Lecture I: From Quark-Gluon Plasma to Neutrino Decoupling

Jan Rafelski
Department of Physics, The University of ARIZONA, Tucson

Krakow School of Theoretical Physics, Zakopane, May 20/21/22, 2012

[i] **Intro remarks; Looking back at beginning of the Universe QGP: Early universe in LHC-ion experiments**

[ii] **Tutorials: Interdisciplinary Quiz; Statistical physics, quark-gluon QCD interactions, QGP, hadron gas, supercooling, chemistry**

supported by a grant from the U.S. Department of Energy, DE-FG03-95ER41318.

PARTICLES IN THE EARLY UNIVERSE

The understanding of equations of state of Quark-Gluon Plasma and Hadron Gas phase allows exploration of the conditions in which matter (protons, neutrons) formed.

We Need Preparation for Following Tasks

1) Describe in quantitative terms the chemical composition of the Universe before hadronization and at hadronization:

$$T \simeq 160\,\mathbf{MeV} \qquad t \simeq 30\mu s,$$

2) Understand the quark-hadron phase transformation dynamics, baryon number distillation;

3) Describe the composition of the Universe during evolution towards the condition of neutrino decoupling

$$T \simeq 1 - 3\,\mathbf{MeV} \qquad t \simeq 10\,s$$

4) Demonstrate that the Universe can be in chemical equilibrium during this period

5) Beyond Theory: we need experimental anchor points

Direct Probe of the Early Universe: Heavy Ion Collisions at LHC

RECREATE THE EARLY UNIVERSE IN LABORATORY:
Recreate and understand the high energy density conditions prevailing in the Universe when matter formed from elementary degrees of freedom (quarks, gluons) at about 30 μs after big bang.

QGP-Universe hadronization led to nearly matter-antimatter symmetric state, ensuing matter-antimatter annihilation yields 10^{-10} *matter asymmetry, the world around us.*

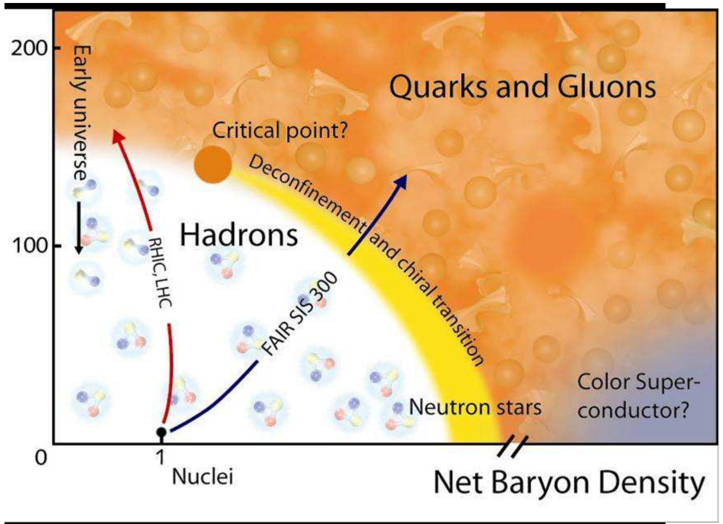
STRUCTURED VACUUM (Einsteins 1920+ Aether/Field/Universe)
Quantum dogma: The global vacuum state determines the prevailing state-dependent laws of nature. Demonstrate by changing from the vacuum of hadronic matter that of quark matter, understanding the changes in laws of physics.

ORIGIN OF MASS OF MATTER –DECONFINEMENT
The confining quark vacuum state is the origin of 99.9% of GRAVITATING visible matter mass, the Higgs mechanism applies to the remaining 0.1%. We need to be sure that the quantum zero-point energy of confined quarks is the mass of matter. First step: we 'melt' the vacuum structure setting quarks free.

What and where is deconfinement?

A domain of (space, time) of modified quantum vacuum much larger than normal hadron size in which color-charged quarks and gluons are propagating, constrained by external 'frozen vacuum' which abhors color. Matter as we know it does not exist. This is the early Universe strongly interacting phase of matter.

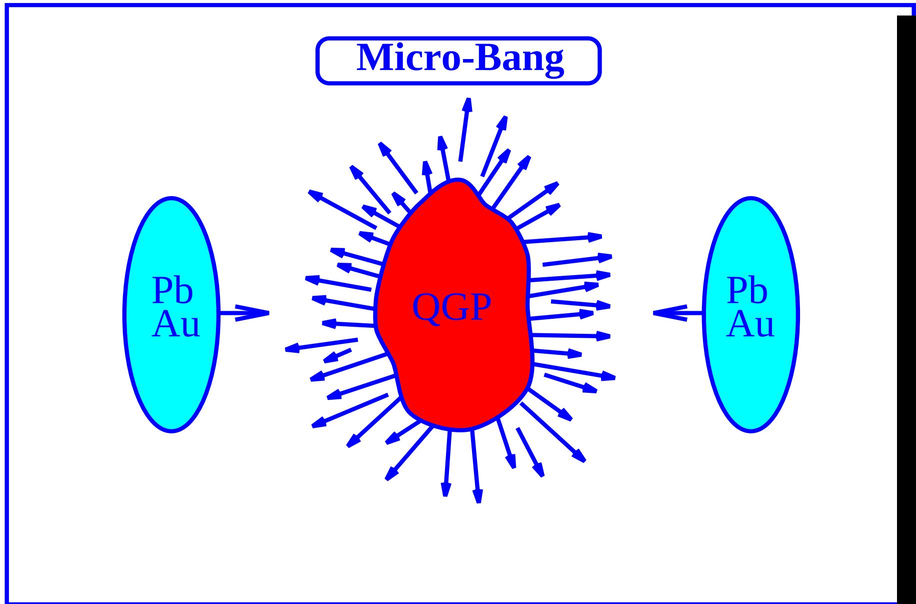
We expect a pronounced boundary in temperature and density between confined and deconfined phases of matter: phase diagram. Deconfinement expected at both: high temperature and at high matter density.



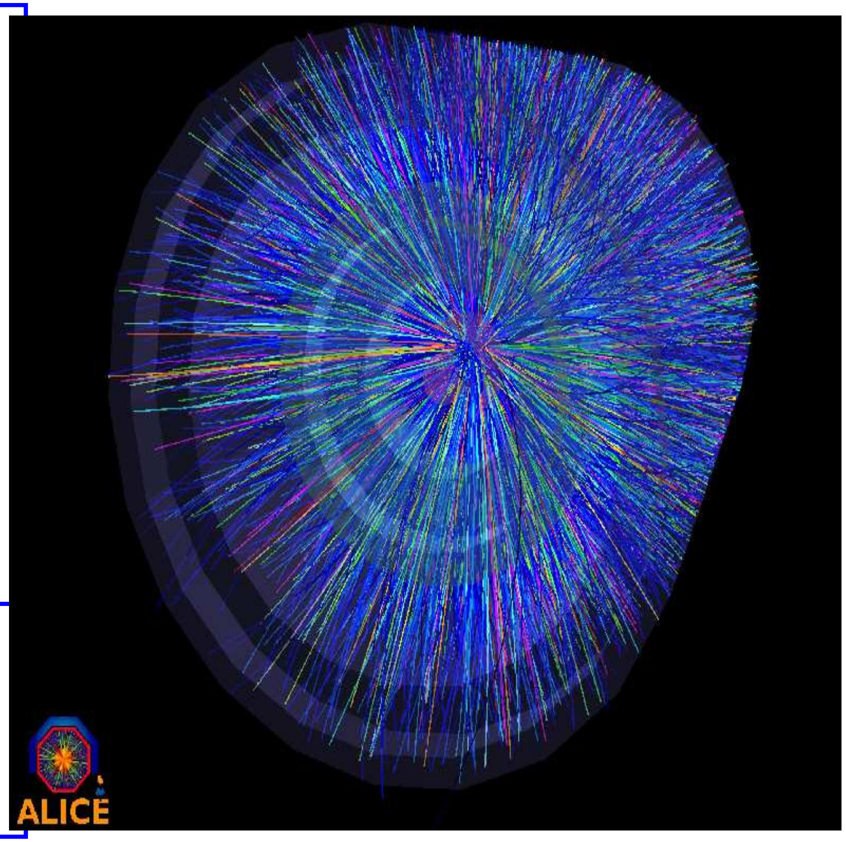
THEORY: What physics we need
Hot QCD in equilibrium (QGP from QCD-lattice) and out of chemical equilibrium

DECONFINEMENT NOT A 'NEW PARTICLE',
there is no good answer to journalists question:
How many new vacuua have you produced today?

