The Determination of Spectral Functions from Lattice QCD

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Outline of the talk

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- Definition of the SPF, what is it good for?
- The Maximum Entropy Method
- Demonstration of the method with mock data analysis
- Preliminary results on J/Ψ (Work in progress)

Brief definitions

Hadronic current operators

$$J_H(\vec{x},t) = \bar{q}(\vec{x},t)\Gamma_i q(\vec{x},t)$$
$$\Gamma_H = 1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5 \quad \text{for} \quad \mathrm{H} = \mathrm{S}, \mathrm{P}, \mathrm{V}, \mathrm{A}$$

Hadronic point-point correlators

$$D_H^{>}(x_0,\vec{x}) = \left\langle J_H(x_0,\vec{x}) J_H(0,\vec{0}) \right\rangle \ , \mathrm{x}_0 > 0$$
$$D_H^{<}(x_0,\vec{x}) = \left\langle J_H(0,\vec{0}) J_H(x_0,\vec{x}) \right\rangle \ , \mathrm{x}_0 > 0$$
$$G(\tau,\vec{p}) = D_H^{>}(-i\tau,\vec{x}) = \int d^3x e^{ipx} \left\langle T_\tau J_H(-i\tau,\vec{x}) J_H(0,\vec{0}) \right\rangle$$

$G(\tau,\vec{p})$ can be calculated on the lattice.

Brief definitions

Spectral function

Imaginary part of the Fourier-transform of the real time retarded correlator.

$$A_H(\omega) = \frac{1}{\pi}\mathrm{Im}\mathrm{D}^{\mathrm{R}}_{\mathrm{H}}(\mathrm{p}_0, \tilde{\mathrm{p}}) = \frac{1}{2\pi}\left(\mathrm{D}^{>}_{\mathrm{H}}(\mathrm{p}_0, \tilde{\mathrm{p}}) - \mathrm{D}^{<}_{\mathrm{H}}(\mathrm{p}_0, \tilde{\mathrm{p}})\right)$$

$$D_H^{>(<)}(p_0, \vec{p}) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} D_H^{>(<)}(x_0, \vec{x})$$

Kubo-Martin-Schwinger condition

At temperature T, the correlators satisfy:

$$D^{>}_{H}(x_0, \vec{x}) = D^{>}_{H}(x_0 + i/T, \vec{x})$$

What is the spectral function good for?

Contributions from different states

Inserting a complete set of states and using the KMS condition:

$$A(p_0, \vec{p}) = \frac{(2\pi)^2}{Z} \sum_{m,n} \left(e^{-E_n/T} \pm e^{-E_m/T} \right) \times$$
$$|\langle n | J_H(0) | m \rangle|^2 \, \delta^{(4)}(p_\mu - k_\mu^n + k_\mu^m)$$

- A stable state gives a δ-function like peak contribution.
- A quasi-particle in matter gives a smeared peak.

What is the spectral function good for?

It is accessible by experiments

E.g. if $J = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + ...$ is the EM current:

- The standard R-ratio in e^+e^- annihilation experiments is proportional to the SPF at $T = 0$.
- The differential thermal cross-section for the production of dilepton pairs in heavy ion experiments is proportional to the SPF at $T \neq 0$.

Kubo-formulas

Transport coefficients are given by the low frequency limit. E.g. if J is the EM current, than the electric conductivity is:

$$\frac{\sigma}{T} = \lim_{\omega \to 0} \frac{A(\omega)}{6\omega T}$$

Determining the spectral function

Relation to the Euclidean time correlator

$$G(\tau, \vec{p}) = \int_0^\infty d\omega A(\omega, \vec{p}) K(\omega, \tau)$$

$$K(\omega, \tau) = \frac{\cosh(\omega(\tau - 1/2T))}{\sinh(\omega/2T)}$$

Regularizing

This inversion is ill-defined. To get a unique answer one has to regularize. A commonly used regularization scheme is the Maximum Entropy Method.

The Maximum Entropy Method

The method in a nutshell

We maximize

$$Q = \alpha S - \frac{1}{2}\chi^2$$

$$S = \int d\omega \left(A(\omega) - m(\omega) - A(\omega) \log\left(\frac{A(\omega)}{m(\omega)}\right)\right)$$

$$\chi^2 = \sum_{i,j} (G_i^{\text{fit}} - G_i^{\text{data}}) C_{ij}^{-1} (G_j^{\text{fit}} - G_j^{\text{data}})$$

where C is the covariance matrix of the data and $m(\omega)$ is a function, summarizing our prior knowledge of the solution. Then we average over α. The conditional probability $P[\alpha|\text{data}, \text{m}]$ is given by Bayes' theorem.

