Kerr Killing tensor and its conserved charge at $\mathcal{I}^+$

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Work done in collaboration with Abhay Ashtekar.

Outline

- Physical problem
- Killing vector fields and conserved quantities
- Asymptotic Kerr Killing tensor and conserved quantity
- Summary and Outlook

Kerr blackhole formation

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- Similarly ∂_ϕ gives Angular momentum.

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  - For non-trivial Killing tensors, NO general prescription!

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- Associate charge to Killing tensor as follows:
  $Q_K = \Sigma_i Q_{v_i} Q_{v_i}$ where $Q_{v_i} =$ Weyl part + Radiation part.

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- For Kerr, $Q_K = 4(Ma)^2$

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Future work

- Charge of Kerr Killing tensor at blackhole horizon.

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- Calculate rate of change of Carter's constant, solve for inspiralling geodesic in EMRI calculations.

THANK YOU!

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