Twisted Geometries and Secondary Constraints

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Table of Contents

Purpose and Program

Twisted Geometries: The Main Problem

Twisted geometries arise as a generalization of the Regge's geometries in a smeared version of GR motivated by LQG. Is there a consistent dynamics for these objects?

Addressing some dynamics' aspects

- It has been argued that the dynamics naturally selects the Regge subcase. We study a simplified hamiltonian dynamics and show that this is indeed the case.
- It this is true there will be important consequences for the spin-foam formalism

Program

1. LQG: basic aspects of the phase space and its twistorial structure
2. Twistor networks and *"twisted"* geometries - basic ideas
3. Toy-model for the study of the secondary constraints, geometrical interpretation

Loop Quantum Gravity
Twisted Geometries

Two roads to loop quantum gravity

Loop Quantum Gravity

↙ ↘

Path Integral

$$\langle q_{ab}|q'_{ab}\rangle = \int_{[g]} \mathcal{D}[g]\, e^{iS_{GR}([g])}$$

Hamiltonian

$$i\hbar \frac{\partial}{\partial t}\Psi(\phi) = \hat{H}\left(\phi, \frac{\delta}{\delta\phi}\right)\Psi(\phi)$$

Bianchi: Spinfoam Gravity: Progress and Perspective

Pawloski: Loop Quantum Gravity & Cosmology: a Primer

Thiemann: Foundations of Loop Quantum Gravity

Phase space of the smeared Loop Gravity

Canonical Analysis

- Thanks to the Dirac-Bergmann formalism, we can treat GR as an Hamiltonian constrained theory, usually starting from the Holst's action
- The arising structure leads to $SL(2,\mathbb{C})$ variables of the (Covariant) Loop Gravity

Conjugate Variables on the spatial hypersurface

$$\left\{\Pi^a_i(p), A^j_b(q)\right\} = \left\{\bar{\Pi}^a_i(p), \bar{A}^j_b(q)\right\} = \delta^a_b \delta^j_i \delta(p,q)$$

Smeared variables: HF Algebra on each link

$h[l] = \mathrm{Pexp}\left[-\int_l A\right] \in SL(2,\mathbb{C})$
$\Pi[l] = \int_{q\in l} h_{q\to p} \Pi_q h^{-1}_{q\to p} \in \mathfrak{sl}(2,\mathbb{C})$
$\Pi[l^{-1}] = -h[l]\Pi[l]h[l]^{-1} \equiv \underset{\sim}{\Pi}[l] \in \mathfrak{sl}(2,\mathbb{C})$

Loop Gravity's Phase Space on each link

$SL(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \cong T^* SL(2,\mathbb{C})$

Twistors and $T^*SL(2,\mathbb{C})$

Definition: a couple of spinors

- $\mathbb{T} := \mathbb{C}^2 \oplus \bar{\mathbb{C}}^{2*}$
- $Z \in \mathbb{T} \;:\; Z = (\omega^A, \bar{\pi}_{\dot{A}})$

$SL(2,\mathbb{C})$ - invariant symplectic structure

$$\{\pi_A, \omega^B\} = \delta_A^B = \{\underset{\sim}{\omega}_A, \underset{\sim}{\pi}^B\}$$

$\mathbb{T}^2$ carries a $T^*SL(2,\mathbb{C})$ representation - Area-Matching symplectic reduction

$$C \equiv \pi_A\omega^A - \underset{\sim}{\pi}_B\underset{\sim}{\omega}^B \overset{!}{\approx} 0 \qquad \Rightarrow \qquad \mathbb{T}^2/\!\!/C \cong T^*SL(2,\mathbb{C})$$

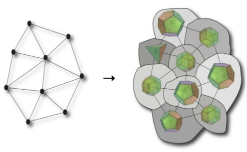
Finally: the twistorial representation of the HF Algebra on $T^*SL(2,\mathbb{C})$

$$\Pi^{AB} = -\frac{1}{2}\pi^{(A}\omega^{B)} \qquad \underset{\sim}{\Pi}^{AB} = \frac{1}{2}\underset{\sim}{\pi}^{(A}\underset{\sim}{\omega}^{B)} \qquad h^A_B = \frac{\underset{\sim}{\omega}^A\pi_B - \underset{\sim}{\pi}^A\omega_B}{\sqrt{\pi\omega}\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}}$$

