Simulations of binary neutron stars and black hole-torus systems in general relativity

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Outline

Lecture 1: General relativistic hydrodynamics equations.
Einstein's equations.
Gravitational radiation.
Numerical methods for conservation laws.

Lecture 2: Binary neutron stars.
Black-hole torus systems.

Motivation:

Multi-messenger astronomy: photons, cosmic rays, neutrinos, gravitational waves.

www.aspera-eu.org

From the ASPERA roadmap (as of Nov 2011)

ASPERA: European network of national agencies responsible for coordinating and funding national research efforts in Astroparticle Physics.

*[...] **a few projects whose funding has to be kept at substantial levels**, be it because they have an impressive momentum that needs to be maintained, because they enter a phase with high discovery potential, because they **go hand in hand with LHC physics**, because they are technologically ready and have a worldwide community behind them, or finally, because a delay of crucial decisions and funding could even jeopardize the project. In this spirit, we recommend the following projects and urge the agencies to join their forces in order to provide effective and substantial support:*

[...]

• ***Gravitational waves**: With Advanced VIRGO, Advanced LIGO and GEO-HF, a discovery in the next five years becomes highly probable. This would open an entirely **new window to the Universe.** **We urge the agencies to continue to substantially support the ongoing and planned upgrades to advanced detectors.***

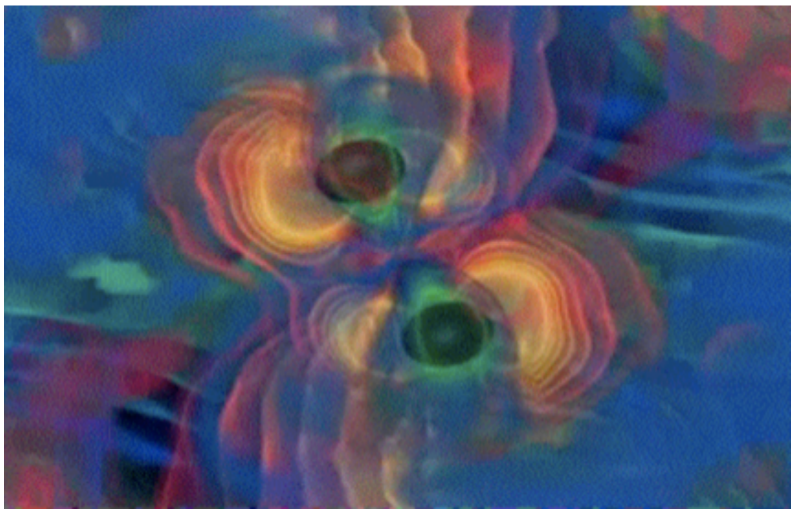
*The path for research in gravitational waves beyond the advanced detectors foresees **two very large-scale projects** (costs on the billion Euro scale): the Earth-bound **Einstein Telescope (E.T.)** and the space-bound **LISA** project. In today's perspective, E.T. construction would start at the end of the decade and after the first detection of gravitational waves with the advanced detectors. We also look forward to the results of LISA-Pathfinder. **We renew our strong support of the LISA mission and preparatory work on E.T.***

ASPERA

The Einstein Telescope project aims to the realization of a crucial research infrastructure in Europe: a third generation Gravitational Wave observatory. Supported as Design Study by the European Commission under the Framework Programme 7 (Grant Agreement 211743).

The ASPERA organization includes ET in the "Magnificent Seven" list in its roadmap.

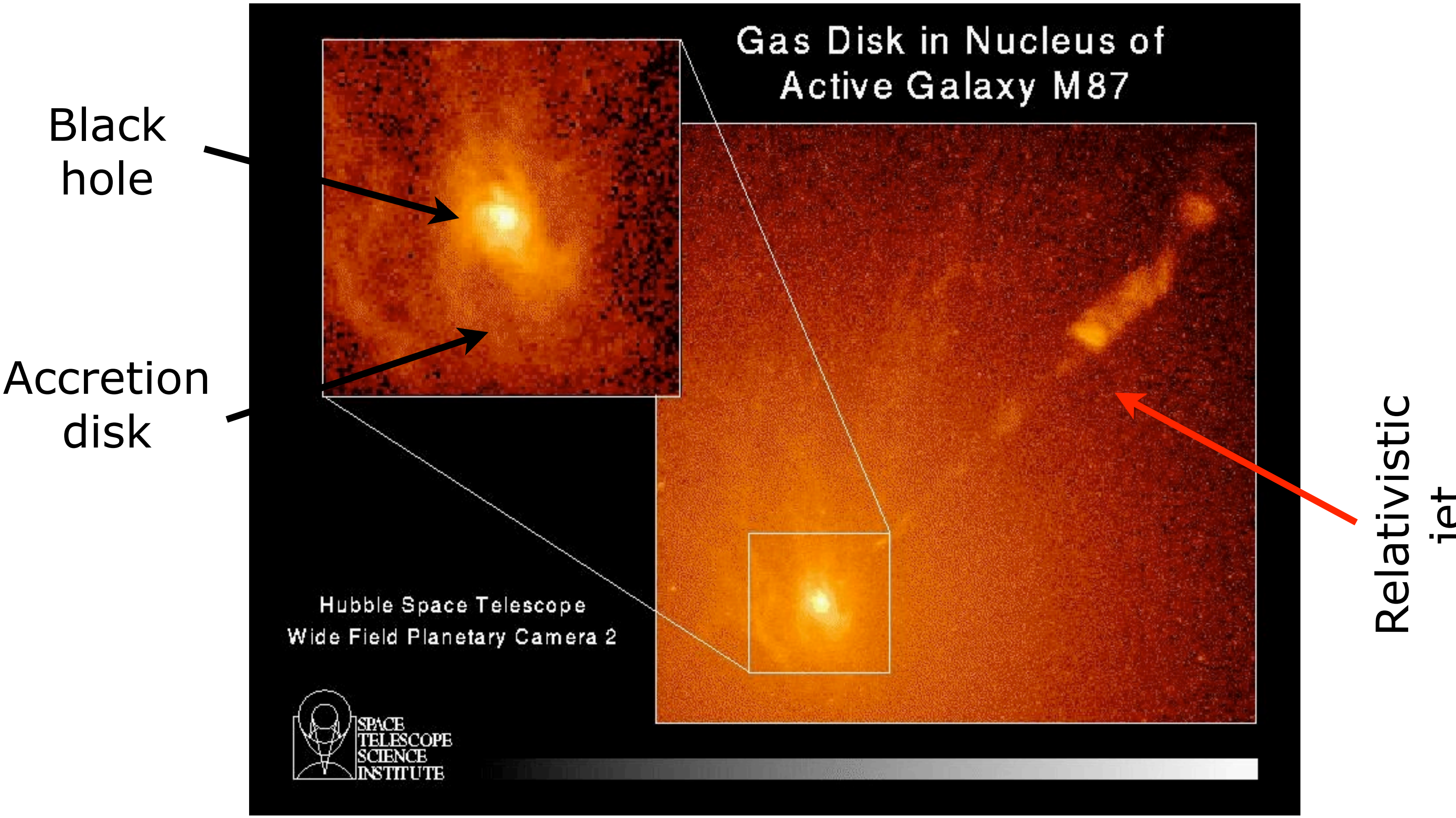
Numerical relativity in non-vacuum spacetimes, which amounts to solving the Einstein-Euler system of equations, is providing the most accurate gravitational waveforms that can be used as templates to help detectors dig out the (tiny) signal buried in the noise.



These two lectures will discuss the theoretical framework to perform numerical relativity simulations, and will focus on a particular important scenario in gravitational wave astronomy - binary neutron stars.

Introduction

Many astrophysical objects of interest (planets, stars, jets, galaxies), as well as the ISM and the IGM can be modelled at a theoretical model as fluids or plasmas (continuum dynamics).



(Compressible) **fluid dynamics** plays a central role in many numerical applications of Computational Astrophysics.

Computational Relativistic Astrophysics

Natural domain of general relativistic hydrodynamics (GRHD) and magneto-hydrodynamics (GRMHD) is the field of relativistic astrophysics. Essential role in the description of **gravitational collapse** and the formation of **compact objects** (neutron stars and black holes).

Scenarios characterised by the **presence of high speed flows** (close to the speed of light), strong **shock waves** (discontinuous solutions), and **strong gravitational fields.**

GRHD/GRMHD equations are nonlinear hyperbolic systems. Solid mathematical foundations and accurate numerical methodology imported from CFD. A "preferred" choice: high-resolution shock-capturing schemes written in conservation form.

Time-dependent evolutions of fluid flow coupled to the spacetime geometry (Einstein's equations) possible through accurate, large-scale numerical simulations. Some scenarios can be described in the test-fluid approximation: GRHD/GRMHD computations in curved backgrounds (highly mature, particularly GRHD case).

Black hole-torus systems are common in the universe:

- The central region of **AGNs** is believed to consist of a supermassive black hole of mass $M_{BH} \sim 10^6$-$10^{10} M_{sun}$ surrounded by a torus. Such systems may form through the **collapse of supermassive stars** (Rees 1984; Shibata and Shapiro 2002).

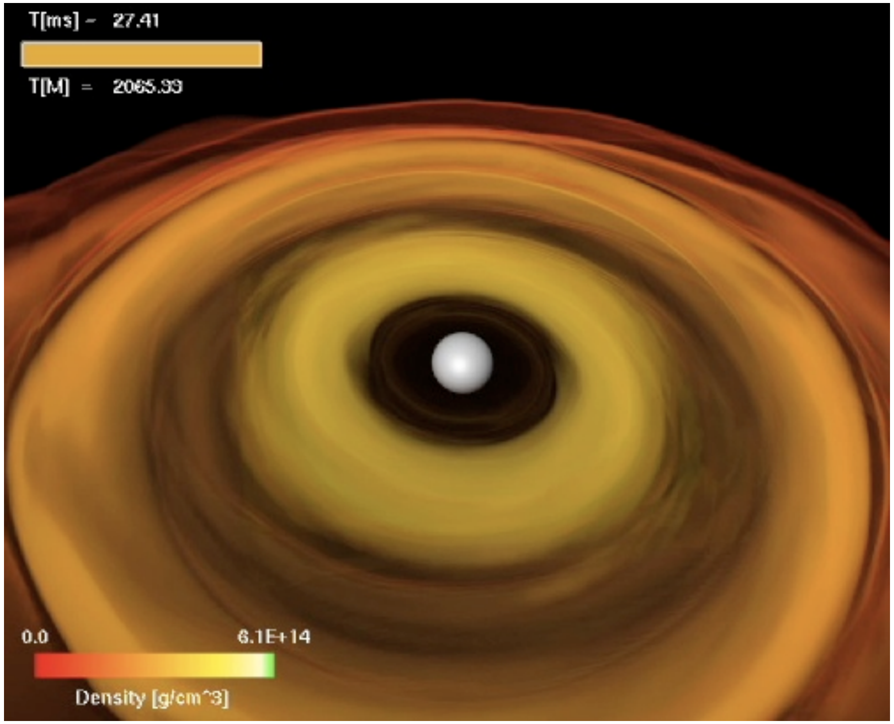
- **Mergers of NS-NS binaries and BH-NS binaries** often result in a BH and a torus (Rezzolla et al 2010, Kiuchi et al 2010).

- Such systems can also be produced at the end of the life of **massive stars** (Heger et al 2003).

- The merger and collapsar scenarios are linked to short- and long-duration gamma-ray bursts (GRBs), respectively (Narayan et al 1992; Woosley 1993).

Numerical relativity simulations of non-vacuum spacetimes have reached a status where accurate descriptions of such distinctive scenarios of relativistic astrophysics are possible.

