Instanton interactions and Borel summability

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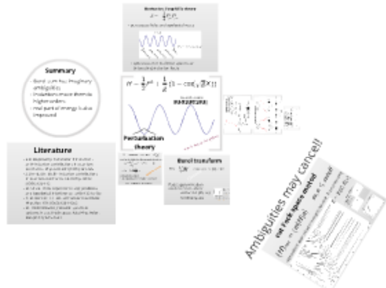


Zakopane 2012

INNOVATIVE ECONOMY
NATIONAL COHESION STRATEGY

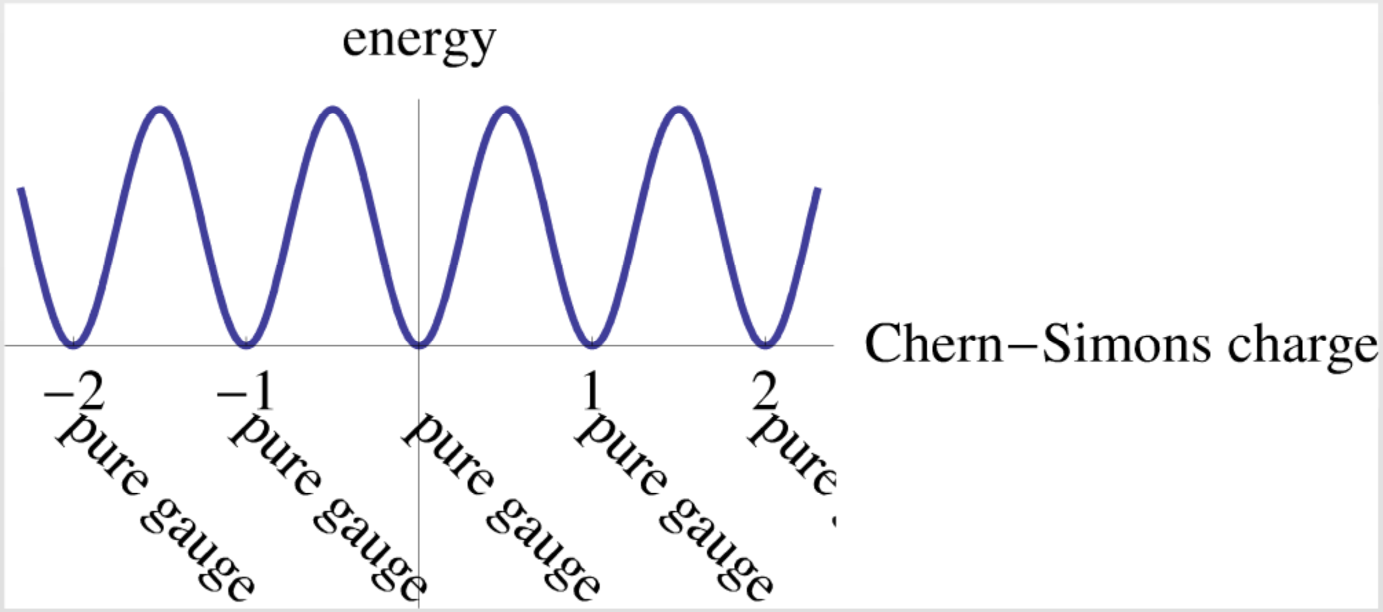
FNP
Foundation
for Polish Science

EUROPEAN UNION
EUROPEAN REGIONAL
DEVELOPMENT FUND

Motivation: Yang-Mills theory

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

- pure gauge fields are topological vacua

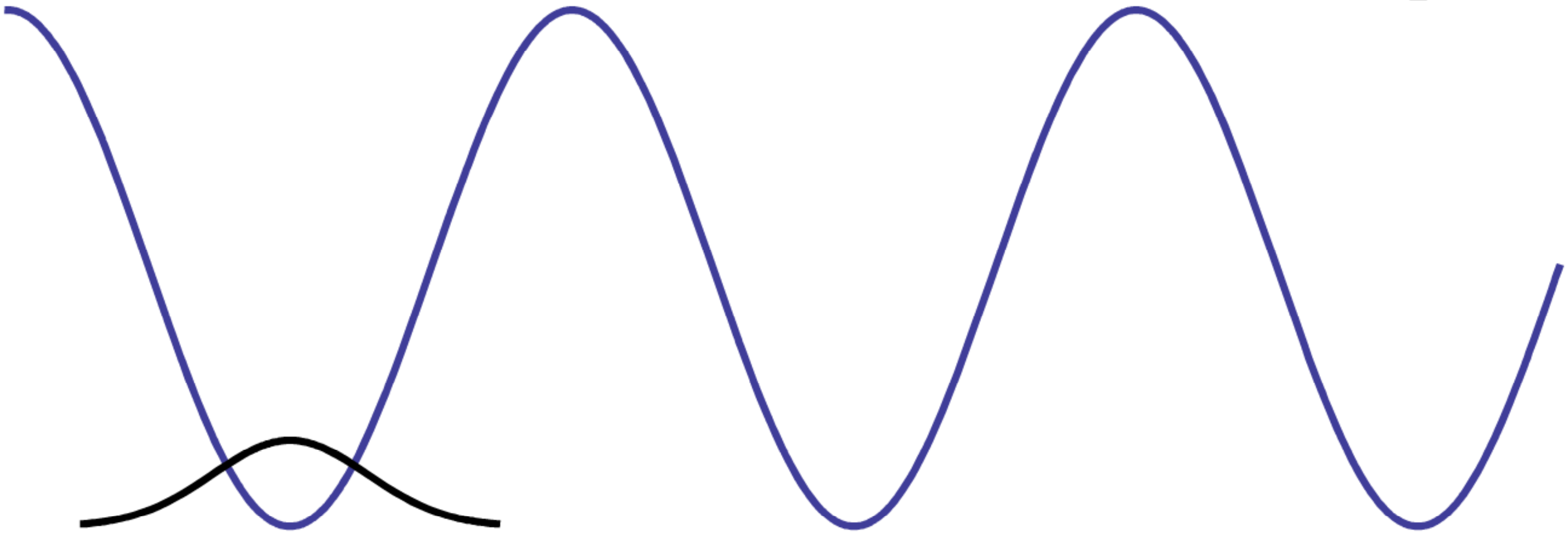


- QCD vacuum in Euclidean space is an (interacting) instanton liquid

$$H = \frac{1}{2}P^2 + \frac{1}{g}\left(1 - \cos(\sqrt{g}X)\right)$$

Instantons

(in dilute gas approximation)



Perturbation theory

How do they work together?

$$H = harmonic\ \ oscillator + \mathcal{O}(g)$$

perform Rayleigh-Schrödinger perturbation theory

$$E_0 = \sum_n \epsilon_n g^n \quad n = 0, \dots, 150, \dots$$

$$\epsilon_n \approx -0.64 \frac{1}{16^n} n!$$

asymptotic series!

common procedure:

stop summing on the smallest coefficient

How to eliminate finite errors?

Borel transform

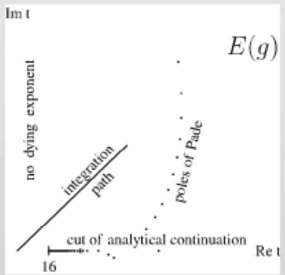
$$\mathcal{B}(t) = \sum_n \frac{\epsilon_n}{n!} t^n$$ convergent for $|t| < 16$

Padé approximation

(approximation of analytical continuation)

