# QCD Thermodynamics With Continuum Extrapolated Wilson Fermions

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# Outline

# Quantum Chromodynamics

## Theory of Strong Nuclear Interaction

- SU(3) gauge theory
- *Asymptotic Freedom*
  - Coupling becomes small in short distance or large momentum transfer processes.
  - PT is successful.
- *Confinement*
  - Coupling becomes large at the low energy regime.
  - PT does not work.
- Non-perturbative methods are necesary:
  - Effective model calculations
  - Lattice field theory

## Path Integral in Eucledian Space-time

- Partition function:

$$Z = \int \mathcal{D}A_\mu \, \mathcal{D}\bar{\psi} \, \mathcal{D}\psi \, e^{-S[A_\mu,\bar{\psi},\psi]}$$

- Action:

$$S = \int \mathrm{d}^4x \left( \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} M \psi \right)$$

- An observable:

$$\langle \mathcal{O}(A_\mu, \bar{\psi}, \psi) \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \, \mathcal{D}\bar{\psi} \, \mathcal{D}\psi \, \mathcal{O}(A_\mu, \bar{\psi}, \psi) \, e^{-S[A_\mu,\bar{\psi},\psi]}$$

- We have to regularize these integrals.

## Discretization of Space-time

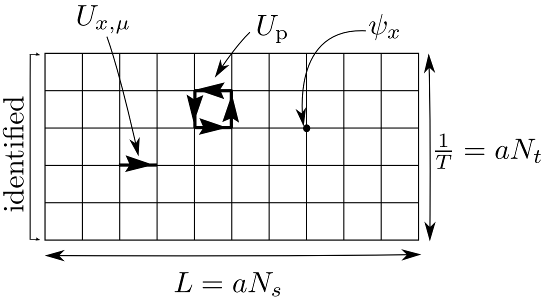
- 4D isotropic hypercubic grid:

$$\Lambda = a\,\mathbb{Z}^4 = \{x \,|\, x_\mu / a \in \mathbb{Z}\}$$

- lattice spacing: $a$, lattice size: $N_s^3 \times N_t$
- $N_t >> N_s$: zero temperature, $N_t < N_s$: finite temperature

### Physical Temperature

$$T = \frac{1}{a\,N_t}$$



## Parallel Transport

In continuum space-time:

$$\psi(x_1) \longrightarrow \mathcal{P}e^{-\int_{x_1}^{x_2} A_\mu(x')\mathrm{d}x'_\mu}\psi(x_1), \quad A_\mu(x) = -ig\sum_{b=1}^{8} A_\mu^b(x)\frac{\lambda^b}{2}$$

Link variables, $U \in \mathrm{SU}(3)$:

$$U(x, x+\hat{\mu}) \equiv U_{x,\mu} = e^{-aA_\mu} = 1 - aA_\mu(x) + \frac{a^2}{2}A_\mu(x)^2 + \ldots$$

$$U(x, x-\hat{\mu}) \equiv U_{x,-\mu} = U^\dagger_{x-\hat{\mu},\mu}$$

The *plaquette* variable is the only relevant operator:

$$U_\mathrm{p} = U_{x,\mu}U_{x+\hat{\mu},\nu}U^\dagger_{x+\hat{\nu},\mu}U^\dagger_{x,\nu}$$

# Gauge Fields on a Lattice

## SU(3) Wilson Gauge Action

$$
\begin{aligned}
S_{\mathrm{g}}^{\mathrm{W}}[U] &= -\beta \sum_{\mathrm{p}} \left\{ \frac{2}{2\,\mathrm{Tr}\,\mathbf{1}} \mathrm{Tr}\left( U_{\mathrm{p}} + U_{\mathrm{p}}^{\dagger} \right) - 1 \right\} \\
&= \beta \sum_{\mathrm{p}} \left( 1 - \frac{1}{3} \mathrm{Re}\,\mathrm{Tr}\, U_{\mathrm{p}} \right), \quad \beta = \frac{6}{g^2} \\
\lim_{a \to 0} S_{\mathrm{g}}^{\mathrm{W}} &= \int \mathrm{d}^4 x F_{\mu\nu} F_{\mu\nu} + \mathcal{O}(a^5)
\end{aligned}
$$