Black hole - torus system



Numerical simulations of NS-NS show that most of the material disappears beyond the event horizon in a few ms.

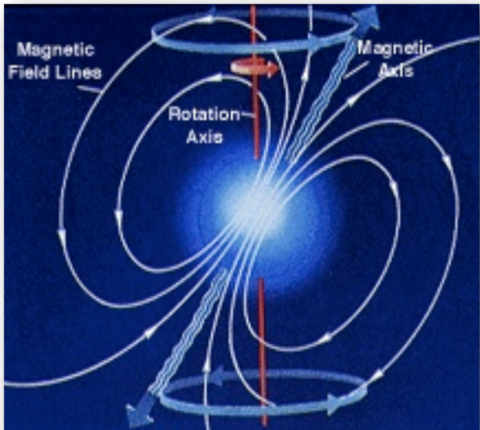
As a result, a thick accreting disk or torus with mass of about 10% of total mass (upper limit) may be formed.

The formation and evolution of BH-torus systems not yet been observed - sites opaque to electromagnetic waves (EWs) due to their intrinsic high density and temperature. **Gravitational waves** are much more transparent than EWs regarding absorption and scattering with matter.

If BH-torus systems emitted detectable GWs, it would be possible to explore their formation and evolution, along with the prevailing hypotheses that associate them to GRB engines.

Gravitational radiation: experimental evidence

The only experimental evidence of the existence of gravitational waves comes from the study of binary pulsars.

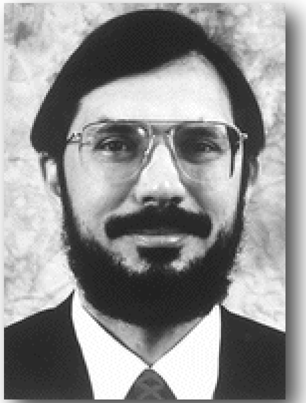


What is a pulsar? (cf. Prof. Blaschke's talk)

- very compact star - as massive as the Sun but with a radius of about 10 km.
- very rapidly-rotating star - some pulsars rotate around the axis once per second but others do it hundreds of times per second.

The first binary pulsar, PSR1913+16, was discovered by radio-astronomers Russell Hulse and Joseph Taylor in 1974.

It is located in the Milky Way, its orbital period is ~7.5 hours and the radio signal is received ~17 times per second.



Hulse

Taylor

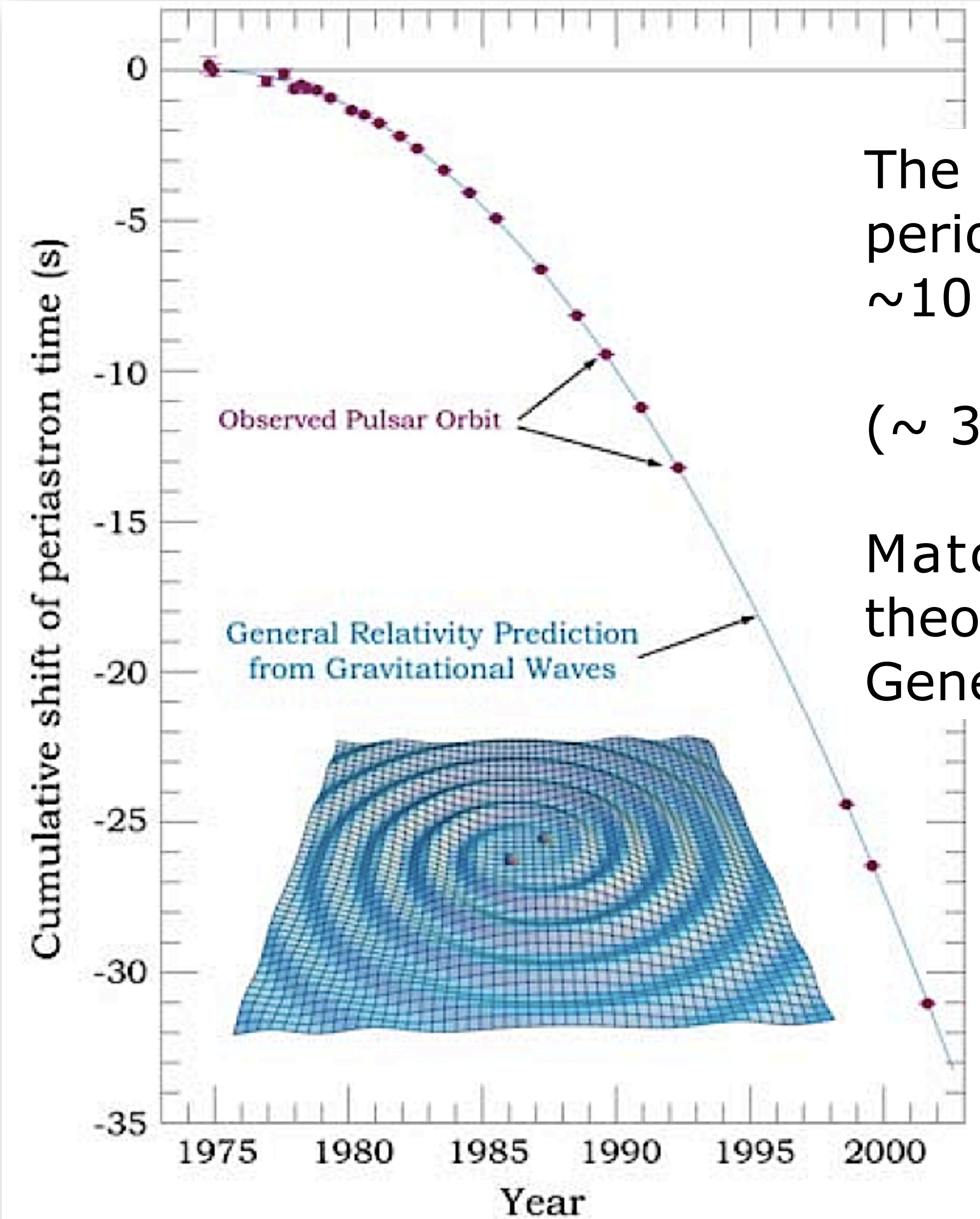
Decrease of the orbital period in the Hulse-Taylor binary pulsar

Two orbiting neutron stars:

- mass of each star ~ 1.4 solar masses.
- orbital period ~ 7.5 hours.
- the stars revolve around the center of mass at a speed of a thousandth of the speed of light.

According to General Relativity the binary pulsar has to radiate gravitational waves. Such emission brings the two stars closer together, decreasing the orbital period and increasing the rotational frequency.

The final fate of the binary pulsar will be the **merger of the two neutron stars,** which will produce a tremendous **"burst" of gravitational radiation** that would be detected by the current instruments.



The decrease of the orbital period of PSR1913+16 is ~10 μs per year.

(~ 3.1 mm in every rotation).

Matches exactly with the theoretical rate predicted by General Relativity.

The merger of the two stars will occur in about 240 million years ...

Astrophysical sources of gravitational waves

- Merger of compact binaries: "chirp" signal

- Gravitational collapse, i.e. supernovae / gamma-ray burts: "burst" signal

- Rotating neutron stars (pulsars): periodic signal

- Cosmological sources.

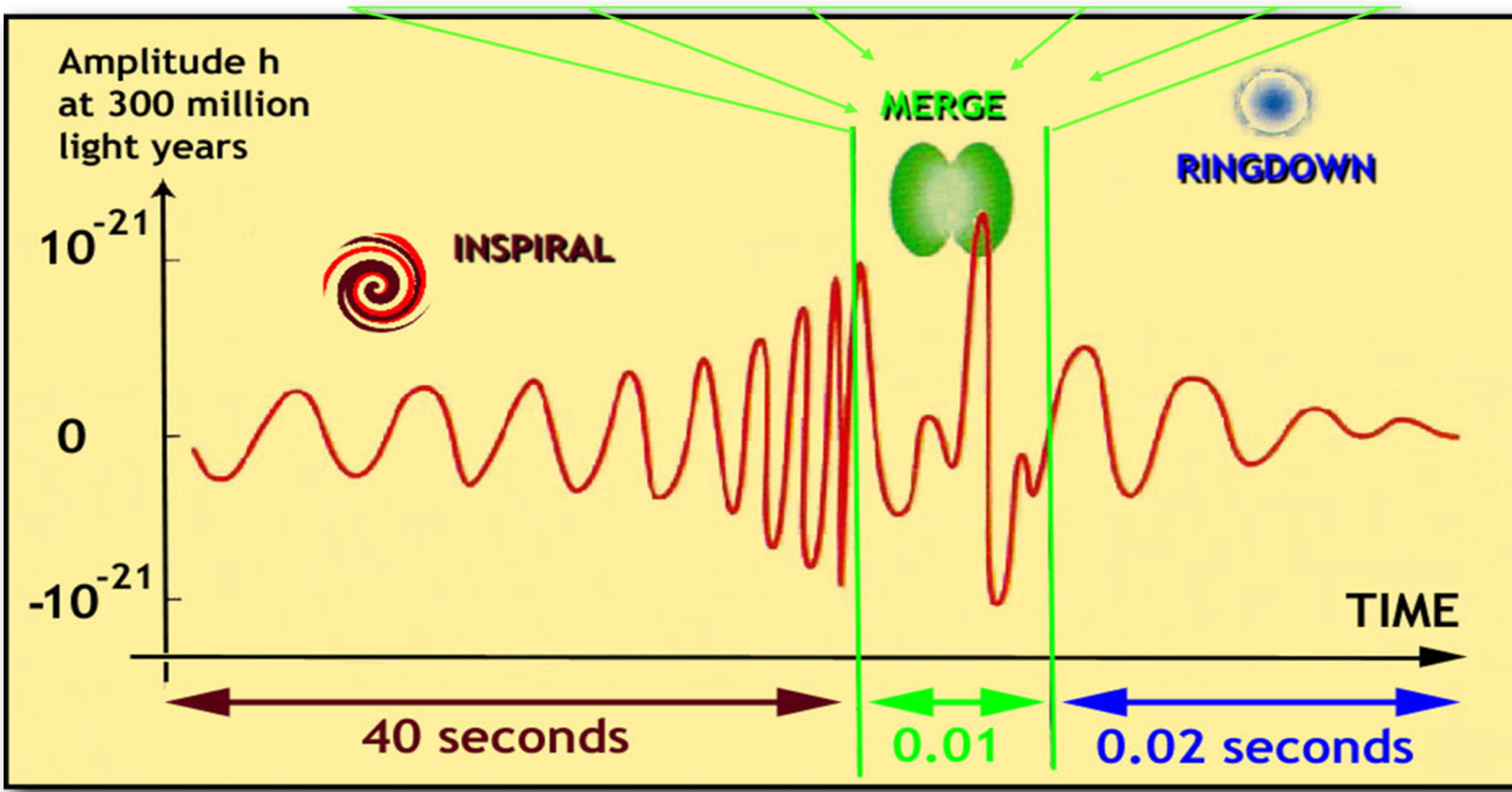
All of these sources, each in its own particular range of frequencies, contribute to the "writing of Einstein's unfinished symphony" ...