cut for $t \in (16, \infty)$

+artificial poles



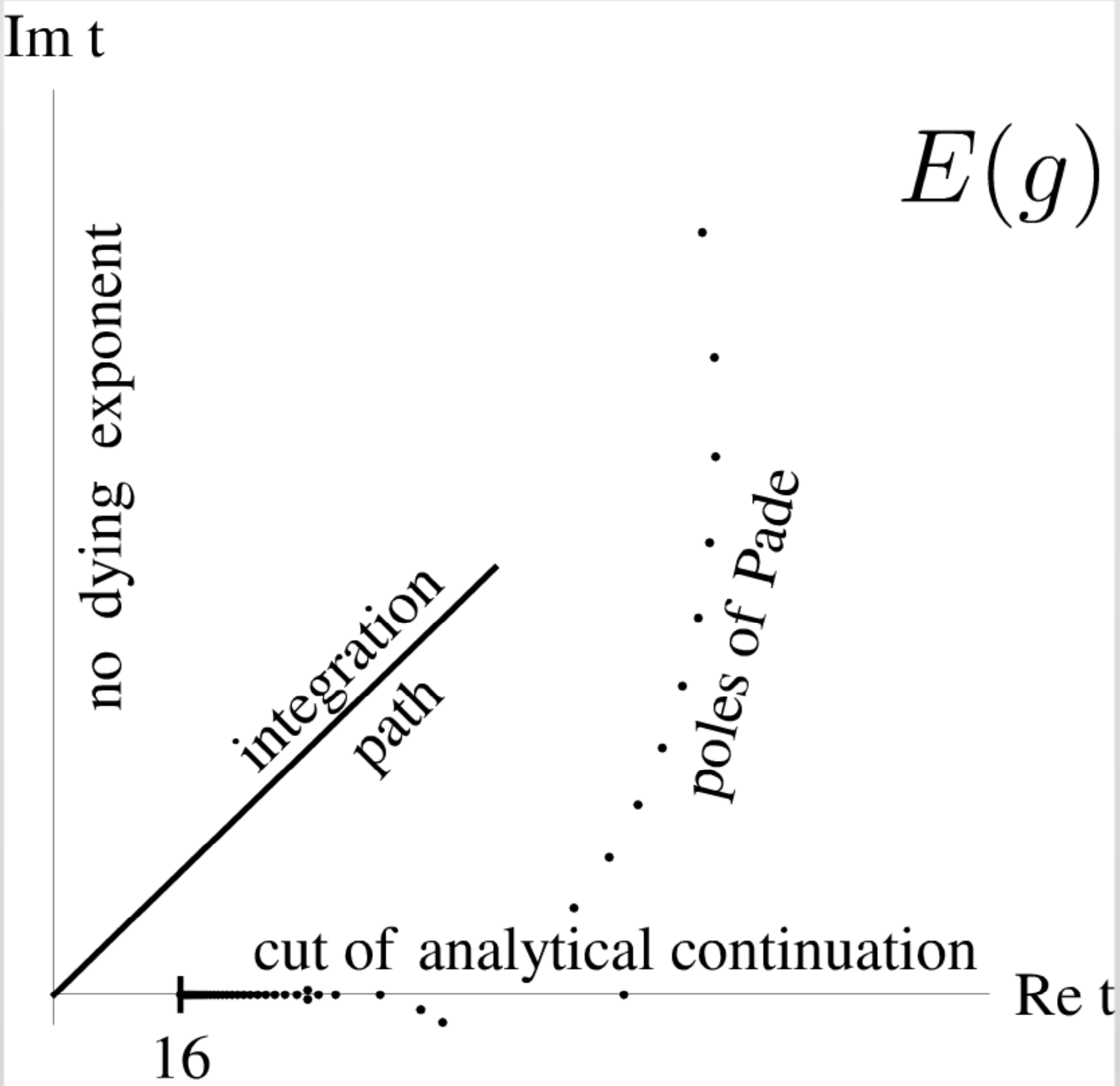
$$E(g) = \frac{1}{g}\int_0^\infty dt e^{-t/g}\mathcal{B}(t)$$

$E(2.5) = 0.4 + 0.01i$

