a = a(x) d\phi$$

2. It follows that $G(x) = a'(x)$

3. $\hat{R}_{x\phi}$ Einstein equation implies $k = \textit{constant}$

4. $k = 0$ corresponds to the static case

Stationary Near-Horizon Geometries

1. Define $A = \Gamma F - k^2 \Gamma^{-1} B$

2. Changing $r \to \Gamma(x) r$, NH metric now takes the form

$$g_{NH} = \Gamma(x)[A r^2 dv^2 + 2 dv dr] + \frac{1}{B} dx^2 + B(d\phi + k r dv)^2$$

3. Using YM equation x component of contracted BI simplifies to

$$B A' = 4 \Gamma \mathrm{Tr}(E[a, G])$$

4. Define obstruction term $T = \mathrm{Tr}(\Gamma E[a, \Gamma G])$

5. Define also $S = \Gamma^2 \mathrm{Tr}(E^2 + G^2)$

Symmetry enhancement

1. YM equation equivalent to

$$BS' = -4T \tag{1}$$
$$BT' = -\Gamma^2 \mathrm{Tr}\left([a, G]^2 + [a, E]^2\right) \tag{2}$$

2. Now define vector field $X = B\frac{\partial}{\partial x}$
3. X is globally defined on S^2 and vanish at the endpoints $x = x_1, x_2$
4. For any smooth function f on H, $X(f)$ is smooth everywhere and vanishes at $x = x_1, x_2$
5. (1) implies $T(x_1) = T(x_2) = 0$
6. (2) says $X(T)|_{x=x_1,x_2} = 0$ and $X(T) \leq 0$

Stationary Near-Horizon Geometries

1. Assume $\exists$ a point in $x_1 < x < x_2$ where $X(T) < 0$ so that $T' < 0$

2. Fundamental theorem of calculus then gives

$$T(x_2) - T(x_1) = \int_{x_1}^{x_2} T' dx < 0$$

3. This is a contradiction and we deduce that $T = 0$ identically and $A(x) = A_0$ is a constant

4. Integrate F Einstein equation to get the sign of A_0. For $\Lambda \leq 0$, $A_0 < 0$ and we prove the AdS_2 symmetry enhancement:

$$g_{NH} = \Gamma(x)[A_0 r^2 dv^2 + 2dvdr] + \frac{1}{B}dx^2 + B(d\phi + krdv)^2$$

Stationary Near-Horizon Geometrie

1. $T = 0$ also implies S is constant. This in turn implies all components in $\mathcal{A}_{NH}$ and $\mathcal{F}_{NH}$ commute
2. Classification reduces to Einstein-Maxwell case
3. Find from R_{ab} Einstein equation $\Gamma(x) = \frac{k^2}{\beta} + \frac{\beta x^2}{4}$ where $\beta > 0$ is a constant
4. Then $B(x) = \frac{P(X)}{\Gamma}$ where $P(x)$ is polynomial of order 4:

$$P(x) = -\frac{\Lambda \beta x^4}{12} + \left(A_0 - \frac{2k^2\Lambda}{\beta} \right) x^2 + \frac{4}{\beta^3}(\Lambda k^4 - A_0 \beta k^2 + S_0 \beta^2)$$

5. For $\Lambda \leq 0$ smoothness at end points implies $P(x)$ contains no odd power
6. Metric is isometric to NH limit of extreme Kerr-Newman(AdS)
7. Can integrate YM equation to determine E, G

Excluding $H = T^2$

1. Now both (x, ϕ) are periodic
2. NH metric takes the same form with the same $\Gamma(x)$ which is quadratic function of x
3. On the other hand $\Gamma(x)$ is globally defined on H therefore must be periodic
4. This is a contradiction thus we rule out T^2 (for any Λ)