Practical implementation

Discretization

$$\omega_i = i\delta\omega \quad (i = 1 ... N_\omega)$$
$$\tau_i = i a_t \quad (i = 1 ... N_{data})$$

Subspace

It can be shown, that the maximum in the N_ω dimensional space lies in an N_{data} dimensional subspace, that can be parametrized as:

$$A(\omega) = m(\omega) \exp\left(\sum_{i=1}^{N_{data}} s_i f_i(\omega) \right)$$

Two choice for basis functions: Bryan (Eur. Biophys J. 18, 165 (1990)) or Jakovác et al (Phys.Rev. D75 014506 (2007), `arXiv:0611017`)

Prior function

Four choices

$$m(\omega) = m_0 \qquad \text{We know nothing}$$
$$m(\omega) = m_0\omega^2 \qquad \text{Motivated by PT, we use this for know}$$
$$m(\omega) = m_0\omega \qquad \text{Kubo – formula}$$
$$m(\omega) = m_0\omega(b+\omega) \qquad \text{Kubo + PT}$$

Analysis with mock data

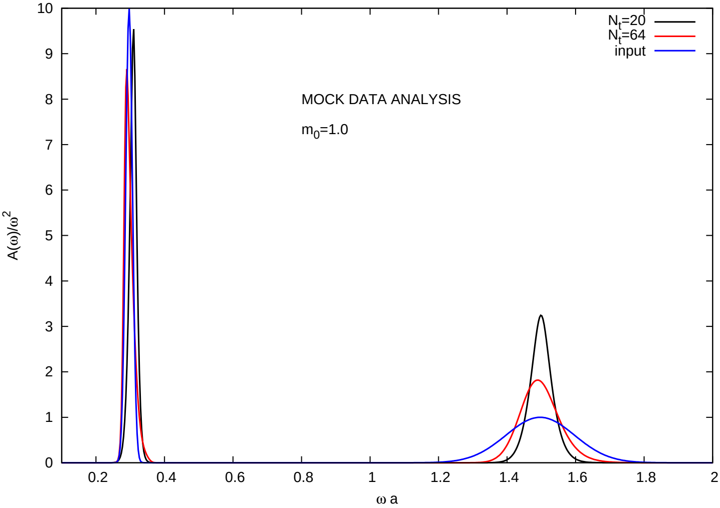
Strategy

- Write down an input spectral function A_{in}
- Generate correlators by integrating

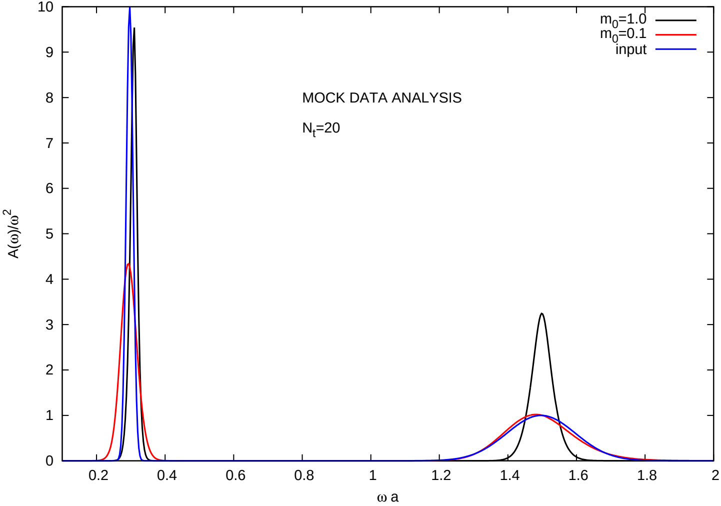
$$G(\tau) = \int K(\tau, \omega) A_{in}(\omega) d\omega$$

- Generate mock data by adding gaussian noise with variance $\sigma(\tau) = \eta \cdot \tau \cdot G_{mock}(\tau)$.
- Reconstruct spectral function with MEM.

