# Topological solitons in a coupled $\phi^4$ model

A. Halavanau, T. Romanczukiewicz, Ya. Shnir

Belarusian State University and Jagellonian University

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# Outline

- Introduction
- Single component $\phi^4$ model
- Extension of $\phi^4$ model
- Solitonic collisions
- Effective Lagrangian
- Modulated dynamics

# Solitons in non-linear physics

*Soliton:* This is a solution of a nonlinear partial differential equation which represents a solitary travelling wave, which:

- Is localized in space
- Has a constant shape
- Does not obey the superposition principle.

**Examples:**

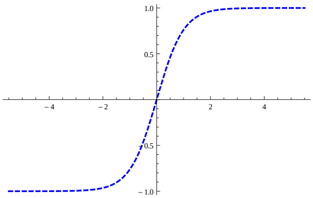
Optical fibres - NLSE

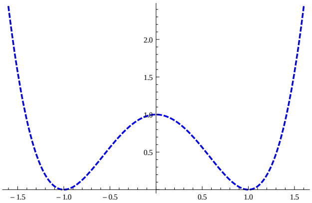
Josephson junctions - sine-Gordon model

Lattice QCD - caloron solutions

Superconductivity - Abrikosov-Nielsen-Olesen model

# Single component $\phi^4$ system




## Lagrangian and equation of motion

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\left(\phi^2 - 1\right)^2$$

$$\partial_{tt}\phi - \partial_{xx}\phi + 2\phi(\phi^2 - 1) = 0$$

## Static solution (kink/antikink)

$$\phi = \tanh(A(\pm x + x_0))$$

# Modes of $\phi^4$ kink

## Linear oscillations on the background

$$\phi = \phi_K + \delta\phi$$

where $\delta\phi$ can be expanded in a set:

$$\delta\phi = \sum_{n=0}^{\infty} C_n(t)\eta_n(x)$$

## Eigenfunctions

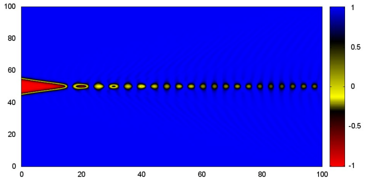
$$\eta_0(x) = \frac{1}{\cosh^2 x}$$
$$\eta_1(x) = \frac{\sinh x}{\cosh^2 x}$$
$$\eta_k(x) = e^{ikx}(3\tanh^2 x - 3ik\tanh x) \quad - e^{ikx}(1+k^2) + \mathrm{C.C.}.$$

# $\phi^4$ kink-antikink collisions

## Properties

- Annihilate or repel
- Produce oscillon state after annihilation
- Have bounce windows structure
- Fractal structure

Bounce windows and fractal structure refers to the energy exchange mechanism between translational and internal modes



# Bounce windows structure

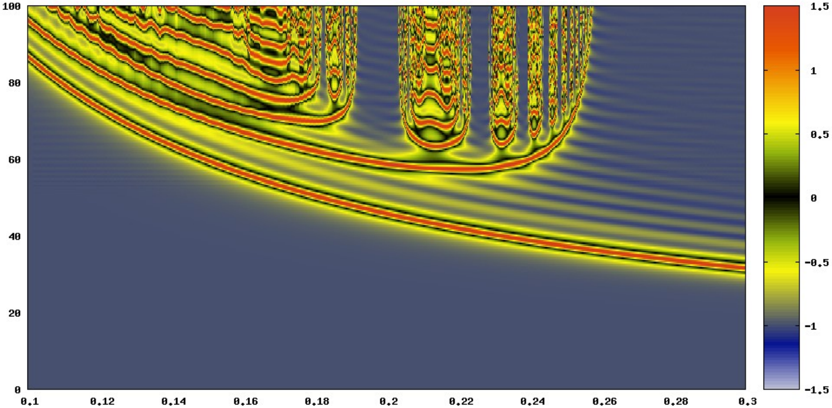



Figure: Bounce window structure of $K\bar{K}$ collision in a usual $\phi^4$ case

# Lagrangian and field equations

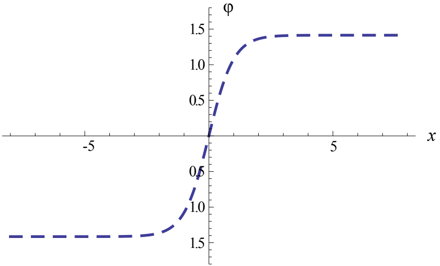
**Lagrangian of the coupled two-component model can be written as:**

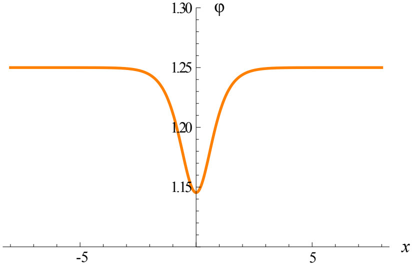
$$L = \frac{1}{2}[(\partial_t\phi_1)^2 - (\partial_x\phi_1)^2 - (\phi_1^2 - 1)^2] + \frac{1}{2}[(\partial_t\phi_2)^2 - (\partial_x\phi_2)^2 - (\phi_2^2 - 1)^2] + \kappa\phi_1^2\phi_2^2$$