The unfolding picture: Covariant Twisted Geometries

Geometrical Interpretation achieved through the closure constraint

- Locally flat polyhedra define a unique discrete metric. Curvature is smeared over the faces of the graph, dual to the edge of the triangulation



Twistor Space
$\downarrow$ *Area-Matching*
"Open" Twisted Geometries $\Leftrightarrow$ Loop Gravity Phase space
$\downarrow$ *Gauss' closure*
"Closed" Twisted Geometries $\Leftrightarrow$ Gauge-Inv Phase space
$\downarrow$ *Shape Matching*
Regge's Phase space

Twisted geometries: two comments on the role of the "mismatch"

- Regge geometries are "too rigid" to represent generic HF configuration. Regge's metric is piecewise-flat but continuous
- *Twisted* Geometries fully represent the *HF* algebra. The counterpart is that twisted geometries give a discrete and <u>discontinuous</u> metric

The unfolding picture: Covariant Twisted Geometries

Fixed Graph Truncation - Physical Meaning

On a fixed graph Γ the phase space of the Covariant Loop Gravity is $T^*SL(2,\mathbb{C})^L$
Speziale and Rovelli showed that fixed graph smearing is a truncation of the full GR to a finite number of degrees of freedom - PRD **82** 044018 (2010)

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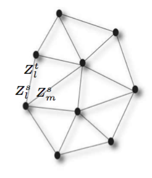
The picture

- So far, we have a graph where we attached a $\mathbb{T}^2$ on each link. These objects are called *Twistor Networks*. Imposing the area-matching we reach the phase space of the covariant loop gravity.



Covariant Loop Gravity's phase space

$\mathbb{T}^2 /\!/ C \simeq T^*SL(2,\mathbb{C})^L$

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- Imposing the Gauss constraint allows to bring in the geometrical interpretation as collection of polyhedra, locally flat.
- The geometries arising from this picture are quite different from the Regge geometries: they lack of the gluing conditions

The point: twistor networks and covariant twisted geometries

Twistorial formalism perfectly suit the LG structure

Twisted geometries are a generalization of the Regge geometries, they lack of the gluing conditions. In a finite d.o.f. truncation of covariant loop gravity, they completely represent the phase-space of the theory.

Secondary constraints and Twisted geometries

Twistors and classical loop gravity

Twistors' space

$$\mathbb{T}^2 \\ \downarrow \\ \mathbb{C} - \textit{Area Matching} \\ \downarrow \\ T^*SL(2,\mathbb{C})$$

Loop Gravity's phase-space

$SL(2,\mathbb{C})$	$\rightarrow$	*Constraints*	$\rightarrow$	*AB variables*
$\Downarrow$		*Smearing*		$\Downarrow$
$T^*SL(2,\mathbb{C})$	$\rightarrow$	*Constraints*	$\rightarrow$	$T^*SU(2)$

Constraints in the continuum

- Uniqueness of the metric structure, simple bi-vectors
- Torsionless constraint providing the embedding in the covariant space, $\Gamma = \Gamma(g)$

Smeared theory - opening the problem

- Primary: *"simple"* twistors, unique locally flat metric: (twisted) geometries
- Consistency conditions are an open question: discrete torsion? embedding of $T^*SU(2)$ in $T^*SL(2,\mathbb{C})$? discrete $\Gamma = \Gamma(E)$?

ArXiv:1207.6348 - Wieland, Speziale   2012 *Class. Quantum Grav.* **29** - Wieland

An issue: torsionless condition and secondary constraints in the discrete

The main question

Covariant twisted geometries represent the phase space of a truncation of LG:

1. Is there a consistent dynamics for these objects?
2. What is its relation with the Regge case? Role of the "mismatch"?