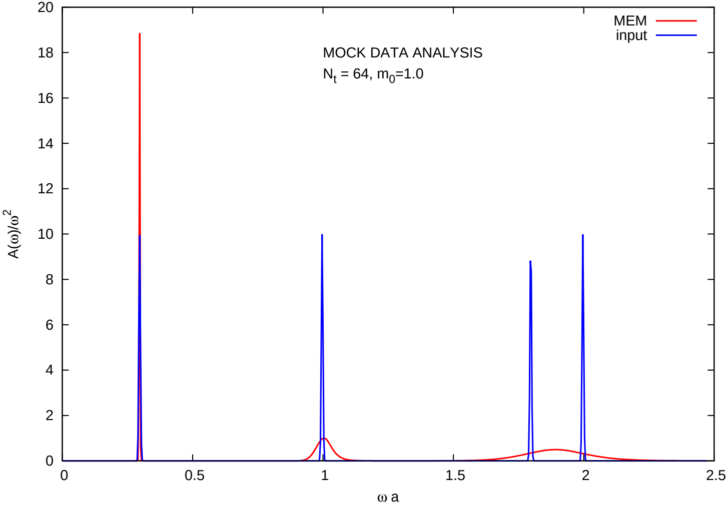
How many data points are enough?



Sensitivity on the prior function



Individual resolution of many peaks



Conclusion of the mock data analysis

Lessons learned

- MEM gives the correct qualitative features of the spectral function, but it is not a precise quantitative method.
- The peak positions agree well with the input, the shapes do not
- As long as the data points are not too noisy, O(10) point are enough for reconstruction.
- Features that remain unchanged by varying the prior are restricted by the data.
- Higher excitations can be merged into one broader peak.

Analysis with lattice QCD data

Lattice details

Gauge action = Symanzik tree-level improved gauge action
Fermion action = 2+1 dynamical Wilson fermions with 6 step stout smearing ($\rho = 0.11$) and tree-level clover improvement

a[fm]	am_{ud}	am_s	m_π	N_s	N_t	$T = \frac{1}{N_t a}$
0.057(1)	-0.00336	0.0050	545MeV	64	64	$\approx$ 0MeV
0.057(1)	-0.00336	0.0050	545MeV	64	28	123MeV
0.057(1)	-0.00336	0.0050	545MeV	64	20	173MeV
0.057(1)	-0.00336	0.0050	545MeV	64	18	192MeV
0.057(1)	-0.00336	0.0050	545MeV	64	16	216MeV
0.057(1)	-0.00336	0.0050	545MeV	64	14	247MeV
0.057(1)	-0.00336	0.0050	545MeV	64	12	288MeV

Analysis with lattice QCD data

Quenched approximation

In our simulation, u,d, and s quarks, are dynamical, but c quarks are not. They are in the quenched approximation. There are no internal charm quark loops, they are only at tree level.

Mass tuning

We had m_s and m_{ud} already tuned (Science 322 (2008) 1224-1227, `arXiv:0906.3599`) We adopted $m_c = 0.3022$, chosen so that the ratios of the masses of D_s and φ mesons are physical.