# Fermion Fields on a Lattice

## Naive Discretization

- $S_{\mathrm{f}}^{\mathrm{cont}} = \int \mathrm{d}^4x\, \bar\psi\, (\not{D} + m)\, \psi$
- $\bar\psi$, $\psi$ Grassmann variables
- $\partial_\mu \longrightarrow \frac{1}{2}\left(\Delta_\mu^{\mathrm{f}} + \Delta_\mu^{\mathrm{b}}\right)$ finite differences
- Naive fermion action:

$$S_{\mathrm{f}}^{\mathrm{N}} = \sum_{x,y\in\Lambda} \bar\psi_x \underbrace{\left(D_{xy} + am\delta_{xy}\right)}_{M_{xy}} \psi_y$$

$$D_{xy} = \frac{1}{2}\sum_{\mu=1}^{4} \gamma_\mu \left(U_{x,\mu}\delta_{x+\hat\mu,y} - U_{x-\hat\mu,\mu}^{\dagger}\delta_{x-\hat\mu,y}\right)$$

# The Doubling Problem

## Free Propagator ($U = 1$)

$$\tilde{G}^{-1}(p) = m + \frac{i}{a}\sum_{\mu} \gamma_\mu \sin p_\mu a$$

It has 16 poles in the Brillouin cell in the limit $a \to 0$.

## Nielsen–Ninomiya Theorem

$\nexists D$ lattice Dirac operator:

1. correct continuum limit ($\tilde{D}(p) = i\not{p} + \mathcal{O}(ap^2), \quad p << \pi/2$)
2. locality ($\|D_{xy}\| \leq c\, e^{-\lambda|x-y|}, \quad c, \lambda > 0$)
3. chiral symmetry ($\{D, \gamma_5\} = 0$)
4. no fermion doublers ($\exists\, \tilde{D}^{-1}(p), \quad p \neq 0$)

stands simultaneously.

# Solutions for the Doubling Problem

## Drop one condition from N–N theorem

- **Wilson discretization:** breaks chiral symmetry, restores in the continuum limit
- **staggered discretizaton:** possibly breaks locality
- **chiral fermions:**

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

(Ginsparg–Wilson relation)

## Computational Costs

staggered fermions $<<_{\times 10}$ Wilson fermions $<<_{\times 100}$ chiral fermions

## Wilson Fermions

- idea: $\bar{\psi}\,(\not{D} + m)\,\psi \;\longrightarrow\; \bar{\psi}\,(\not{D} + a\Box + m)\,\psi$
- Wilson action:

$$S_{\mathrm{f}}^{\mathrm{W}} = \sum_{x\in\Lambda}\left\{\bar{\psi}_x\psi_x - \kappa\sum_{\mu=\pm1}^{\pm4}\left[\bar{\psi}_{x+\hat{\mu}}\,(1+\gamma_\mu)\,U_{x,\mu}\psi_x\right]\right\},$$

$$\kappa = 1/(2am+8)$$

## Free Wilson Propagator ($U = 1$)

$$\tilde{G}^{-1}(p) = \frac{1}{a}\left(1 - 2\kappa\sum_\mu \cos p_\mu a + 2i\kappa\sum_\mu \gamma_\mu \sin p_\mu a\right)$$

15 doubler masses $\to\infty$ in the limit $a\to 0$.

## Staggered Fermions

- idea: “Half the Brillouin cell and double the lattice spacing.”
- staggered action:

$$S_{\mathrm{f}}^{\mathrm{stagg}} = \sum_{x\in\Lambda}\Big\{ am\bar{\psi}_x\psi_x + \frac{1}{2}\sum_{\mu=1}^{4}\alpha_{x,\mu}\left[\bar{\psi}_x U^{\dagger}_{x,\mu}\psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}}U_{x,\mu}\psi_x\right]\Big\}$$

- We still have 4 doublers.
- Partition function of $N_f$ flavour, locality!

