Emergent electrodynamics in a ^{3}He-like system

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Introduction

- Non-relativistic (quantum) fermionic particles in interaction. Inspired by superfluid ^{3}He.
- Our goal: To analyze the low-energy excitations of the system (suitably defined) to check if they can be consistently described by a relativistic field theory, as it is suggested by the work of Grigori Volovik in ^{3}He.
- Techniques which lead to good descriptions of complex condensed matter systems.
- Warm-up to a more complex situation: gravitational interaction.

Plan

The model

Non-relativistic system composed by two different families of massless fermions in interaction:

$$\hat{\psi}_\alpha(t, x), \qquad \alpha = 1, 2$$

- Finite temperature T (Sec. 5): Euclidean time in an interval $[0, \beta] := [0, \hbar / k_B T]$.

Hamiltonian of the system in the second-quantization formalism in momentum representation:

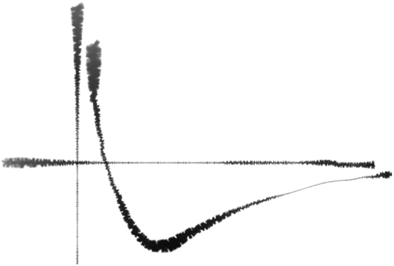
$$\hat{H} - \mu \hat{N} := \sum_{p,\alpha} \left(\frac{p^2}{2m} - \mu \right) a^{+}_{p\alpha} \hat{a}_{p\alpha} + \frac{1}{2} \frac{g}{p_{\mathrm{F}}^2} \sum_{p,p',\alpha,\beta} (p \cdot p') \hat{a}^{+}_{-p'\beta} \hat{a}^{+}_{p'\alpha} \hat{a}_{p\alpha} \hat{a}_{-p\beta}$$

- Annihilation and creation operators:

$$\hat{\psi}_{\alpha} = \sum_{p} \hat{a}_{p\alpha} e^{ip\cdot x}; \qquad \hat{a}_{p,\alpha}, \qquad \hat{a}^{+}_{p,\alpha}$$

- Chemical potential μ as independent variable instead of the number of particles N by practical convenience.
- Symmetry group: $\mathrm{U}(1) \times \mathrm{SO}(3) \times \mathrm{SU}(2)$.

- The second term represents the interaction between fermions. Inspired by ^{3}He, with a repulsive hard core and an attractive tail.



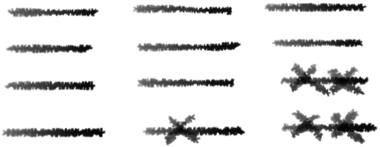
- A general interaction invariant under SO(3) is written in terms of Legendre polynomials:

$$V(|p - p'|) = \sum_l V_l(p, p') P_l(\hat{p} \cdot \hat{p}')$$

We have taken $V_{l \neq 1} = 0$.

The ground state of the free theory, $g = 0$, is the so-called Fermi sea:

- Filling up the one-particle energy levels:



- Fermi momentum: $p_{\mathrm{F}} := \sqrt{2m\mu}$.
- Excitations: Particle-hole pairs.

Interacting theory: $g \neq 0$.

- Cooper's problem: in we put two fermions over the Fermi sea, any small attractive interaction permits the formation of bound states. Compare with the situation in free space.

- Condensation of these bound states. When condensation occurs the system will exhibit anomalous mean values of pairs of fermionic operators such this one:

$$\left\langle \sum_p p \hat{a}_{p\alpha} \hat{a}_{-p\beta} \right\rangle \neq 0$$

- This quantity contains the information about the condensed part of the system. By the relevance of this condensation in what follows, we are going to devote the following section to study it.

Order parameter

The occurrence of a condensed phase implies a spontaneous symmetry breaking of the original symmetry of the Hamiltonian. Order parameter:

$$\Psi^i_{\alpha\beta} := \frac{g}{p_{\mathrm{F}}} \left\langle \sum_p p^i \hat{a}_{p\alpha} \hat{a}_{-p\beta} \right\rangle$$

- This quantity is symmetric in the internal indices, so it can be written without loss of generality in terms of Pauli matrices, and a quantity with two indices d^{ai},

$$\Psi^i_{\alpha\beta} = i(\sigma_a \sigma_2)_{\alpha\beta} d^{ai}$$

  d^{ai} is a complex vector in both internal and position space.
- The possible structures of the order parameter can be found by a minimization principle.