**Field equations:**

$$\begin{cases} \partial_t^2\phi_1 - \partial_x^2\phi_1 + 2\phi_1(\phi_1^2 - 1) - 2\kappa\phi_1\phi_2^2 = 0 \\ \partial_t^2\phi_2 - \partial_x^2\phi_2 + 2\phi_2(\phi_2^2 - 1) - 2\kappa\phi_2\phi_1^2 = 0 \end{cases}$$

Wess-Zumino model with two coupled Majorana spinor fields
Montonen-Sarker-Trullinger-Bishop model with two scalar coupled fields


φ
1.5
1.0
0.5
-5
5
x
0.5
1.0
1.5


1.30
φ
1.25
1.20
1.15
-5
5
x

# Perturbative expansion

## Zero order approximation

$$\phi_1^{(0)} = \tanh(x)/\sqrt{1-\kappa}; \qquad \phi_2^{(0)} = 1/\sqrt{1-\kappa}$$

## Expansion of the fields

$$\phi_1 = \phi_1^{(0)} + \kappa\phi_1^{(1)} + \kappa^2\phi_1^{(2)} + \dots; \qquad \phi_2 = \phi_2^{(0)} + \kappa\phi_2^{(1)} + \kappa^2\phi_2^{(2)} + \dots;,$$

$$\phi_1^{(1)} = \sum_{n=0}^{\infty} A_n \tanh^{2n+1} x, \qquad \phi_2^{(1)} = \sum_{n=0}^{\infty} \frac{B_n}{\cosh^{2n} x}$$

$$\phi_1^{(1)} = \frac{16}{15}\tanh x - \frac{1}{3}\tanh^3 x - \frac{4}{75}\tanh^5 x - \frac{1}{75}\tanh^7 x + \cdots,$$
$$\phi_2^{(1)} = \frac{1}{2} - \frac{1}{6\cosh^2 x} - \frac{1}{28\cosh^4 x} - \frac{5}{728\cosh^6 x} - \frac{45}{11284\cosh^8 x} + \dots$$

# Different collision channels

## Topological charge

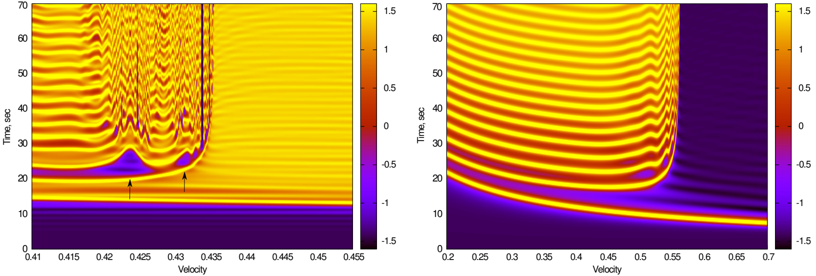
$$Q_i = \frac{1}{2}\sqrt{1-\kappa}\int\limits_{-\infty}^{\infty} dx\,\frac{\partial \phi_i}{\partial x}, \qquad i = 1,2$$

## Interesting channels

$$(-1,0)+(1,0) \to \begin{cases} (0,0) \\ (0,-1)+(0,1) \\ (-1,0)+(1,0) \end{cases}$$

$$(-1,1)+(-1,1) \to \begin{cases} (0,0) \\ (-1,1)+(-1,1) \end{cases}$$

# Topological flipping and double kink configuration



# Effective Lagrangian

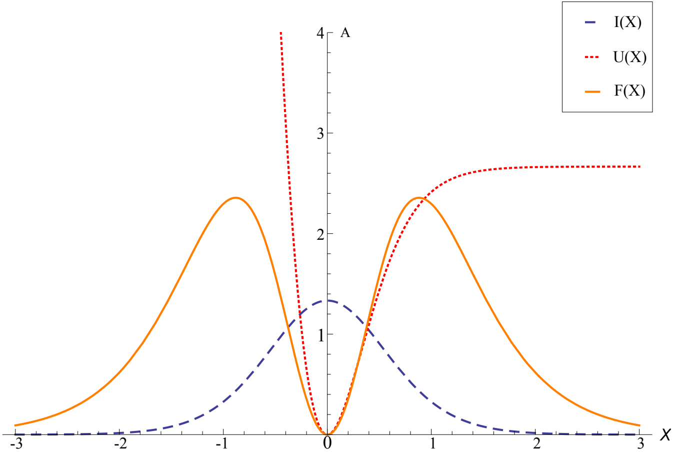
X(t) - mutual distance; A(t),B(t) - internal mode perturbation

$$\phi_{ansatz} = 2\left[\frac{\tanh(x+X(t)) - \tanh(x-X(t)) - 1}{\sqrt{1-\kappa}} + \frac{1}{2}A(t)(\eta(x+X(t)) - \eta(x-X(t))) + \frac{1}{2}B(t)(\eta(x+X(t)) + \eta(x-X(t)))\right]$$