complex energy!

if $arg(t) = -\frac{\pi}{4}$ then $E(2.5) = 0.4 - 0.01i$

ambiguity of energy!



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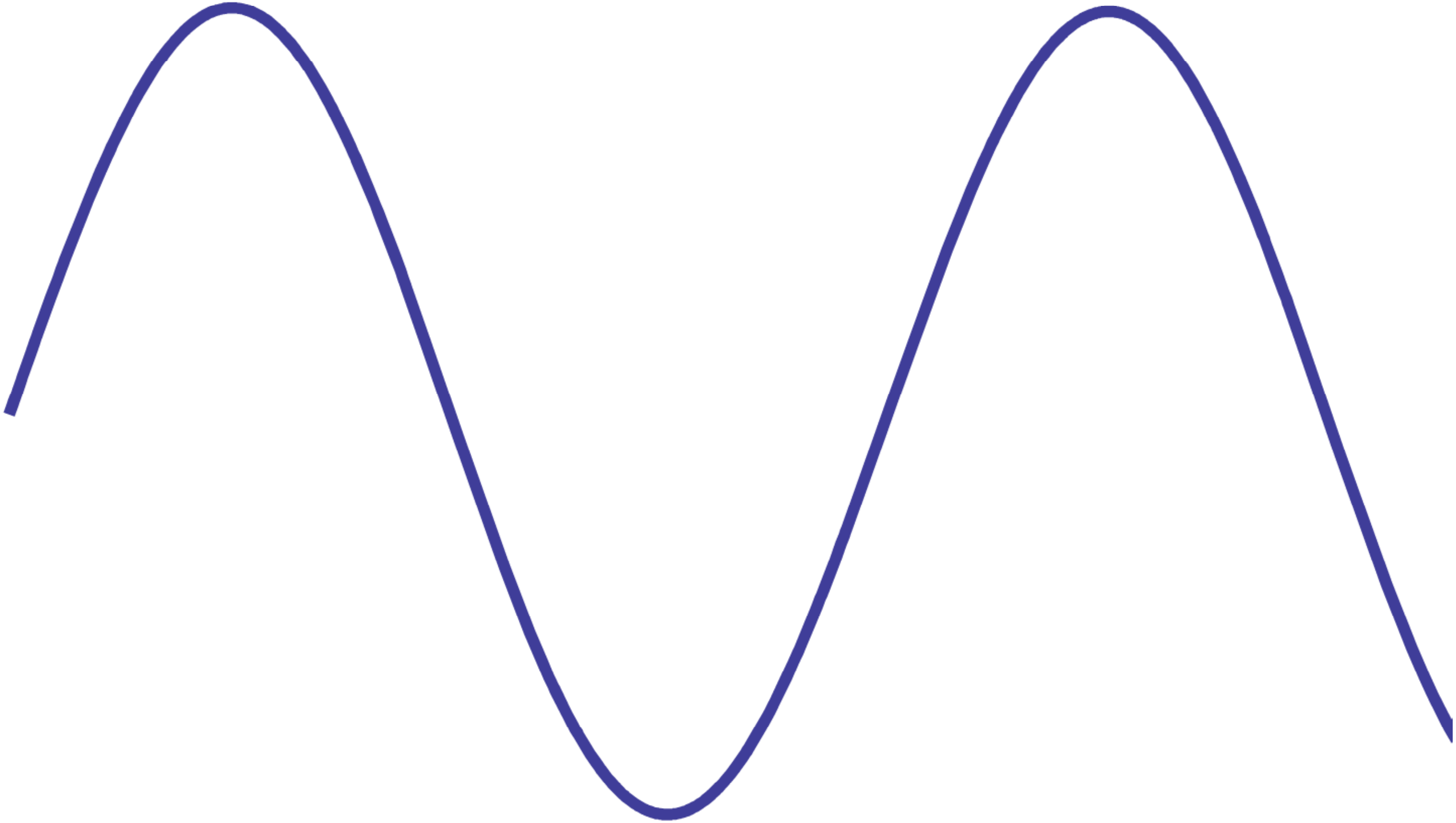