The order parameter for these two phases is given by the following expressions:

$$d^{ai}_{\text{planar}}(T) := \Delta(T)(\hat{s}^a \hat{m}^i + \hat{s}'^a \hat{n}^i)$$

$$d^{ai}_{\text{ABM}}(T) := \Delta(T)\hat{s}^a(\hat{m}^i + i\hat{n}^i)$$

- $\hat{m}$, $\hat{n}$, $\hat{s}$ and $\hat{s}'$ are unit vectors in position and spin space. Orthogonality conditions:

$$\hat{m} \cdot \hat{n} = 0, \qquad \hat{s}^* \cdot \hat{s}' = 0$$

- The scalar function $\Delta(T)$ is the gap parameter which contains the temperature dependence of the order parameter. At zero temperature its value is approximately:

$$\Delta_0 := \Delta(0) \simeq k_{\text{B}} T_{\text{C}}$$

Quasiparticle excitations

Heisenberg equations of motion of annihilation-creation operators under the influence of the condensed phase under a mean-field ansatz.

- Mean field:

$$\frac{1}{2}\frac{g}{p_{\mathrm{F}}^2}\sum_{p,p',\alpha,\beta}(p\cdot p')\hat{a}^+_{-p'\beta}\hat{a}^+_{p'\alpha}\hat{a}_{p\alpha}\hat{a}_{-p\beta}\simeq\frac{1}{2}\sum_{p',\alpha,\beta}\Psi^i_{\alpha\beta}p'_i\hat{a}^+_{-p'\beta}\hat{a}^+_{p'\alpha}+\mathrm{H.c.}$$

$$\frac{g}{p_{\mathrm{F}}}\sum_{p}p^i\hat{a}_{p\alpha}\hat{a}_{-p\beta}\simeq\Psi^i_{\alpha\beta}$$

Under this simplification the equations of motion are linear:

$$i\hbar\dot{a}_{p,\alpha} = \left(\frac{p^2}{2m} - \mu\right) a_{p,\alpha} + \frac{1}{p_{\mathrm{F}}} p \cdot \boldsymbol{\Psi}_{\alpha}{}^{\beta} a^{+}_{-p,\beta}$$

$$i\hbar\dot{a}^{+}_{-p,\alpha} = -\left(\frac{p^2}{2m} - \mu\right) a^{+}_{-p,\alpha} + \frac{1}{p_{\mathrm{F}}} p \cdot (\boldsymbol{\Psi}_{\alpha}{}^{\beta})^{*} a_{p,\beta}$$

- Notice the notation:

$$p \cdot \boldsymbol{\Psi}_{\alpha\beta} := p_i \Psi^{i}_{\alpha\beta}$$

- In this section we are going to analyze these equations for both planar and ABM states.

For the planar state, the evolution of the two fermionic families is decoupled. So let us take one of the indices first to perform the analysis of the equations:

$$i\hbar\dot{a}_{p,1} = \left(\frac{p^2}{2m} - \mu\right) a_{p,1} - c_\perp p \cdot (\hat{m} - i\hat{n}) a^+_{-p,1}$$
$$i\hbar\dot{a}^+_{-p,1} = -\left(\frac{p^2}{2m} - \mu\right) a^+_{-p,1} - c_\perp p(\hat{m} + i\hat{n}) a_{p,1}$$

- We have defined the orthogonal velocity $c_\perp = \Delta_0 / p_\mathrm{F}$.
- These equations can be written in compact form in terms of Pauli matrices:

$$i\hbar\partial_t \begin{pmatrix} a_{p,1} \\ a^+_{-p,1} \end{pmatrix} =$$
$$= \left(\frac{p^2}{2m} - \mu\right) \sigma_3 \begin{pmatrix} a_{p,1} \\ a^+_{-p,1} \end{pmatrix} - c_\perp p^1 \sigma_1 \begin{pmatrix} a_{p,1} \\ a^+_{-p,1} \end{pmatrix} - c_\perp p^2 \sigma_2 \begin{pmatrix} a_{p,1} \\ a^+_{-p,1} \end{pmatrix}$$