RECREATING THE EARLY UNIVERSE: ENERGY TO MATTER



Big-Bang	Micro-Bang
$\tau \simeq 10\mu s$ $N_B / N \simeq 10^{-10}$	$\tau \simeq 4\ 10^{-23} s$ $N_B / N \simeq 0.1$

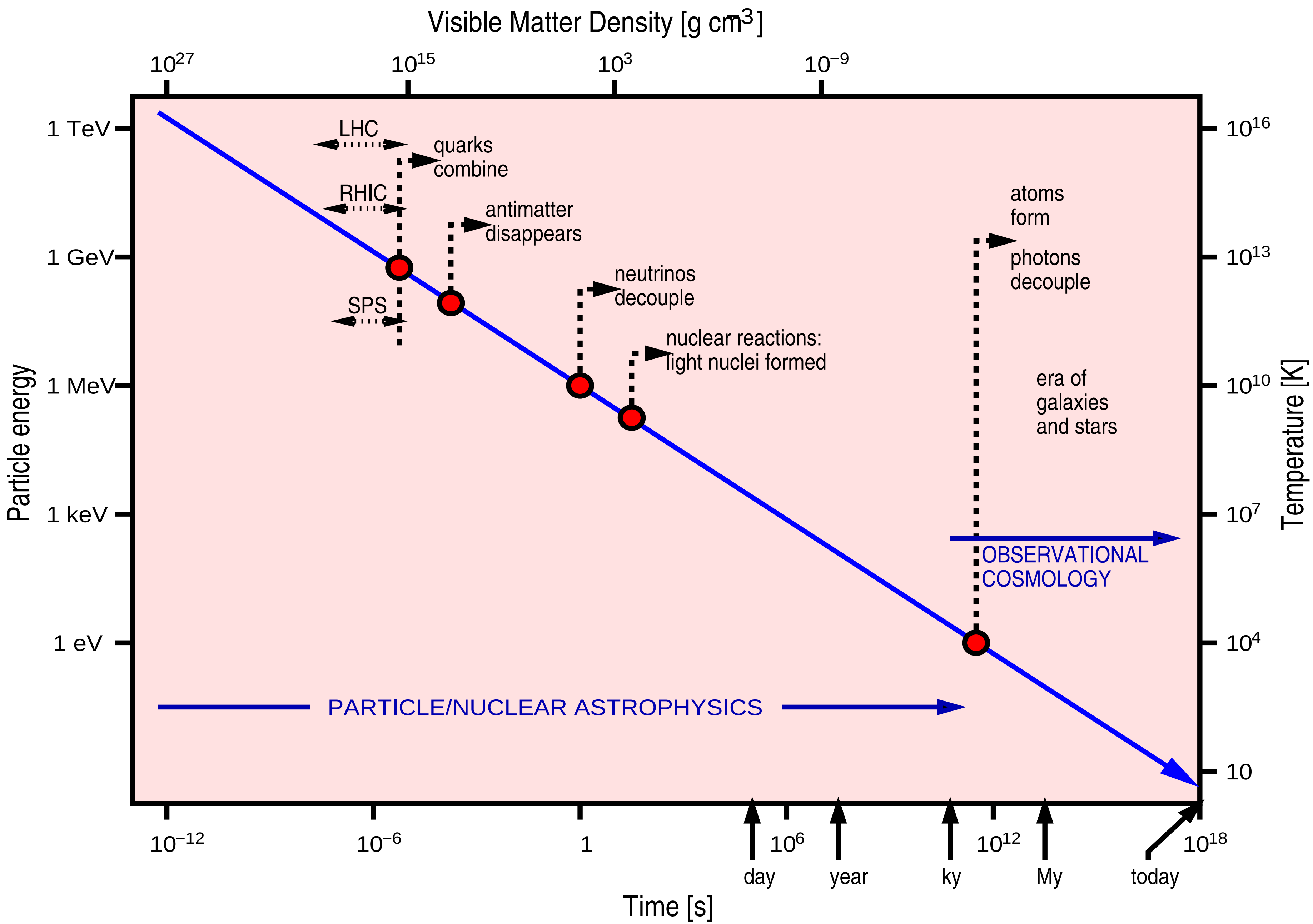


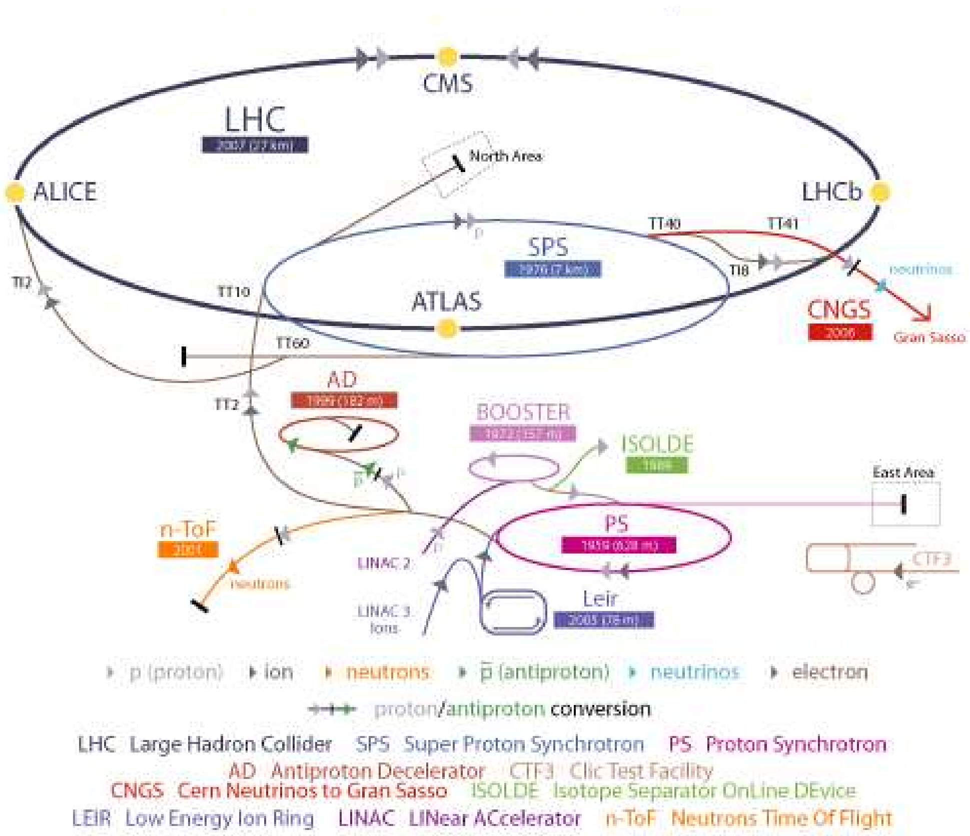
ALICE at LHC

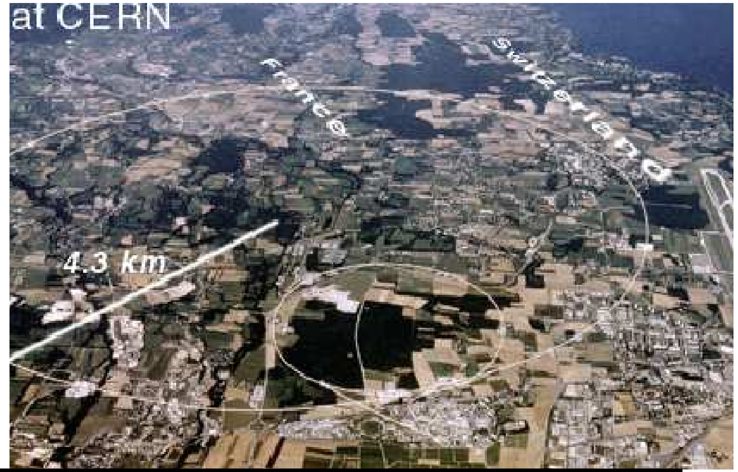
Orders of Magnitude

ENERGY density	ϵ	$\simeq 1$–**50GeV/fm**$^3 = 0.18$–$9\ 10^{16}$**g/cc**
Latent vacuum heat	B	$\simeq$**0.4GeV/fm**$^3 = (234\mathrm{MeV})^4 = 0.64\ 10^{29}$**J/cc**
PRESSURE	P	$= \frac{1}{3}\epsilon = (0.52 - 26)\ 10^{30}$ **bar**
TEMPERATURE	T_0, T_f	**500, 160 MeV;** **300MeV**$\simeq$**3.5** 10^{12}**K**

Stages in the evolution of the Universe







CERN: One lab, many opportunities

ii: Interdisciplinary Topics - Tutorials

STUDENTS: ARE YOU PREPARED?

This is a interdisciplinary subject matter:
attempt to answer these trivia questions quantitatively:

- **Relativity: is Lorentz-contraction of space or body of matter; what is proper time, rapidity?**
- **Relativistic Statistical Physics: Is $3P \geq \varepsilon$ or $3P \leq \varepsilon$?**
- **Nuclear Physics: what is quark content of a hyperon?**
- **Particle Physics:**
  what is evidence that gluons are charged, confined particles?
- **Quantum Field Theory (for pedestrians):**
  How strong is strong interaction α_s in quark-gluon plasma?
- **Cosmology:**
  when did quark Universe hadronizes – what fixes the time scale!
- **Astrophysics: why is not every neutron star a quark star, and conversely are any quark stars around?**

Statistical Physics Tutorial

Independent quantum (quasi)particles

$$\hat{H}|i\rangle = E_i|i\rangle; \qquad [\hat{b},\hat{H}]=0; \qquad \hat{b}|i,b\rangle = b|i,b\rangle$$

The grand-canonical partition function, can be written as:

$$\mathcal{Z} \equiv \sum_{i,b} \langle i,b|\gamma e^{-\beta(\hat{H}-\mu\hat{b})}|i,b\rangle = \mathbf{Tr}\,\gamma\, e^{-\beta(\hat{H}-\mu\hat{b})} \equiv \sum_n \langle n|e^{-\beta(\hat{H}-\mu\hat{b}-\beta^{-1}\ln\gamma)}|n\rangle .$$

The trace of a quantum operator is representation-independent; that is, any complete set of microscopic basis states $|n\rangle$ may be used to find the (quantum) canonical or grand-canonical partition function. This allows us to obtain the physical properties of quantum gases in the, often useful, approximation that they consist of independent (quasi)particles, and, eventually, to incorporate any remaining interactions by means of a perturbative expansion.

$$\mathcal{Z} = \sum_n e^{-\sum_{i=1}^{\infty} n_i\beta(\varepsilon_i-\mu b_i-\beta^{-1}\ln\gamma)} = \sum_n \prod_i e^{-n_i\beta(\varepsilon_i-\mu b_i-\beta^{-1}\ln\gamma)} = \prod_i \sum_{n_i=0,1\ldots} e^{-n_i\beta(\varepsilon_i-\mu b_i-\beta^{-1}\ln\gamma)} .$$

To show last equality, one considers whether all the terms on the left-hand side are included on the right hand side, where the sum is not over all the sets of occupation numbers n, but over all the allowed values of occupation numbers n_i.For fermions (F,) we can have only $n_i=0,1$,whereas for bosons (Bs) $n_i=0,1,\ldots,\infty$. The resulting sums are easily carried out analytically:

$$\ln \mathcal{Z}_{\mathrm{F/B}} = \ln \prod_i \left(1 \pm \gamma e^{-\beta(\varepsilon_i-\mu b_i)}\right)^{\pm 1} = \pm \sum_i \ln(1 \pm \gamma\lambda_i^{b} e^{-\beta\varepsilon_i}).$$