Instantons

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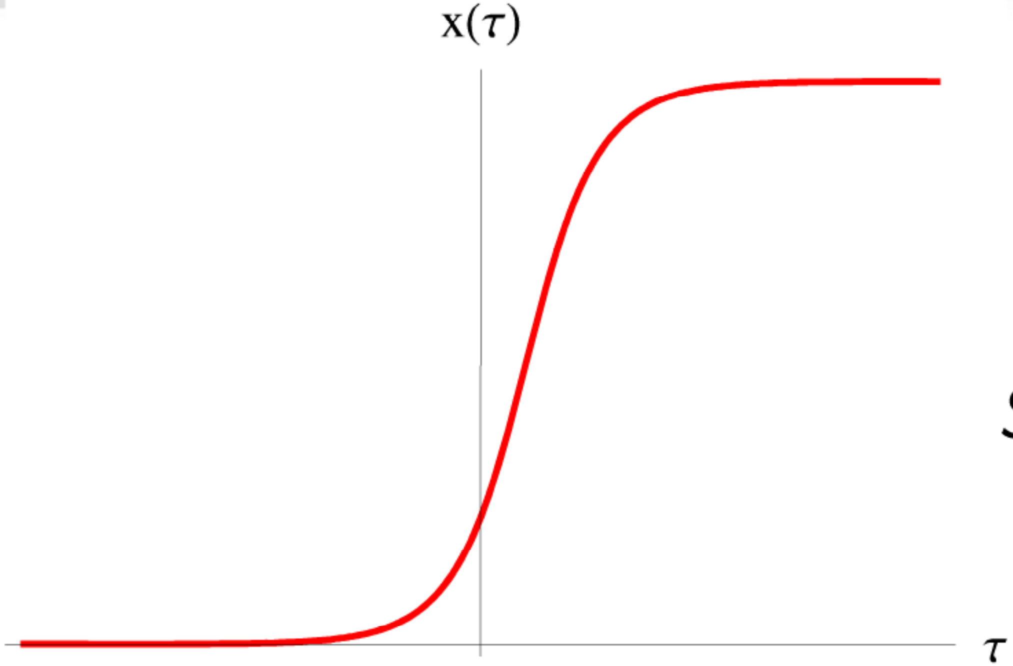


Euclidean action
(inverted potential)

$$S[x] = \int_{-T/2}^{T/2} d\tau \left(\frac{1}{2}\dot{x}^2 + V(x) \right)$$

$$S_0 = S[instanton]$$

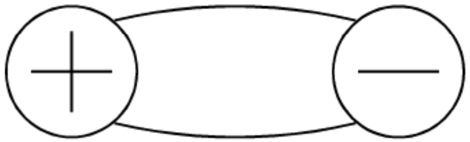
$$e^{-ET} = \int \mathcal{D}[x] e^{-S[x]}$$