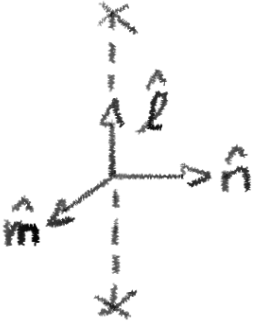
- Dispersion relation of plane-wave quasiparticles:

$$E^2(p) = \left(\frac{p^2}{2m} - \mu\right)^2 + c_\perp[(p_1)^2 + (p_2)^2] =$$

$$= \left(\frac{p^2}{2m} - \mu\right)^2 + c_\perp(p \times \hat{l})^2$$

- Two Fermi points:

$$p_\pm := \pm p_F \hat{l} \qquad\qquad \hat{l} := \hat{m} \times \hat{n}$$



No matter how low is the energy, there will be always excited quasiparticles near the Fermi points. We want to describe these low-energy excitations, whose momentum can be written in terms of the deviations from the Fermi points:

$$\pm p_{\mathrm{F}} \hat{l} + p$$

- The corresponding creation and annihilation operators are:

$$\alpha_{p,1,+} := a_{p_{\mathrm{F}} \hat{l} + p} \qquad \alpha_{p,1,-} := a_{-p_{\mathrm{F}} \hat{l} + p}$$

- Linearization around Fermi points:

$$\frac{p^2}{2m} - \mu \simeq c_{\|} \hat{l} \cdot (p - p_{\mathrm{F}} \hat{l}) \qquad c_{\|} := \frac{p_{\mathrm{F}}}{m}$$

- With the definitions:

$$\chi_{p,1} := \begin{pmatrix} \alpha_{p,1,+} \\ \alpha^{+}_{-p,1,-} \end{pmatrix}, \qquad \mathcal{H}_{p,1} := c_{\parallel} p^3 \sigma_3 - c_{\perp} p^2 \sigma_2 - c_{\perp} p^1 \sigma_1$$

  the linearized equations of motion are written as:

$$E_{p,1}\chi_{p,1} = \mathcal{H}_{p,1}\chi_{p,1}$$

- For the other index $\alpha = 2$ the situation is similar. The two indices can be combined in a composite field,

$$\chi := \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

  such that the equations of motion can be written as:

$$E_p \chi_p = e^a{}_b M^b \chi_p$$

$$E_p\chi_p = e^a{}_b M^b p_a \chi_p$$

- The only nonzero components of the tetrad are in the diagonal:

$$e^1{}_1 := c_\perp \qquad e^2{}_2 := c_\perp \qquad e^3{}_3 := c_\parallel$$

- The evolution equation contains 4×4 matrices which are written in terms of Pauli matrices:

$$M^1 = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad M^2 = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad M^3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

Now it is easy to find a matrix X such that the set

$$\{X, XM^1, XM^2, XM^3\}$$

is a representation of the Dirac matrices.

- A particular solution is given by:

$$X := \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$$

Then the low-energy evolution equations are equivalent a Dirac equation:

$$e^{\mu}{}_{I} \gamma^I p^{\mu} \chi_p = 0$$

We have defined:

$$p^0 := E_p \qquad e^0{}_0 := 1$$

To summarize the content of this section:

- In the planar state the occurrence of Fermi points imply that the low-energy excitations (those which are near the Fermi points) are described by a free, massless Dirac equation.

$$e^{\mu}{}_{I}\gamma^{I}p^{\mu}\chi_{p} = 0$$

- The components of the tetrad are constant so the corresponding spatiotemporal metric is flat.
- By performing a similar analysis, it can be shown that, in the ABM state, the situation is completely equivalent. Differences arise in the inhomogeneous situation.

Low-energy emergent properties

- Lorentz invariance:
  This emergent property is obtained when linearizing the equations of motion. The linearization

$$\frac{p^2}{2m} - \mu \simeq c_{\parallel} \hat{l} \cdot (p - p_{\mathrm{F}} \hat{l})$$

is a good approximation well below the following energy scale:

$$E \ll E_{\mathrm{L}} := m c_{\perp}^2$$

- Chirality:
  In the representation of the Dirac matrices which has been obtained, the chirality operator is given by

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

  One only needs to remember the definition of the Fermi field,

$$\chi := \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

  to realize that the two chiralities in the low-energy theory are nothing but the two families of fermions considered in the original theory.

- Charge:
  This is the conserved charge associated to global U(1) transformations of the fermionic fields. The concrete expression for the charge operator is given by

$$\hat{Q} := \hat{N}_+ - \hat{N}_-$$

  - $\hat{N}_\pm :=$ number operator of particles near the Fermi point $\pm p_F \hat{l}$
  - The notion of charge is a combination of the symmetries of the original theory and the nontrivial topology of its vacuum (Fermi points).