The idea by Dittrich and Ryan

- Matching conditions as secondary constraints. Mismatch could encode torsion and dynamics is Regge-type. *ArXiv:* 1209.4892 - Dittrich, Ryan
- They derive them through the discretization of the continuum theory, rather then from the study of a discrete Hamiltonian

A counterargument from Marseille

- The torsionless equation is about the connection, which in principle has nothing to do with the geometry or with the matching conditions
- Mismatch $\neq$ Torsion: *twisted* Levi-Civita connection
  PRD **87** 024038 (2013) Haggard, Rovelli, Wieland, Vidotto

An issue: torsionless condition and secondary constraints in the discrete

A conundrum arise

Do the twisted geometries have a consistent dynamics, or it is just a "kinematical" parametrization and the dynamics deal just with Regge geometries?

Is it so hard to solve it?

- *Pseudo*-constraints arise after the smearing of the theory
  Dittrich and Bahr (2009)
- Only the dynamics will have the last word

Our strategy

- Even in the discrete, if there is no curvature, the evolution is given by a constraint
- Search for secondary constraints in a toy-model imposing flatness

The model: ingredients

The model - Smearing over a graph with triangular faces

$$\mathcal{H} = \underbrace{\sum_{l} a_l C_l}_{Area\ Matching} + \underbrace{\sum_{l} \lambda_l D_l + b_l F_l^{(2)} + \underset{\sim}{b}_l \underset{\sim}{F}_l^{(2)}}_{Simplicity} + \underbrace{\sum_{k} g_k \vec{\mathcal{G}}_k}_{Gauss} + \underbrace{\sum_{f} N_f H_f}_{Hamiltonian}$$

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Primary Constraints

- Area-Matching

Physical meaning

- $\mathbb{T}^2 \rightarrow T^*SL(2,\mathbb{C})$

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Primary Constraints

- Area-Matching
- Simplicity Constraints
- Gauss Law - Closure

Physical meaning

- $\mathbb{T}^2 \rightarrow T^*SL(2,\mathbb{C})$
- "Simple" Twistors - Bivectors
- Polyhedra - Gauge Invariance

The "toy" part: scalar constraint

$$H_f = \Re\left[\mathrm{Tr}\left\{h_f - \mathbb{I}\right\}\right]$$

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The "toy" part: scalar constraint

$$H_f = \Re\left[\mathrm{Tr}\left\{h_f - \mathbb{I}\right\}\right]$$

We ask for zero (discrete) scalar curvature

$$h_f = h_{\alpha_{ab}} \approx \mathbb{I} + \frac{1}{2}\epsilon^2 F^i_{ab}\tau_i + \mathcal{O}(\epsilon^4)$$

Canonical analysis - Poisson Algebra

Dirac-Bergmann stability procedure: the logic

- Constraints' equations must hold through the evolution: *consistency conditions*

First-Class - Gauge generators

- $\vec{\mathcal{G}}_k$—Internal Gauge
- C_l - Conformal transformation

Second-Class

- $D_l \Leftrightarrow \{D_l, H_f\} \not\approx 0$
- $H_f \Leftrightarrow \{D_l, H_f\} \not\approx 0$
- $F_l^{(2)} \Leftrightarrow \left\{F_l^{(2)}, \bar{F}_l^{(2)}\right\} \not\approx 0$

Consistency conditions

Some constraints are second class. They may not be preserved under evolution

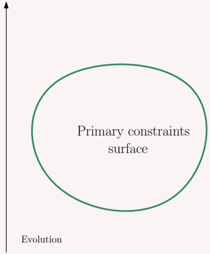
Canonical Analysis - Secondary constraints

Secondary constraints and simplicity constraints

Interesting secondary constraints arise from the consistency conditions of the diagonal part of the simplicity constraints, that is $\dot{D}_l \overset{!}{\approx} 0$

Secondary constraints: the standard guess

Often they are overlooked. One hope that imposing the primary constraints in some consistent way will assure they are preserved through the evolution.