$$Z = \int \prod_{x,\mu} \mathrm{d}U_{x,\mu}\; (\det M)^{N_f/4}\; e^{-S_{\mathrm{g}}}$$

## Comparison of staggered and Wilson discretization

| | **Wilson** | **staggered** |
|---|---|---|
| discretization effetcs | $\mathcal{O}(a)$ | $\mathcal{O}(a^2)$ |
| chiral symmetry | $\varnothing$ | U(1) |
| $m \to -m$ | not a symmetry | symmetry |
| quark mass renormalization | both additive and multiplicative | multiplicative |
| number of fermions | 1 | 4 |

# Simulation Points and Techniques

## Actions

- Symanzik tree-level improved gauge action.
- Six step stout smeared $\mathcal{O}(a)$ clover improved Wilson fermion action.

## Algorithms for $N_f = 2 + 1$ flavour simulations

- $m_u = m_d \equiv m_{ud}$ light quarks $\longrightarrow$ HMC
- $m_s$ strange quark $\longrightarrow$ RHMC

## Simulation Points

- 4 different lattice spacings
- Bare masses were tuned to: $m_\pi/m_\Omega = 0.326(4)$, $m_K/m_\Omega = 0.366(4)$
- $m_\pi \approx 545\,\text{MeV}$ and $m_K \approx 614\,\text{MeV}$
- Lattice sizes: $m_\pi L \gtrsim 8$
- The temperature was controlled with $N_t$.

## Simulation Parameters and Lattice Sizes

| $a$ [fm] | $am_{ud}$ | $am_s$ | $N_s$ | $N_t$ |
|---|---|---|---|---|
| 0.139(1) | −0.0985 | −0.0710 | 32 | 4 − 16, 32 |
| 0.093(1) | −0.0260 | −0.0115 | 32 | 4 − 16, 64 |
| 0.070(1) | −0.0111 | −0.0 | 48 | 8 − 28, 48 |
| 0.057(1) | −0.00336 | −0.0050 | 64 | 12 − 28, 64 |

Science **322** (2008) 1224-1227, `arXiv:0906.3599`

# Chiral Condensate of the Light Quarks

## Definition

$$\langle \bar{\psi}\psi \rangle = \frac{T}{V} \frac{\partial \ln Z}{\partial m_{ud}}$$

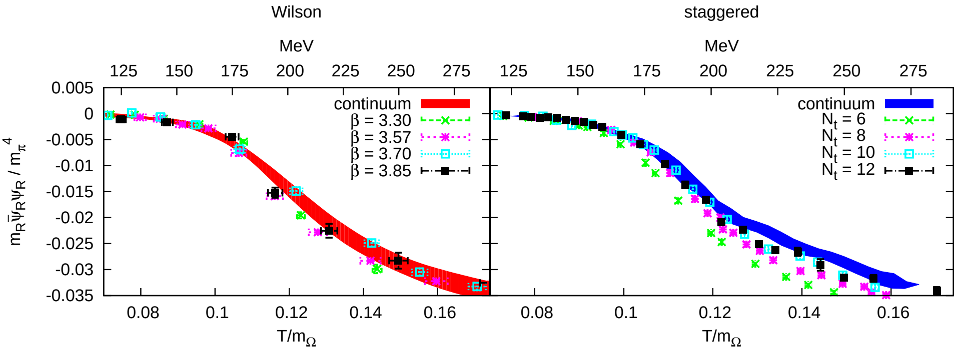
## Properties

- This is the order parameter of the chiral phase transition in the chiral limit.
- There are both additive (power-like) and multiplicative (logarithmic) divergences.