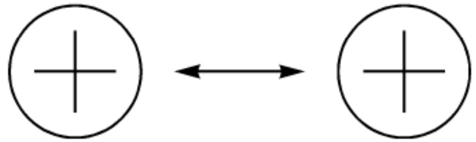
at large T

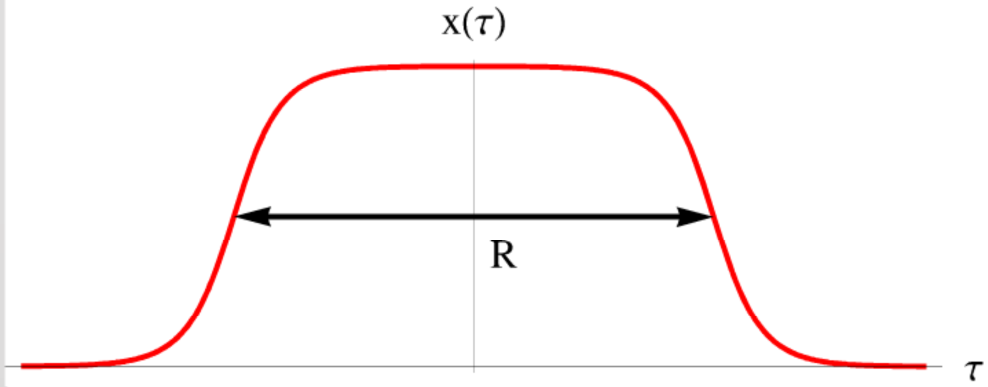
$$= \int_0^{\pi} d\theta e^{-E(\theta)T}$$

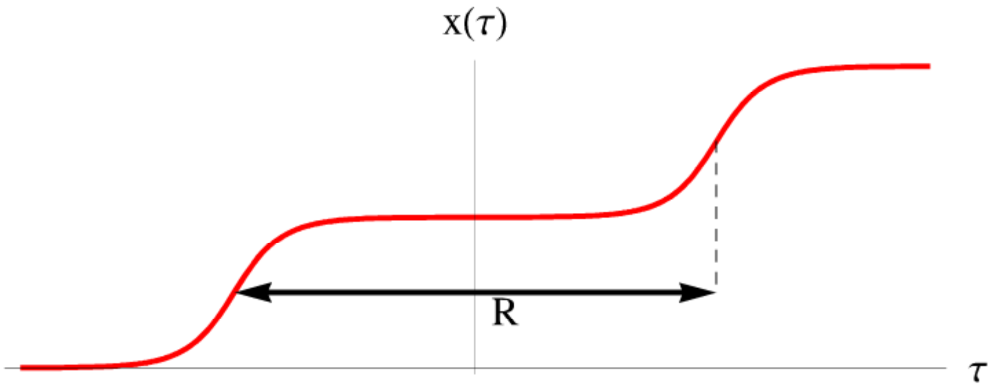
$$E(\theta) = \frac{1}{2} - \frac{4\sqrt{2}}{\pi\sqrt{g}} e^{-S_0} \cos\theta$$

Instanton interactions









$$S[x] = 2S_0 - \frac{16}{g}e^{-R}$$

$$Z' = \int dR \left(e^{\frac{16}{g}e^{-R}} - 1\right)$$ main contribution from small R -WRONG

$$= \int dR \left(e^{-\frac{16}{g}e^{-R}} - 1\right)\Big|_{g\to -g}$$ main contribution from large R -OK

$$S[x] = 2S_0 + \frac{16}{g}e^{-R}$$

no such problem

$$E = \frac{1}{2} - \frac{8}{\sqrt{\pi g}}e^{-S_0}\cos\theta - \frac{32}{\pi g}\left(-\gamma + \log\left(-\frac{g}{32}\right)\right)e^{-2S_0} - \frac{32}{\pi g}\left(-\gamma + \log\left(\frac{g}{32}\right)\right)e^{-2S_0}\cos(2\theta)$$