Suggested reading: Marcia Bartusiak, *Einstein's unfinished symphony: Listening to the sounds of space-time.*

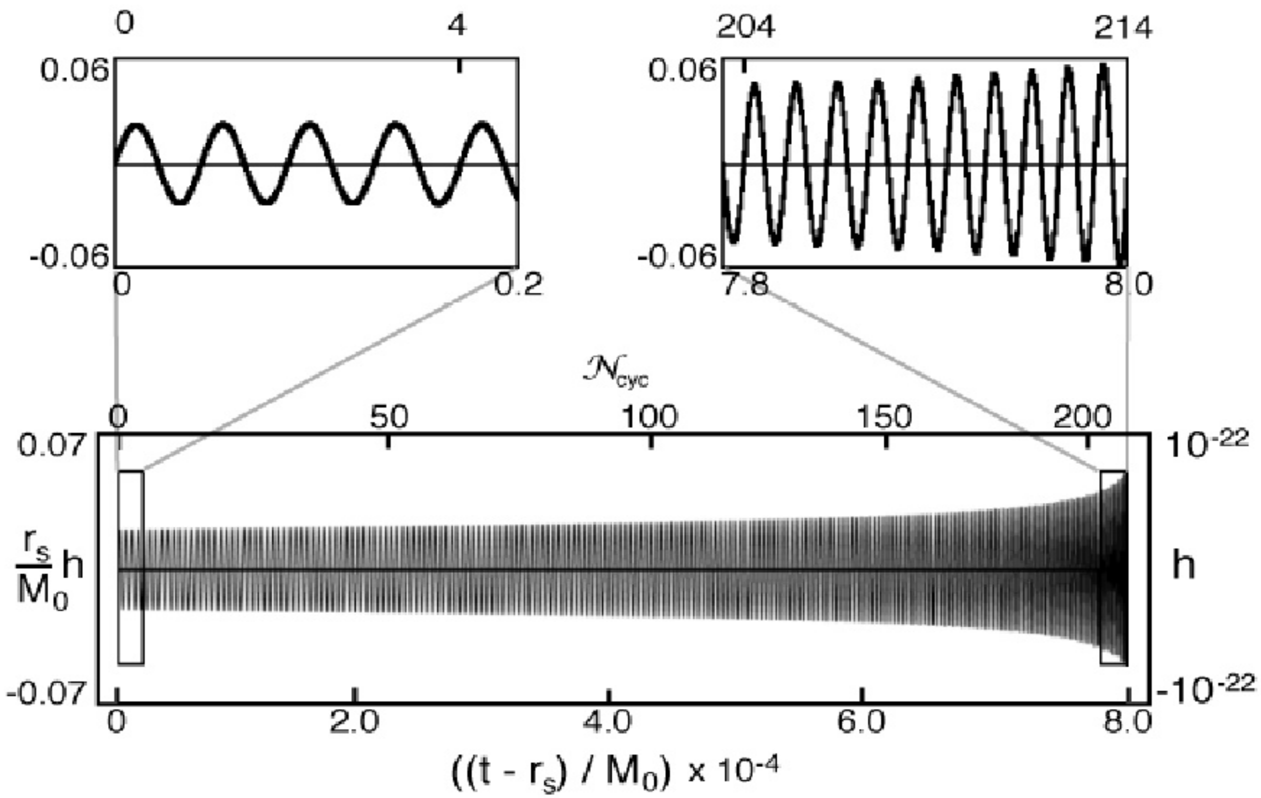
www.marciabartusiak.com

Merger of compact binaries

Compact binary systems formed by two neutron stars, two black holes, or a mixed configuration.



Duez et al (2001)



The gravitational signal **increases** both in **frequency** and in **amplitude** as the two objects are brought closer together.

Accurate templates of the gravitational radiation produced in the moment of largest intensity (the merger) can only be obtained solving numerically the Einstein's equations coupled to the hydrodynamics equations: **Numerical Relativity.**

Basic theoretical model

Our best approximation to "reality"

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \quad (\text{field eqs}: \ 6+6+3+1)$$

$$\nabla_\mu T^{\mu\nu} = 0 \,, \quad (\text{cons. en./mom.}: \ 3+1)$$

$$\nabla_\mu (\rho u^\mu) = 0 \,, \quad (\text{cons. of baryon no}: \ 1)$$

$$p = p(\rho, \epsilon, \ldots) \,. \quad (\text{EoS}: \ 1 + \ldots)$$

Still very crude, though it can be improved: microphysics for the EoS, magnetic fields, dissipative fluids, radiative transfer, ...

$$\nabla^*_\nu F^{\mu\nu} = 0, \ (\text{Maxwell eqs.}: \ \text{induction}, \ \text{zero div.})$$

$$T_{\mu\nu} = T^{\text{fluid}}_{\mu\nu} + T^{\text{em}}_{\mu\nu} + \ldots$$

General relativistic hydrodynamics equations

F. Banyuls et al, "Numerical 3+1 general relativistic hydrodynamics: a local characteristic approach", Astrophysical Journal, 476, 221 (1997)

J.A. Font, "Numerical hydrodynamics and magnetohydrodynamics in general relativity", Living Reviews in Relativity (2008) (www.livingreviews.org)

(Classical) hydrodynamics equations

Conservation laws of mass, momentum and energy. First-order hyperbolic system of conservation laws.

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f^i}}{\partial x^i} = \vec{s}(\vec{u})$$

$$\begin{aligned} \vec{u} &= (\rho, \rho v^j, e) && \text{state vector} \\ \vec{f^i} &= (\rho v^i . \rho v^i v^i + p\delta^{ij}, (e+p)v^i) && \text{fluxes} \\ \vec{s} &= \left(0, -\rho\frac{\partial \Phi}{\partial x^j} + Q^j_M, -\rho v^i \frac{\partial \Phi}{\partial x^i} + Q_E + v^i Q^i_M\right) && \text{sources} \end{aligned}$$

$\vec{g}$ conservative external force field (e.g. gravity) $\quad \vec{g} = -\nabla\Phi, \quad \Delta\Phi = 4\pi G\rho$

Q^i_M, Q_E source terms in momentum and energy eqs. due to coupling between matter and radiation (transport phenomena).

Hyperbolic equations have finite propagation speed: information cannot travel with speed higher than that given by the largest characteristic curves of the system.

Range of influence of solution bounded by eigenvalues of Jacobian matrix of the system.

$$A = \frac{\partial \vec{f^i}}{\partial \vec{u}} \Rightarrow \lambda_0 = v_i, \; \lambda_\pm = v_i \pm c_s$$

In general relativity the hydrodynamics equations are obtained from the **local conservation laws of the stress-energy tensor and of the matter current density** (continuity equation):

$$\nabla_\mu(\rho u^\mu) = 0 \qquad \nabla_\mu T^{\mu\nu} = 0$$

Equations of motion

∇_μ is the covariant derivative associated with the 4-metric $g_{\mu\nu}$. The density current is given by $J^\mu=\rho u^\mu$, u^μ representing the fluid 4-velocity and ρ the rest-mass density in a locally inertial reference frame.

The stress-energy tensor for a **non-perfect fluid** is defined as:

$$T^{\mu\nu} = \rho(1+\varepsilon)u^\mu u^\nu + (p - \mu\Theta)h^{\mu\nu} - 2\xi\sigma^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu$$

where ε is the specific internal energy density, p is the pressure, $h^{\mu\nu}$ is the spatial projection tensor, $h^{\mu\nu}=u^\mu u^\nu+g^{\mu\nu}$, and q^μ is the energy flux.

In addition, μ and ξ are the shear and bulk viscosity coefficients. The expansion, Θ, describing the divergence or convergence of the fluid world lines is defined as $\Theta=\nabla_\mu u^\nu$. The symmetric, trace-free, and spatial shear tensor $\sigma^{\mu\nu}$ is defined by:

$$\sigma^{\mu\nu} = \frac{1}{2}(\nabla_\alpha u^\mu h^{\alpha\nu} + \nabla_\alpha u^\nu h^{\alpha\mu}) - \frac{1}{3}\Theta h^{\mu\nu}$$

In the following we will neglect non-adiabatic effects, such as viscosity or heat transfer, assuming the stress-energy tensor to be that of a **perfect fluid:**

$$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

where we have introduced the relativistic specific enthalpy, defined as:

$$h = 1 + \varepsilon + \frac{p}{\rho}$$

Introducing an explicit coordinate chart, the previous conservation equations read:

$$\boxed{\begin{aligned} \frac{\partial}{\partial x^\mu}(\sqrt{-g}\rho u^\mu) &= 0 \\ \frac{\partial}{\partial x^\mu}(\sqrt{-g}T^{\mu\nu}) &= \sqrt{-g}\Gamma^\nu_{\mu\lambda}T^{\mu\lambda} \end{aligned}}$$

x^0 foliation of the spacetime with hypersurfaces (coordinatised by x^i).

volume element

$$\sqrt{-g}, \;\; g = \det(g_{\mu\nu})$$

Christoffel symbols

$$\Gamma^\nu_{\mu\lambda}$$

The system formed by the equations of motion and the continuity equation must be supplemented with an **equation of state** (EOS) relating the pressure to some fundamental thermodynamical quantities.

$$p = p(\rho, \varepsilon)$$

Ideal gas: $$p = (\Gamma - 1)\rho\varepsilon$$

Polytrope: $$p = \kappa\rho^{\Gamma}, \quad \Gamma = 1 + \frac{1}{N}$$

In the "test-fluid" approximation (fluid's self-gravity neglected), the dynamics of the matter fields is fully described by the previous conservation laws and the EOS.

When such approximation does not hold, the previous equations must be solved in conjunction with Einstein's equations for the gravitational field which describe the evolution of a dynamical spacetime:

$$\nabla_\mu(\rho u^\mu) = 0$$

$$\nabla_\mu T^{\mu\nu} = 0$$

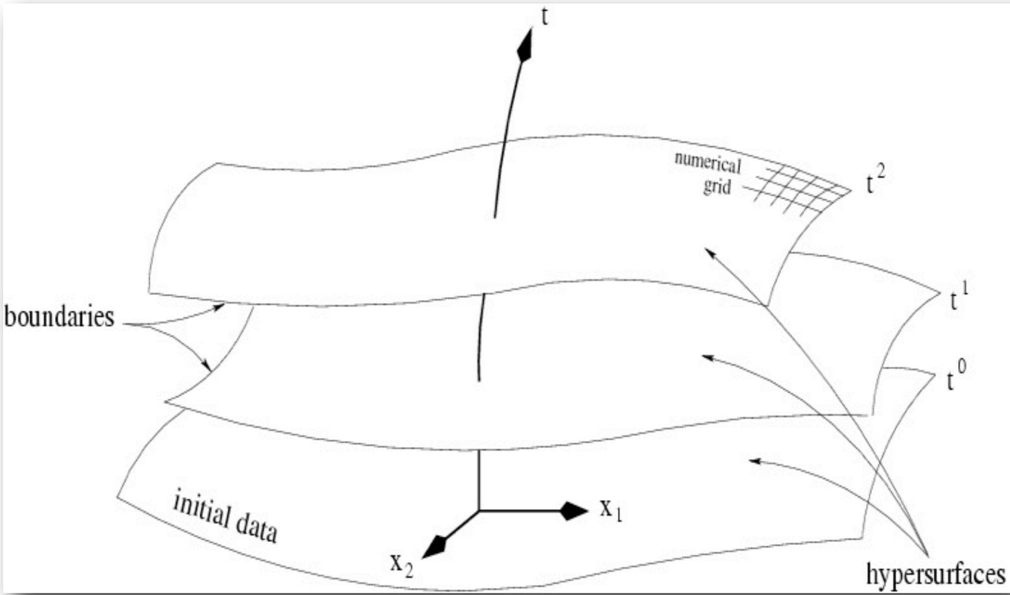
$$p = p(\rho, \varepsilon)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

Einstein's equations

(Newtonian analogy: Euler equations + Poisson equation)

The most widely used approach to solve Einstein’s equations in Numerical Relativity is the so-called Cauchy or 3+1 formulation (IVP).



Spacetime is foliated with a set of non-intersecting spacelike hypersurfaces Σ.

Within a given surface distances are measured with the spatial 3-metric.