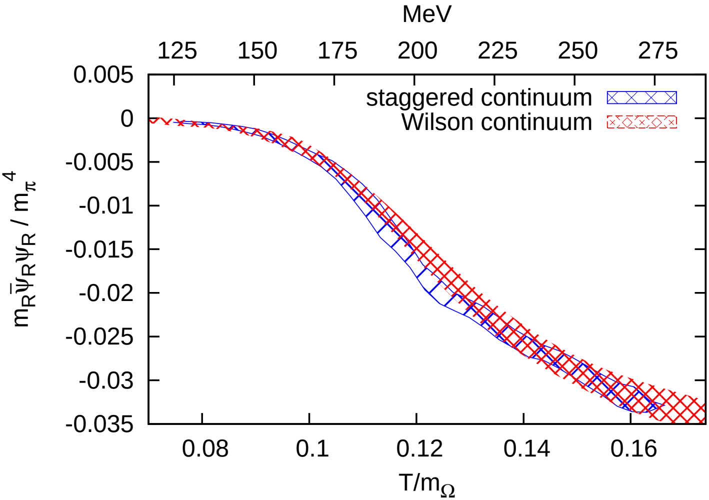
## Renormalization Group Invariant Quantities

$$\begin{aligned} m_{\mathrm{R}} \langle \bar{\psi}\psi \rangle_{\mathrm{R}}(T) &= 2N_f m_{\mathrm{PCAC}}^2 Z_A^2 \Delta_{PP}(T) \\ m_{\mathrm{R}} \langle \bar{\psi}\psi \rangle_{\mathrm{R}}(T) &= m_{\mathrm{PCAC}} Z_A \Delta_{\bar{\psi}\psi}(T) \end{aligned}$$

# Renormalized Chiral Condensate



# Continuum Limit of the Chiral Condensate



# Strange Quark Number Susceptibility

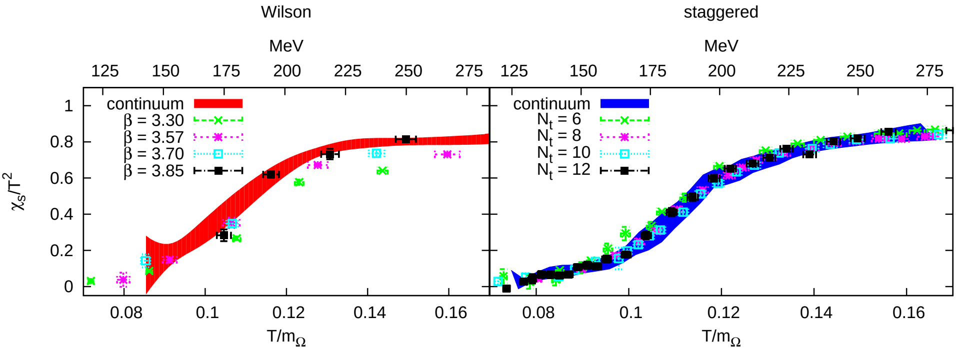
## Definition

$$\frac{\chi_s}{T^2} \equiv \frac{1}{T^2}\frac{\partial n_s}{\partial \mu_s} = \frac{1}{TV}\frac{\partial^2 \ln Z}{\partial \mu_s^2}\bigg|_{\mu_s=0}$$

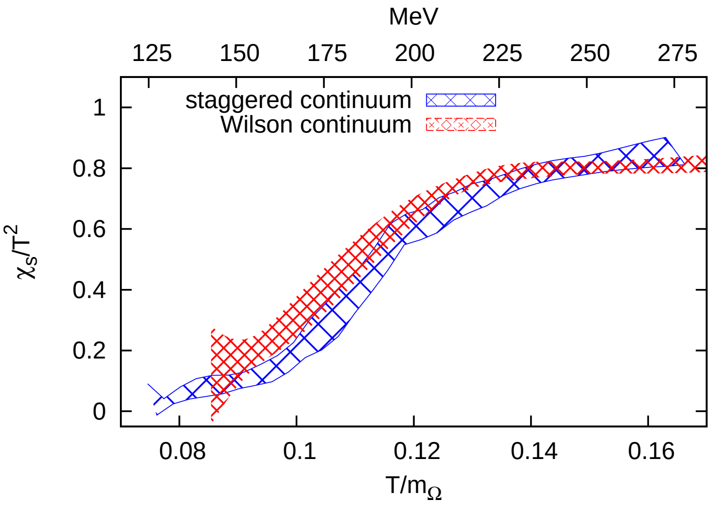
## Properties

- It measures strangeness fluctuations.
- There is no need for renormalization, the continuum limit is straightforward.