independent attracting repelling

$$\mp i\pi$$

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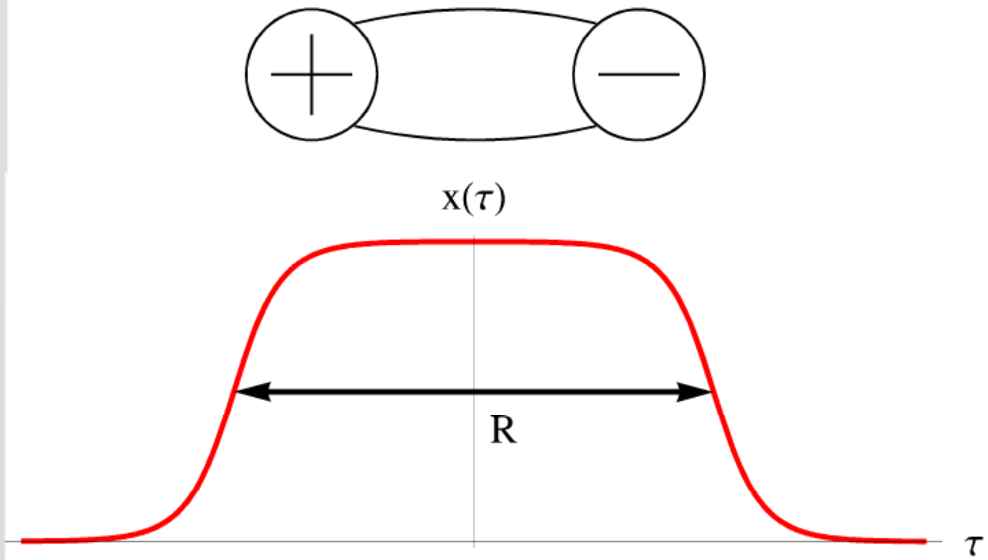
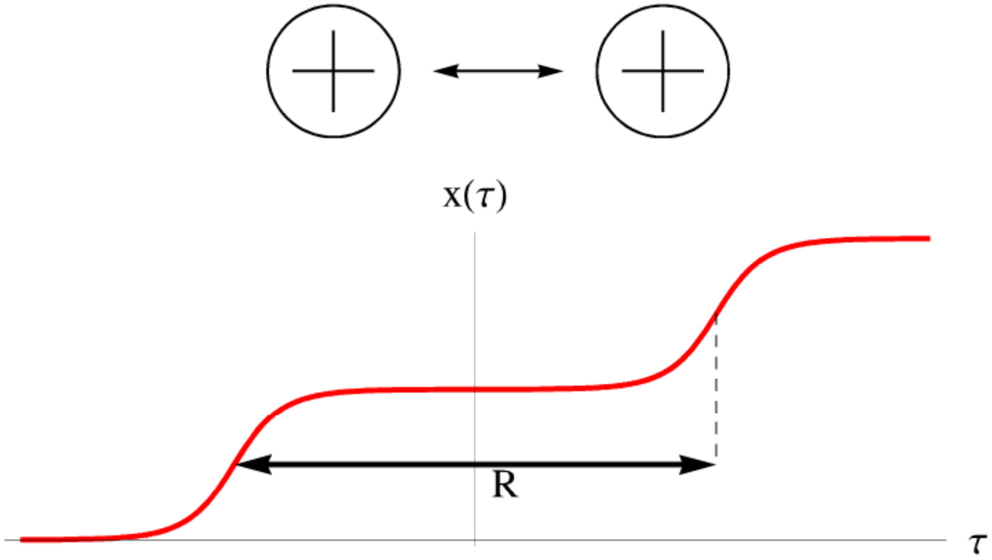
Instanton interactions