- **For antiparticles, the eigenvalue of $\hat{b}$ is the negative of the particle value, the fugacity $\lambda_{\bar{\mathrm{f}}}$ for antiparticles $\lambda_{\bar{\mathrm{f}}} = \lambda_{\mathrm{f}}^{-1}$. This implies $\mu_{\mathrm{f}} = -\mu_{\bar{\mathrm{f}}}$.**

- **level sum $\sum_i$: If energy is the only controlling factor then we carry out this summation in terms of the single particle level density $\sigma_1(\varepsilon, V)$. Taking quantum levels in a box in the limit of infinite volume of the system we find the phase-space integral:**

$$\sum_i \to g \int \frac{d^3x\, d^3p}{(2\pi)^3}.$$

- **Independent particle energy $\varepsilon_i = \sqrt{m_i^2 + \vec{p}^{\,2}}$.**

$$\ln \mathcal{Z}_{\mathrm{F/B}}(V,\beta,\lambda,\gamma) = \pm gV \int \frac{d^3p}{(2\pi)^3}\left[\ln(1 \pm \gamma\lambda e^{-\beta\sqrt{p^2+m^2}}) + \ln(1 \pm \gamma\lambda^{-1} e^{-\beta\sqrt{p^2+m^2}})\right];$$

Boltzmann limit:

$$\ln \mathcal{Z}_{\mathrm{cl}}(V,\beta,\lambda,\gamma) = gV \int \frac{d^3p}{(2\pi)^3}\,\gamma(\lambda+\lambda^{-1})e^{-\beta\sqrt{p^2+m^2}}.$$ **for Fermi and Bose**

Single particle phase space occupancy:

$$\overline{w_i} \equiv \frac{\bar{n}_i}{N} = \frac{e^{-\beta E_i}}{\sum_j e^{-\beta E_j}}$$

$$= -\frac{1}{\beta}\frac{\partial}{\partial E_i}\left(\ln \sum_j \gamma e^{-\beta E_j}\right) = -\frac{1}{\beta}\frac{\partial}{\partial E_i}\ln Z \to \frac{1}{\gamma^{-1}\lambda^{-1}e^{\beta E_i} \pm 1}$$

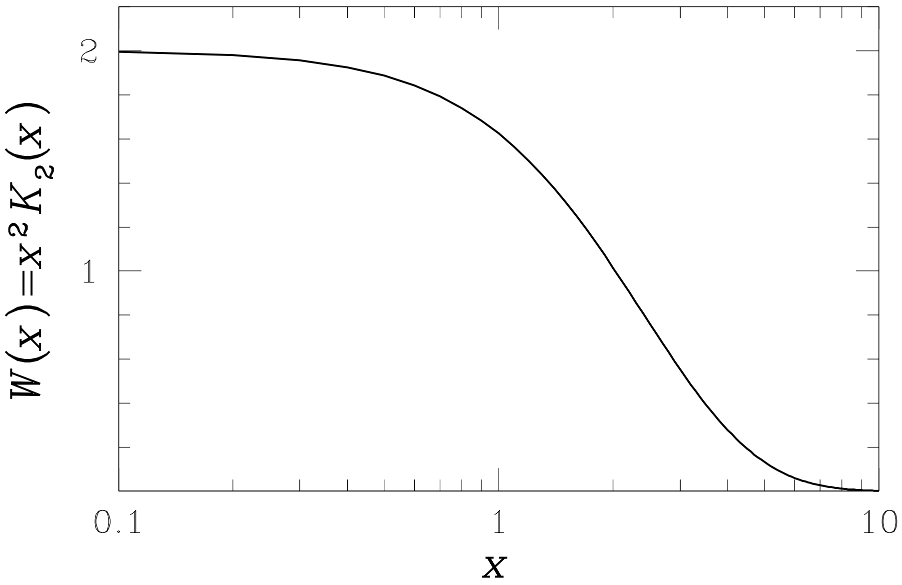
$$= \pm \sum_{n=1}^{\infty}\left(\pm\gamma\lambda e^{-\beta E_i}\right)^n \to \gamma\lambda e^{-\beta E_i},$$

we recognize FERMI, BOSE, BOLTZMANN distributions.
To evaluate a statistical physics property weight a distribution with suitable factor, for example, energy density results using as the weight single particle energy; particle density has weight 1 (integral of distribution).

Classical relativistic gas

Relativistic Boltzmann gas – a useful integral, for $\varepsilon = \sqrt{m^2 + p^2}$:

$$W(\beta m) \equiv \beta^3 \int e^{-\beta\varepsilon} p^2\, dp = (\beta m)^2 K_2(\beta m), \quad \to 2, \ \textbf{for } m \to 0, \quad \to \sqrt{\frac{\pi m^3}{2T^3}} e^{-m/T}, \ \textbf{for } m \gg T$$



$$\ln \mathcal{Z}_{\rm cl} \equiv Z^{(1)} = \sum_i \gamma_i(\lambda_i + \lambda_i^{-1}) Z_i^{(1)},$$

$$Z_i^{(1)} = g_{\rm f} V \int \frac{d^3p}{(2\pi)^3} e^{-\beta\varepsilon(p)}$$

$$= g_i \frac{\beta^{-3} V}{2\pi^2} W(\beta m_i).$$

Statistical and thermal physics relations

$$\beta P = \frac{\partial \ln \mathcal{Z}(V,\beta,\mu)}{\partial V}, \qquad E = -\frac{\partial \ln \mathcal{Z}(V,\beta,\mu)}{\partial \beta},$$

$$\mathcal{F}(V,T,\mu) \equiv EI(S,b) - ST - \mu b = -P(T,\mu)V,$$

$$S = -\frac{d}{dT}\mathcal{F}(V,T,\mu) = \frac{d}{dT}T \ln \tilde{\mathcal{Z}}(V,T,\mu) = \frac{dP}{dT}\,|_\mu$$

Statistical physics Gibbs–Duham relation

$$P = T\sigma + \mu\nu - \epsilon, \quad \sigma = \frac{S}{V}, \quad \nu = \frac{b}{V}, \quad \epsilon = \frac{E}{V},$$

is more powerful than the 1st law of thermodynamics:

$$dE(V,S,b) = -P\,dV + T\,dS + \mu\,db, \qquad d\mathcal{F} = -P\,dV - S\,dT - b\,d\mu,$$

Quark gas

Fermi gas, with phase space occupancy $\gamma = 1$

$$\ln \mathcal{Z}_{\rm F} = g_{\rm F} V \int \frac{d^3p}{(2\pi)^3} [\ln(1+e^{-\beta(\varepsilon-\mu)}) + \ln(1+e^{-\beta(\varepsilon+\mu)})],$$

$d^3p \rightarrow 4\pi p^2 dp$; **integrate by parts:**

$$3\frac{T}{V}\ln \mathcal{Z}_{\rm F} = g_{\rm F}\frac{\beta}{3}\int \frac{d^3p}{(2\pi)^3}\frac{\vec{p}^{\,2}}{\varepsilon}\left(\frac{1}{e^{\beta(\varepsilon-\mu)}+1} + \frac{1}{e^{\beta(\varepsilon+\mu)}+1}\right).$$

Substitute the arguments of f and $\bar{f}$ with $x = \beta(\varepsilon \pm \mu)$:

$$3\frac{T}{V}\ln \mathcal{Z}_{\rm F} = \frac{g_{\rm F}}{2\pi^2}T^4\Big(\int_{\beta(m-\mu)}^{\infty} dx\, \frac{[(x+\mu/T)^2-(m/T)^2]^{3/2}}{e^x+1} + (\mu \rightarrow -\mu)\Big).$$

For $m \rightarrow 0$ $[(x \pm \mu/T)^2 - (m/T)^2]^{3/2} \rightarrow (|x \pm \beta\mu|)^3$, integrals split to be from $\pm\beta\mu \rightarrow 0$ and from $0 \rightarrow \infty$. The finite-range terms:

$$\int_{-\beta\mu}^{0} dx\, \frac{|x+\beta\mu|^3}{1+e^x} - \int_0^{\beta\mu} dx \frac{(x-\beta\mu)^3}{1+e^x} = \int_0^{\beta\mu} dx\, \frac{(\beta\mu - x)^3}{1+e^{-x}} + \int_0^{\beta\mu} dx\, \frac{(\beta\mu-x)^3}{1+e^x}$$

$$= \int_0^{\beta\mu} dx\, (\beta\mu - x)^3 = \frac{(\beta\mu)^4}{4},$$

The reminder evaluated expanding $(e^x+1)^{-1} = \sum_{n=1}^{\infty} e^{-nx}$.

$$\ln \mathcal{Z}_{\rm F}|_{m=0} = \frac{\frac{7}{4}g_{\rm F}V\beta^{-3}\pi^2}{90}\left(1 + \frac{30}{7\pi^2}\ln^2\lambda + \frac{15}{7\pi^4}\ln^4\lambda\right).$$

Degeneracy: $g_{\rm F} = n_f 2_s 3_c$: for each flavor of quarks (u,d,s,c,b,t) we have 2-spins, 3-colors, so $g_{u,d,s} = 6$, keep in mind the doubling due to particle-antiparticle symmetry and factor 7/8 compared to Bosons which produces coefficient 7/4.