# Strange Quark Number Susceptibility



# Continuum Limit of the Quark Number Susceptibility



# Polyakov Loop

## Definition

$$L = \frac{1}{N_s^3} \sum_{x|x_4=0} \mathrm{Tr}\left( U_{x,4} U_{x+\hat{4},4} \cdots U_{x+(N_t-1)\hat{4},4} \right)$$

## Properties

- Pure gauge theory has a $Z_3$ symmetry which spontaneously breaks at high temperature. The Polyakov loop is the order parameter of the appropriate phase transition.
- Connection with the free energy of a quark-antiquark pair and with confinement:

$$|\langle L \rangle|^2 \propto e^{-\Delta F_{q\bar{q}}(r\to\infty)/T}$$

# Polyakov Loop

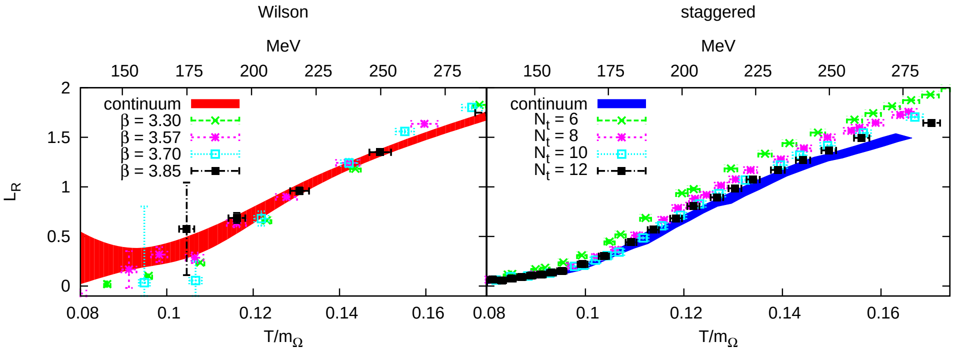
## Renormalization

- Additive divergences in $\Delta F_{q\bar{q}} \implies$ multiplicative divergences in $L$.
- Demand a fix $L_*$ value for fixed $T_* > T_\mathrm{c}$:

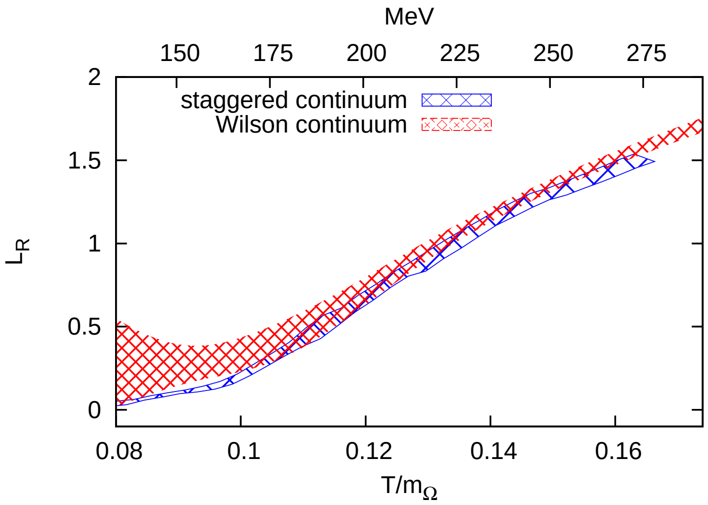
$$L_\mathrm{R}(T) = \left(\frac{L_*}{L_0(T_*)}\right)^{\frac{T_*}{T}} L_0(T)$$

We chose $T_* = 0.143 m_\Omega$ and $L_* = 1.2$.

# Renormalized Polyakov Loop



# Continuum Limit of the Polyakov Loop



# Summary

- Continuum extrapolation of results with Wilson fermions are possible.
- There is an agreement with the appropriate staggered results ($\approx 1\sigma$).

- Outlook
  - Lowering the light quark mass towards their physical value.
  - Testing our staggered and Wilson results with chiral fermion results. (`arXiv:1204.4089 [hep-lat]`)