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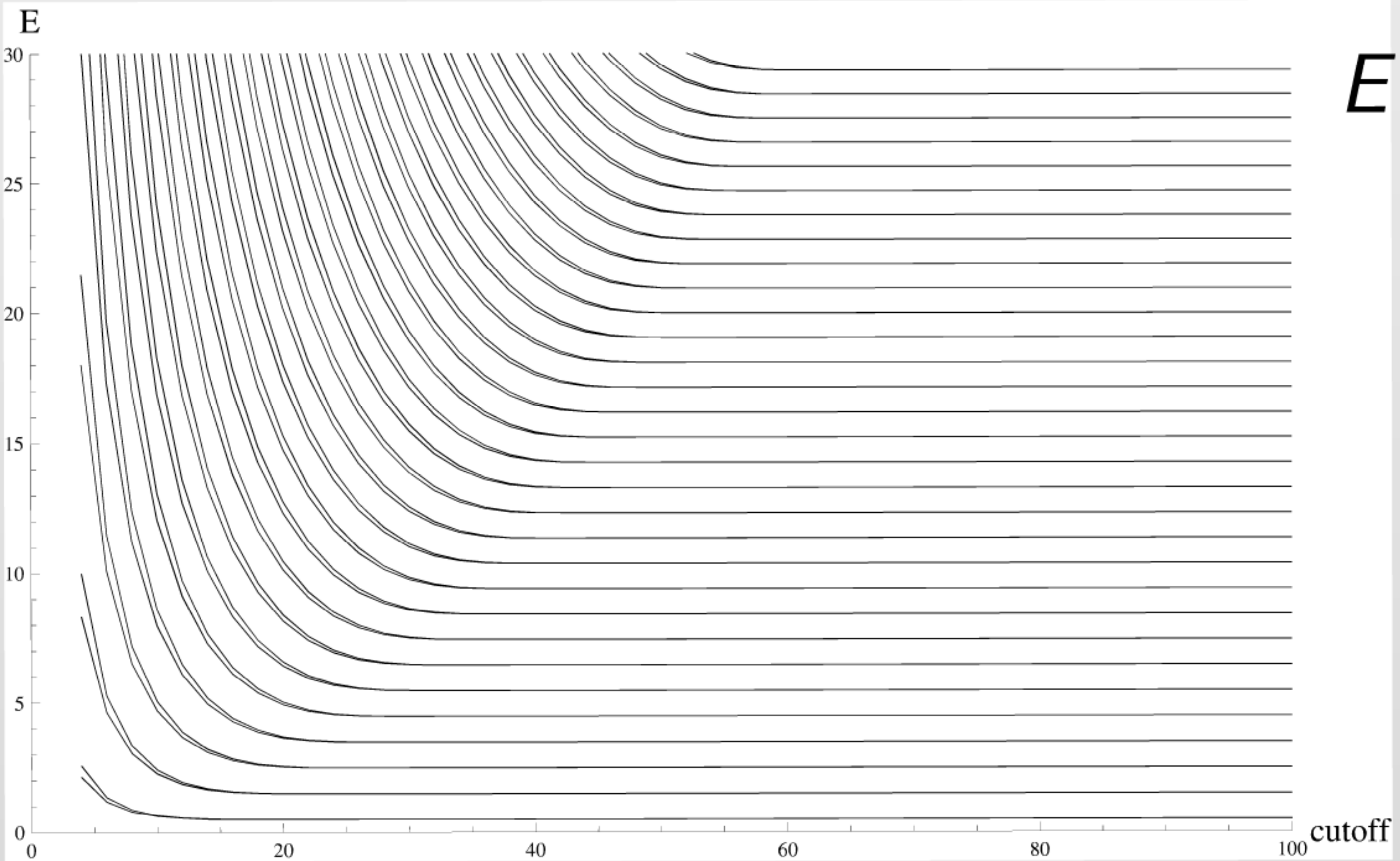
cut Fock space metod

$$(H)_{mn} = \langle m|H|n\rangle \qquad m, n \leq cutoff$$

eigenvalues approximate energies [Wosiek, Trzetrzelewski]

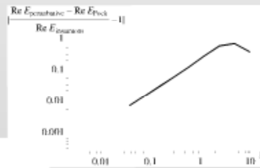


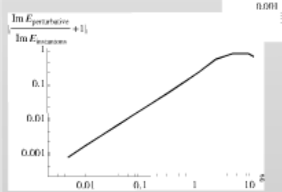
$$E = E(0), E(\pi)$$

Relative difference

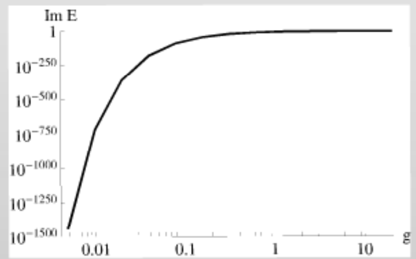
$$E = \frac{1}{2}(E(0) + E(\pi))$$

(cancel effects of independent instantons)