Relativistic Boson Gas

Relativistic Bose gas, e.g. photon, gluons, pions: We expand:

$$f(\varepsilon) = \frac{1}{\gamma^{-1}e^{\beta\varepsilon} - 1} = \sum_{n=1}^{\infty} \gamma^n e^{-n\beta\varepsilon}, \qquad \gamma < e^{\beta m}.$$

or for the partition function

$$\ln \mathcal{Z} = -gV \int \frac{dp^3}{(2\pi)^3} \ln(1 - \gamma e^{-\beta\varepsilon}) = \frac{gV}{2\pi^2} \int_0^\infty dp p^2 \sum_{n=1}^{\infty} \frac{\gamma^n}{n} e^{-n\beta\varepsilon}, \qquad \gamma < e^{\beta m, \varepsilon \geq m}.$$

Exchange integral and sum! As we see, each term in the sum differs by $\beta \to n\beta$ and all we have to do it so make sure that we have the right power of $1/n$ in the final expression from substitution:

$$\ln \mathcal{Z} = \frac{gVT^3}{2\pi^2} \sum_{n=1}^{\infty} \frac{\gamma^n}{n^4} \int_0^\infty dx\, x^2 e^{-\sqrt{(nm/T)^2 + x^2}} = \frac{gVT^3}{2\pi^2} \sum_{n=1}^{\infty} \frac{\gamma^n}{n^4} (n\beta m)^2 K_2(n\beta m) \to \frac{gT^4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^4}$$

For last limit recall Riemann zeta function to recognize the Stefan-Boltzmann law:

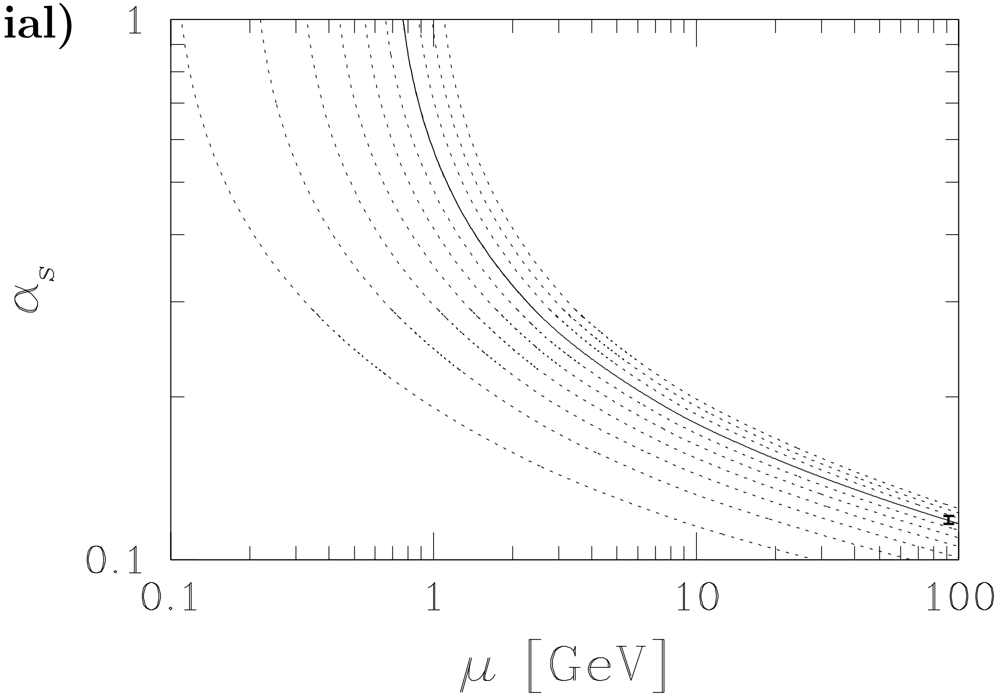
$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}, \ \zeta(2) = \frac{\pi^2}{6}, \ \zeta(3) \simeq 1.202, \ \zeta(4) = \frac{\pi^4}{90}. \qquad \qquad /\!\!\to \frac{gT^4\pi^2}{90}.$$

For a Fermi occupation function, the signs of the terms in the sums are alternating, which leads to the eta function, and the factor $7/8$ reduction in Fermi degrees of freedom compared to Bosons

$$\eta(k) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^k} = (1 - 2^{1-k})\zeta(k), \ \eta(3) = \frac{3}{4}\zeta(3) = 0.901\,5, \ \eta(4) = \frac{7}{8}\zeta(4) = \frac{7}{720}\pi^4.$$

QCD PERTURBATIVE EFFECTS

An essential pre-requirement for the perturbative QCD theory to be applicable in 'soft' quark domain of interest to us, is the relatively small experimental value $\alpha_s(M_Z) \simeq 0.118$. that is $\alpha_s(M_Z)/\pi = 0.0376$ (note that here μ is scale of energy NOT chemical potential)

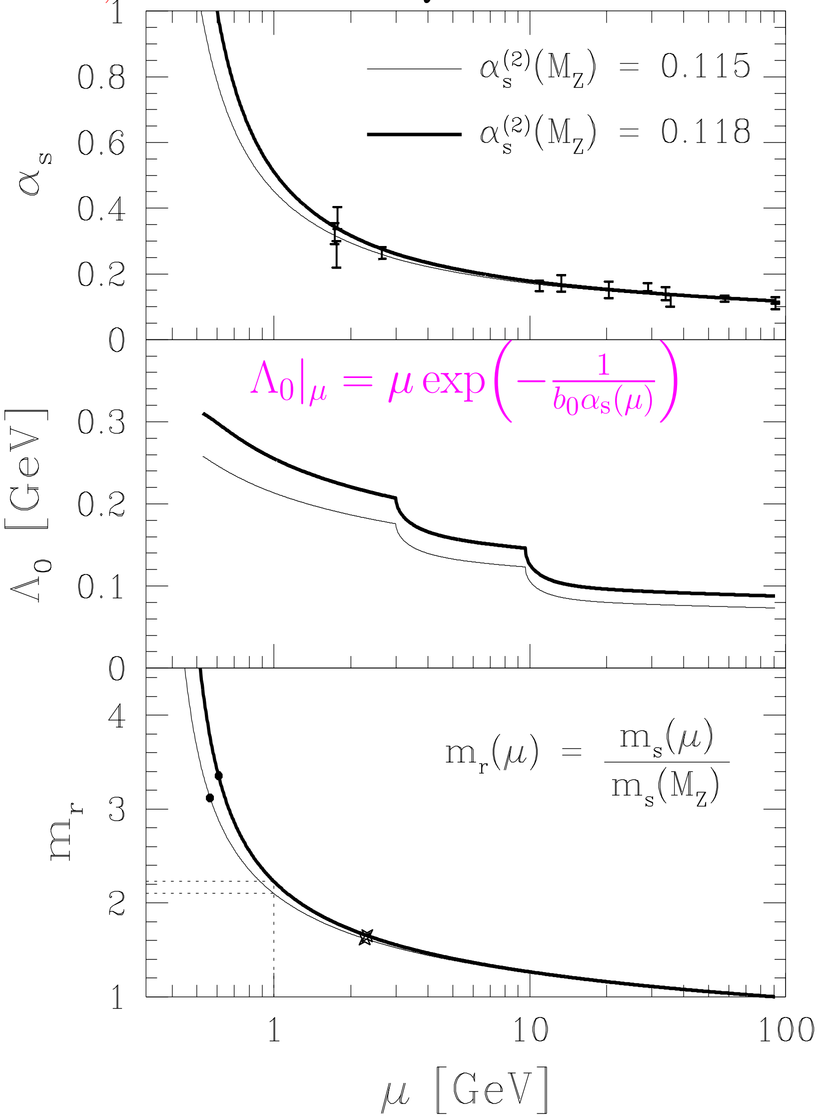


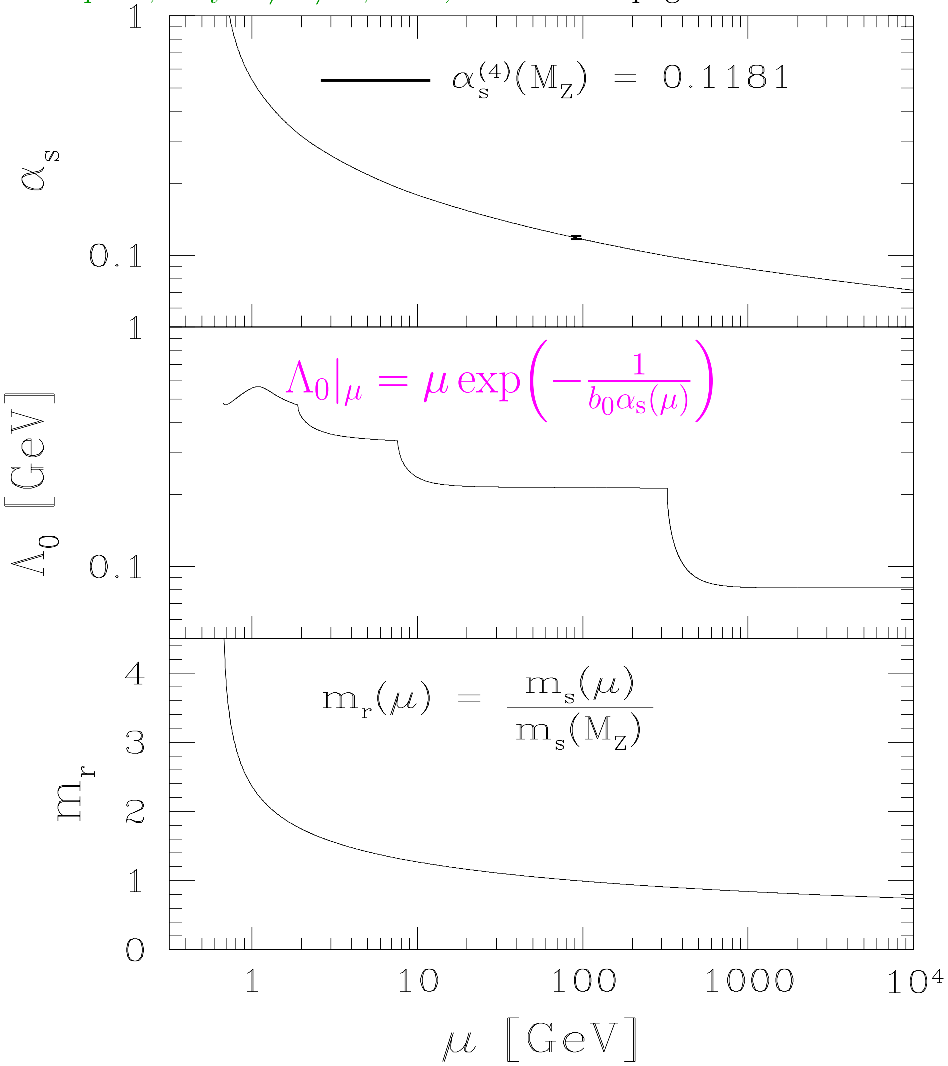
$\alpha_s^{(4)}(\mu)$ as function of energy scale μ for a variety of initial conditions. Solid line: $\alpha_s(M_Z) = 0.1182$ (experimental point, includes the error bar at $\mu = M_Z$). Result of integration of renormalization group equation (RGE).