Two kinematical variables describe the evolution from one hypersurface to the next: the lapse function α which describes the rate of proper time along a timelike unit vector n^μ normal to the hypersurface, and the shift vector β^i, spatial vector which describes the movement of coordinates in the hypersurface.

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

3+1 GR Hydro equations: formulations

$$\frac{\partial}{\partial x^\mu}(\sqrt{-g}\rho u^\mu) = 0$$

$$\frac{\partial}{\partial x^\mu}(\sqrt{-g}T^{\mu\nu}) = \sqrt{-g}\Gamma^\nu_{\mu\lambda}T^{\mu\lambda}$$

There exist different formulations depending on:

1. Choice of slicing: level surfaces of x^0 can be spatial (3+1) or null (CIVP)

2. Choice of physical (primitive) variables (ρ, ε, u^i …)

Wilson (1972) wrote the system as a set of advection equation within the 3+1 formalism. Non-conservative.

Conservative formulations well-adapted to numerical methodology were developed in the 1990s:

- Martí, Ibáñez & Miralles (1991): 1+1, general EOS
- Eulderink & Mellema (1995): covariant, perfect fluid
- Banyuls et al (1997): 3+1, general EOS
- Papadopoulos & Font (2000): covariant, general EOS

Non-conservative formulation

The use of Eulerian coordinates in multidimensional numerical relativistic hydrodynamics started with the seminal work of Wilson (1972).

Introducing the basic dynamical variables D, S_μ, and E, i.e. the relativistic density, momenta, and energy, respectively, defined as:

$$D = \rho u^0, \quad S_\mu = \rho h u_\mu u^0, \quad E = \rho \varepsilon u^0$$

The equations of motion in Wilson's formulation are:

$$\boxed{\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}D)}{\partial x^0} + \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}DV^i)}{\partial x^i} &= 0 \\
\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}S_\mu)}{\partial x^0} + \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}S_\mu V^i)}{\partial x^i} &+ \frac{\partial p}{\partial x^\mu} + \frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\mu}\frac{S_\alpha S_\beta}{S^0} = 0 \\
\frac{\partial(\sqrt{-g}E)}{\partial x^0} + \frac{\partial(\sqrt{-g}EV^i)}{\partial x^i} &+ p\frac{\partial(\sqrt{-g}u^0V^\mu)}{\partial x^\mu} = 0
\end{aligned}}$$

with the "transport velocity" given by $V^\mu = \dfrac{u^\mu}{u^0}$

A direct inspection of the system shows that the equations are written as a coupled set of **advection equations.**

Conservation of mass:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \quad \text{if } u(x,t) = a \Rightarrow \frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0}$$

(linear advection eq.)

Sidesteps an important guideline for the formulation of nonlinear hyperbolic systems of eqs, the preservation of their conservation form.

This is a necessary feature to guarantee correct evolution in regions of entropy generation (i.e. shocks). As a result, some amount of numerical dissipation (artificial viscosity) must be used to stabilize the numerical solution across discontinuities.

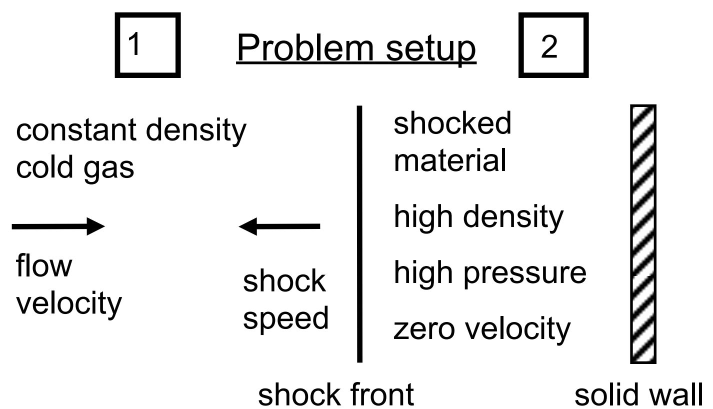
Formulation showed some limitations in dealing with situations involving ultrarelativistic flows, as first pointed out by Centrella & Wilson (1984).

Norman & Winkler, in their 1986 paper "*Why ultrarelativistic hydrodynamics is difficult?*" performed a comprehensive numerical study of such formulation in the special relativistic limit.

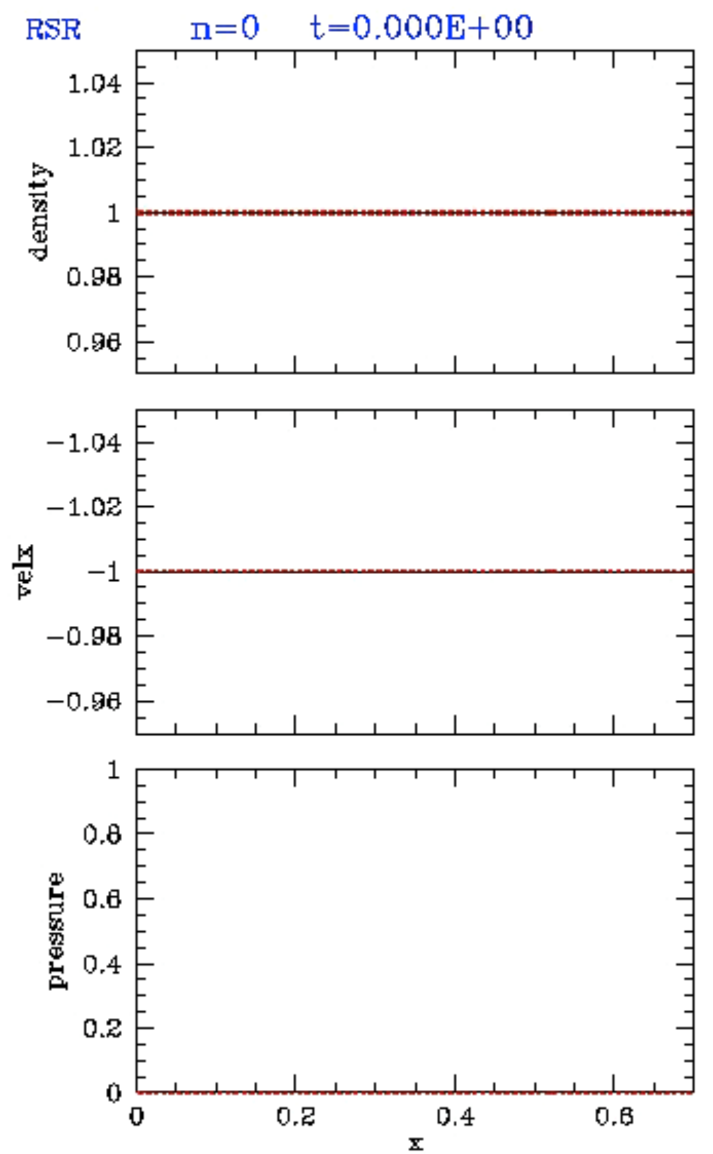
Relativistic shock reflection test

The relativistic shock reflection problem was among the 1D tests considered by Norman & Winkler (1986).

This is a demanding test involving the heating of a cold gas which impacts at relativistic speed with a solid wall creating a shock which propagates off the wall.



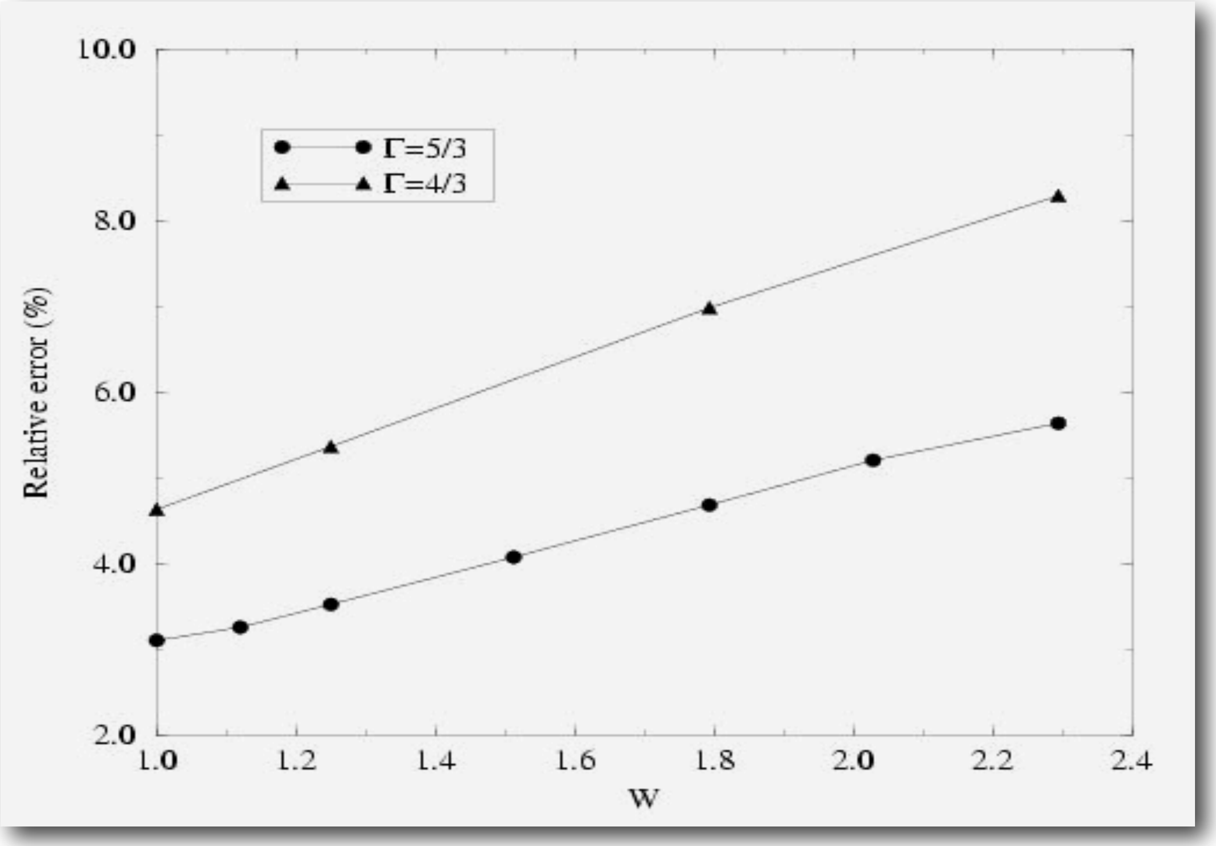
Analytic solution: $\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1} + \frac{\gamma}{\gamma - 1}\varepsilon_2, \quad \varepsilon_2 = W_1 - 1$



(Martí & Müller, 2003)

Non-conservative formulations show limitations to handle ultrarelativistic glows (Centrella & Wilson 1984, Norman & Winkler 1986).

Relative error relativistic shock reflection test as a function of Lorentz factor W of incoming gas. For W≈2 (v≈0.86c), 5-7% error (depending on EOS adiabatic index); shows linear increase with W.



Ultrarrelativistic flows could only be simulated resorting to conservative formulations. (Martí, Ibáñez & Miralles 1991; Marquina et al 1992)

Conservative formulation (Banyuls et al 1997)

Numerically, the hyperbolic and conservative nature of the GRHD equations allows to design a solution procedure based on the characteristic speeds and fields of the system, translating to relativistic hydrodynamics existing tools of CFD.

This procedure departs from earlier approaches, most notably in avoiding the need for artificial dissipation terms to handle discontinuous solutions as well as implicit schemes as proposed by Norman & Winkler (1986).