The inhomogeneous situation

Up to now we have been working in situations in which the order parameter is homogeneous. However, one can consider local variations of the order parameter, as long as these variations develop over a scale which is large compared to the effective size of a Cooper pair, or healing length.

$$\xi_0 := \frac{\hbar p_{\mathrm{F}}}{\pi m k_{\mathrm{B}} T_{\mathrm{C}}}$$

Quasiparticle evolution equations in the inhomogeneous situation. Differential equation in position space:

$$i\hbar\partial_t\chi_1 = \mathcal{H}_1\chi_1$$

$$\mathcal{H}_1 := c_{\|}\sigma_3\hat{l}\cdot(-i\hbar\nabla - p_{\mathrm{F}}\hat{l}) - c_{\perp}\sigma_1\hat{m}\cdot(-i\hbar\nabla) - c_{\perp}\sigma_2\hat{n}\cdot(-i\hbar\nabla)$$

- In this expression, the vectors $\hat{m}(x), \hat{n}(x), \hat{l}(x)$ are perturbations with respect to an homogeneous vacuum state $\hat{m}_0, \hat{n}_0, \hat{l}_0$. The perturbation satisfies an orthogonality condition

  $$\delta\hat{l} := \hat{l} - \hat{l}_0 \ ; \qquad \delta\hat{l}\cdot\hat{l}_0 \simeq 0; \qquad \delta\hat{l}\cdot\hat{l} \simeq 0$$

  The same applies to $\delta\hat{m}$ and $\delta\hat{n}$.
- Also p_{F} varies from p_{F}^0, which can be interpreted as fluctuations of the local density (sound waves).

- Let us define a spinor field such that it contains only wavelengths associated to the deviations of momentum with respect to the Fermi points:

$$\tilde{\chi}_{\uparrow} =: \chi_{\uparrow} \exp[-i p_{\mathrm{F}}^{0} \hat{l}_{0} \cdot x]$$

- If we introduce this spinor in the evolution equation and keep only the first order in the perturbations and the deviations from the Fermi momentum, the only change in the equations of motion is the apparition of the covariant derivative

$$-i\hbar\boldsymbol{\nabla} + \nu A$$

where we have defined the vector field

$$A = \frac{1}{\nu} p_{\mathrm{F}} (\delta \hat{m} \cdot \hat{l}_0) \hat{m}_0 + \frac{1}{\nu} p_{\mathrm{F}} (\delta \hat{n} \cdot \hat{l}_0) \hat{n}_0 - \frac{1}{\nu} \delta p_{\mathrm{F}} \hat{l}_0$$

One can perform exactly the same exercise with the other internal index, $\alpha = 2$, and again combine the two components into a massless Dirac equation but, now, in presence of a gauge vector field A_μ:

$$i\hbar e^{\mu}{}_{I}\gamma^{I}\partial_{\mu}\tilde{\chi} - v e^{\mu}{}_{I}\gamma^{I}A_{\mu}\tilde{\chi} = 0, \qquad A_0 = 0$$

- In this way we see that the first effect of inhomogeneities in the low energy excitations is equivalent to the introduction of a gauge field.

- For the ABM state one has a similar result, but the coupling is now axial:

$$i\hbar e^{\mu}{}_{I}\gamma^{I}\partial_{\mu}\tilde{\chi} - \nu e^{\mu}{}_{I}\gamma^{I}\gamma^{5}B_{\mu}\tilde{\chi} = 0, \qquad B_{0} = 0$$

- Now we should study the dynamics of these inhomogeneities. Usually this is the delicate point in condensed-matter analogies. The natural candidate in the relativistic low-energy theory is the mechanism of induction of dynamics of Sakharov, which leads to standard electrodynamics as was discussed by Zel'dovich. Work in progress.

Conclusions

- We started with a non-relativistic theory of interacting fermions. We analyzed it in the regime in which condensation occurs by using usual techniques in condensed matter physics.

- The low-energy fermionic quasiparticles have a relativistic dispersion relation. The original fermionic degrees of freedom are rearranged in the form of a Dirac spinor, and the evolution equations in presence of inhomogeneities are written as a Dirac equation in presence of a vector potential. In this model relativistic invariance, chirality and gauge invariance are low-energy properties associated to the presence of Fermi points.

- The mechanism of induction of dynamics proposed by Sakharov points that the low-energy effective theory should ve equivalent to a relativistic theory of fermionic fields in interaction with gauge fields, in both planar and ABM states. The fermionic degrees of freedom are given by the quasiparticles, while the gauge fields are the Goldstone modes produced in the spontaneous symmetry breaking associated to the condensation.

- Additional work is needed to show that this picture is consistent. Specially, to justify that the mechanism proposed by Sakharov captures the relevant dynamics of the system in some regime.

Thank you for your attention.