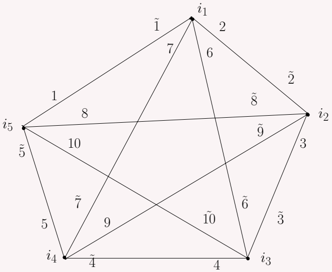
Secondary constraints

$$\dot{D}_l = \{\mathcal{H}, D_l\} \approx \sum_f N_f \{H_f, D_l\} \overset{!}{\approx} 0 \quad \Longleftrightarrow \quad \forall\, l, f \quad \{H_f, D_l\} \overset{!}{\approx} 0$$

Secondary Constraints - Solution

Strategy

1. Fix the graph for the smearing. We picked up the simplest: a 4-Simplex



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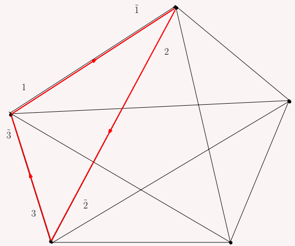
Strategy

1. Fix the graph for the smearing. We picked up the simplest: a 4-Simplex
2. It has 10 triangular independent faces. On each face there is a system of three equation coming from $\{H_f, D_l\}$ where $l = 1, 2, 3 \in \partial f$



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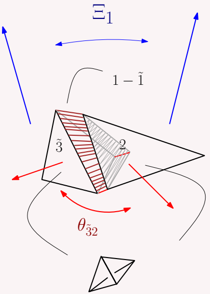

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Here the solution for Ξ_1, arising from the secondary constraints on the face $1-2-3$

$$\Xi_1 = \mathrm{Acosh}\left[\frac{\cosh\theta_{\underset{\sim}{23}} + \cosh\theta_{\underset{\sim}{31}}\cosh\theta_{\underset{\sim}{12}}}{\sinh\theta_{\underset{\sim}{31}}\sinh\theta_{\underset{\sim}{12}}}\right]$$

Reconstruction formula

Secondary Constraints - Solution

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Each link l is in the boundary of tree independent faces

$$\Xi_l^{(A)} = \Xi_l^{(B)} = \Xi_l^{(C)} \qquad \Longrightarrow \qquad \underline{\text{Shape} - \text{matching} \ \ \text{conditions}}$$

Final remarks

Twistor networks: summary

1. Gauge inv. phase space $\longleftrightarrow$ Twisted geometries
2. Piecewise-flat and discontinuous $3D$ geometries
3. Is there a dynamics, different from the Regge's one?

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Final statement

In a flatness toy-model, the twisted geometries correctly parametrize LQG phase-space **BUT** the dynamics select the Regge solutions through the secondary constraints

Resonance with Eugenio's lectures: EPRL and Spinfoam

Final remarks

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These are just preliminary results

1. Quantum theory
2. Improve the model

Thanks

Thank you!

Secondary Constraints - Geometry from twistors

Equations: just a taste, for the sake of understanding

$$\{H_{123}, D_1\} = \mathrm{Tr}\left[h_3 h_2 \widehat{h}_1\right] + \frac{\gamma+i}{\gamma-i}\overline{\mathrm{Tr}\left[h_3 h_2 \widehat{h}_1\right]} \overset{!}{\approx} 0 \qquad \widehat{h}_1 \equiv \{h_1, D_1\}$$

Geometry from twistors variables

We need to extract the geometrical information from an algebraic expression. This information will be used as an hint for solving the secondary constraints

$4D$ Geometry - Ξ_l angles

$$h_l\Big|_{F=0} = \frac{e^{-\frac{(1+i\gamma)}{2}\Xi}|z_l\rangle\langle z_l| + e^{\frac{(1+i\gamma)}{2}\Xi}|\tilde{z}_l][z_l|}{\sqrt{\langle z_l|z_l\rangle}\sqrt{\langle \tilde{z}_l|\tilde{z}_l\rangle}}$$

$2D$ and $3D$ Geometry - α^i_j and θ_{ij} angles

$$[z_i|z_j\rangle = \sqrt{\langle z_i|z_i\rangle\langle z_j|z_j\rangle}\,\sin\frac{\theta_{ij}}{2}\,e^{\frac{i}{2}\left(\alpha^i_j+\alpha^j_i\right)}$$

$$[z_i|z_j] = \sqrt{\langle z_i|z_i\rangle\langle z_j|z_j\rangle}\,\cos\frac{\theta_{ij}}{2}\,e^{\frac{i}{2}\left(\alpha^i_j-\alpha^j_i\right)}$$

ArXiv: 1305.3326 - Freidel, Hnybida