3+1: spacetime foliation with constant *t* spatial hypersurfaces Σ_t

Line element: $$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

Eulerian observer: at rest on the hypersurface; moves from Σ_t to $\Sigma_{t+\Delta t}$ along the normal to the hypersurface, with velocity:

$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$

The extension of high-resolution shock-capturing (HRSC) schemes from classical fluid dynamics to relativistic hydrodynamics accomplished in three steps:

1. Casting the GRHD equations as a system of conservation laws.
2. Identifying the suitable vector of unknowns.
3. Building up an approximate Riemann solver.

The associated numerical scheme had to meet a key prerequisite – being written in conservation form, as this automatically guarantees the correct propagation of discontinuities as well as the correct Rankine-Hugoniot (jump) conditions across discontinuities (the shock-capturing property).

In 1991 Martí, Ibáñez, and Miralles presented a new formulation of the general relativistic hydrodynamics equations, in 1+1, aimed at taking advantage of their hyperbolic character. Corresponding 3+1 extension presented in Font et al (1994) in special relativity, and in Banyuls et al (1997) in general relativity.

Replace the "primitive variables" in terms of the "conserved variables" :

$$\vec{w} = (\rho, \varepsilon, v^i) \Rightarrow \begin{aligned} D &= \rho W \\ S_j &= \rho h W^2 v_j \\ \tau &= \rho h W^2 - p - D \end{aligned}$$

$$W^2 = \frac{1}{1 - v^j v_j}$$

Lorentz factor

$$h = 1 + \varepsilon + \frac{p}{\rho}$$

specific enthalpy

Conservative formulations well-adapted to numerical methodology:

- Banyuls et al (1997); Font et al (2000): 3+1, general EOS

Hyperbolic system:

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{U}}{\partial x^0}+\frac{\partial\sqrt{-g}\mathbf{F}^i}{\partial x^i}\right)=\mathbf{S}$$

$$\begin{aligned}
\mathbf{U} &= (D, S_j, \tau)\\
\mathbf{F}^i &= \left(D\left(v^i-\frac{\beta^i}{\alpha}\right), S_j\left(v^i-\frac{\beta^i}{\alpha}\right)+p\delta^i_j, \tau\left(v^i-\frac{\beta^i}{\alpha}\right)+pv^i\right)\\
\mathbf{S} &= \left(0, T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu}-\Gamma^\delta_{\nu\mu}g_{\delta j}\right), \alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)\right)
\end{aligned}$$

First-order flux-conservative hyperbolic system

$$\begin{aligned}
D &= \rho W\\
S_j &= \rho h W^2 v_j\\
\tau &= \rho h W^2 - p - D
\end{aligned}$$

Solved using HRSC schemes

(either upwind or central)

Hydrodynamics equations

Newton

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v^i}{\partial x^i} = 0$$

$$\frac{\partial \rho v^j}{\partial t} + \frac{\partial (\rho v^i v^j + p\delta^{ij})}{\partial x^i} = 0$$

$$\frac{\partial \left(\rho\varepsilon + \frac{1}{2}\rho v^2\right)}{\partial t} + \frac{\partial \left(\rho\varepsilon + \frac{1}{2}\rho v^2 + p\right) v^i}{\partial x^i} = 0$$

Hyperbolic systems of conservation laws

Minkowski

$$\frac{\partial \rho W}{\partial t} + \frac{\partial \rho W v^i}{\partial x^i} = 0$$

$$\frac{\partial \rho h W^2 v^j}{\partial t} + \frac{\partial (\rho h W^2 v^i v^j + p\delta^{ij})}{\partial x^i} = 0$$

$$\frac{\partial (\rho h W^2 - p - \rho W)}{\partial t} + \frac{\partial (\rho h W^2 - \rho W) v^i}{\partial x^i} = 0$$

General Relativity

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial \sqrt{\gamma}\rho W}{\partial t} + \frac{\partial \sqrt{-g}\rho W v^i}{\partial x^i}\right) = 0$$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial \sqrt{\gamma}\rho h W^2 v^j}{\partial t} + \frac{\partial \sqrt{-g}(\rho h W^2 v^i v^j + p\delta^{ij})}{\partial x^i}\right) = T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu} - \Gamma^\delta_{\nu\mu} g_{\delta j}\right)$$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial \sqrt{\gamma}(\rho h W^2 - p - \rho W)}{\partial t} + \frac{\partial \sqrt{-g}(\rho h W^2 - \rho W) v^i}{\partial x^i}\right) = \alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu} - T^{\mu\nu}\Gamma^0_{\nu\mu}\right)$$

HRSC schemes based on approximate Riemann solvers use the local characteristic structure of the hyperbolic system of equations. For the previous system, this information was presented in Banyuls et al (1997).

The eigenvalues (characteristic speeds) are all real (but not distinct, one showing a threefold degeneracy), and a complete set of right-eigenvectors exists. The above system satisfies, hence, the definition of hiperbolicity.

Eigenvalues (along the x direction)

$$\lambda_0 = \alpha v^x - \beta^x \quad \text{(triple)}$$

$$\lambda_\pm = \frac{\alpha}{1-v^2c_s^2}\left\{v^x(1-c_s^2)\pm c_s\sqrt{(1-v^2)[\gamma^{xx}(1-v^2c_s^2)-v^xv^x(1-c_s^2)]}\right\} - \beta^x$$

Right-eigenvectors

$$\mathbf{r}_{0,1} = \begin{bmatrix} \frac{\mathcal{K}}{hW} \\ v_x \\ v_y \\ v_z \\ 1-\frac{\mathcal{K}}{hW} \end{bmatrix} \quad \mathbf{r}_{0,2} = \begin{bmatrix} Wv_y \\ h\left(\gamma_{xy}+2W^2v_xv_y\right) \\ h\left(\gamma_{yy}+2W^2v_yv_y\right) \\ h\left(\gamma_{zy}+2W^2v_zv_y\right) \\ Wv_y(2hW-1) \end{bmatrix} \quad \mathbf{r}_{0,3} = \begin{bmatrix} Wv_z \\ h\left(\gamma_{xz}+2W^2v_xv_z\right) \\ h\left(\gamma_{yz}+2W^2v_yv_z\right) \\ h\left(\gamma_{zz}+2W^2v_zv_z\right) \\ Wv_z(2hW-1) \end{bmatrix} \quad \mathbf{r}_\pm = \begin{bmatrix} 1 \\ hW\left(v_x - \frac{v^x-\Lambda_\pm^x}{\gamma^{xx}-v^x\Lambda_\pm^x}\right) \\ hWv_y \\ hWv_z \\ \frac{hW(\gamma^{xx}-v^xv^x)}{\gamma^{xx}-v^x\Lambda_\pm^x}-1 \end{bmatrix}$$

Special relativistic limit (along x-direction)

$$\lambda_0 = v^x$$

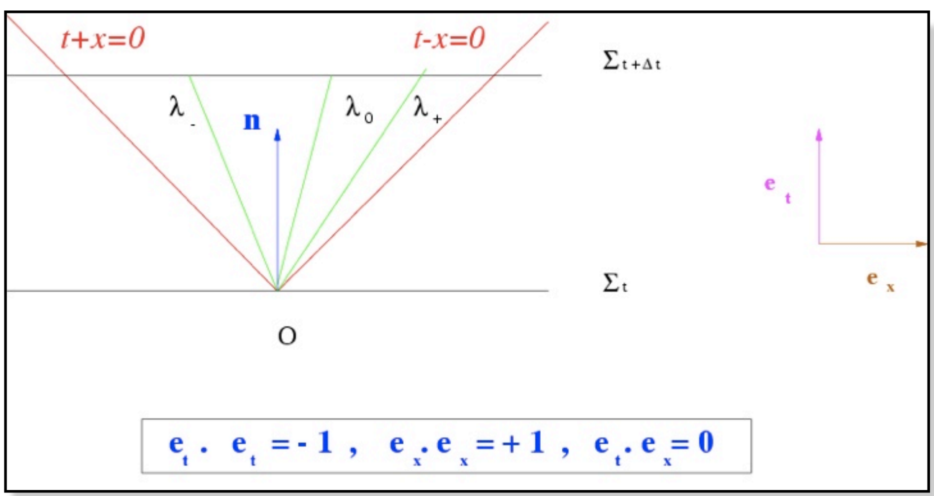
$$\lambda_\pm = \frac{1}{1 - v^2 c_s^2}\left[v^x(1-c_s^2) \pm c_s\sqrt{(1-v^2)(1-v^2c_s^2 - v^x v^x(1-c_s^2))}\right]$$

coupling with transversal components of the velocity
(important difference with Newtonian case)

Even in the purely 1D case:

$$\vec{v} = (v^x, 0, 0) \Rightarrow \lambda_0 = v^x, \ \lambda_\pm = \frac{v^x \pm c_s}{1 \pm v^x c_s}$$

For causal EOS the sound cone lies within the light cone



Recall Newtonian (1D) case:

$$\lambda_0 = v^x, \ \lambda_\pm = v^x \pm c_s$$

Recovering the primitive variables

A distinctive feature of the numerical solution of the relativistic hydrodynamics equations is that while the numerical algorithm updates the vector of conserved quantities, the numerical code makes extensive use of the primitive variables.

Those would appear repeatedly in the solution procedure, e.g. in the characteristic fields, in the solution of the Riemann problem, and in the computation of the numerical fluxes.

For spacelike foliations of the spacetime (3+1) the relation between the two sets of variables is implicit. Hence, iterative (root-finding) algorithms are required. Those have been developed for all existing formulations.

This feature, which is distinctive of the equations of general (and special) relativistic hydrodynamics (and also in GRMHD) – not existing in the Newtonian case – may lead to accuracy losses in regions of low density and small velocities, apart from being computationally inefficient.

Example of primitive recovery

Newtonian hydro: explicit to obtain "primitive" variables from state vector.

3+1 GR hydro: root-finding procedure. The expressions relating the primitive variables to the state vector depend explicitly on the EOS. Simple expressions are only obtained for simple EOS, i.e. ideal gas.

One can build a function of pressure whose zero represents the pressure in the physical state (other choices possible):

$$\vec{w} = (\rho, \varepsilon, v^i) \Rightarrow \begin{aligned} D &= \rho W \\ S_j &= \rho h W^2 v_j \\ \tau &= \rho h W^2 - p - D \end{aligned}$$

$$\boxed{f(\bar{p}) = p(\rho_*(\bar{p}), \varepsilon_*(\bar{p})) - \bar{p}}$$

$$\rho_*(\bar{p}) = \frac{D}{W_*(\bar{p})} \qquad \varepsilon_*(\bar{p}) = \frac{\tau + D[1 - W_*(\bar{p})] + \bar{p}[1 - W_*(\bar{p})^2]}{DW_*(\bar{p})}$$

$$W_*(\bar{p}) = \frac{1}{\sqrt{1 - v_*^i(\bar{p})\, v_{*i}(\bar{p})}} \qquad v_*^i(\bar{p}) = \frac{S^i}{\tau + D + \bar{p}}$$

The root of the above function can be obtained by means of a nonlinear root-finder (e.g. Newton-Raphson method).

Einstein's Equations

Einstein's equations and Numerical Relativity

The dynamics of the gravitational field is described by Einstein's field equations:

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}}$$

These equations relate the **spacetime geometry** (left-hand side) with the **distribution of matter and energy** (right-hand side): *"Matter tells spacetime how to curve, and spacetime tells matter how to move."*

Einstein's equations are a system of 10 nonlinear, coupled, partial differential equations in 4 dimensions.

When written with respect to a general coordinate system they may contain hundreds of terms ...

There's plenty of **exact solutions** of Einstein's equations, but very **few of such solutions have astrophysical significance.** Due to their complexity exact solutions of such equations have only been found when adopting simplifying symmetries:

- **Schwarzschild solution** (static and spherically symmetric)
- **Kerr solution** (stationary and axisymmetric)
- **Cosmological solution** (isotropic, homogeneous, or both)

When studying more complex systems with astrophysical significance (gravitational collapse, supernovae, mergers of compact binaries) is unfeasible to solve Einstein's equations in an exact way.