Simplified by dropping out high-order terms

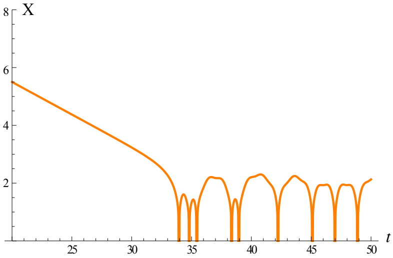
$$L_{eff} = \frac{2}{1-\kappa}\left[(M + I(X))\dot{X}^2 - U(X) + F(X)A(t)\sqrt{1-\kappa} + F(X)B(t)\sqrt{1-\kappa}\right] + \frac{4}{3}\dot{A}^2 - 8\omega^2 A^2 + \frac{4}{3}\dot{B}^2 - 8\omega^2 B^2$$

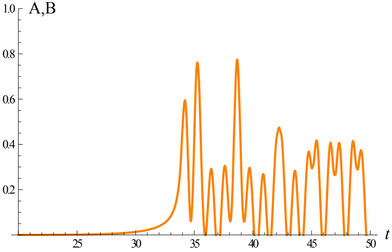
# Potential and coupling functions




Note: $U(X)$ is similar to Lennard-Jones potential. Instantonic liquid?

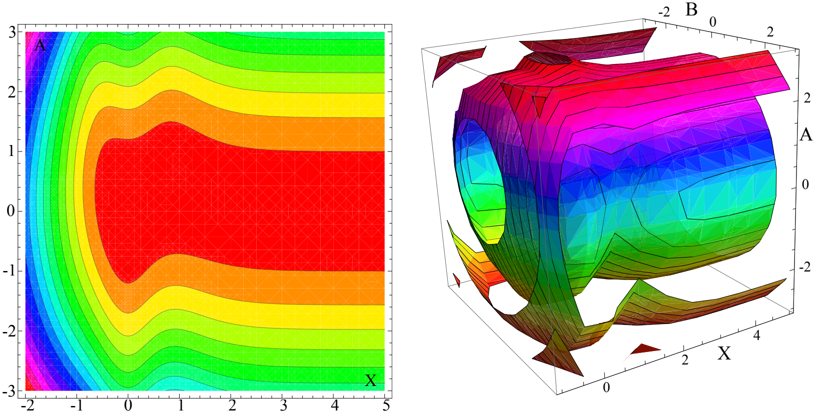
# Solutions for X and A,B




Note: Annihilation is represented as many-bouncing state

# Effective potential

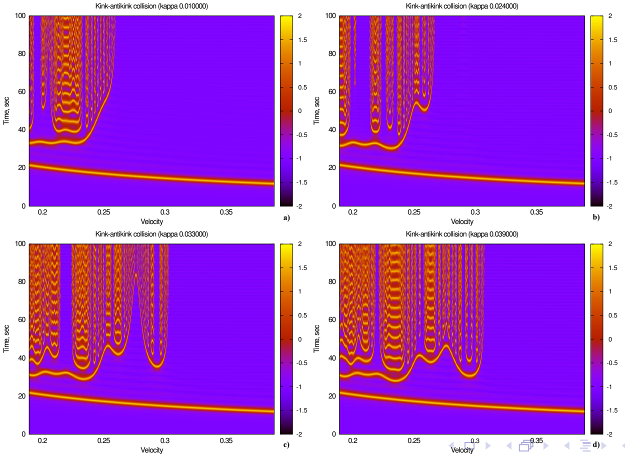



Contour plots for $V_{eff}$ from $H_{eff} = \frac{P_X^2}{(M+I(X))} + P_A^2 + V_{eff}$ for single-component (left) and double-component (right)

# Modulated $K\bar{K}$ dynamics

## Initial configuration

$$\phi_1 = \tanh(x)/\sqrt{1-\kappa}; \qquad \phi_2 = 1/\sqrt{1-\kappa} + A\sin(kx+\omega t)$$



# Conclusions and outlook

- There are two different types of solitons, the double kink and the lump kink, in both cases our numerical simulations of collisions between the various types of solitons reveal some chaotic resonance behavior
- The resonant scattering structure in the $K\bar{K}$ collisions is related with energy exchange between the translational mode and the inner mode of the kink-antikink system
- Effective Lagrangian which can describe the main properties of the model is proposed
- Coupling of the model with harmonic perturbation in the second sector allows us manipulate the fine structure of the process providing harmonic time dependency of the translational and inner modes of the kinks
- Perturbative analytical analyse is quite consistent with numerical results