$$\mu\frac{\partial \alpha_s}{\partial \mu} = -b_0\alpha_s^2 - b_1\alpha_s^3 + \ldots \equiv \beta_2^{\rm pert}\,, \qquad b_0 = \frac{11 - 2n_{\rm f}/3}{2\pi}\,, \quad b_1 = \frac{51 - 19n_{\rm f}/3}{4\pi^2}\,.$$

Λ_0: integration constant when solving the lowest order RGE:

$$\frac{1}{-b_0\alpha_s} = \int\frac{d\mu}{\mu}, \quad \alpha_s = \frac{1}{b_0\ln(\mu/\Lambda_0)} \quad \text{fixed by} \quad \Lambda_0|_\mu = \mu\exp\left(-\frac{1}{b_0\alpha_s(\mu)}\right)$$



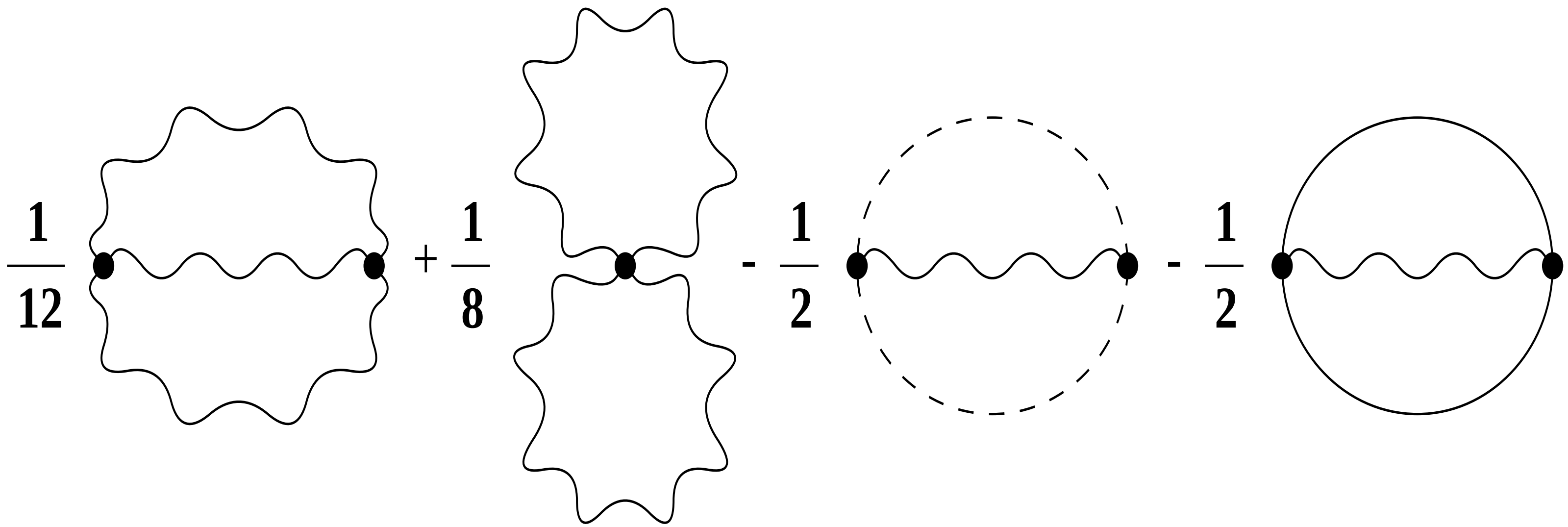


$$\mu \frac{\partial \alpha}{\partial \mu} \equiv \beta(\alpha_s) \qquad \beta^{\rm pert} = -\alpha_s^2 \left[\, b_0 + b_1\alpha_s + b_2\alpha_s^2 + \cdots \,\right]$$

$$-\frac{\mu}{m}\frac{\partial m}{\partial \mu} \equiv \gamma(\alpha_s), \qquad \gamma_m^{\rm pert} = \alpha_s \left[\, w_0 + w_1\alpha_s + w_2\alpha_s^2 + \cdots \,\right]$$

QCD perturbative interaction

Evaluation in thermal field theory of the Feynman diagrams in order α_s shows that on average QCD perturbative interaction reduce is ATTRACTIVE and some of the many degrees of freedom 'freeze'. This result is upheld in all orders when this study is done on the lattice. Feynman diagrams contributing are of the type:



Wavy lines represent gluons, solid lines represent quarks, and dashed lines denote the ghost subtractions of non-physical degrees of freedom. Full discussion beyond scope of this lecture.

Perturbative QCD and QGP

$$\frac{T}{V}\ln \mathcal{Z}_{QGP} = -\mathcal{B} + \frac{8}{45\pi^2}c_1(\pi T)^4 + \sum_{i=u,d,s}\frac{n_i}{15\pi^2}\left[\frac{7}{4}c_2(\pi T)^4 + \frac{15}{2}c_3\left(\mu_i^2(\pi T)^2 + \frac{1}{2}\mu_i^4\right)\right]$$

$$c_1 = 1 - \frac{15\alpha_s}{4\pi}\,, \qquad c_2 = 1 - \frac{50\alpha_s}{21\pi}\,, \qquad c_3 = 1 - \frac{2\alpha_s}{\pi}\,.$$

We recall that $\mu_b = 3\mu_q$ and $\lambda_q = e^{\mu_q/T}$. The temperature dependence $\alpha_s(T)$ is estimated to be $\mu = 2\pi T$ that is use $\alpha_s(2\pi T)$ with lowest order perturbative correction, which works well. At finite chemical potential $\mu = 2\sqrt{(\pi T)^2 + \mu_q^2} = 2\pi T\sqrt{1 + \frac{1}{\pi^2}\ln^2\lambda_q}$. A convenient way to obtain entropy and baryon density uses the thermodynamic potential $\mathcal{F}$:

$$\frac{\mathcal{F}(T,\mu_q,V)}{V} = -\frac{T}{V}\ln \mathcal{Z}(\beta,\lambda_q,V)_{QGP} = -P_{QGP}\,.$$

The entropy density is:

$$s_{QGP} = -\frac{d\mathcal{F}}{VdT} = \frac{32\pi^2}{45}c_1 T^3 + \frac{n_f 7\pi^2}{15}c_2 T^3 + n_f c_3 \mu_q^2 T + A\frac{\pi^2 T}{\pi^2 T^2 + \mu_q^2}\,.$$

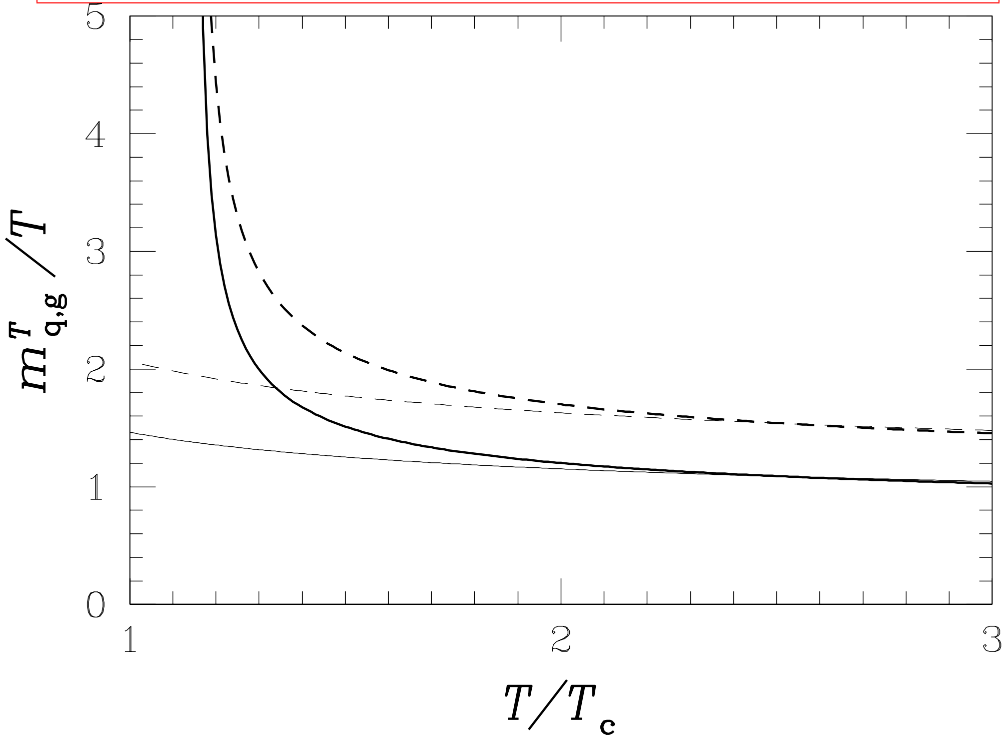
Noting that baryon density is 1/3 of quark density, we have:

$$\rho_B = -\frac{1}{3}\frac{d\mathcal{F}}{Vd\mu_q} = \frac{n_f}{3}c_3\left\{\mu_q T^2 + \frac{1}{\pi^2}\mu_q^3\right\} + \frac{1}{3}A\frac{\mu_q}{\pi^2T^2 + \mu_q^2}\,.$$

$$A = A_g + A_q + A_s; \qquad A_g = (b_0\alpha_s^2 + b_1\alpha_s^3)\frac{2\pi}{3}T^4$$

$$A_{i=q,s} = (b_0\alpha_s^2 + b_1\alpha_s^3)\left[\frac{n_i 5\pi}{18}T^4 + \frac{n_i}{\pi}\left\{\mu_i^2 T^2 + \frac{1}{2\pi^2}\mu_i^4\right\}\right]\,.$$