The field of **Numerical Relativity** emerged in the mid 1960s from the need to study such kind of problems, aiming at trying to solve the field equations with supercomputers using numerical approximations.

Numerical Relativity's main goal: provide templates of the gravitational produced in astrophysical sources to facilitate its detection (and the analysis of the available data; LIGO, VIRGO).

Numerical Relativity: dynamical spacetimes

Numerical Relativity is the field of research of General Relativity devoted to seeking numerical solutions of Einstein's equations through (super)computer simulations.

- Mathematical difficulties:
  - PDEs highly non-linear with hundreds of terms.
  - Hyperbolic and elliptic character.
  - Coordinates and gauge conditions (gauge freedom).
  - Boundary conditions.
- Numerical difficulties:
  - Formulation of the equations.
  - Development of numerical schemes.
  - Stability and efficiency.
  - Possible formation of curvature singularities (collapse to black hole).
- Huge computational resources needed (3D).

Intrinsic and extrinsic curvature of spatial hypersurfaces:

Intrinsic curvature given by the 3-dimensional Riemann tensor defined in terms of the 3-metric γ_{ij} .

Extrinsic curvature K_{ij} measures the change of the vector normal to the hypersurface as it is parallel-transported from one point in the hypersurface to another.

Projection operator: $P^\alpha_\beta \equiv \delta^\alpha_\beta + n^\alpha n_\beta$

Unit normal vector: $n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right), \;\; n_\mu = (-\alpha, 0), \;\; n^\mu n_\mu = -1$

$$K_{\alpha\beta} \equiv -P^\mu_\alpha P^\nu_\beta \nabla_\mu n_\nu = -(\nabla_\alpha n_\beta + n_\alpha n^\mu \nabla_\mu n_\beta))$$

Substituting the form of the normal vector in the definition of the extrinsic curvature, we get:

$$K_{ij} = \frac{1}{2\alpha}(-\partial_t \gamma_{ij} + \nabla_i \beta_j + \nabla_j \beta_i)$$

Einstein's equations in 3+1 form

Using the projection operator and the normal vector, Einstein's equations can be separated in **three groups:**

- Normal projection (1 equation; energy or Hamiltonian constraint)

$$n^\alpha n^\beta (G_{\alpha\beta} - 8\pi T_{\alpha\beta}) = 0$$

- Mixed projections (3 equations; momentum constraints)

$$P[n^\alpha (G_{\alpha\beta} - 8\pi T_{\alpha\beta})] = 0$$

- Projection onto the hypersurface (6 equations; evolution of the extrinsic curvature)

$$P(G_{\alpha\beta} - 8\pi T_{\alpha\beta}) = 0$$

3+1 Formulation (Cauchy)

Lichnerowicz (1944); Choquet-Bruhat (1962); Arnowitt, Deser & Misner (1962); York (1979)

Evolution equations:

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \quad (*) \\
\partial_t K_{ij} &= -\nabla_i \nabla_j \alpha + \alpha \left(R_{ij} + K\ K_{ij} - 2K_{im} K^m_j \right) + \beta^m \nabla_m K_{ij} \\
&\quad + K_{im} \nabla_j \beta^m + K_{mj} \nabla_i \beta^m - 8\pi\alpha \left(T_{ij} - \frac{1}{2}\gamma_{ij} T^m_m + \frac{1}{2}\rho\gamma_{ij} \right)
\end{aligned}$$

Constraint equations:

$$\begin{aligned}
R + K^2 - K^{ij} K_{ij} &= 16\pi\rho \\
\nabla_i \left(K^{ij} - \gamma^{ij} K \right) &= 8\pi S^j
\end{aligned}$$

(*) $(\partial_t - \mathcal{L}_\beta)\gamma_{ij} = -2\alpha K_{ij}$

Cauchy problem (IVP):

- Specify γ_{ij} , K_{ij} at t=0 subjected to the constraint equations.
- Specify coordinates through α , β^i
- Evolve the data using EE and the definition of K_{ij}

Definitions:

∇_i Covariant derivative wrt the induced 3-metric

Ricci tensor $$R_{ij} = \partial_n \Gamma^n_{ij} - \partial_j \Gamma^n_{in} + \Gamma^n_{mn}\Gamma^m_{ij} - \Gamma^n_{jm}\Gamma^m_{in}$$

Christoffel symbols $$\Gamma^i_{jk} = \frac{1}{2}\gamma^{in}\left(\frac{\partial \gamma_{nj}}{\partial x^k} + \frac{\partial \gamma_{nk}}{\partial x^j} - \frac{\partial \gamma_{jk}}{\partial x^n}\right)$$

Scalar curvature $$R = R_{ij}\gamma^{ij}$$

Trace of extrinsic curvature $$K = K_{ij}\gamma^{ij}$$

Matter fields $$\begin{cases} \rho & \equiv & T^{\mu\nu} n_\mu n_\nu = \rho h W^2 - P \\ S^i & \equiv & -\perp^i_\mu T^{\mu\nu} n_\nu = \rho h W^2 v^i \\ S_{ij} & \equiv & \perp^\mu_i \perp^\nu_j T^{\mu\nu} = \rho h W^2 v_i v_j + \gamma_{ij} P \\ S & \equiv & \rho h W^2 v_i v^i + 3P \end{cases}$$

BSSN formulation

Kojima, Nakamura & Oohara (1987); Shibata & Nakamura (1995); Baumgarte & Shapiro (1999)

Idea: Remove mixed second derivatives in the Ricci tensor by introducing auxiliary variables. Evolution equations start to look like wave equations for 3-metric and extrinsic curvature (idea goes back to De Donder 1921; Choquet-Bruhat 1952; Fischer & Marsden 1972).

- Conformal decomposition of the 3-metric:

$$\tilde{\gamma}_{ij} = \psi^4 \gamma_{ij} \qquad \det \tilde{\gamma}_{ij} = 1$$

- BSSN evolution variables (trace of extrinsic curvature is a separate variable):

$$\phi = \frac{1}{4} \log \psi \qquad \tilde{\gamma}_{ij} = e^{-4\phi} \gamma_{ij}$$

$$K = \gamma^{ij} K_{ij} \qquad \tilde{A}_{ij} = e^{-4\phi} \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right)$$

- Introduce evolution variables (gauge source functions):

$$\tilde{\Gamma}^a = \tilde{\gamma}^{ij} \tilde{\Gamma}^a_{ij} = -\partial_i \tilde{\gamma}^{ai}$$

Evolution equations in BSSN formulation

Kojima, Nakamura & Oohara (1987); Shibata & Nakamura (1995); Baumgarte & Shapiro (1999)

$$
\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} \\
(\partial_t - \mathcal{L}_\beta)\phi &= -\frac{1}{6}\alpha K \\
(\partial_t - \mathcal{L}_\beta)K &= -\gamma^{ij}D_iD_j\alpha + \alpha\left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + \frac{1}{2}(\rho + S)\right] \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= e^{-4\phi}\left[-D_iD_j\alpha + \alpha\left(R_{ij} - S_{ij}\right)\right]^{\mathrm{TF}} + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) \\
(\partial_t - \mathcal{L}_\beta)\tilde{\Gamma}^i &= -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_jK - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\right) \\
&\quad + \partial_j\left(\beta^l\tilde{\partial}_l\gamma^{ij} - 2\tilde{\gamma}^{m(j}\partial_m\beta^{i)} + \frac{2}{3}\tilde{\gamma}^{ij}\partial_l\beta^l\right)
\end{aligned}
$$

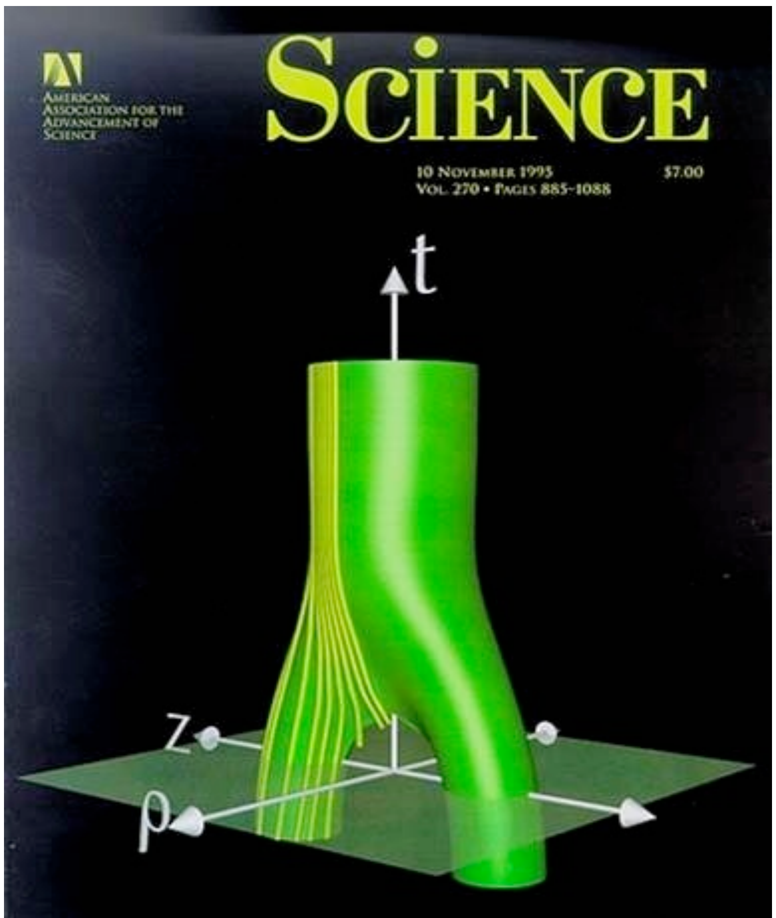
Can be solved numerically!

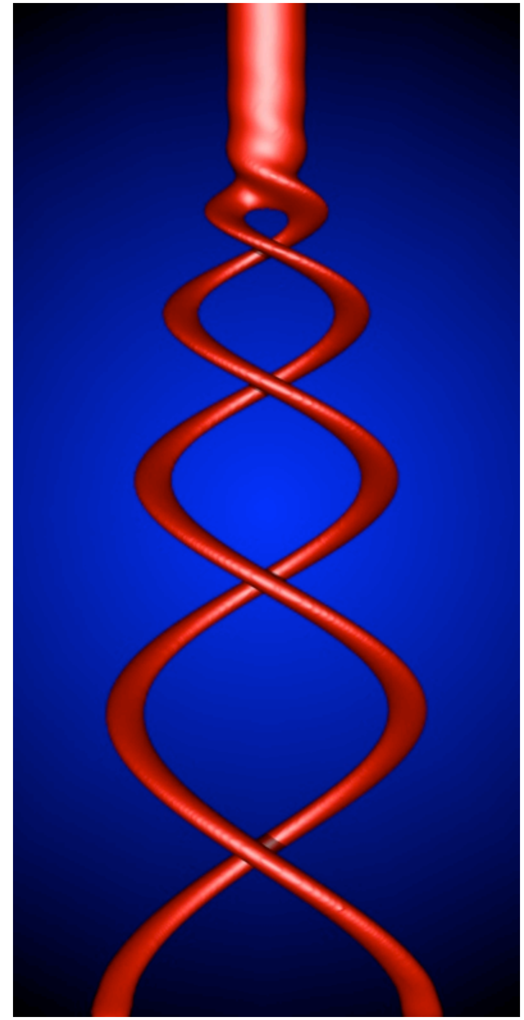
BSSN is currently the standard 3+1 formulation in Numerical Relativity. Long-term stable applications include strongly-gravitating systems such as neutron stars (isolated and binaries) and single and **binary black holes!**