Absolute value of ambiguity



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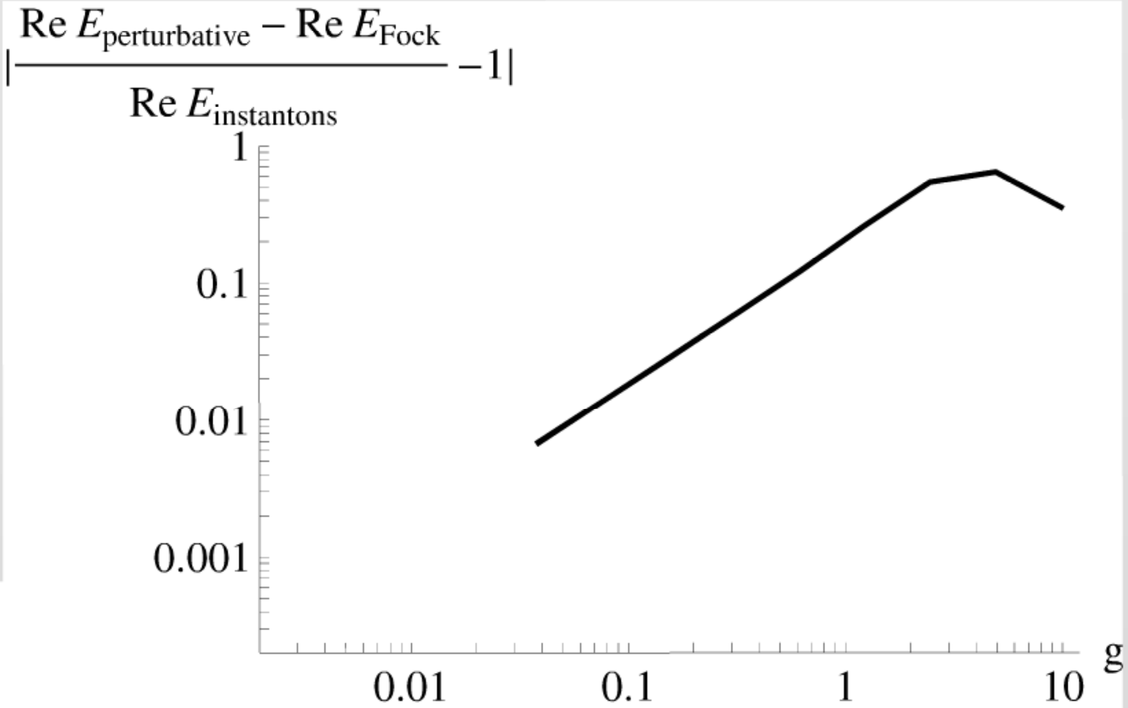

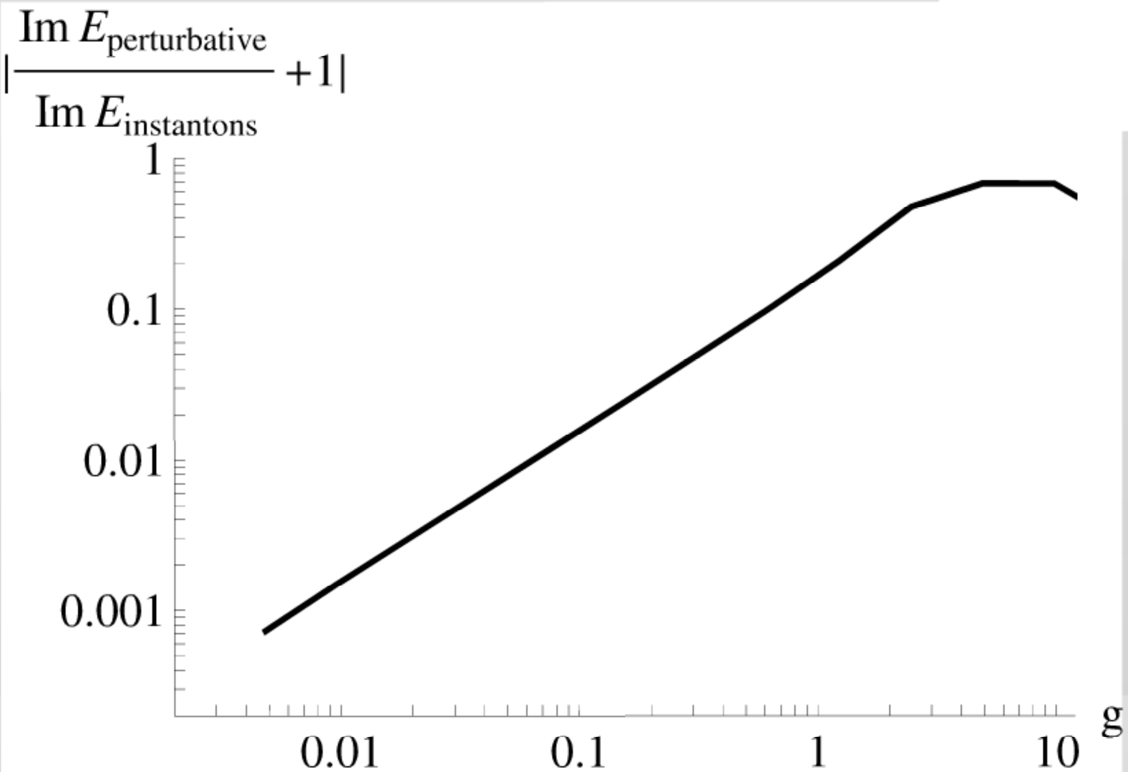

Summary

- Borel sum has imaginary ambiguities
- instantons move them to higher orders
- real part of energy is also improved

Literature

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Thank you for your attention