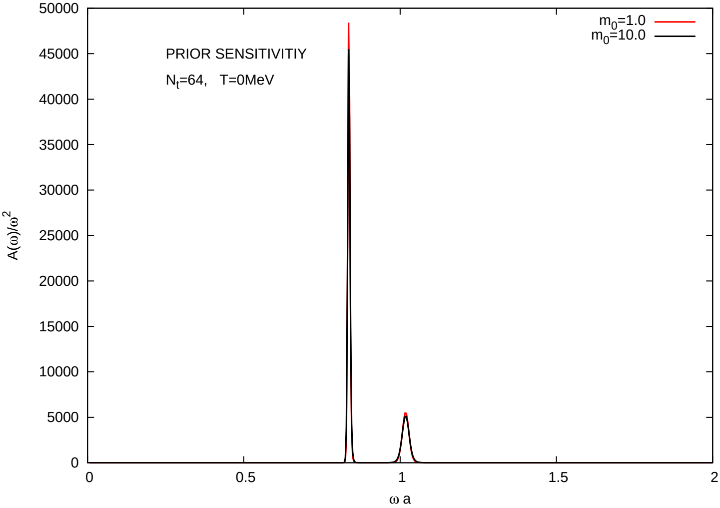
Preliminary results on J/Ψ

Expectations of heavy ion physicists

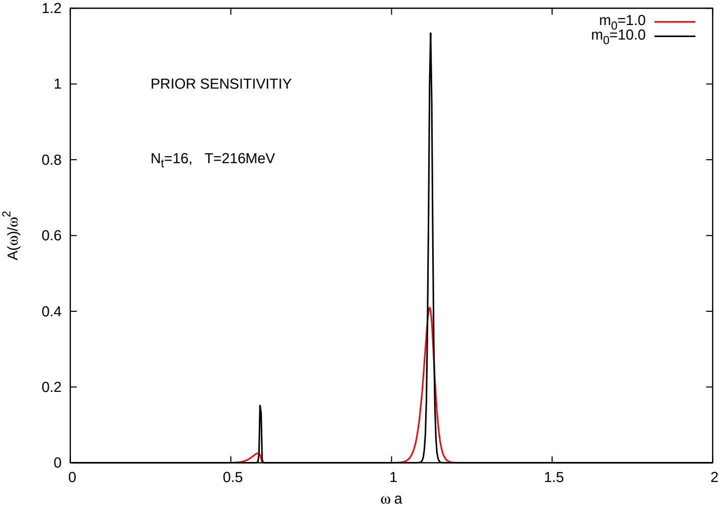
- At sufficiently high temperature, QCD undergoes a transition to a deconfined phase.
- Unlike light mesons, heavy mesons like J/Ψ may survive in the hot medium up to higher temperatures, before dissociating because of colour screening, and collisions within the medium.
- Their supression may be a good experimental signal on the formation of QGP.

Original paper on the idea:
T. Matsui, H. Satz, Phys. Lett. B178, 416 (1986).

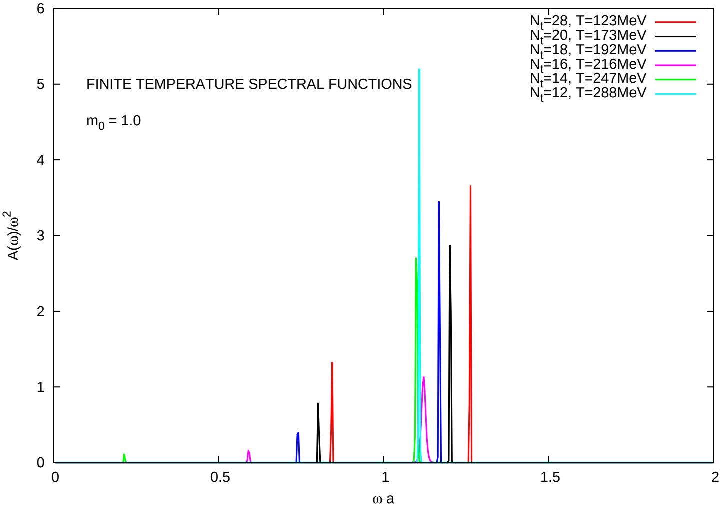
Cold lattice, prior function sensitivity



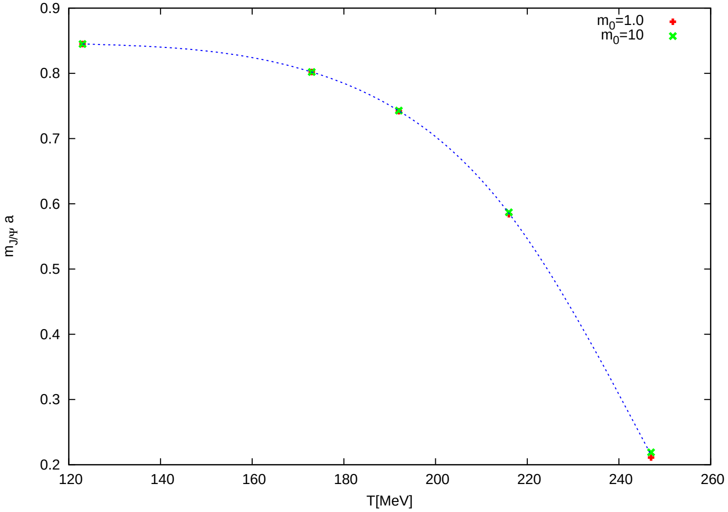
Hot lattice, prior function sensitivity



Temperature dependence



J/Ψ mass as a function of temperature



Future work

What's next?

- Doing the same analysis for η_c, D_s, ... mesons
- Doing the same analysis for bottonium states
- Checking the conclusions on the first peak by doing the same analysis with smeared operators
- Determining the masses of the first two-three states by diagonalizing cross-correlators
- Determining transport coefficients by looking at the $\omega \to 0$ limit
- Error bars!
- Continuum extrapolation?
- Anisotropic lattice calculations with dynamical light quarks