BBH simulations: State of the art

1995: Pair of pants
(Head-on collision)

BBH Grand Challenge Alliance





FSU-Jena Numerical Relativity Group

2007: Pair of twisted pants
(spiral & merge)

Useful approximations: CFC equations

In the CFC approximation (Isenberg 1985; Wilson & Mathews 1996) the ADM 3+1 equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i$$

$$\partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha \left(R_{ij} + K\ K_{ij} - 2K_{im}K^m_j \right) + \beta^m \nabla_m K_{ij} + K_{im}\nabla_j \beta^m + K_{mj}\nabla_i \beta^m - 8\pi\alpha \left(T_{ij} - \frac{1}{2}\gamma_{ij} T^m_m + \frac{1}{2}\rho\gamma_{ij} \right)$$

$$R + K^2 - K^{ij}K_{ij} = 16\pi\rho$$

$$\nabla_i \left(K^{ij} - \gamma^{ij} K \right) = 8\pi S^j$$

reduce to a system of five coupled, nonlinear elliptic equations for the lapse function, conformal factor, and the shift vector:

CFC approximation

$$\gamma_{ij} = \phi^4 \delta_{ij}$$

$$\boxed{\begin{aligned} \hat{\Delta}\phi &= -2\pi\phi^5 \left(\rho h W^2 - P + \frac{K_{ij}K^{ij}}{16\pi} \right) \\ \hat{\Delta}(\alpha\phi) &= 2\pi\alpha\phi^5 \left(\rho h (3W^2 - 2) + 5P + \frac{7K_{ij}K^{ij}}{16\pi} \right) \\ \hat{\Delta}\beta^i &= 16\pi\alpha\phi^4 S^i + 2\phi^{10} K^{ij} \hat{\nabla}_j \left(\frac{\alpha}{\phi^6} \right) - \frac{1}{3}\hat{\nabla}^i \hat{\nabla}_k \beta^k \end{aligned}}$$

Numerical methods for conservation laws

R.J. LeVeque, "Numerical methods for conservation laws"
Birkhäuser, Basel (1992)

E.F. Toro, "Riemann solvers and numerical methods for fluid dynamics"
Springer Verlag, Berlin (1997)

Numerical methods in Astrophysical Fluid Dynamics

Main numerical schemes to solve the equations of a compressible fluid:

• **Finite difference methods.** Require numerical viscosity to stabilize the solution in regions where discontinuities develop.

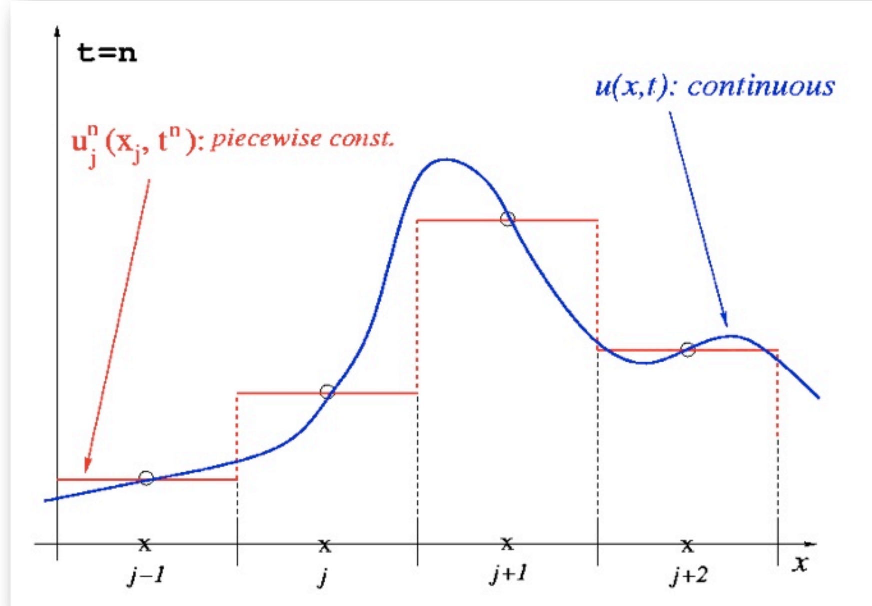
• **Finite volume methods.** Conservation form. Use Riemann solvers to solve the equations in the presence of discontinuities **(Godunov 1959)**. **HRSC schemes.**

• **Symmetric methods.** Conservation form. Centred finite differences and high spatial order.

• **Particle methods**. Smoothed Particle Hydrodynamics (Monaghan 1992). Integrate movement of discrete particles to describe the flow. Diffusive.

For hyperbolic systems of conservation laws, schemes written in conservation form guarantee that the convergence (if it exists) is to one of the weak solutions of the system of equations (Lax-Wendroff theorem 1960).

When a Cauchy problem described by a set of continuous PDEs is solved in a **discretized form** the numerical solution is **piecewise constant** (collection of local Riemann problems).

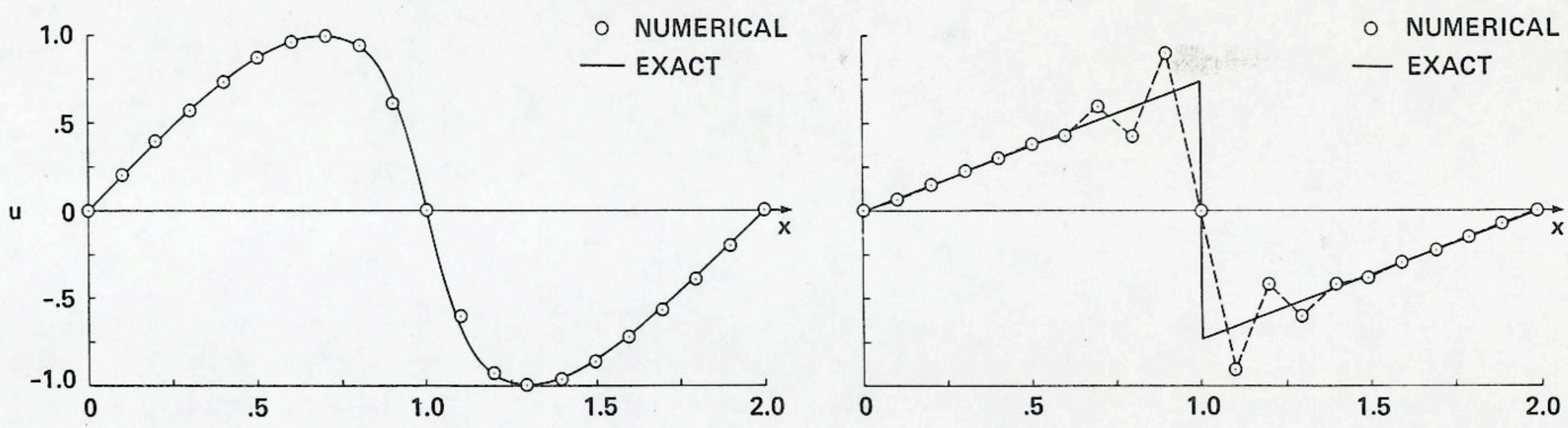


This is particularly problematic when solving the hydrodynamic equations (either Newtonian or relativistic) for compressible fluids.

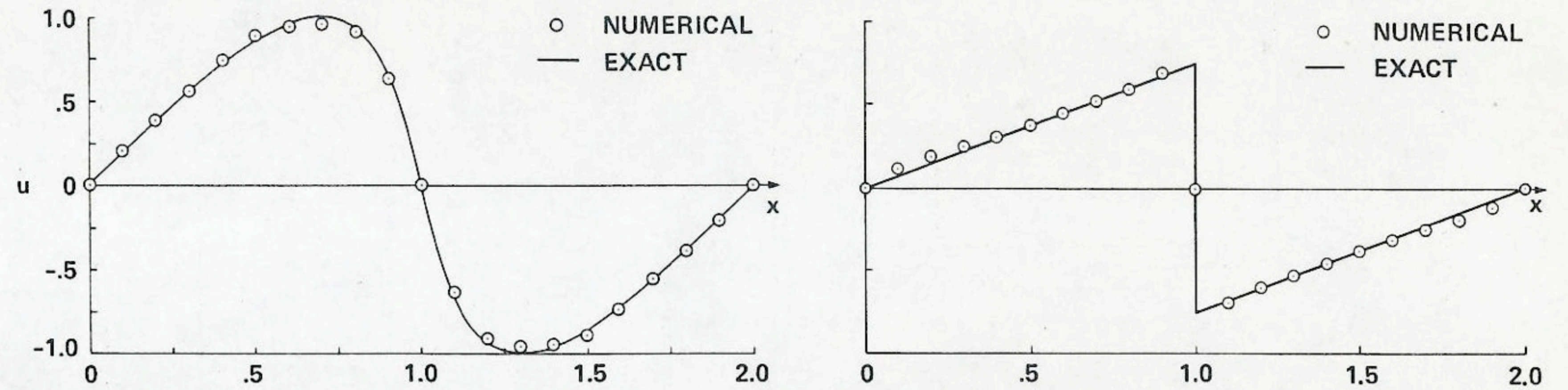
Their hyperbolic, nonlinear character produces discontinuous solutions in a finite time (shock waves, contact discontinuities) even from smooth initial data!

Any FD scheme must be able to handle discontinuities in a satisfactory way.

1. **1st order accurate schemes (Lax-Friedrich):** Non-oscillatory but inaccurate across discontinuities (excessive diffusion)
2. **(standard) 2nd order accurate schemes (Lax-Wendroff):** Oscillatory across discontinuities
3. 2nd order accurate schemes with artificial viscosity
4. Godunov-type schemes (upwind High Resolution Shock Capturing schemes)