Other way: massive quasi-particles



Thermal masses fitted to reproduce Lattice-QCD results

Thick solid line for quarks, and thick dashed line for gluons. Thin lines, perturbative QCD masses for $\alpha_s(\mu = 2\pi T)$.

$$(m_{\rm q}^{\rm T})^2 = \frac{4\pi}{3}\alpha_s T^2\,, \qquad (m_{\rm g}^{\rm T})^2 = 2\pi\alpha_s T^2\left(1+\frac{n_{\rm f}}{6}\right)\,,$$

The thermal masses required to describe the reduction of the number of degrees of freedom for $T > 2T_{\rm c}$ are outside of the range of the vacuum structure influence ($\mathcal{B}$) the perturbative QCD result. This means that thermal masses express, in a different way, the effect of perturbative QCD

Next: Hadron Gas Phase

We can use the methods we proposed for quarks and gluons to describe the gas of pions, nucleons and the rest – to be precise – many thousand hadronic particles. Hadron Gas phase. A topic in itself, so I will just glance at it and its properties.
bla

This was research area of intense interest before we understood the world of quarks, gluons and confinement.

Hadron gas is a phase of hot matter in which we have not only a few degrees of freedom but many particles, following Hagedorn we must take all including resonances.

Abundance of particles decreases as $e^{-m/T}$ but how many particles are there at mass m? Hagedorn: exponentially many so that $\rho(m)e^{-m/T}$ controlled by power law.

Theoretical models were developed to evaluate ρ but it seems more appropriate to simply work with what we "see".

Hot Hadron Tutorial: Limiting Hagedorn Temperature

A gas of hadrons with exponentially rising mass spectrum:

$$\ln \mathcal{Z}_{\rm HG}^{\rm cl} = cV\left(\frac{T}{2\pi}\right)^{3/2}\int_M^\infty m^a e^{m/T_{\rm H}} m^{3/2} e^{-m/T}\, dm + D(T,M),$$

Cutoff $M > m_a > T_{\rm H}$ is arbitrary, its role is to separate off $D(T,M) < \infty$. Because of the exponential factor, the first integral can be divergent for $T > T_{\rm H}$, and the partition function is singular for $T \to T_{\rm H}$ for a range of a:

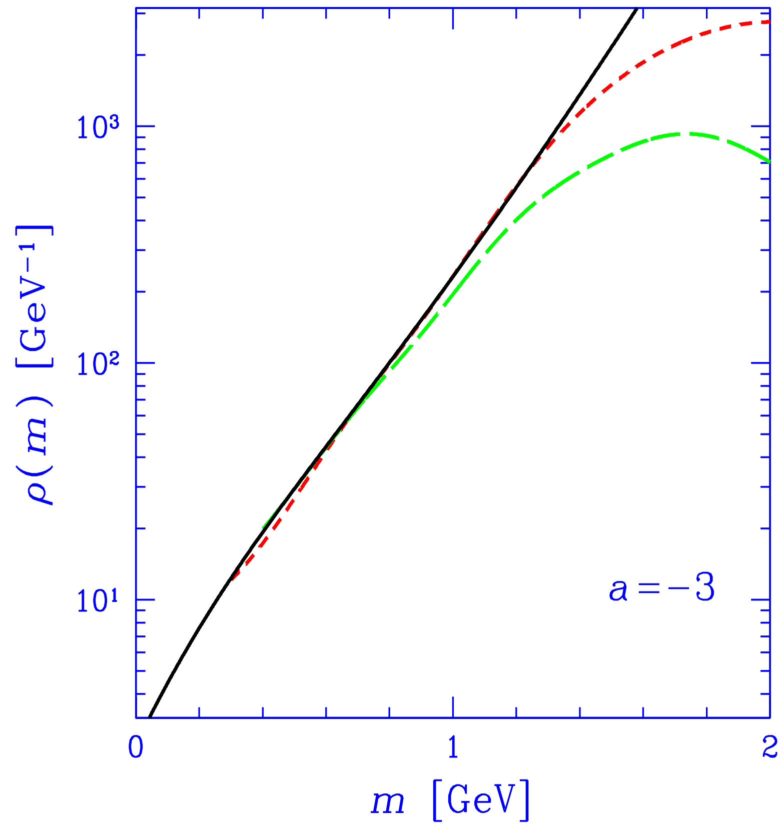
$$P(T) \to \begin{cases} \left(\frac{1}{T} - \frac{1}{T_{\rm H}}\right)^{-(a+5/2)}, & \textbf{for} \quad a > -\frac{5}{2}, \\ \ln\left(\frac{1}{T} - \frac{1}{T_{\rm H}}\right), & \textbf{for} \quad a = -\frac{5}{2}, \\ \textbf{constant}, & \textbf{for} \quad a < -\frac{5}{2}; \end{cases} \qquad \epsilon \to \begin{cases} \left(\frac{1}{T} - \frac{1}{T_{\rm H}}\right)^{-(a+7/2)}, & \textbf{for} \quad a > -\frac{7}{2}, \\ \ln\left(\frac{1}{T} - \frac{1}{T_{\rm H}}\right), & \textbf{for} \quad a = -\frac{7}{2}, \\ \textbf{constant}, & \textbf{for} \quad a < -\frac{7}{2}. \end{cases}$$

The energy density ϵ goes to infinity for $a \geq -\frac{7}{2}$, when $T \to T_{\rm H}$.

Mass spectrum slope $T_{\rm H}$ appears as the limiting Hagedorn temperature beyond which we cannot heat a system which can have an infinite energy density. The partition function can be singular even when $V < \infty$.

Exponential Hadron Mass Spectrum

RH discovered that the exponential growth of the hadronic mass spectrum could lead to an understanding of the limiting hadron temperature $T_\mathrm{H} \simeq 160$ MeV,



The solid line is the fit:

$$\rho(m) \approx c(m_a^2 + m^2)^{a/2} \exp(m/T_\mathrm{H})$$

with $a = -3$, $m_a = 0.66$ GeV, $T_\mathrm{H} = 0.158$ GeV.
Long-dashed line: 1411 states of 1967.
Short-dashed line: 4627 states of 1996.

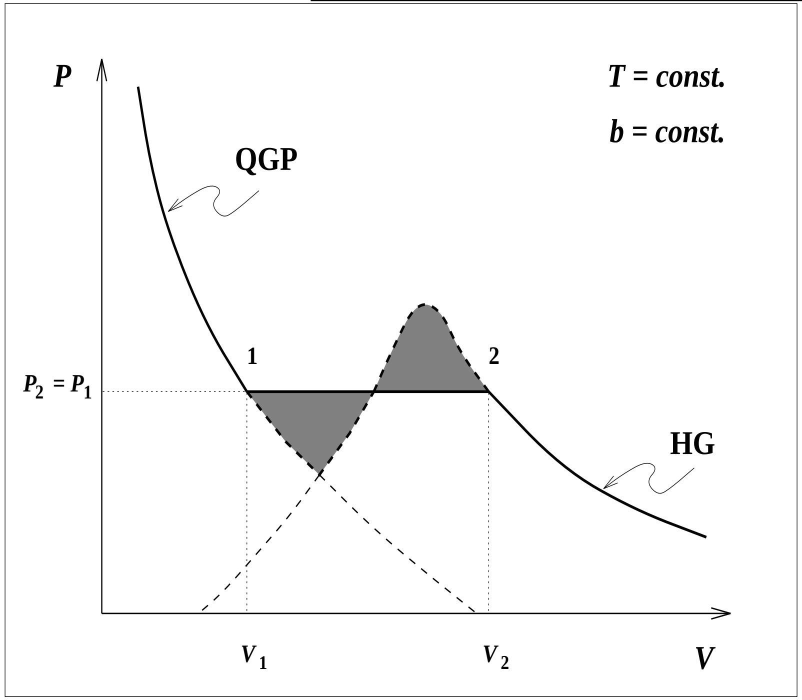
Experimental lines include Gaussian smoothing:

$$\rho(m) = \sum_{m^*=m_\pi, m_\rho,\ldots} \frac{g_{m^*}}{\sqrt{2\pi}\sigma_{m^*}} \exp\left(-\frac{(m-m^*)^2}{2\sigma_{m^*}^2}\right).$$

$\sigma = \Gamma/2$, $\Gamma = \mathcal{O}(200)$ MeV is the assumed width of the resonance, excluding the 'stable' pion, a special case.

Note the missing resonances at $m > 1.4 GeV$.

Equilibrium – Phase Transition Tutorial



The P–V diagram for the QGP–HG system, shown at fixed temperature and baryon number; dashed lines indicate unstable domains of overheated and undercooled phases. Darkened area: Maxwell construction, connecting the volumes $V_1 = b/\rho_1$ and $V_2 = b/\rho_2$, such that work done along the metastable branches vanishes:

$$\int_{V_1}^{V_2} (P - P_{12})\, dV = 0.$$

Construction can be repeated for different values of b and T, the set of resulting points 1 and 2 forms then two phase-boundary lines.

Between V_1 and V_2 is the mixed phase comprising a mixture of hadrons and drops of QGP. Was such a phase formed in early Universe?