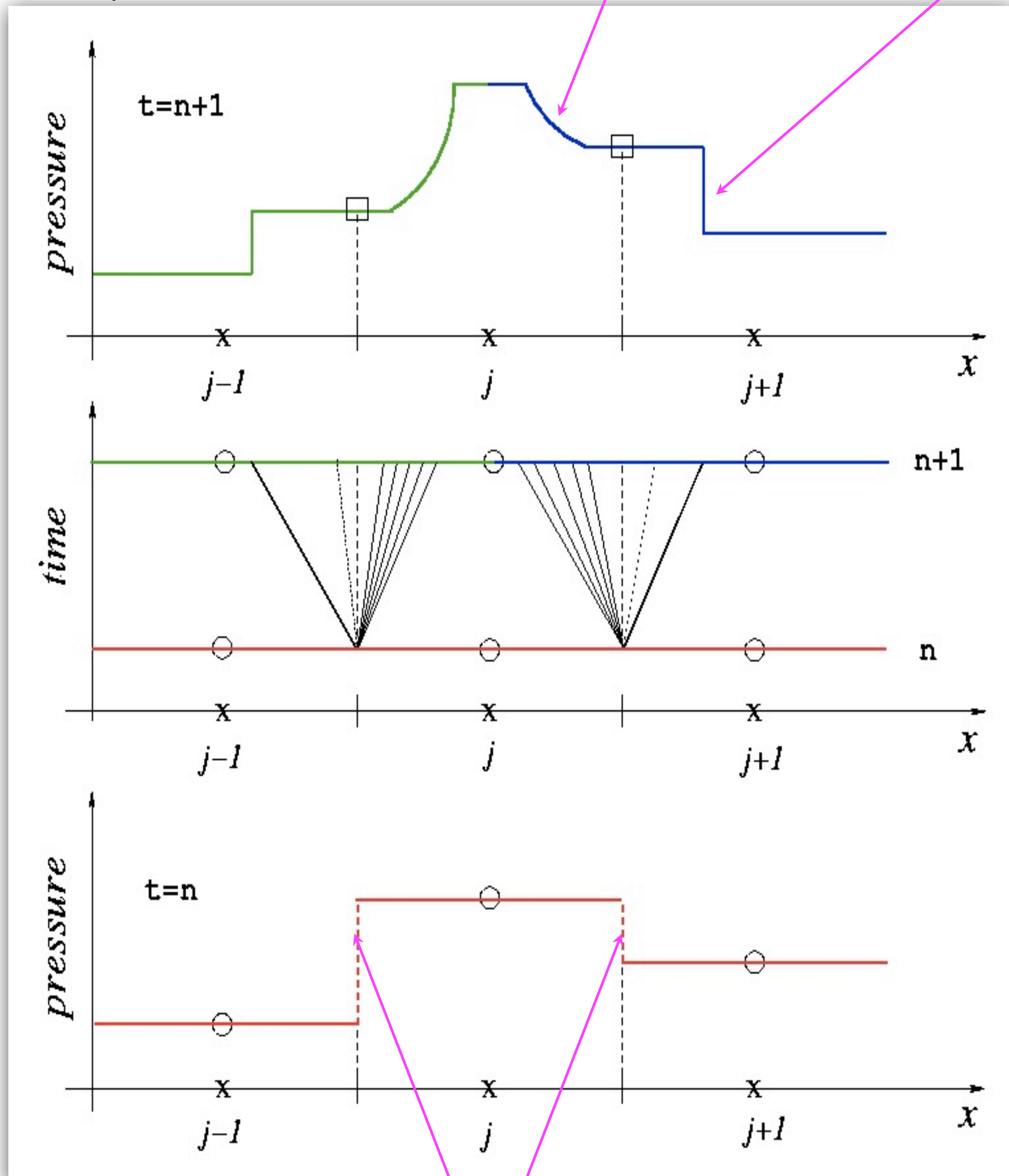
Lax-Wendroff numerical solution of Burger's equation at t=0.2 (left) and t=1.0 (right)



2nd order TVD numerical solution of Burger's equation at t=0.2 (left) and t=1.0 (right) rarefaction wave

shock front

courtesy of L. Rezzolla



Solution at time n+1 of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$

Spacetime evolution of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$. Each problem leads to a shock wave and a rarefaction wave moving in opposite directions

Initial data at time n for the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}\left(\hat{\mathbf{f}}^n_{j+\frac{1}{2}} - \hat{\mathbf{f}}^n_{j-\frac{1}{2}}\right)$$

cell boundaries where fluxes are required

Approximate Riemann solvers

In Godunov's method the structure of the Riemann solution is "lost" in the cell averaging process (1st order in space).

The exact solution of a Riemann problem is computationally expensive, particularly in multidimensions and for complicated EoS.

Relativistic multidimensional problems: coupling of all flow velocity components through the Lorentz factor.

- Shocks: increase in the number of algebraic jump (RH) conditions.
- Rarefactions: solving a system of ODEs.

This motivated development of approximate (linearized) Riemann solvers.

Based on the exact solution of Riemann problems corresponding to a new system of equations obtained by a linearization of the original one (quasi-linear form). The spectral decomposition of the Jacobian matrices is on the basis of all solvers ("extending" ideas used for linear hyperbolic systems).

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f}}{\partial x} = 0 \Rightarrow \frac{\partial \vec{u}}{\partial t} + A\frac{\partial \vec{u}}{\partial x} = 0\,, \quad A = \frac{\partial \vec{f}}{\partial \vec{u}} \qquad \text{(Jacobian matrix)}$$

Approach followed by an important subset of shock-capturing schemes, the so-called **Godunov-type methods** (Harten & Lax 1983; Einfeldt 1988).

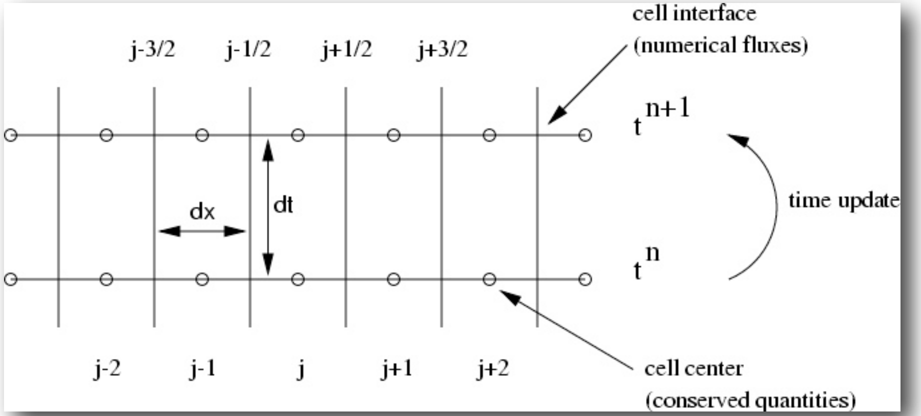
Standard implementation of a HRSC scheme

1. Time update:

Algorithm in conservation form

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}\left(\hat{\mathbf{f}}^n_{j+\frac{1}{2}} - \hat{\mathbf{f}}^n_{j-\frac{1}{2}}\right)$$

In practice: 2nd or 3rd order time accurate, conservative Runge-Kutta schemes (Shu & Osher 1989; MoL)

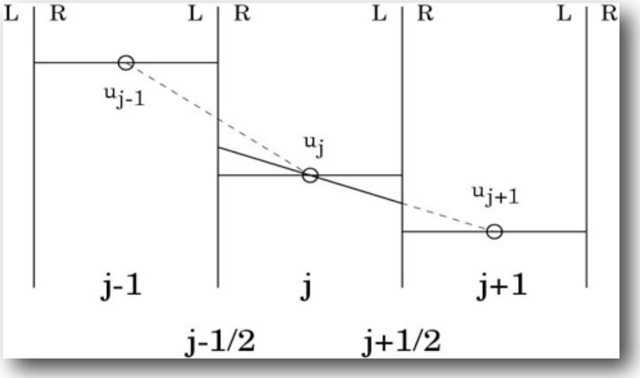


2. Cell reconstruction: Piecewise constant (Godunov), linear (MUSCL, MC, van Leer), parabolic (PPM, Colella & Woodward) interpolation procedures of state-vector variables from cell centers to cell interfaces.

3. Numerical fluxes: Approximate Riemann solvers (Roe, HLLE, Marquina). Explicit use of the spectral information of the system

$$\hat{\vec{f}}_i = \frac{1}{2}\left[\vec{f}_i(w_R) + \vec{f}_i(w_L) - \sum_{n=1}^{5}\left|\widetilde{\lambda}_n\right|\Delta\widetilde{\omega}_n\widetilde{R}_n\right]$$

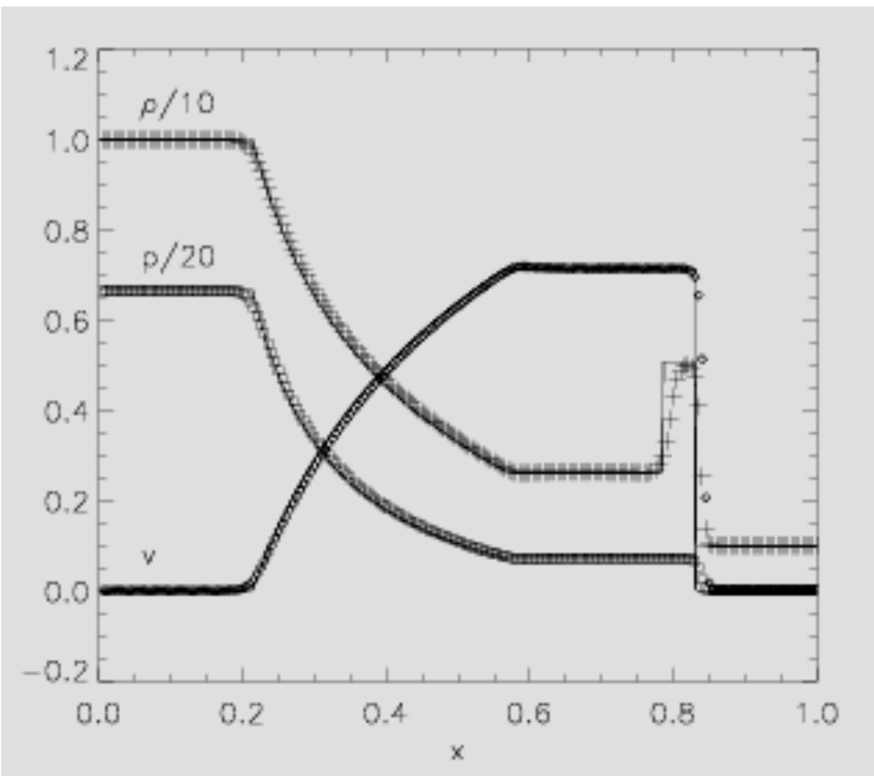
$$\mathrm{U}(w_R) - \mathrm{U}(w_L) = \sum_{n=1}^{5}\Delta\widetilde{\omega}_n\widetilde{R}_n$$



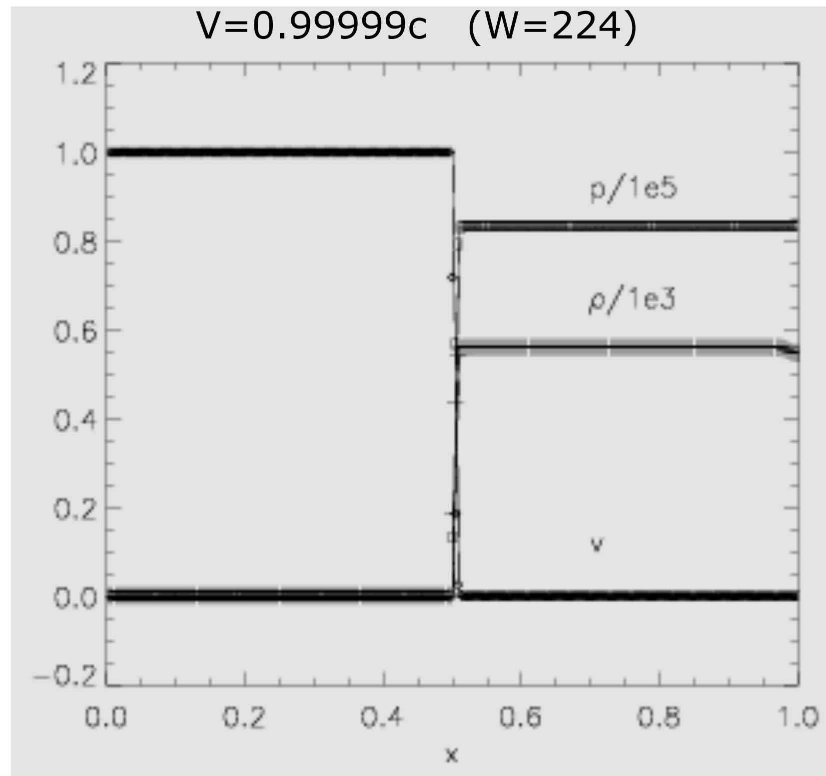
High-resolution shock-capturing schemes

- **Stable and accurate shock profiles**
- **Accurate propagation speed of discontinuities**
- **Accurate numerical resolution of nonlinear features:** discontinuites, rarefaction waves, vortices, turbulence, etc

Shock tube test



Relativistic shock reflection



The theoretical tools outlined in this lecture will be used in the simulations that will be discussed in tomorrow's lecture:

Lecture 2: Binary neutron stars.
Black-hole torus systems.

Motivation: gravitational waveforms.