Hagedorn Temperature is:

1. **The intrinsic temperature at which hadronic particles are formed, in *pp* interactions seen as the inverse slope of hadron spectra.**
2. **This boiling point of hadrons which is the (inverse) slope of exponentially rising hadron mass spectrum.**
3. **The boundary value of temperature at which finite size hadrons coalesces into one cluster consisting of a new phase comprising hadron constituents.**

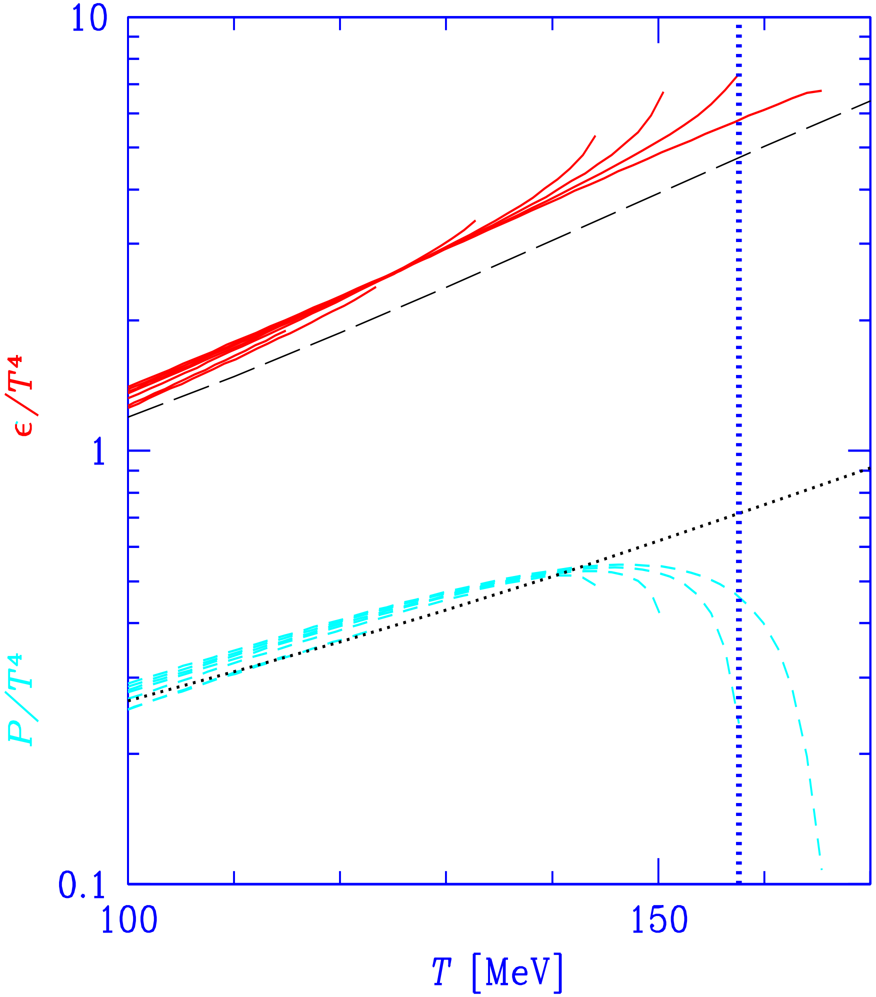
Statistical Bootstrap Model is:

1. **A connection between hadronic particle momentum distribution and properties of hadronic interactions dominated by resonant scattering, and exponentially rising mass spectrum.**
2. **A theoretical framework for study of the properties of the equations of state of dense and hot baryonic matter (nuclear matter at finite temperature).**
3. **It is not a fundamental dynamical theory, in fact Statistical Bootstrap Model (SBM) is to be motivated today in terms of properties of the fundamental dynamical approach (QCD).**

Finite Volume Hadron Gas Model

The gas of finite size hadrons with exponential mass spectrum has nearly the same properties as a gas of point hadrons with today experimentally observed mass spectrum. That is why 'statistical hadronization works'.

Point hadron gas in free available volume Δ to have the properties of finite size hadron gas in total mean volume $\langle V\rangle$ (RH/JR 1978+)



$$\ln \mathcal{Z}_{\rm pt}(T,\Delta,\lambda) \equiv \ln \mathcal{Z}(T,\langle V\rangle,\lambda)$$

Proper particle volume in the rest frame is assumed to be proportional to mass. For a gas of moving hadrons, in gas rest frame: $\langle V\rangle = \Delta + \langle E\rangle/4\mathcal{B}$.

$$\langle E\rangle = \langle V\rangle \epsilon(\beta,\lambda) = -\frac{\partial}{\partial\beta}\ln \mathcal{Z}(\beta,\langle V\rangle,\lambda) =$$

$$= -\frac{\partial}{\partial\beta}\ln \mathcal{Z}_{\rm pt}(\beta,\Delta,\lambda) = \Delta\epsilon_{\rm pt}(\beta,\lambda)$$

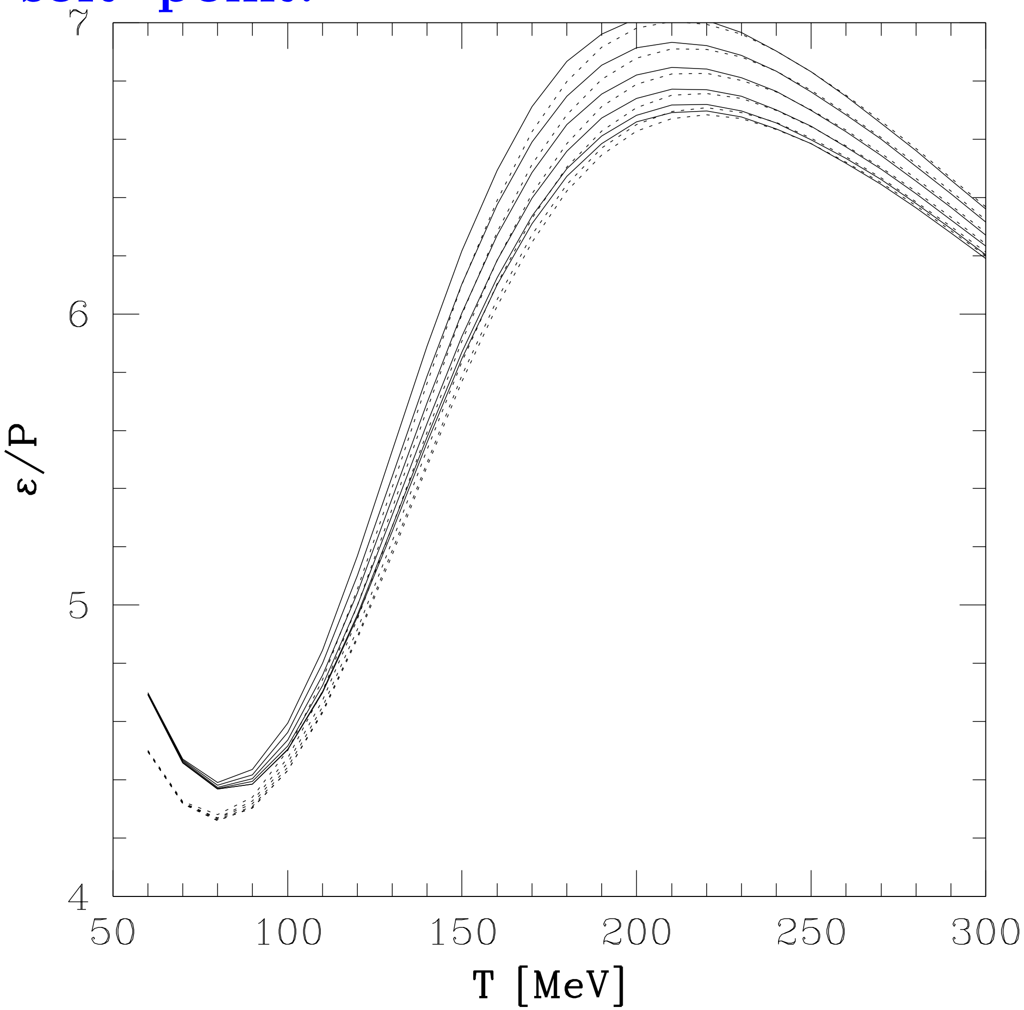
$$\langle V\rangle = \Delta\big(1+\epsilon_{\rm pt}(\beta,\lambda)/4\mathcal{B}\big),$$

$$\frac{\langle E\rangle}{\langle V\rangle} \equiv \epsilon(\beta,\lambda) = \frac{\epsilon_{\rm pt}(\beta,\lambda)}{1+\epsilon_{\rm pt}(\beta,\lambda)/(4\mathcal{B})},$$

$$P = \frac{P_{\rm pt}(\beta,\lambda)}{1+\epsilon_{\rm pt}(\beta,\lambda)/4\mathcal{B}}.$$

Inertia to Force Ratio

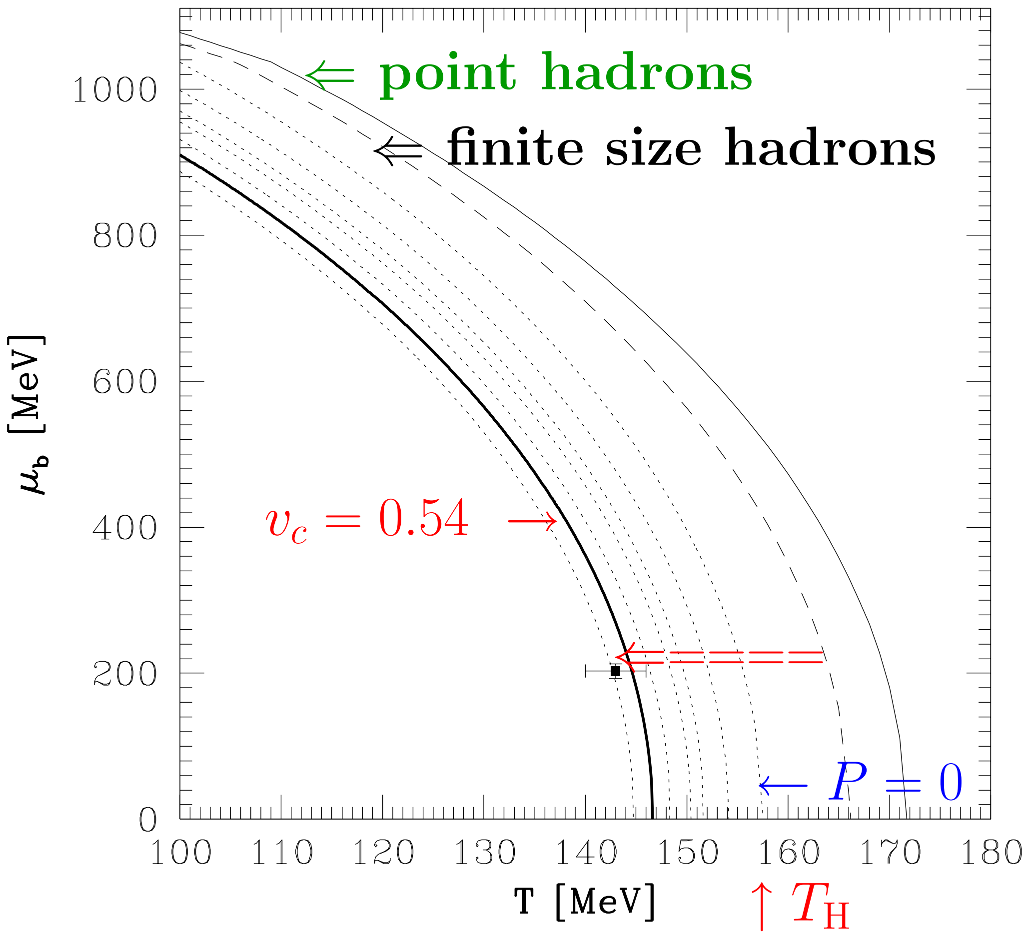
For study of flow of matter one of most relevant quantities is the rigidity of the matter. Hadrons are heavy thus their pressure is less, hence unlike for relativistic matter, $\epsilon/P > 3$ But there is a 'soft' point.



The energy density over pressure for a hadronic gas with statistical parameters $\lambda_\mathrm{s} = 1.1$ and $\gamma_\mathrm{s}/\gamma_\mathrm{q} = 0.8$, with $\lambda_\mathrm{q} = 1$ to 2 in steps of 0.2 from bottom to top and $\gamma_\mathrm{q} = 1$ (dashed lines), or $\gamma_\mathrm{q} = e^{m_\pi/(2T)}$ (full lines).

Hadronic gas flows very different from quark-gluon plasma.

Phase boundary

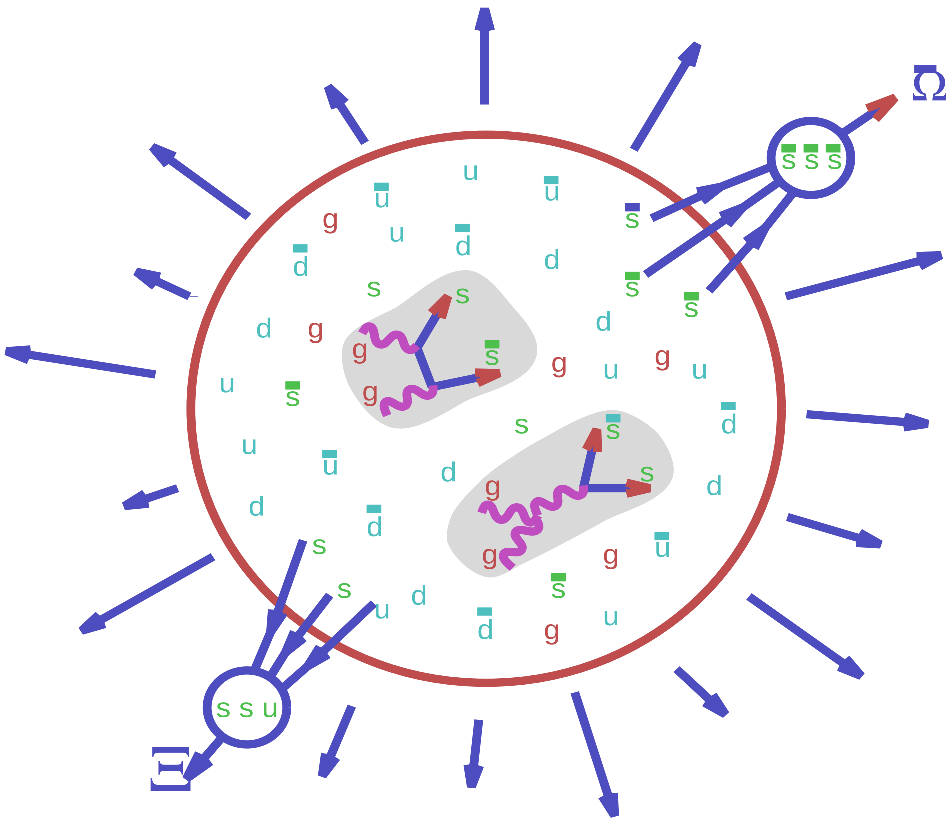


Solid: point hadrons T_p
Dashed: finite size
Dotted: $T_c(\mu_b)|_{P_{eff}-\mathcal{B}=0}$ for $v^2 = 0, 1/10, 1/6, 1/5, 1/4, 1/3$ **.**
Thick solid: breakup with $v = 0.54$ $(\kappa = 0.6)$
PRL 85 (2000) 4695
DEEP SUPERCOOLING by 20 MeV

$T_H = 158$ **MeV Hagedorn temperature where** $P = 0$**, no hadron** P
$T_f \simeq 0.9T_H \simeq 143$ **MeV is where supercooled QGP fireball breaks up**
equilibrium phase transformation is at $\simeq 166$**.**

From QGP to Hadrons: Statistical Hadronization Model

= recombinant quark hadronization, main consequence: enhancement of flavored (strange, charm, bottom) antibaryons progressing with 'exotic' flavor content. Anomalous meson to baryon relative yields. Proposed 25 years ago, see review
See: P. Koch, B. Muller and J. Rafelski, *Strangeness In Relativistic Heavy Ion Collisions,* **Phys. Rept. 142, 167 (1986), and references therein.**



1. **$GG \to s\bar{s}$ (thermal gluons collide)**
   $GG \to c\bar{c}$ (initial parton collision)
   $GG \to b\bar{b}$ (initial parton collision)
   gluon dominated reactions
2. **RECOMBINATION of pre-formed**
   s, $\bar{s}$, c, $\bar{c}$, b, $\bar{b}$ quarks

Formation of complex rarely produced multi flavor (exotic) (anti)particles enabled by coalescence between s, $\bar{s}$, c, $\bar{c}$, b, $\bar{b}$ quarks made in different microscopic reactions; this is signature of quark mobility and independent action, thus of deconfinement. Moreover, strangeness enhancement = gluon mobility.

SHM is FERMI MODEL with QUARK CHEMISTRY

If QGP near/at chemical equilibrium prior to fast hadronization we expect that emerging hadron multiplicities to be governed by parameters of ABSOLUTE chemical non-equilibrium described by phase space occupancy γ; Boltzmann gas: $\gamma \equiv \frac{\rho(T,\mu)}{\rho^{\rm eq}(T,\mu)}$

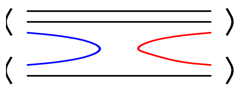
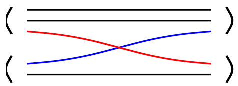
DISTINGUISH: hadron 'h' phase space and QGP phase parameters: micro-canonical variables such as baryon number, strangeness, charm, bottom, etc flavors are continuous, and entropy is almost continuous across phase boundary:

$$\gamma_s^{\rm QGP}\rho_{\rm eq}^{\rm QGP}V^{\rm QGP} = \gamma_s^{\rm h}\rho_{\rm eq}^{\rm h}V^{\rm h}$$

Equilibrium distributions are different in two phases and hence are densities:

$$\rho_{\rm eq}^{\rm QGP} = \int f_{\rm eq}^{\rm QGP}(p)dp \neq \rho_{\rm eq}^{\rm h} = \int f_{\rm eq}^{\rm h}(p)dp$$

Another RELATIVE equilibrium: FOUR QUARKS: $s,\bar{s},q,\bar{q}$ → FOUR CHEMIC

γ_i controls overall abundance of quark $(i=q,s)$ pairs	**Absolute chemical equilibrium**	**HG production** 
$\lambda_i = e^{\mu_i/T}$ controls difference between strange and light quarks $(i=q,s)$	**Relative chemical equilibrium**	**HG exchange** 

See Physics Reports 1986 Koch, Müller, JR

Example of counting hadronic particles

The counting of hadrons is conveniently done by counting the valence quark content $(u, d, s, \dots \lambda_q^2 = \lambda_u \lambda_d,\ \lambda_{I3} = \lambda_u/\lambda_d)$:

$$\Upsilon_i \equiv \Pi_i \gamma_i^{n_i} \lambda_i^{k_i} = e^{\sigma_i/T}; \qquad \lambda_q \equiv e^{\frac{\mu_q}{T}} = e^{\frac{\mu_b}{3T}}, \qquad \lambda_s \equiv e^{\frac{\mu_s}{T}} = e^{\frac{[\mu_b/3-\mu_S]}{T}}$$

Example of NUCLEONS $\gamma_N = \gamma_q^3$**:**

$$\Upsilon_N = \gamma_N e^{\frac{\mu_b}{T}}, \qquad \Upsilon_{\overline{N}} = \gamma_N e^{\frac{-\mu_b}{T}};$$

$$\sigma_N \equiv \mu_b + T \ln \gamma_N, \qquad \sigma_{\overline{N}} \equiv -\mu_b + T \ln \gamma_N$$

Meaning of parameters from e.g. the first law of thermodynamics:

$$\begin{aligned} dE + P\,dV - T\,dS &= \sigma_N\,dN + \sigma_{\overline{N}}\,d\overline{N} \\ &= \mu_b(dN - d\overline{N}) + T \ln \gamma_N (dN + d\overline{N}). \end{aligned}$$

NOTE: For $\gamma_N \to 1$ the pair terms vanishes, the μ_b term remains, it costs $dE = \mu_B$ to add to baryon number.